

## Research Article

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# Weighted Hermite-Hadamard inequalities for $r$ -times differentiable preinvex functions for $k$ -fractional integrals

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**Abstract:** In this article, we have established some new bounds of Fejér-type Hermite-Hadamard inequality for  $k$ -fractional integrals involving  $r$ -times differentiable preinvex functions. It is noteworthy that in the past, there was no weighted version of the left and right sides of the Hermite-Hadamard inequality for  $k$ -fractional integrals for generalized convex functions available in the literature.

**Keywords:** Hermite-Hadamard-Fejér inequality,  $k$ -fractional integral, Preinvex function,  $r$ -times differentiable function

**MSC 2020:** 26D10, 26A51, 26A33

## 1 Introduction

In various disciplines of science, the Hermite-Hadamard inequality for convex functions is studied as it develops a link between convex function theory and integral inequalities. Many generalizations of convex functions have been discovered recently. Researchers have also shown a lot of interest in generalizing this concept for preinvex functions. These inequalities have applications in a variety of areas, including optimization, numerical analysis and statistics. He Chengtian's inequality and the Hermite-Hadamard inequality are frequently used in engineering, particularly in 3D printing technology, to estimate the maximal and lowest printing velocity because it is hard to predict the velocity precisely, see, e.g., [1,2].

Let  $\psi$  be a convex function such that  $\psi : \Omega \subseteq \mathbb{R} \rightarrow \mathbb{R}$  and  $\sigma_a, \sigma_b \in \Omega$  with  $\sigma_a < \sigma_b$ , then

$$\psi\left(\frac{\sigma_a + \sigma_b}{2}\right) \leq \frac{1}{\sigma_b - \sigma_a} \int_{\sigma_a}^{\sigma_b} \psi(t) dt \leq \frac{\psi(\sigma_a) + \psi(\sigma_b)}{2} \quad (1)$$

is the well-known Hermite-Hadamard inequality for convex functions.

The generalization of inequality (1) is given by Fejér [3] as follows:

$$\psi\left(\frac{\sigma_a + \sigma_b}{2}\right) \int_{\sigma_a}^{\sigma_b} \phi(t) dt \leq \int_{\sigma_a}^{\sigma_b} \phi(t) \psi(t) dt \leq \frac{\psi(\sigma_a) + \psi(\sigma_b)}{2} \int_{\sigma_a}^{\sigma_b} \phi(t) dt, \quad (2)$$

where  $\phi : [\sigma_a, \sigma_b] \rightarrow \mathbb{R}$  is a nonnegative, integrable function and symmetric about  $t = \frac{\sigma_a + \sigma_b}{2}$ .

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Due to the wide applications of fractional calculus and Hermite-Hadamard inequalities in different fields of sciences, researchers are working on Hermite-Hadamard-type inequalities for fractional and generalized fractional integrals, see, e.g., [4–9].

In [10], Sarikaya et al. proposed the following inequalities as follows.

**Theorem 1.** Let  $\psi : [\sigma_a, \sigma_b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$  with  $0 \leq \sigma_a < \sigma_b$  and  $\psi \in L[\sigma_a, \sigma_b]$  be a convex function. If  $\psi$  is a positive, integrable function on  $[\sigma_a, \sigma_b]$ , then the following inequalities hold:

$$\psi\left(\frac{\sigma_a + \sigma_b}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(\sigma_b - \sigma_a)^\alpha} [J_{\sigma_b^-}^\alpha \psi(\sigma_a) + J_{\sigma_a^+}^\alpha \psi(\sigma_b)] \leq \frac{\psi(\sigma_a) + \psi(\sigma_b)}{2},$$

with  $\alpha > 0$ .

In [11], Kilbas et al. defined left- and right-sided Riemann-Liouville fractional integrals as follows:

$$J_{\sigma_a^+}^{\mathfrak{p}} \psi(t) = \frac{1}{\Gamma(\mathfrak{p})} \int_{\sigma_a}^t (t - s)^{\mathfrak{p}-1} \psi(s) ds, \quad 0 \leq \sigma_a < t \leq \sigma_b,$$

and

$$J_{\sigma_b^-}^{\mathfrak{p}} \psi(t) = \frac{1}{\Gamma(\mathfrak{p})} \int_t^{\sigma_b} (s - t)^{\mathfrak{p}-1} \psi(s) ds, \quad 0 \leq \sigma_a \leq t < \sigma_b,$$

where  $J_{\sigma_a^+}^{\mathfrak{p}}$  and  $J_{\sigma_b^-}^{\mathfrak{p}}$  represent the left- and right-sided Riemann-Liouville fractional integrals, respectively, of the order  $\mathfrak{p} \in \mathbb{R}^+$ .

We now give the definition of  $k$ -fractional integral, which is mainly due to [12].

**Definition 1.** The left-sided Riemann-Liouville  $k$ -fractional integral of order  $\mathfrak{p}$ ,  $k > 0$  is defined as follows:

$$J_{\sigma_a^+}^{\mathfrak{p},k} \psi(t) = \frac{1}{k\Gamma_k(\mathfrak{p})} \int_{\sigma_a}^t (t - s)^{\frac{\mathfrak{p}}{k}-1} \psi(s) ds, \quad 0 \leq \sigma_a < t \leq \sigma_b,$$

and the right-sided Riemann-Liouville  $k$ -fractional integral of order  $\mathfrak{p}$ ,  $k > 0$  is defined as follows:

$$J_{\sigma_b^-}^{\mathfrak{p},k} \psi(t) = \frac{1}{k\Gamma_k(\mathfrak{p})} \int_t^{\sigma_b} (s - t)^{\frac{\mathfrak{p}}{k}-1} \psi(s) ds, \quad 0 \leq \sigma_a \leq t < \sigma_b,$$

where  $\psi \in L_1([\sigma_a, \sigma_b])$

For  $k = 1$ , the Riemann-Liouville  $k$ -fractional integrals become the Riemann-Liouville fractional integrals (see [11]).

Antczak [13] gave the idea of invex sets as:

**Definition 2.** A set  $\Omega \subseteq \mathbb{R}$  be an invex w.r.t. the map  $\kappa : \Omega \times \Omega \rightarrow \mathbb{R}$  if, for every  $\sigma_a, \sigma_b \in \Omega$  and  $s \in [0, 1]$ ,  $\sigma_b + s\kappa(\sigma_a, \sigma_b) \in \Omega$ .

The generalization of convex functions is given by Weir and Mond [14].

**Definition 3.** Let  $\Omega \subseteq \mathbb{R}$  is an invex set and  $\psi : \Omega \rightarrow \mathbb{R}$  is called a preinvex function w.r.t.  $\kappa$  if

$$\psi(\sigma_b + s\kappa(\sigma_a, \sigma_b)) \leq t\psi(\sigma_a) + (1 - s)\psi(\sigma_b)$$

$\forall \sigma_a, \sigma_b \in \Omega$  and  $s \in [0, 1]$ .

If  $\kappa(\sigma_a, \sigma_b) = \sigma_a - \sigma_b$ , then in the classical sense, the preinvex functions become convex functions.

The situation of  $n$ -times differentiable preinvex functions has been added to the scope of these inequalities. Preinvex functions have constraints on their integrals that rely on the values of the function and its

derivatives at the ends of the interval. These bounds are provided by the Hermite-Hadamard and Fejér inequalities, see, e.g., [15–21].

The following lemma for  $n$ -times differentiable preinvex functions is proposed by Mehmood et al. (see [18]).

**Lemma 1.** Let  $\Omega \subseteq [0, \infty)$  be an open invex subset with respect to  $\kappa : \Omega \times \Omega \rightarrow \mathbb{R}$ . Suppose  $\psi : \Omega \rightarrow \mathbb{R}$  is a function such that  $\psi^{(n)}$  exists on  $\Omega$  and  $\psi^{(n)}$  is integrable on  $[\sigma_a, \sigma_a + \kappa(\sigma_b, \sigma_a)]$  for  $n \in \mathbb{N}$ ,  $n \geq 1$ , then for every  $\sigma_a, \sigma_b \in \Omega$  with  $\kappa(\sigma_b, \sigma_a) > 0$ , the following equality holds:

$$\begin{aligned} & \frac{\psi(\sigma_a) + \psi(\sigma_a + \kappa(\sigma_b, \sigma_a))}{2} - \frac{\Gamma(\mathfrak{p}+1)}{2(\kappa(\sigma_b, \sigma_a))^{\mathfrak{p}}} [J_{\sigma_a}^{\mathfrak{p},k} \psi(\sigma_a + \kappa(\sigma_b, \sigma_a)) + J_{(\sigma_a + \kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{p},k} \psi(\sigma_a)] \\ &= \sum_{\kappa=1}^{n-1} \frac{\Gamma(\mathfrak{p}+1)(\kappa(\sigma_b, \sigma_a))^{\kappa}}{2 \Gamma(\mathfrak{p}+\kappa+1)} [(-1)^{\kappa-1} \psi^{(\kappa)}(\sigma_a + \kappa(\sigma_b, \sigma_a)) - \psi^{(\kappa)}(\sigma_a)] - \frac{(\kappa(\sigma_b, \sigma_a))^n \Gamma(\mathfrak{p}+1)}{2 \Gamma(\mathfrak{p}+n)} \\ & \quad \times \int_0^1 [(1-s)^{\mathfrak{p}+n-1} + (-1)^n s^{\mathfrak{p}+n-1}] \psi^{(n)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds, \end{aligned}$$

where  $\mathfrak{p} > 0$  and  $n \geq 1$ .

In this article, we have developed new Fejér-type Hermite-Hadamard identities for higher-order differentiable generalized convex functions for  $k$ -fractional integrals. Then, we have developed both the left- and right-hand sides of weighted Hermite-Hadamard inequalities.

## 2 Main results

In the main section, we make the assumption  $\|\phi\|_{\infty} = \sup_{t \in [\sigma_a, \sigma_a + \kappa(\sigma_b, \sigma_a)]} |\phi(t)|$ , where  $\phi : [\sigma_a, \sigma_a + \kappa(\sigma_b, \sigma_a)] \rightarrow \mathbb{R}$  is a continuous function,  $\psi^{(r)}$  is the  $r$ th derivative of  $\psi$  w.r.t. variable  $s$  and  $L[\sigma_a, \sigma_b]$  is the collection of all real-valued Riemann-integrable functions defined on the interval  $[\sigma_a, \sigma_b]$ .

**Lemma 2.** Let  $\Omega \subseteq \mathbb{R}$  be an open invex set and  $\kappa$  be a function such that  $\kappa : \Omega \times \Omega \rightarrow \mathbb{R}$ . Suppose,  $\psi : \Omega \rightarrow \mathbb{R}$  is a differentiable mapping such that  $\psi^{(r)} \in L[\sigma_a, \sigma_a + \kappa(\sigma_b, \sigma_a)]$ , where  $\kappa(\sigma_b, \sigma_a) > 0$ . If  $w : [\sigma_a, \sigma_a + \kappa(\sigma_b, \sigma_a)] \rightarrow [0, \infty)$  is an integrable mapping, then  $\forall \sigma_a, \sigma_b \in \Omega$ , we have the following equality:

$$\begin{aligned} & \sum_{m=0}^{r-1} \frac{\psi^{(m)}\left(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)\right)}{2^m \kappa(\sigma_b, \sigma_a)^{\frac{\mathfrak{p}}{k} + r - m}} \left[ (-1)^{r-m-1} J_{\left(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)\right)^-}^{\mathfrak{p},k} \phi(\sigma_a) + (-1)^{r+1} J_{\left(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)\right)^+}^{\mathfrak{p},k} \phi(\sigma_a + \kappa(\sigma_b, \sigma_a)) \right] \\ & + (-1)^r \frac{1}{(\kappa(\sigma_b, \sigma_a))^{\frac{\mathfrak{p}}{k} + r}} \left[ J_{\left(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)\right)^-}^{\mathfrak{p},k} (\phi\psi)(\sigma_a) + J_{\left(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)\right)^+}^{\mathfrak{p},k} (\phi\psi)(\sigma_a + \kappa(\sigma_b, \sigma_a)) \right] \\ &= \frac{1}{k\Gamma_k(\mathfrak{p})} \int_0^1 w(s) \psi^{(r)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds, \end{aligned} \quad (3)$$

where

$$w(s) = \begin{cases} \underbrace{\int_0^s \dots \int_0^s}_{r \text{ integrals}} u^{\frac{\mathfrak{p}}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^r, & s \in \left[0, \frac{1}{2}\right] \\ \underbrace{\int_s^{\sigma_a + \kappa(\sigma_b, \sigma_a)} \dots \int_s^{\sigma_a + \kappa(\sigma_b, \sigma_a)}}_{r \text{ integrals}} (1-u)^{\frac{\mathfrak{p}}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^r, & s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

**Proof.** First, we consider

$$\begin{aligned} \int_0^1 w(s) \psi^{(r)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds &= \int_0^{\frac{1}{2}} \left( \underbrace{\int_0^s \int_0^s \dots \int_0^s}_{r \text{ integrals}} u^{\frac{p}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^r \right) \times \psi^{(r)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \\ &\quad + \int_{\frac{1}{2}}^1 \left( \underbrace{\int_0^s \int_0^s \dots \int_0^s}_{r \text{ integrals}} (1-u)^{\frac{p}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^r \right) \psi^{(r)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \\ &= I_1 + I_2. \end{aligned} \quad (4)$$

From the first integral, we have

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{2}} \left( \underbrace{\int_0^s \int_0^s \dots \int_0^s}_{r \text{ integrals}} u^{\frac{p}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^r \right) \psi^{(r)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \\ &= \frac{1}{\kappa(\sigma_b, \sigma_a)} \left( \underbrace{\int_0^s \int_0^s \dots \int_0^s}_{r \text{ integrals}} u^{\frac{p}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^r \right) \psi^{(r-1)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) \Big|_0^{\frac{1}{2}} \\ &\quad - \frac{1}{\kappa(\sigma_b, \sigma_a)} \int_0^{\frac{1}{2}} \left( \underbrace{\int_0^s \int_0^s \dots \int_0^s}_{r-1 \text{ integrals}} u^{\frac{p}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^{r-1} \right) \psi^{(r-1)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds. \\ I_1 &= \frac{\psi^{(r-1)}(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))}{\kappa(\sigma_b, \sigma_a)} \left( \underbrace{\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \dots \int_0^{\frac{1}{2}}}_{r \text{ integrals}} s^{\frac{p}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) (ds)^r \right) \\ &\quad - \frac{1}{\kappa(\sigma_b, \sigma_a)} \int_0^{\frac{1}{2}} \left( \underbrace{\int_0^s \int_0^s \dots \int_0^s}_{r-1 \text{ integrals}} u^{\frac{p}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^{r-1} \right) \psi^{(r-1)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds. \end{aligned}$$

After generalization, we obtain

$$\begin{aligned}
I_1 = & \frac{\psi^{(r-1)}\left(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)\right)}{\kappa(\sigma_b, \sigma_a)} \left[ \underbrace{\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \dots \int_0^{\frac{1}{2}}}_{r \text{ integrals}} s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) (ds)^r \right] \\
& - \frac{\psi^{(r-2)}(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))}{(\kappa(\sigma_b, \sigma_a))^2} \left[ \underbrace{\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \dots \int_0^{\frac{1}{2}}}_{r-1 \text{ integrals}} s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) (ds)^{r-1} \right] \\
& \vdots \\
& \vdots \\
& + (-1)^{r-2} \frac{\psi'(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))}{(\kappa(\sigma_b, \sigma_a))^{r-1}} \left[ \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) (ds)^2 \right] \\
& + (-1)^{r-1} \frac{\psi(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))}{(\kappa(\sigma_b, \sigma_a))^r} \left[ \int_0^{\frac{1}{2}} s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \right] \\
& + (-1)^r \frac{1}{(\kappa(\sigma_b, \sigma_a))^r} \left[ \int_0^{\frac{1}{2}} s^{\frac{b}{k}-1} \psi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \right].
\end{aligned}$$

After simplification, we obtain

$$\begin{aligned}
I_1 = & \frac{\psi^{(r-1)}\left(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)\right)}{2^{r-1}\kappa(\sigma_b, \sigma_a)} \left[ \int_0^{\frac{1}{2}} s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \right] \\
& - \frac{\psi^{(r-2)}(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))}{2^{r-2}(\kappa(\sigma_b, \sigma_a))^2} \left[ \int_0^{\frac{1}{2}} s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \right] \\
& \vdots \\
& \vdots \\
& + (-1)^{r-2} \frac{\psi'(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))}{2(\kappa(\sigma_b, \sigma_a))^{r-1}} \left[ \int_0^{\frac{1}{2}} s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \right] \\
& + (-1)^{r-1} \frac{\psi\left(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)\right)}{(\kappa(\sigma_b, \sigma_a))^r} \left[ \int_0^{\frac{1}{2}} s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \right] \\
& + (-1)^r \frac{1}{(\kappa(\sigma_b, \sigma_a))^r} \left[ \int_0^{\frac{1}{2}} s^{\frac{b}{k}-1} \psi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \right].
\end{aligned}$$

On substituting  $t = \sigma_a + s\kappa(\sigma_b, \sigma_a)$ , we obtain

$$\begin{aligned}
I_1 = & \frac{\psi^{(r-1)}\left(\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)\right)}{2^{r-1}\mathcal{N}(\sigma_b, \sigma_a)^{\frac{b}{k}+1}} \int_{\sigma_a}^{\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)} (t - \sigma_a)^{\frac{b}{k}-1} \phi(t) dt \\
& - \frac{\psi^{(r-2)}\left(\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)\right)}{2^{r-2}(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+2}} \int_{\sigma_a}^{\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)} (t - \sigma_a)^{\frac{b}{k}-1} \phi(t) dt \\
& \vdots \\
& + (-1)^{r-2} \frac{\psi'\left(\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)\right)}{2(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+r-1}} \int_{\sigma_a}^{\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)} (t - \sigma_a)^{\frac{b}{k}-1} \phi(t) dt \\
& + (-1)^{r-1} \frac{\psi\left(\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)\right)}{(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+r}} \int_{\sigma_a}^{\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)} (t - \sigma_a)^{\frac{b}{k}-1} \phi(t) dt \\
& + (-1)^r \frac{1}{(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+r}} \int_{\sigma_a}^{\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)} (t - \sigma_a)^{\frac{b}{k}-1} \psi(t) \phi(t) dt.
\end{aligned}$$

From the definition of  $k$ -fractional integrals, we have

$$\begin{aligned}
I_1 = & \frac{\psi^{(r-1)}\left(\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)\right) k\Gamma_k(\mathfrak{p})}{2^{r-1}\mathcal{N}(\sigma_b, \sigma_a)^{\frac{b}{k}+1}} J_{\left[\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)\right]^-}^{\mathfrak{p}, k} \phi(\sigma_a) \\
& - \frac{\psi^{(r-2)}\left(\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)\right) k\Gamma_k(\mathfrak{p})}{2^{r-2}(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+2}} J_{\left[\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)\right]^-}^{\mathfrak{p}, k} \phi(\sigma_a) \\
& \vdots \\
& + (-1)^{r-2} \frac{\psi'\left(\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)\right) k\Gamma_k(\mathfrak{p})}{2(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+r-1}} J_{\left(\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)\right)^-}^{\mathfrak{p}, k} \phi(\sigma_a) \\
& + (-1)^{r-1} \frac{\psi\left(\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)\right) k\Gamma_k(\mathfrak{p})}{(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+r}} J_{\left(\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)\right)^-}^{\mathfrak{p}, k} \phi(\sigma_a) \\
& + (-1)^r \frac{k\Gamma_k(\mathfrak{p})}{(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+r}} J_{\left(\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)\right)^-}^{\mathfrak{p}, k} (\phi\psi)(\sigma_a).
\end{aligned}$$

After summing the above series, we obtain

$$\begin{aligned}
I_1 = & \sum_{m=0}^{r-1} (-1)^{r-m-1} \frac{\psi^{(m)}\left(\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)\right)}{2^m \mathcal{N}(\sigma_b, \sigma_a)^{\frac{b}{k}+r-m}} k\Gamma_k(\mathfrak{p}) J_{\left(\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)\right)^-}^{\mathfrak{p}, k} \phi(\sigma_a) \\
& + (-1)^r \frac{1}{(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+r}} k\Gamma_k(\mathfrak{p}) J_{\left(\sigma_a + \frac{1}{2}\mathcal{N}(\sigma_b, \sigma_a)\right)^-}^{\mathfrak{p}, k} (\phi\psi)(\sigma_a).
\end{aligned} \tag{5}$$

Similarly from the second integral, we have

$$\begin{aligned}
I_2 = & (-1)^{r+1} \sum_{m=0}^{r-1} \frac{\psi^{(m)}\left(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)\right)}{2^m \kappa(\sigma_b, \sigma_a)^{\frac{p}{k}+r-m}} k\Gamma_k(p) J_{\left(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)\right)^+}^{p,k} \phi(\sigma_a + \kappa(\sigma_b, \sigma_a)) \\
& + (-1)^r \frac{1}{(\kappa(\sigma_b, \sigma_a))^{\frac{p}{k}+r}} k\Gamma_k(p) J_{\left(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)\right)^+}^{p,k} (\phi\psi)(\sigma_a + \kappa(\sigma_b, \sigma_a)).
\end{aligned} \quad (6)$$

On utilizing equations (3) and (5) in equation (4), we obtain the required result.  $\square$

**Lemma 3.** For  $k = r = 1$  in equation (3), we obtain Lemma 1 of [5].

$$\begin{aligned}
& \frac{\psi\left(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)\right)}{\kappa(\sigma_b, \sigma_a)^{p+1}} \left[ J_{\left(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)\right)^-}^p \phi(\sigma_a) + J_{\left(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)\right)^+}^p \phi(\sigma_a + \kappa(\sigma_b, \sigma_a)) \right] \\
& - \frac{1}{(\kappa(\sigma_b, \sigma_a))^{p+1}} \left[ J_{\left(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)\right)^-}^p (\phi\psi)(\sigma_a) + J_{\left(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)\right)^+}^p (\phi\psi)(\sigma_a + \kappa(\sigma_b, \sigma_a)) \right] \\
& = \frac{1}{\Gamma(p)} \int_0^1 w(s) \psi'(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds,
\end{aligned}$$

where

$$w(s) = \begin{cases} \int_0^s u^{p-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) du, & s \in \left[0, \frac{1}{2}\right] \\ \int_1^s (1-u)^{p-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) du, & s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

**Lemma 4.** If  $\psi : [\sigma_a, \sigma_a + \kappa(\sigma_b, \sigma_a)] \rightarrow \mathbb{R}$  is an integrable function, which is symmetric about  $\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)$  with  $\sigma_a < \sigma_a + \kappa(\sigma_b, \sigma_a)$ , then we have

$$J_{\sigma_a^+}^{p,k} \psi(\sigma_a + \kappa(\sigma_b, \sigma_a)) = J_{(\sigma_a + \kappa(\sigma_b, \sigma_a))^-}^{p,k} \psi(\sigma_a) = \frac{1}{2} [J_{\sigma_a^+}^{p,k} \psi(\sigma_a + \kappa(\sigma_b, \sigma_a)) + J_{(\sigma_a + \kappa(\sigma_b, \sigma_a))^-}^{p,k} \psi(\sigma_a)], \quad (7)$$

where  $p > 0$

**Proof.** Since  $\psi$  is symmetric about  $\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)$ , we have  $\psi(2\sigma_a + \kappa(\sigma_b, \sigma_a) - t) = \psi(t)$ , for all  $t \in [\sigma_a, \sigma_a + \kappa(\sigma_b, \sigma_a)]$ . Taking  $2\sigma_a + \kappa(\sigma_b, \sigma_a) - s = t$

$$\begin{aligned}
J_{\sigma_a^+}^{p,k} \psi(\sigma_a + \kappa(\sigma_b, \sigma_a)) &= \frac{1}{k\Gamma_k(p)} \int_{\sigma_a}^{\sigma_a + \kappa(\sigma_b, \sigma_a)} [(\sigma_a + \kappa(\sigma_b, \sigma_a)) - s]^{\frac{p}{k}-1} \psi(s) ds \\
&= \frac{1}{k\Gamma_k(p)} \int_{\sigma_a}^{\sigma_a + \kappa(\sigma_b, \sigma_a)} (t - \sigma_a)^{\frac{p}{k}-1} \psi(2\sigma_a + \kappa(\sigma_b, \sigma_a) - t) dt \\
&= \frac{1}{k\Gamma_k(p)} \int_{\sigma_a}^{\sigma_a + \kappa(\sigma_b, \sigma_a)} (t - \sigma_a)^{\frac{p}{k}-1} \psi(t) dt \\
&= J_{(\sigma_a + \kappa(\sigma_b, \sigma_a))^-}^{p,k} \psi(\sigma_a) \\
&= \frac{1}{2} [J_{\sigma_a^+}^{p,k} \psi(\sigma_a + \kappa(\sigma_b, \sigma_a)) + J_{(\sigma_a + \kappa(\sigma_b, \sigma_a))^-}^{p,k} \psi(\sigma_a)].
\end{aligned} \quad \square$$

**Lemma 5.** Let  $\Omega \subseteq \mathbb{R}$  be an open invex set and  $\kappa$  be a function such that  $\kappa : \Omega \times \Omega \rightarrow \mathbb{R}$ . Suppose  $\psi : \Omega \rightarrow \mathbb{R}$  is a differentiable mapping such that  $\psi^{(r)} \in L[\sigma_a, \sigma_a + \kappa(\sigma_b, \sigma_a)]$ , where  $\kappa(\sigma_b, \sigma_a) > 0$ . If  $w : [\sigma_a, \sigma_a + \kappa(\sigma_b, \sigma_a)] \rightarrow [0, \infty)$  is an integrable mapping, then  $\forall \sigma_a, \sigma_b \in \Omega$ , we have the following equality:

$$\begin{aligned}
& \sum_{m=0}^{r-1} \left[ \frac{(-1)^{r+1} \psi^{(m)}(\sigma_a) + (-1)^{r-m-1} \psi^{(m)}(\sigma_a + \kappa(\sigma_b, \sigma_a))}{2(\kappa(\sigma_b, \sigma_a))^{\mathfrak{b}+r-m}} \right] \\
& \quad \times [J_{(\sigma_a + \kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{b}, k} \phi(\sigma_a) + J_{\sigma_a^+}^{\mathfrak{b}, k} \phi(\sigma_a + \kappa(\sigma_b, \sigma_a))] \\
& \quad + (-1)^r \left[ \frac{J_{(\sigma_a + \kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{b}, k} (\phi\psi)(\sigma_a) + J_{\sigma_a^+}^{\mathfrak{b}, k} (\phi\psi)(\sigma_a + \kappa(\sigma_b, \sigma_a))}{(\kappa(\sigma_b, \sigma_a))^{\mathfrak{b}+r}} \right] \\
& = \frac{1}{k\Gamma_k(\mathfrak{b})} \int_0^1 v(s) \psi^{(r)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds,
\end{aligned} \tag{8}$$

where

$$v(s) = \underbrace{\int_0^s \int_0^s \dots \int_0^s u^{\frac{\mathfrak{b}}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^r}_{r \text{ integrals}} + \underbrace{\int_0^s \int_0^s \dots \int_0^s (1-u)^{\frac{\mathfrak{b}}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^r}_{r \text{ integrals}}, \quad s \in [0, 1].$$

**Proof.** First, we consider

$$\begin{aligned}
& \int_0^1 v(s) \psi^{(r)}(\sigma_a, \sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \\
& = \int_0^1 \left( \underbrace{\int_0^s \int_0^s \dots \int_0^s u^{\frac{\mathfrak{b}}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^r}_{r \text{ integrals}} \right) \psi^{(r)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \\
& \quad + \int_0^1 \left( \underbrace{\int_0^s \int_0^s \dots \int_0^s (1-u)^{\frac{\mathfrak{b}}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^r}_{r \text{ integrals}} \right) \psi^{(r)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \\
& = I_1 + I_2.
\end{aligned} \tag{9}$$

From the first integral, we have

$$\begin{aligned}
I_1 & = \int_0^1 \left( \underbrace{\int_0^s \int_0^s \dots \int_0^s u^{\frac{\mathfrak{b}}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^r}_{r \text{ integrals}} \right) \psi^{(r)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \\
& = \frac{1}{\kappa(\sigma_b, \sigma_a)} \left( \underbrace{\int_0^s \int_0^s \dots \int_0^s u^{\frac{\mathfrak{b}}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^r}_{r \text{ integrals}} \right) \psi^{(r-1)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) \Big|_0^1 \\
& \quad - \frac{1}{\kappa(\sigma_b, \sigma_a)} \int_0^1 \left( \underbrace{\int_0^s \int_0^s \dots \int_0^s u^{\frac{\mathfrak{b}}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^r}_{r-1 \text{ integrals}} \right) \psi^{(r-1)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds
\end{aligned}$$

$$I_1 = \frac{\psi^{(r-1)}(\sigma_a + \kappa(\sigma_b, \sigma_a))}{\kappa(\sigma_b, \sigma_a)} \left( \underbrace{\int_0^1 \int_0^1 \dots \int_0^1}_{r \text{ integrals}} s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) (ds)^r \right) \\ - \frac{1}{\kappa(\sigma_b, \sigma_a)} \int_0^1 \left( \underbrace{\int_0^s \int_0^s \dots \int_0^s}_{r-1 \text{ integrals}} u^{\frac{b}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^{r-1} \right) \psi^{(r-1)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds.$$

On generalizing the result, we have

$$I_1 = \frac{\psi^{(r-1)}(\sigma_a + \kappa(\sigma_b, \sigma_a))}{\kappa(\sigma_b, \sigma_a)} \left( \underbrace{\int_0^1 \int_0^1 \dots \int_0^1}_{r \text{ integrals}} s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) (ds)^r \right) \\ - \frac{\psi^{(r-2)}(\sigma_a + \kappa(\sigma_b, \sigma_a))}{(\kappa(\sigma_b, \sigma_a))^2} \left( \underbrace{\int_0^1 \int_0^1 \dots \int_0^1}_{r-1 \text{ integrals}} s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) (ds)^{r-1} \right) \\ + \frac{\psi^{(r-3)}(\sigma_a + \kappa(\sigma_b, \sigma_a))}{(\kappa(\sigma_b, \sigma_a))^3} \left( \underbrace{\int_0^1 \int_0^1 \dots \int_0^1}_{r-2 \text{ integrals}} s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) (ds)^{r-2} \right) \\ \vdots \\ \vdots \\ + (-1)^{r-2} \frac{\psi'(\sigma_a + \kappa(\sigma_b, \sigma_a))}{(\kappa(\sigma_b, \sigma_a))^{r-1}} \left( \int_0^1 \int_0^1 s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) (ds)^2 \right) \\ + (-1)^{r-1} \frac{\psi(\sigma_a + \kappa(\sigma_b, \sigma_a))}{(\kappa(\sigma_b, \sigma_a))^r} \left( \int_0^1 s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \right) \\ + (-1)^r \frac{1}{(\kappa(\sigma_b, \sigma_a))^r} \left( \int_0^1 s^{\frac{b}{k}-1} \psi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \right).$$

After simplification, we obtain

$$I_1 = \frac{\psi^{(r-1)}(\sigma_a + \kappa(\sigma_b, \sigma_a))}{\kappa(\sigma_b, \sigma_a)} \left( \int_0^1 s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \right) \\ - \frac{\psi^{(r-2)}(\sigma_a + \kappa(\sigma_b, \sigma_a))}{(\kappa(\sigma_b, \sigma_a))^2} \left( \int_0^1 s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \right) \\ + \frac{\psi^{(r-3)}(\sigma_a + \kappa(\sigma_b, \sigma_a))}{(\kappa(\sigma_b, \sigma_a))^3} \left( \int_0^1 s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \right) \\ \vdots \\ \vdots \\ + (-1)^{r-2} \frac{\psi'(\sigma_a + \kappa(\sigma_b, \sigma_a))}{(\kappa(\sigma_b, \sigma_a))^{r-1}} \left( \int_0^1 s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \right) \\ + (-1)^{r-1} \frac{\psi(\sigma_a + \kappa(\sigma_b, \sigma_a))}{(\kappa(\sigma_b, \sigma_a))^r} \left( \int_0^1 s^{\frac{b}{k}-1} \phi(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \right)$$

$$+ (-1)^r \frac{1}{(\mathcal{N}(\sigma_b, \sigma_a))^r} \left[ \int_0^1 s^{\frac{b}{k}-1} \psi(\sigma_a + s\mathcal{N}(\sigma_b, \sigma_a)) \phi(\sigma_a + s\mathcal{N}(\sigma_b, \sigma_a)) ds \right].$$

After substituting  $t = \sigma_a + s\mathcal{N}(\sigma_b, \sigma_a)$ , we have

$$\begin{aligned} I_1 &= \frac{\psi^{(r-1)}(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a))}{\mathcal{N}(\sigma_b, \sigma_a)^{\frac{b}{k}+1}} \int_{\sigma_a}^{\sigma_a + \mathcal{N}(\sigma_b, \sigma_a)} (t - \sigma_a)^{\frac{b}{k}-1} \phi(t) dt \\ &\quad - \frac{\psi^{(r-2)}(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a))}{(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+2}} \int_{\sigma_a}^{\sigma_a + \mathcal{N}(\sigma_b, \sigma_a)} (t - \sigma_a)^{\frac{b}{k}-1} \phi(t) dt \\ &\quad + \frac{\psi^{(r-3)}(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a))}{(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+3}} \int_{\sigma_a}^{\sigma_a + \mathcal{N}(\sigma_b, \sigma_a)} (t - \sigma_a)^{\frac{b}{k}-1} \phi(t) dt \\ &\quad \vdots \\ &\quad \vdots \\ &\quad \vdots \\ &\quad + (-1)^{r-2} \frac{\psi'(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a))}{(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+r-1}} \int_{\sigma_a}^{\sigma_a + \mathcal{N}(\sigma_b, \sigma_a)} (t - \sigma_a)^{\frac{b}{k}-1} \phi(t) dt \\ &\quad + (-1)^{r-1} \frac{\psi(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a))}{(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+r}} \int_{\sigma_a}^{\sigma_a + \mathcal{N}(\sigma_b, \sigma_a)} (t - \sigma_a)^{\frac{b}{k}-1} \phi(t) dt \\ &\quad + (-1)^r \frac{1}{(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+r}} \int_{\sigma_a}^{\sigma_a + \mathcal{N}(\sigma_b, \sigma_a)} (t - \sigma_a)^{\frac{b}{k}-1} \psi(t) \phi(t) dt. \end{aligned}$$

Using the definition of  $k$ -fractional integral,

$$\begin{aligned} I_1 &= \frac{\psi^{(r-1)}(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a)) k \Gamma_k(\frac{b}{k})}{\mathcal{N}(\sigma_b, \sigma_a)^{\frac{b}{k}+1}} J_{(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a))-, \frac{b}{k}}^{\frac{b}{k}, k} \phi(\sigma_a) \\ &\quad - \frac{\psi^{(r-2)}(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a)) k \Gamma_k(\frac{b}{k})}{(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+2}} J_{(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a))-, \frac{b}{k}}^{\frac{b}{k}, k} \phi(\sigma_a) \\ &\quad + \frac{\psi^{(r-3)}(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a)) k \Gamma_k(\frac{b}{k})}{(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+3}} J_{(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a))-, \frac{b}{k}}^{\frac{b}{k}, k} \phi(\sigma_a) \\ &\quad \vdots \\ &\quad \vdots \\ &\quad \vdots \\ &\quad + (-1)^{r-2} \frac{\psi'(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a)) k \Gamma_k(\frac{b}{k})}{(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+r-1}} J_{(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a))-, \frac{b}{k}}^{\frac{b}{k}, k} \phi(\sigma_a) \\ &\quad + (-1)^{r-1} \frac{\psi(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a)) k \Gamma_k(\frac{b}{k})}{(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+r}} J_{(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a))-, \frac{b}{k}}^{\frac{b}{k}, k} \phi(\sigma_a) \\ &\quad + (-1)^r \frac{k \Gamma_k(\frac{b}{k})}{(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+r}} J_{(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a))-, \frac{b}{k}}^{\frac{b}{k}, k} (\phi \psi)(\sigma_a). \end{aligned}$$

After summing the above series, we obtain

$$\begin{aligned} I_1 &= \sum_{m=0}^{r-1} (-1)^{r-m-1} \frac{\psi^{(m)}(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a))}{\mathcal{N}(\sigma_b, \sigma_a)^{\frac{b}{k}+r-m}} k \Gamma_k(\frac{b}{k}) J_{(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a))-, \frac{b}{k}}^{\frac{b}{k}, k} \phi(\sigma_a) \\ &\quad + (-1)^r \frac{1}{(\mathcal{N}(\sigma_b, \sigma_a))^{\frac{b}{k}+r}} k \Gamma_k(\frac{b}{k}) J_{(\sigma_a + \mathcal{N}(\sigma_b, \sigma_a))-, \frac{b}{k}}^{\frac{b}{k}, k} (\phi \psi)(\sigma_a). \end{aligned} \quad (10)$$

Similarly, from the second integral, we obtain

$$\begin{aligned}
I_2 = & (-1)^{r+1} \sum_{m=0}^{r-1} \frac{\psi^{(m)}(\sigma_a)}{\kappa(\sigma_b, \sigma_a)^{\frac{b}{k}+r-m}} k\Gamma_k(\mathfrak{p}) J_{\sigma_a^+}^{b,k} \phi(\sigma_a + \kappa(\sigma_b, \sigma_a)) \\
& + (-1)^r \frac{1}{(\kappa(\sigma_b, \sigma_a))^{\frac{b}{k}+r}} k\Gamma_k(\mathfrak{p}) J_{\sigma_a^+}^{b,k} (\phi\psi)(\sigma_a + \kappa(\sigma_b, \sigma_a)).
\end{aligned} \quad (11)$$

On utilizing equations (10) and (11) in equation (9) and using Lemma 4, we obtain the required result.  $\square$

**Remark 1.** For  $k = r = 1$  in equation (8), we obtain Lemma 3 of [5].

$$\begin{aligned}
& \left[ \frac{\psi(\sigma_a) + \psi(\sigma_a + \kappa(\sigma_b, \sigma_a))}{2(\kappa(\sigma_b, \sigma_a))^{\mathfrak{b}+1}} \left[ J_{(\sigma_a + \kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{b}} \phi(\sigma_a) + J_{\sigma_a^+}^{\mathfrak{b}} \phi(\sigma_a + \kappa(\sigma_b, \sigma_a)) \right] \right. \\
& \quad \left. - \left[ \frac{J_{(\sigma_a + \kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{b}} (\phi\psi)(\sigma_a) + J_{\sigma_a^+}^{\mathfrak{b}} (\phi\psi)(\sigma_a + \kappa(\sigma_b, \sigma_a))}{(\kappa(\sigma_b, \sigma_a))^{\mathfrak{b}+1}} \right] \right] \\
& = \frac{1}{\Gamma(\mathfrak{b})} \int_0^1 v(s) \psi'(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds,
\end{aligned}$$

where

$$v(s) = \begin{cases} \int_0^s u^{\mathfrak{b}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) du \\ + \int_1^s (1-u)^{\mathfrak{b}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) du, \quad s \in [0, 1]. \end{cases}$$

**Theorem 2.** Let  $\Omega \subseteq \mathbb{R}$  be an open invex set and  $\kappa$  be a function such that  $\kappa : \Omega \times \Omega \rightarrow \mathbb{R}^{\mathfrak{b}}$ . Suppose,  $\psi : \Omega \rightarrow \mathbb{R}$  is a differentiable mapping such that  $\psi^{(r)} \in L[\sigma_a, \sigma_a + \kappa(\sigma_b, \sigma_a)]$ , where  $\kappa(\sigma_b, \sigma_a) > 0$ . If there is an integral mapping such that  $\phi : [\sigma_a, \sigma_a + \kappa(\sigma_b, \sigma_a)] \rightarrow [0, \infty)$  and it is also symmetric with respect to  $\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)$ . Let  $|\psi^{(r)}|$  be a preinvex function on  $\Omega$ , then  $\forall \sigma_a, \sigma_b \in \Omega$ , we have the following inequality:

$$\begin{aligned}
& \left| \sum_{m=0}^{r-1} \frac{\psi^{(m)}(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))}{2^m \kappa(\sigma_b, \sigma_a)^{\frac{b}{k}+r-m}} \left[ (-1)^{r-m-1} J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^-}^{b,k} \phi(\sigma_a) + (-1)^{r+1} J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^+}^{b,k} \phi(\sigma_a + \kappa(\sigma_b, \sigma_a)) \right] \right. \\
& \quad \left. + (-1)^r \frac{1}{(\kappa(\sigma_b, \sigma_a))^{\frac{b}{k}+r}} \left[ J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^-}^{b,k} (\phi\psi)(\sigma_a) + J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^+}^{b,k} (\phi\psi)(\sigma_a + \kappa(\sigma_b, \sigma_a)) \right] \right| \\
& \leq \frac{[|\psi^{(r)}(\sigma_a)| + |\psi^{(r)}(\sigma_b)|] \|\phi\|_{\infty} k^r \Gamma(\mathfrak{p})}{2^{\frac{b}{k}+r} (\mathfrak{p}+k(-1+r)) (\mathfrak{p}+kr) \Gamma_k(\mathfrak{p}) \Gamma(\mathfrak{p}+r-1)}.
\end{aligned} \quad (12)$$

**Proof.** On taking the modulus of both sides of equation (3), we obtain

$$\begin{aligned}
& \left| \sum_{m=0}^{r-1} \frac{\psi^{(m)}(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))}{2^m \kappa(\sigma_b, \sigma_a)^{\frac{b}{k}+r-m}} \right. \\
& \quad \times \left[ (-1)^{r-m-1} J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^-}^{b,k} \phi(\sigma_a) + (-1)^{r+1} J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^+}^{b,k} \phi(\sigma_a + \kappa(\sigma_b, \sigma_a)) \right] \\
& \quad \left. + (-1)^r \frac{1}{(\kappa(\sigma_b, \sigma_a))^{\frac{b}{k}+r}} \left[ J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{b}} (\phi\psi)(\sigma_a) + J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^+}^{\mathfrak{b}} (\phi\psi)(\sigma_a + \kappa(\sigma_b, \sigma_a)) \right] \right| \\
& = \frac{1}{k\Gamma_k(\mathfrak{p})} \left| \int_0^{1/2} w(s) \psi^{(r)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds + \frac{1}{k\Gamma_k(\mathfrak{p})} \int_{1/2}^1 w(s) \psi^{(r)}(\sigma_a + s\kappa(\sigma_b, \sigma_a)) ds \right|.
\end{aligned}$$

From preinvexity of  $|\psi^{(r)}|$  on  $\Omega$  and using the fact  $\|\phi\|_\infty = \sup_{t \in [\sigma_a, \sigma_b]} |\phi(t)|$ , we have

$$\begin{aligned}
 & \left| \sum_{m=0}^{r-1} \frac{\psi^{(m)}(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))}{2^m \kappa(\sigma_b, \sigma_a)^{\frac{p}{k} + r - m}} \left[ (-1)^{r-m-1} J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^-}^{p,k} \phi(\sigma_a) + (-1)^{r+1} J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^+}^{p,k} \phi(\sigma_a + \kappa(\sigma_b, \sigma_a)) \right] \right. \\
 & \quad \left. + (-1)^r \frac{1}{(\kappa(\sigma_b, \sigma_a))^{\frac{p}{k} + r}} \left[ J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^-}^{p,k} (\phi\psi)(\sigma_a) + J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^+}^{p,k} (\phi\psi)(\sigma_a + \kappa(\sigma_b, \sigma_a)) \right] \right| \\
 & \leq \frac{\|\phi\|_\infty}{k\Gamma_k(p)} \int_0^{1/2} \left| \underbrace{\int_0^s \int_0^s \dots \int_0^s u^{\frac{p}{k}-1} (du)^r}_{r \text{ integrals}} \right| [(1-s)|\psi^{(r)}(\sigma_a)| + s|\psi^{(r)}(\sigma_b)|] ds \\
 & \quad + \frac{\|\phi\|_\infty}{k\Gamma_k(p)} \int_{1/2}^1 \left| \underbrace{\int_1^s \int_1^s \dots \int_1^s (1-u)^{\frac{p}{k}-1} (du)^r}_{r \text{ integrals}} \right| [(1-s)|\psi^{(r)}(\sigma_a)| + s|\psi^{(r)}(\sigma_b)|] ds \\
 & = I_1 + I_2.
 \end{aligned} \tag{13}$$

From the first term of equation (13), we have

$$\begin{aligned}
 I_1 &= \frac{\|\phi\|_\infty}{k\Gamma_k(p)} \int_0^{1/2} \left| \underbrace{\int_0^s \int_0^s \dots \int_0^s u^{\frac{p}{k}-1} (du)^r}_{r \text{ integrals}} \right| [(1-s)|\psi^{(r)}(\sigma_a)| + s|\psi^{(r)}(\sigma_b)|] ds \\
 &= \frac{k^{r-2} \|\phi\|_\infty \Gamma(p)}{\Gamma_k(p) \Gamma(p+r-1)} \int_0^{1/2} u^{\frac{p}{k} + r - 2} \int_u^{1/2} [(1-s)|\psi^{(r)}(\sigma_a)| + s|\psi^{(r)}(\sigma_b)|] ds du \\
 &= \frac{k^{r-2} \|\phi\|_\infty \Gamma(p)}{\Gamma_k(p) \Gamma(p+r-1)} \int_0^{1/2} u^{\frac{p}{k} + r - 2} \left[ |\psi^{(r)}(\sigma_a)| \left( \frac{(1-u)^2}{2} - \frac{1}{8} \right) + |\psi^{(r)}(\sigma_b)| \left( \frac{1}{8} - \frac{u^2}{2} \right) \right] du.
 \end{aligned}$$

Making the change of variable  $t = \sigma_a + u\kappa(\sigma_b, \sigma_a)$  for  $u \in [0, 1]$ ,

$$\begin{aligned}
 I_1 &= \frac{k^{r-2} \|\phi\|_\infty \Gamma(p)}{\Gamma_k(p) \Gamma(p+r-1)} \frac{|\psi^{(r)}(\sigma_a)|}{\kappa(\sigma_b, \sigma_a)} \times \int_{\sigma_a}^{\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)} \left( \frac{1}{2} \left( 1 - \frac{t - \sigma_a}{\kappa(\sigma_b, \sigma_a)} \right)^2 - \frac{1}{8} \left( \frac{t - \sigma_a}{\kappa(\sigma_b, \sigma_a)} \right)^{\frac{p}{k} + r - 2} \right) dt \\
 & \quad + \frac{k^{r-2} \|\phi\|_\infty \Gamma(p)}{\Gamma_k(p) \Gamma(p+r-1)} \frac{|\psi^{(r)}(\sigma_b)|}{\kappa(\sigma_b, \sigma_a)} \times \int_{\sigma_a}^{\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)} \left( \frac{1}{8} - \frac{1}{2} \left( \frac{t - \sigma_a}{\kappa(\sigma_b, \sigma_a)} \right)^2 \right) \left( \frac{t - \sigma_a}{\kappa(\sigma_b, \sigma_a)} \right)^{\frac{p}{k} + r - 2} dt.
 \end{aligned} \tag{14}$$

From the second term of equation (13), we have

$$\begin{aligned}
 I_2 &= \frac{\|\phi\|_\infty}{k\Gamma_k(p)} \int_{1/2}^1 \left| \underbrace{\int_1^s \int_1^s \dots \int_1^s (1-u)^{\frac{p}{k}-1} (du)^r}_{r \text{ integrals}} \right| [(1-s)|\psi^{(r)}(\sigma_a)| + s|\psi^{(r)}(\sigma_b)|] ds \\
 &= \frac{k^{r-2} \|\phi\|_\infty \Gamma(p)}{\Gamma_k(p) \Gamma(p+r-1)} \int_{1/2}^1 \int_1^s (1-u)^{\frac{p}{k} + r - 2} du [(1-s)|\psi^{(r)}(\sigma_a)| + s|\psi^{(r)}(\sigma_b)|] ds \\
 &= \frac{k^{r-2} \|\phi\|_\infty \Gamma(p)}{\Gamma_k(p) \Gamma(p+r-1)} \int_{1/2}^1 (1-u)^{\frac{p}{k} + r - 2} \left( \int_{1/2}^u [(1-s)|\psi^{(r)}(\sigma_a)| + s|\psi^{(r)}(\sigma_b)|] ds \right) du \\
 &= \frac{k^{r-2} \|\phi\|_\infty \Gamma(p)}{\Gamma_k(p) \Gamma(p+r-1)} \int_{1/2}^1 (1-u)^{\frac{p}{k} + r - 2} \left[ |\psi^{(r)}(\sigma_a)| \left( \frac{1}{8} - \frac{(1-u)^2}{2} \right) + |\psi^{(r)}(\sigma_b)| \left( \frac{u^2}{2} - \frac{1}{8} \right) \right] du.
 \end{aligned}$$

By the change of variable  $t = \sigma_a + (1 - u)\kappa(\sigma_b, \sigma_a)$ ,

$$I_2 = \frac{k^{r-2} \|\phi\|_\infty \Gamma(\mathfrak{p})}{\Gamma_k(\mathfrak{p})\Gamma(\mathfrak{p}+r-1)} \frac{|\psi^{(r)}(\sigma_a)|}{\kappa(\sigma_b, \sigma_a)} \times \int_{\sigma_a}^{\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)} \left( \frac{1}{8} - \frac{1}{2} \left( \frac{t - \sigma_a}{\kappa(\sigma_b, \sigma_a)} \right)^2 \right) \left( \frac{t - \sigma_a}{\kappa(\sigma_b, \sigma_a)} \right)^{\frac{\mathfrak{p}}{k} + r - 2} dt$$

$$+ \frac{k^{r-2} \|\phi\|_\infty \Gamma(\mathfrak{p})}{\Gamma_k(\mathfrak{p})\Gamma(\mathfrak{p}+r-1)} \frac{|\psi^{(r)}(\sigma_b)|}{\kappa(\sigma_b, \sigma_a)} \times \int_{\sigma_a}^{\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)} \left( \frac{1}{2} \left( 1 - \frac{t - \sigma_a}{\kappa(\sigma_b, \sigma_a)} \right)^2 - \frac{1}{8} \left( \frac{t - \sigma_a}{\kappa(\sigma_b, \sigma_a)} \right)^{\frac{\mathfrak{p}}{k} + r - 2} \right) dt. \quad (15)$$

Adding equations (14) and (15) based on equation (13), we obtain the required result as follows:

$$\left| \sum_{m=0}^{r-1} \frac{\psi^{(m)} \left( \sigma_a + \frac{1}{2} \kappa(\sigma_b, \sigma_a) \right)}{2^m \kappa(\sigma_b, \sigma_a)^{\frac{\mathfrak{p}}{k} + r - m}} \right.$$

$$\times \left[ (-1)^{r-m-1} J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{p}, k} \phi(\sigma_a) + (-1)^{r+1} J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^+}^{\mathfrak{p}, k} \phi(\sigma_a + \kappa(\sigma_b, \sigma_a)) \right]$$

$$+ (-1)^r \frac{1}{(\kappa(\sigma_b, \sigma_a))^{\frac{\mathfrak{p}}{k} + r}} \left[ J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{p}, k} (\phi\psi)(\sigma_a) + J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^+}^{\mathfrak{p}, k} (\phi\psi)(\sigma_a + \kappa(\sigma_b, \sigma_a)) \right] \Bigg|$$

$$\leq \frac{[|\psi^{(r)}(\sigma_a)| + |\psi^{(r)}(\sigma_b)|] \|\phi\|_\infty k^r \Gamma(\mathfrak{p})}{2^{\frac{\mathfrak{p}}{k} + r} (\mathfrak{p} + k(-1 + r)) (\mathfrak{p} + kr) \Gamma_k(\mathfrak{p}) \Gamma(\mathfrak{p} + r - 1)}.$$

□

**Remark 2.** For  $k = r = 1$  in equation (12), we obtain Theorem 3 of [5].

$$\left| \frac{\psi(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))}{\kappa(\sigma_b, \sigma_a)^{\mathfrak{p}+1}} \left[ J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{p}} \phi(\sigma_a) + J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^+}^{\mathfrak{p}} \phi(\sigma_a + \kappa(\sigma_b, \sigma_a)) \right] \right.$$

$$+ \frac{1}{(\kappa(\sigma_b, \sigma_a))^{\mathfrak{p}+1}} \left[ J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{p}} (\phi\psi)(\sigma_a) + J_{(\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a))^+}^{\mathfrak{p}} (\phi\psi)(\sigma_a + \kappa(\sigma_b, \sigma_a)) \right] \Bigg|$$

$$\leq \frac{\|\phi\|_\infty}{\Gamma(\mathfrak{p}+2)} \frac{1}{2^{\mathfrak{p}+1}} [|\psi'(\sigma_a)| + |\psi'(\sigma_b)|].$$

**Remark 3.** For  $k = \mathfrak{p} = 1$ ,  $\phi = 1$ ,  $\kappa(\sigma_b, \sigma_a) = \sigma_b - \sigma_a$  and  $r = 2$  in equation (12), we obtain the special case

$$\left| \psi \left( \frac{\sigma_a + \sigma_b}{2} \right) - (\sigma_b - \sigma_a) \int_{\sigma_a}^{\sigma_b} \psi(t) dt \right| \leq (\sigma_b - \sigma_a)^2 \frac{[|\psi^{(2)}(\sigma_a)| + |\psi^{(2)}(\sigma_b)|]}{48}. \quad (16)$$

**Theorem 3.** Let  $\Omega \subseteq \mathbb{R}$  be an open invex set and  $\kappa$  be a function such that  $\kappa : \Omega \times \Omega \rightarrow \mathbb{R}^{\mathfrak{p}}$ . Suppose,  $\psi : \Omega \rightarrow \mathbb{R}$  is a differentiable mapping such that  $\psi^{(r)} \in L[\sigma_a, \sigma_a + \kappa(\sigma_b, \sigma_a)]$ , where  $\kappa(\sigma_b, \sigma_a) > 0$ . If there is an integral mapping such that  $\phi : [\sigma_a, \sigma_a + \kappa(\sigma_b, \sigma_a)] \rightarrow [0, \infty)$  and it is also symmetric with respect to  $\sigma_a + \frac{1}{2}\kappa(\sigma_b, \sigma_a)$ . Let  $|\psi^{(r)}|$  be a preinvex function on  $\Omega$ , then  $\forall \sigma_a, \sigma_b \in \Omega$ , we have the following inequality:

$$J = \left| \sum_{m=0}^{r-1} \left[ \frac{(-1)^{r+1} \psi^{(m)}(\sigma_a) + (-1)^{r-m-1} \psi^{(m)}(\sigma_a + \kappa(\sigma_b, \sigma_a))}{2(\kappa(\sigma_b, \sigma_a))^{\mathfrak{p}+r-m}} \right] \right.$$

$$\times [J_{(\sigma_a + \kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{p}, k} \phi(\sigma_a) + J_{\sigma_a + \kappa(\sigma_b, \sigma_a)}^{\mathfrak{p}, k} \phi(\sigma_a + \kappa(\sigma_b, \sigma_a))]$$

$$+ (-1)^r \left[ \frac{J_{(\sigma_a + \kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{p}, k} (\phi\psi)(\sigma_a) + J_{\sigma_a + \kappa(\sigma_b, \sigma_a)}^{\mathfrak{p}, k} (\phi\psi)(\sigma_a + \kappa(\sigma_b, \sigma_a))}{(\kappa(\sigma_b, \sigma_a))^{\mathfrak{p}+r}} \right] \Bigg|$$

$$\leq \frac{\|\phi\|_\infty}{(\mathfrak{p}+r)\Gamma(\mathfrak{p}+r-1)} [|\psi^{(r)}(\sigma_a)| + |\psi^{(r)}(\sigma_b)|]. \quad (17)$$

**Proof.** Applying modulus on both sides of equation (8),

$$\begin{aligned}
 & \left| \sum_{m=0}^{r-1} \left[ \frac{(-1)^{r+1} \psi^{(m)}(\sigma_a) + (-1)^{r-m-1} \psi^{(m)}(\sigma_a + \kappa(\sigma_b, \sigma_a))}{2(\kappa(\sigma_b, \sigma_a))^{\mathfrak{p}+r-m}} \right] \right. \\
 & \quad \times [J_{(\sigma_a + \kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{p}, k} \phi(\sigma_a) + J_{\sigma_a^+}^{\mathfrak{p}, k} \phi(\sigma_a + \kappa(\sigma_b, \sigma_a))] \\
 & \quad \left. + (-1)^r \left[ \frac{J_{(\sigma_a + \kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{p}, k} (\phi\psi)(\sigma_a) + J_{\sigma_a^+}^{\mathfrak{p}, k} (\phi\psi)(\sigma_a + \kappa(\sigma_b, \sigma_a))}{(\kappa(\sigma_b, \sigma_a))^{\mathfrak{p}+r}} \right] \right| \\
 &= \left| \frac{1}{k\Gamma_k(\mathfrak{p})} \int_0^1 \underbrace{\int_0^s \int_0^s \dots \int_0^s}_{r \text{ integrals}} u^{\frac{\mathfrak{p}}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^r + \underbrace{\int_0^s \int_0^s \dots \int_0^s}_{r \text{ integrals}} (1-u)^{\frac{\mathfrak{p}}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^r \right. \\
 & \quad \left. + s\kappa(\sigma_b, \sigma_a) ds \right|.
 \end{aligned}$$

From preinvexity of  $|\psi^{(r)}|$  on  $\Omega$ , we have

$$\begin{aligned}
 J &= \left| \sum_{\kappa=0}^{r-1} \left[ \frac{(-1)^{r+1} \psi^{(\kappa)}(\sigma_a) + (-1)^{r-\kappa-1} \psi^{(\kappa)}(\sigma_a + \kappa(\sigma_b, \sigma_a))}{2(\kappa(\sigma_b, \sigma_a))^{\mathfrak{p}+r-\kappa}} \right] \right. \\
 & \quad \times [J_{(\sigma_a + \kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{p}} \phi(\sigma_a) + J_{\sigma_a^+}^{\mathfrak{p}} \phi(\sigma_a + \kappa(\sigma_b, \sigma_a))] \\
 & \quad \left. + (-1)^r \left[ \frac{J_{(\sigma_a + \kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{p}} (\phi\psi)(\sigma_a) + J_{\sigma_a^+}^{\mathfrak{p}} (\phi\psi)(\sigma_a + \kappa(\sigma_b, \sigma_a))}{(\kappa(\sigma_b, \sigma_a))^{\mathfrak{p}+r}} \right] \right| \\
 &= \frac{1}{k\Gamma_k(\mathfrak{p})} \int_0^1 \left| \underbrace{\int_0^s \int_0^s \dots \int_0^s}_{r \text{ integrals}} u^{\frac{\mathfrak{p}}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^r \right. \\
 & \quad \left. + \underbrace{\int_0^s \int_0^s \dots \int_0^s}_{r \text{ integrals}} (1-u)^{\frac{\mathfrak{p}}{k}-1} \phi(\sigma_a + u\kappa(\sigma_b, \sigma_a)) (du)^r \right| ((1-s)|\psi^{(r)}(\sigma_a)| + s|\psi^{(r)}(\sigma_b)|) ds.
 \end{aligned}$$

After simplification and letting  $\|\phi\|_{\infty} = \sup_{t \in [\sigma_a, \sigma_b]} |\phi(t)|$ , we obtain the required result

$$\begin{aligned}
 J &\leq \frac{k^{r-2} \|\phi\|_{\infty} \Gamma(\mathfrak{p})}{\Gamma_k(\mathfrak{p}) \Gamma(\mathfrak{p}+r-1)} \int_0^1 u^{\frac{\mathfrak{p}}{k}+r-2} \left( \int_0^u ((1-s)|\psi^{(r)}(\sigma_a)| + s|\psi^{(r)}(\sigma_b)|) ds \right) du \\
 &\quad + \frac{k^{r-2} \|\phi\|_{\infty} \Gamma(\mathfrak{p})}{\Gamma_k(\mathfrak{p}) \Gamma(\mathfrak{p}+r-1)} \int_0^1 (1-u)^{\frac{\mathfrak{p}}{k}+r-2} \left( \int_u^1 ((1-s)|\psi^{(r)}(\sigma_a)| + s|\psi^{(r)}(\sigma_b)|) ds \right) du \\
 &\leq \frac{\|\phi\|_{\infty} k^{r-1} \Gamma(\mathfrak{p})}{(\mathfrak{p}+kr) \Gamma(\mathfrak{p}+r-1) \Gamma_k(\mathfrak{p})} [|\psi^{(r)}(\sigma_a)| + |\psi^{(r)}(\sigma_b)|].
 \end{aligned}$$

□

**Remark 4.** For  $k = r = 1$  in equation (17), we have

$$\begin{aligned} & \left| \left[ \frac{\psi(\sigma_a) + \psi(\sigma_a + \kappa(\sigma_b, \sigma_a))}{2(\kappa(\sigma_b, \sigma_a))^{\mathfrak{p}+1}} \right] \left[ J_{(\sigma_a + \kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{p}} \phi(\sigma_a) + J_{\sigma_a^+}^{\mathfrak{p}} \phi(\sigma_a + \kappa(\sigma_b, \sigma_a)) \right] \right. \\ & \quad \left. - \left[ \frac{J_{(\sigma_a + \kappa(\sigma_b, \sigma_a))^-}^{\mathfrak{p}} (\phi\psi)(\sigma_a) + J_{\sigma_a^+}^{\mathfrak{p}} (\phi\psi)(\sigma_a + \kappa(\sigma_b, \sigma_a))}{(\kappa(\sigma_b, \sigma_a))^{\mathfrak{p}+1}} \right] \right| \\ & \leq \frac{\|\phi\|_{\infty}}{(\mathfrak{p} + 1)\Gamma(\mathfrak{p})} [|\psi'(\sigma_a)| + |\psi'(\sigma_b)|]. \end{aligned}$$

**Remark 5.** For  $k = 1$ ,  $\mathfrak{p} = 1$ ,  $\phi = 1$  and  $r = 2$  in equation (17), we obtain the special case

$$J = \left| \left[ \frac{\psi(\sigma_a) + \psi(\sigma_b)}{2(\kappa(\sigma_b, \sigma_a))^2} \right] + \left[ \frac{\psi^{(1)}(\sigma_a) - \psi^{(1)}(\sigma_b)}{2(\kappa(\sigma_b, \sigma_a))} \right] - \frac{1}{(\kappa(\sigma_b, \sigma_a))^3} \int_{\sigma_a}^{\sigma_b} \phi(t) dt \right| \leq \frac{|\psi^{(2)}(\sigma_a)| + |\psi^{(2)}(\sigma_b)|}{3}. \quad (18)$$

### 3 Applications

In this section, some examples in the framework of special functions and special means are selected to fulfil the applicability of our obtained results.

#### 3.1 Special functions

Let the function  $F_{\bar{\nu}}(z) : \mathbb{R} \rightarrow [1, \infty]$  be defined by Waston [22] as follows:

$$F_{\bar{\nu}}(z) = 2^{\bar{\nu}} \Gamma(\bar{\nu} + 1) z^{-\nu} G_{\bar{\nu}}(z), \quad z \in \mathbb{R}.$$

We consider the first kind of modified Bessel function denoted by  $\mathfrak{r}_{\bar{\nu}}(z)$ , which is given by [22] as follows:

$$G_{\bar{\nu}}(z) = \sum_{n=0}^{n=\infty} \frac{\left(\frac{z}{2}\right)^{\bar{\nu}+2n}}{n! \Gamma(\bar{\nu} + n + 1)}.$$

Then, the first- and second-order derivatives of  $F_{\bar{\nu}}(z)$  are given as follows:

$$\begin{aligned} F'_{\bar{\nu}}(z) &= \frac{z}{2(\bar{\nu} + 1)} F_{\bar{\nu}+1}(z). \\ F'_{\bar{\nu}}(z) &= \frac{1}{4(\bar{\nu} + 1)} \left[ \frac{z^2}{\bar{\nu} + 2} F_{\bar{\nu}+2}(z) + 2F_{\bar{\nu}+1}(z) \right]. \\ F''_{\bar{\nu}}(z) &= \frac{1}{2(\bar{\nu} + 1)(\bar{\nu} + 2)} \left[ \frac{z^3}{\bar{\nu} + 3} F_{\bar{\nu}+3}(z) + 6zF_{\bar{\nu}+2}(z) \right]. \end{aligned} \quad (19)$$

Let  $\psi = F'_{\bar{\nu}}(z)$ . Then, with the help of Remark given in equation (4) and using the three identities in equation (19), we can deduce

$$\begin{aligned} & \left| \frac{\sigma_a + \sigma_b}{4(\bar{\nu} + 1)} F_{\bar{\nu}+1} \left( \frac{\sigma_a + \sigma_b}{2} \right) - (\sigma_b - \sigma_a) [F_{\bar{\nu}}(\sigma_b) - F_{\bar{\nu}}(\sigma_a)] \right| \\ & \leq \frac{(\sigma_b - \sigma_a)^2}{56(\bar{\nu} + 1)(\bar{\nu} + 2)} \left[ \left( \frac{\sigma_a^3}{\bar{\nu} + 3} F_{\bar{\nu}+3}(\sigma_a) + 6\sigma_a F_{\bar{\nu}+2}(\sigma_a) \right) + \left( \frac{\sigma_b^3}{\bar{\nu} + 3} F_{\bar{\nu}+3}(\sigma_b) + 6\sigma_b F_{\bar{\nu}+2}(\sigma_b) \right) \right]. \end{aligned}$$

### 3.2 Special means

We give the following special means for positive numbers  $\sigma_a$  and  $\sigma_b$ , where  $\sigma_a < \sigma_b$

(1) Arithmetic mean is denoted by  $A(\sigma_a, \sigma_b)$ , and it is defined as follows:

$$A(\sigma_a, \sigma_b) = \frac{\sigma_a + \sigma_b}{2}.$$

(2) Harmonic mean is denoted by  $H(\sigma_a, \sigma_b)$ , and it is defined as follows:

$$H(\sigma_a, \sigma_b) = \frac{2}{\frac{1}{\sigma_a} + \frac{1}{\sigma_b}}.$$

(3) Logarithmic mean is denoted by  $L(\sigma_a, \sigma_b)$ , and it is defined as follows:

$$L(\sigma_a, \sigma_b) = \frac{\sigma_b - \sigma_a}{\ln \sigma_b - \ln \sigma_a}, \sigma_a \neq \sigma_b.$$

(4) Logarithmic mean is denoted by  $L_n(\sigma_a, \sigma_b)$ , and it is defined as follows:

$$L_n(\sigma_a, \sigma_b) = \left( \frac{\sigma_b - \sigma_a}{(n+1)(\sigma_b - \sigma_a)} \right)^{\frac{1}{n}}.$$

**Proposition 1.** For  $\sigma_a, \sigma_b \in \mathbb{R}_+$ , where  $\sigma_a < \sigma_b$  and  $m \in \mathbb{N}$ ,  $m > 1$ , then the following inequality holds:

$$\begin{aligned} J &= \left| [A(\sigma_a^{m+1}, \sigma_b^{m+1}) + (\sigma_b - \sigma_a)(m+1)A(\sigma_a^{m+1}, -\sigma_b^{m+1})] - \frac{1}{(\sigma_b - \sigma_a)(m+2)} A(-\sigma_a^{m+2}, \sigma_b^{m+2}) \right| \\ &\leq \frac{2}{3} (\sigma_b - \sigma_a)^2 m(m+1) A(\sigma_a^{m-1}, \sigma_b^{m-1}). \end{aligned}$$

**Proof.** For  $\psi(v) = v^{m+1}$ ,  $|\psi'(v)| = (m+1)v^m$  and  $|\psi''(v)| = m(m+1)v^{m-1}$  is a convex function on  $\mathbb{R}_+$ , where  $v \in \mathbb{R}_+$  and  $m \in \mathbb{N}$ ,  $m \geq 2$ .

From equation (18), we obtain the required inequality.  $\square$

## 4 Conclusion

Finally, it can be said that research on weighted integral inequalities for  $r$ -times differentiable preinvex functions for  $k$ -fractional integrals has produced significant findings in the field of mathematical analysis. The findings of this study have shown that preinvex functions have the potential to be very useful mathematical tools, especially in the area of fractional calculus. Additional knowledge about the characteristics of preinvex functions and their uses in other mathematical domains is anticipated to be gained through further study in this area. Overall, the findings in this article have opened up new study directions and laid a strong platform for the future.

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