**Research Article**

Pradip Ramesh Patle, Moosa Gabeleh*, and Manuel De La Sen

# Global optimum solutions for a system of ($k$, $\psi$)-Hilfer fractional differential equations: Best proximity point approach

**Abstract:** In this article, a class of cyclic (noncyclic) operators are defined on Banach spaces via concept of measure of noncompactness using some abstract functions. The best proximity point (pair) results are manifested for the said operators. The obtained main results are applied to demonstrate the existence of optimum solutions of a system of fractional differential equations involving ($k$, $\psi$)-Hilfer fractional derivatives.

## 1 Introduction and basic concepts

Fixed point theory serves a incontestable purpose in the development of different branches of sciences. With the advent of new results in the theory, the new applications have been coined out. One such all-time significant application is in proving the existence of solutions to various kinds of equations *viz.* differential equations, integral equations, fractional differential and integral equations, integro-differential equations, functional equations, etc.

When a mapping has no fixed points, we search for the points that are most close to the fixed points, which are called as best proximity points. In the last three decades, there has been significant development in the field of best proximity point (pair) results. The application of such results lies in establishing the existence of optimum solutions for a system of equations. Let us recall the concept of best proximity points (pairs) in brief (see [1–3] for more details).

Throughout this article the following notations are used:

$\mathbb{N}$ – set of natural numbers,

$\mathbb{R}$ – set of real numbers,

$\mathbb{R}_+$ – the set $[0, \infty)$,

$\mathcal{B}(a, \rho)$ – closed ball of radius $\rho$ with center $a$,

$\overline{\mathcal{D}}$ – closure of the set $\mathcal{D}$,

$\overline{\mathrm{con}}(\mathcal{D})$ – convex and closed hull of $\mathcal{D}$,

$\mathrm{diam}(\mathcal{D})$ – diameter of the set $\mathcal{D}$.

* **Corresponding author: Moosa Gabeleh,** Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran, e-mail: gabeleh@abru.ac.ir, gab.moo@gmail.com

**Pradip Ramesh Patle:** Department of Mathematics, School of Advance Sciences, VIT-AP University, Amravati 522237, India, e-mail: pradip.patle12@gmail.com

**Manuel De La Sen:** Institute of Research and Development of Processes, University of the Basque Country, 48940 Leioa, Bizkaia, Spain, e-mail: manuel.delasen@ehu.eus

Let $\mathbb{B}(X)$ be a collection of bounded subsets in a metric space $X$.

Let us take two nonempty subsets $P$ and $Q$ of a normed linear space ($\mathcal{NLS}$) $X$. We consider that a pair $(P, Q)$ satisfies a property, if both $P$ and $Q$ individually satisfy that property. For example, we say a pair $(\mathcal{P}, \mathcal{Q})$ is nonempty if and only if $P$ and $Q$ are nonempty. For the pair $(P, Q)$, we will define,

$$P_0 = \{a \in P : \exists b' \in Q | \|a - b'\| = \mathrm{dist}(\mathcal{P}, \mathcal{Q})\},$$
$$Q_0 = \{b \in Q : \exists a' \in P | \|a' - b\| = \mathrm{dist}(\mathcal{P}, \mathcal{Q})\}.$$

In Banach space $X$, $(P_0, Q_0)$ is a nonempty, convex and weakly compact pair if $(\mathcal{P}, \mathcal{Q})$ is nonempty, convex and weakly compact. If $P_0 = P$ and $Q_0 = Q$, then the pair $(\mathcal{P}, \mathcal{Q})$ of nonempty subsets in a $\mathcal{NLS}$ $X$ is called proximinal.

**Definition 1.1.** A mapping $T : P \cup Q \to P \cup Q$ is called
  (i)   cyclic if $T(P) \subseteq Q$ and $T(Q) \subseteq P$;
 (ii)   noncyclic if $T(P) \subseteq P$ and $T(Q) \subseteq Q$;
(iii)   relatively nonexpansive if it satisfies $\|Ta - Tb\| \le \|a - b\|$ whenever $a \in P$ and $b \in Q$;
(iv)   nonexpansive if $\|Ta - Tb\| \le \|a - b\|$ whenever $a \in P$ and $b \in Q$ and $P = Q$;
 (v)   compact if $(\overline{T(P)}, \overline{T(Q)})$ is compact.

We consider a best proximity point for a cyclic mapping $T$, which is defined as, a point $w^* \in P \cup Q$ satisfying

$$\|w^* - Tw^*\| = \mathrm{dist}(P, Q) = \inf\{\|a - b\| : a \in P, b \in Q\}.$$

In case of a noncyclic mapping $T$, we consider the existence of a pair $(b, a) \in (P, Q)$ for which $b = Tb, a = Ta$ and $\|b - a\| = \mathrm{dist}(P, Q)$. Such pairs are called best proximity pairs.

The notion of cyclic (noncyclic) relatively nonexpansive mappings is presented by Eldred et al. in [1] and the best proximity point (pair) results in Banach spaces are obtained. The existence of best proximity point is manifested in [1] using a concept of proximal normal structure (PNS). Gabeleh [4] proved that every nonempty, convex and compact pair in a Banach space has PNS. This fact enabled Gabeleh to prove the following results.

**Theorem 1.2.** [4] *A relatively nonexpansive cyclic mapping $T : P \cup Q \to P \cup Q$ has a best proximity point if $T$ is compact and $P_0$ is nonempty, where $(P, Q)$ is a nonempty, bounded, closed and convex ($\mathcal{NBCC}$) pair in a Banach space $X$.*

Next result is for noncyclic mappings on a strictly convex Banach space. A Banach space $X$ is strictly convex if for $a, b, x \in X$ and $\Lambda > 0$, the following holds

$$[\|a - x\| \le \Lambda, \|b - x\| \le \Lambda, a \ne b] \Rightarrow \left\| \frac{a + b}{2} - x \right\| < \Lambda.$$

The $L^p$ space ($1 < p < \infty$) and Hilbert space are examples of strictly convex Banach spaces.

**Theorem 1.3.** [4] *Let $X$ be a strictly convex Banach space and $(P, Q)$ is an $\mathcal{NBCC}$ pair in $X$. Then, a relatively nonexpansive noncyclic mapping $T : P \cup Q \to P \cup Q$ admits a best proximity pair, provided it is compact and $P_0$ is nonempty.*

These results (Theorems 1.2 and 1.3) can be considered as extensions of Schauder fixed point theorem for best proximity point (pair). The condition of compactness on the mapping $T$ is a strong one. The Schauder's fixed point theorem is generalized by Darbo [5] and Sadovskii [6] using the concept of measure of noncompactness (MNC) which is defined axiomatically as follows (see Definition 1.4). One of the important aspects about MNC is that it facilitates to choose a class of mappings which are more general than compact operators.

**Definition 1.4.** [7–9] An MNC is a mapping $\mu : \mathbb{B}(X) \to \mathbb{R}^+$ satisfying the following axioms:

(1)  $\mu(P) = 0$ if and only if $P$ is relatively compact,

(2)  $\mu(P) = \mu(\overline{P})$, $P \in \mathbb{B}(X)$,

(3)  $\mu(P \cup Q) = \max\{\mu(P), \mu(Q)\}$, where $P, Q \in \mathbb{B}(X)$.

An MNC $\mu$ on $\mathbb{B}(X)$ satisfies the following properties.

(a)  $P \subset Q$ implies $\mu(P) \leq \mu(Q)$.

(b)  $\mu(P) = 0$ if $P$ is a finite set.

(c)  $\mu(P \cap Q) = \min\{\mu(P), \mu(Q)\}$, for all $P, Q \in \mathbb{B}(X)$.

(d)  If $\lim_{n \to \infty} \mu(P_n) = 0$ for a nonincreasing sequence $\{P_n\}$ of nonempty, bounded and closed subsets of $X$, then $P_\infty = \cap_{n \geq 1} P_n$ is nonempty and compact.

On a Banach space $X$, $\mu$ has the following properties.

 (i)  $\mu(\overline{\mathrm{con}}(Q)) = \mu(Q)$, for all $Q \in \mathbb{B}(X)$.

 (ii)  $\mu(\lambda Q) = |\lambda| \mu(Q)$ for any number $\lambda$ and $Q \in \mathbb{B}(X)$.

(iii)  $\mu(P + Q) \leq \mu(P) + \mu(Q)$.

**Example 1.5.** [8] The non-negative numbers

$$\alpha(C) = \inf\{r > 0 : C \subset \cup_{i=1}^{N} S_i, \mathrm{diam}(S_i) \leq r, i = 1, 2, \dots, N\}$$

and

$$\beta(C) = \inf\{r > 0 : C \subset \cup_{i=1}^{N} B(x_i, r), x_i \in X, i = 1, \dots, N\},$$

assigned with a bounded subset $C$ of a metric space $X$ are called Kuratowski MNC (K-MNC) and Hausdorff MNC (H-MNC), respectively.

Schauder fixed point theorem is generalized using MNC by Darbo [5] and Sadovskii [6]. We present the combined statement of both the theorems as follows:

**Theorem 1.6.** *Let $T$ be a continuous self-mapping on a $\mathcal{NBCC}$ subset $C$ of a Banach space $X$, for every $M \subset C$ satisfying one of the followings*:

(D)  *There exists a $0 \leq \lambda < 1$ such that $\mu(T(M)) \leq \lambda \mu(M)$,*

(S)  $\mu(M) > 0$, $\mu(T(M)) < \mu(M)$.

*Then $T$ has at least one fixed point.*

A mapping satisfying condition (*D*) is called $\lambda$-set contraction (due to Darbo [5]) whereas satisfying (*S*) is called as $\mu$-condensing (due to Sadovskii [6]).

On the line of Darbo fixed point theorem, Gabeleh and Markin in [10] generalized Theorems 1.2 and 1.3 by relaxing the condition of compactness on the operator $T$ by using the concept MNC and applied the obtained results to actualize the optimum solutions of a system of differential equations. Recently, the results of [10] have been generalized further in different directions in [11–16] in which best proximity point (pairs) results are obtained using MNC.

In this article, we prove best proximity point (pair) theorems for a new class of cyclic (non-cyclic) operators facilitated by MNC and some abstract functions. We apply the obtained results to prove the existence of optimal solutions of system of fractional differential equation (FDE) with initial value involving $(k, \psi)$-Hilfer fractional derivative. This is achieved by means of defining an operator from integral equations equivalent to the system of differential equations and proving that this operator has at least one best proximity point.

# 2 Main results

In this section, we present our main results for the existence of best proximity point (pair) for new classes of cyclic and noncyclic operators. We consider the following class of mappings introduced in [11], which will be used to define the new classes of condensing operators.

Let us denote by $\mathcal{H}$ the collection of all functions $\kappa : [0, \infty) \times [0, \infty) \to [0, \infty)$ that satisfy the following conditions:

(1) $\max\{a, b\} \leq \kappa(a, b)$ for $a, b \geq 0$,
(2) $\kappa$ is continuous,
(3) $\kappa(a + b, c + d) \leq \kappa(a, c) + \kappa(b, d)$.

For example, if we take $\kappa(x, y) = x + y$, then it is clear that $k \in \mathcal{H}$.

The following theorem is our first main result. Some part of the proof and the concept of $T$-invariant pair is adopted from [12].

**Theorem 2.1.** *Let $(P, Q)$ be a nonempty and convex pair in a Banach space $E$ with $P_0$ being nonempty and $\mu$ an MNC on $E$. A relatively nonexpansive cyclic mapping $T : P \cup Q \to P \cup Q$ has at least one best proximity point if for every $\mathcal{NBCC}$, proximinal and $T$ invariant pair $(M_1, M_2) \subseteq (P, Q)$ with $\mathrm{dist}(M_1, M_2) = \mathrm{dist}(P, Q)$ and for continuous mappings $\Delta_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i \in \{1, 2, 3\}$ such that for $t > 0$, $\Delta_1(t) > \Delta_2(t) - \Delta_3(t)$, $T$ satisfies*

$$
\begin{aligned}
&\Delta_1(\kappa(\mu(T(M_1) \cup T(M_2)), \varphi(\mu(T(M_1) \cup T(M_2))))) \\
&\leq \Delta_2(\kappa(\mu(M_1 \cup M_2), \varphi(\mu(M_1 \cup M_2)))) - \Delta_3(\kappa(\mu(M_1 \cup M_2), \varphi(\mu(M_1 \cup M_2)))),
\end{aligned}
\tag{1}
$$

*where $\kappa \in \mathcal{H}$ and $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing and continuous function.*

**Proof.** As $P_0$ is nonempty, $(P_0, Q_0)$ is nonempty. Also one can show that $(P_0, Q_0)$ is convex, closed, $T$-invariant and proximinal pair considering the conditions on $T$ (for more details see [12]). For $a \in P_0$, there is a $b \in Q_0$ such that $\|a - b\| = \mathrm{dist}(P, Q)$. Since $T$ is relatively nonexpansive cyclic mapping,

$$\|Ta - Tb\| \leq \|a - b\| = \mathrm{dist}(P, Q),$$

which gives $Ta \in Q_0$, that is, $T(P_0) \subseteq Q_0$. Similarly, $T(Q_0) \subseteq P_0$ and so $T$ is cyclic on $P_0 \cup Q_0$.

Let us define a pair $(G_n, H_n)$ as $G_n = \overline{\mathrm{con}}(T(G_{n-1}))$ and $H_n = \overline{\mathrm{con}}(T(H_{n-1}))$, $n \geq 1$ with $G_0 = P_0$ and $H_0 = Q_0$. We claim that $G_{n+1} \subseteq H_n$ and $H_n \subseteq G_{n-1}$ for all $n \in \mathbb{N}$. We have $H_1 = \overline{\mathrm{con}}(T(H_0)) = \overline{\mathrm{con}}(TQ_0) = \overline{\mathrm{con}}(P_0) \subseteq P_0 = G_0$. Therefore, $T(H_1) \subseteq T(G_0)$. So $H_2 = \overline{\mathrm{con}}(T(H_1)) \subseteq \overline{\mathrm{con}}(T(G_0)) = G_1$. Continuing this pattern, we obtain $H_n \subseteq G_{n-1}$ by using induction. Similarly, we can see that $G_{n+1} \subseteq H_n$ for all $n \in \mathbb{N}$. Thus, $G_{n+2} \subseteq H_{n+1} \subseteq G_n \subseteq H_{n-1}$ for all $n \in \mathbb{N}$. Hence, we obtain a decreasing sequence $\{(G_{2n}, H_{2n})\}$ of nonempty, closed and convex pairs in $P_0 \times Q_0$. Moreover, $T(H_{2n}) \subseteq T(G_{2n-1}) \subseteq \overline{\mathrm{con}}(T(G_{2n-1})) = G_{2n}$ and $T(G_{2n}) \subseteq T(H_{2n-1}) \subseteq \overline{\mathrm{con}}(T(H_{2n-1})) = H_{2n}$. Therefore, for all $n \in \mathbb{N}$, the pair $(G_{2n}, H_{2n})$ is $T$-invariant.

Now if $(u, v) \in P_0 \times Q_0$ is a proximinal pair, then

$$\mathrm{dist}(G_{2n}, H_{2n}) \leq \|T^{2n}u - T^{2n}v\| \leq \|u - v\| = \mathrm{dist}(P, Q).$$

Next, we show that the pair $(G_n, H_n)$ is proximinal using mathematical induction. Obviously for $n = 0$, the pair $(G_0, H_0)$ is proximinal. Suppose that $(G_k, H_k)$ is proximinal, we show that $(G_{k+1}, H_{k+1})$ is also proximinal. Let $x$ be an arbitrary member in $G_{k+1} = \overline{\mathrm{con}}(T(G_k))$. Then, it is represented as $x = \sum_{l=1}^{m} \lambda_l T(x_l)$ with $x_l \in G_k$, $m \in \mathbb{N}$, $\lambda_l \geq 0$ and $\sum_{l=1}^{m} \lambda_l = 1$. Due to proximinality of the pair $(G_k, H_k)$, there exists $y_l \in H_k$ for $1 \leq l \leq m$ such that $\|x_l - y_l\| = \mathrm{dist}(G_k, H_k) = \mathrm{dist}(P, Q)$. Take $y = \sum_{l=1}^{m} \lambda_l T(y_l)$. Then, $y \in \overline{\mathrm{con}}(T(H_k)) = H_{k+1}$ and

$$\|x - y\| = \left\| \sum_{l=1}^{m} \lambda_l T(x_l) - \sum_{l=1}^{m} \lambda_l T(y_l) \right\| \leq \sum_{l=1}^{m} \lambda_l \|x_l - y_l\| = \mathrm{dist}(P, Q).$$

This means that the pair $(G_{k+1}, H_{k+1})$ is proximinal and induction does the rest to prove $(G_n, H_n)$ is proximinal for all $n \in \mathbb{N}$.

Now if possible, let us take $\kappa(\mu(G_{2n} \cup H_{2n}), \varphi(\mu(G_{2n} \cup H_{2n}))) > 0$ for all $n \in \mathbb{N}$. As $\kappa$ is nondecreasing and $\{(G_{2n}, H_{2n})\}$ is a decreasing sequence,

$$\kappa(\mu(G_{2n+2} \cup H_{2n+2}), \varphi(\mu(G_{2n+2} \cup H_{2n+2}))) \leq \kappa(\mu(G_{2n} \cup H_{2n}), \varphi(\mu(G_{2n} \cup H_{2n}))),$$

holds for all $n \in \mathbb{N}$. Therefore, the sequence $\{\kappa(\mu(G_{2n} \cup H_{2n}), \varphi(\mu(G_{2n} \cup H_{2n})))\}$ is nonnegative, bounded below and decreasing. Thus, there exists $a \geq 0$ such that

$$\lim_{n \to \infty} \kappa(\mu(G_{2n} \cup H_{2n}), \varphi(\mu(G_{2n} \cup H_{2n}))) = a.$$

Now if possible, let $a > 0$. From the assumed hypothesis on $T$, we have

$$\begin{aligned}
&\Delta_1(\kappa(\mu(G_{2n+2} \cup H_{2n+2}), \varphi(\mu(G_{2n+2} \cup H_{2n+2})))) \\
&= \Delta_1(\kappa(\max\{\mu(G_{2n+2}), \mu(H_{2n+2})\}, \varphi(\max\{\mu(G_{2n+2}), \mu(H_{2n+2})\}))) \\
&\leq \Delta_1(\kappa(\max\{\mu(H_{2n+1}), \mu(G_{2n+1})\}, \varphi(\max\{\mu(H_{2n+1}), \mu(G_{2n+1})\}))) \\
&= \Delta_1(\kappa(\max\{\mu(\overline{\mathrm{con}}(T(H_{2n}))), \mu(\overline{\mathrm{con}}(T(G_{2n})))\}, \varphi(\max\{\mu(\overline{\mathrm{con}}(T(H_{2n}))), \mu(\overline{\mathrm{con}}(T(G_{2n})))\}))) \\
&= \Delta_1(\kappa(\max\{\mu(T(H_{2n})), \mu(T(G_{2n}))\}, \varphi(\max\{\mu(T(H_{2n})), \mu(T(G_{2n}))\}))) \\
&= \Delta_1(\kappa(\mu(T(G_{2n}) \cup T(H_{2n})), \varphi(\mu(T(G_{2n}) \cup T(H_{2n}))))) \\
&\leq \Delta_2(\kappa(\mu(G_{2n} \cup H_{2n}), \varphi(\mu(G_{2n} \cup H_{2n})))) - \Delta_3(\kappa(\mu(G_{2n} \cup H_{2n}), \varphi(\mu(G_{2n} \cup H_{2n})))) \quad \text{for all } n > 0.
\end{aligned}$$

As $n \to \infty$, we have $\Delta_1(a) \leq \Delta_2(a) - \Delta_3(a)$, which is a contradiction to the assumption on mapping $\Delta_1, \Delta_2$ and $\Delta_3$. This means that $a = 0$. Therefore,

$$\lim_{n \to \infty} \kappa(\mu(G_{2n} \cup H_{2n}), \varphi(\mu(G_{2n} \cup H_{2n}))) = 0.$$

By definition of mapping $\kappa$, we deduce that

$$\lim_{n \to \infty} \mu(G_{2n} \cup H_{2n}) = \lim_{n \to \infty} \varphi(\mu(G_{2n} \cup H_{2n})) = 0.$$

That is, $\lim_{n \to \infty} \mu(G_{2n} \cup H_{2n}) = \max\{\lim_{n \to \infty} \mu(G_{2n}), \lim_{n \to \infty} \mu(H_{2n})\} = 0$. Now let $G_\infty = \cap_{n=0}^{\infty} G_{2n}$ and $H_\infty = \cap_{n=0}^{\infty} H_{2n}$. By property $(d)$ of MNC, the pair $(G_\infty, H_\infty)$ is nonempty, convex, compact and $T$-invariant with $\mathrm{dist}(G_\infty, H_\infty) = \mathrm{dist}(P, Q)$. Now we obtain, $T : G_\infty \cup H_\infty \to G_\infty \cup H_\infty$, which is a cyclic, relatively nonexpansive mapping on a nonempty, convex, compact and $T$-invariant pair $(G_\infty, H_\infty)$ with $\mathrm{dist}(G_\infty, H_\infty) = \mathrm{dist}(P, Q)$. Therefore, Theorem 1.2 ensures that $T$ admits a best proximity point. □

Next result is an analogous of the above theorem for relatively nonexpansive noncyclic mapping which constitutes second main result of the section.

**Theorem 2.2.** *Let $(P, Q)$ be a nonempty and convex pair in a strictly convex Banach space $E$ with $P_0$ nonempty and $\mu$ an MNC on $E$. A relatively nonexpansive noncyclic mapping $T : P \cup Q \to P \cup Q$ has at least one best proximity pair if for every $\mathcal{NBCC}$, proximinal and $T$ invariant pair $(M_1, M_2) \subseteq (P, Q)$ with $\mathrm{dist}(M_1, M_2) = \mathrm{dist}(P, Q)$ equation (1) is satisfied.*

**Proof.** It is clear that $(P_0, Q_0)$ is $\mathcal{NBCC}$ pair which is proximinal and $T$-invariant (see [12] for more details on proof). Let $(a, b) \in P_0 \times Q_0$ be such that $\|a - b\| = \mathrm{dist}(P, Q)$. Since $T$ is relatively nonexpansive noncyclic mapping,

$$\|Ta - Tb\| \leq \|a - b\| = \mathrm{dist}(P, Q),$$

which gives $Ta \in P_0$, that is, $T(P_0) \subseteq P_0$. Similarly, $T(Q_0) \subseteq Q_0$ and so $T$ is noncyclic on $P_0 \cup Q_0$.

Let us define a pair $(G_n, H_n)$ as $G_n = \overline{\mathrm{con}}(T(G_{n-1}))$ and $H_n = \overline{\mathrm{con}}(T(H_{n-1}))$, $n \geq 1$ with $G_0 = P_0$ and $H_0 = Q_0$. We have $H_1 = \overline{\mathrm{con}}(T(H_0)) = \overline{\mathrm{con}}(T(Q_0)) \subseteq Q_0 = H_0$. Therefore, $T(H_1) \subseteq T(H_0)$. Thus, $H_2 = \overline{\mathrm{con}}(T(H_1)) \subseteq \overline{\mathrm{con}}(T(H_0)) = H_1$. Continuing this pattern, we obtain $H_n \subseteq H_{n-1}$ by using induction. Similarly, we can see that $G_{n+1} \subseteq G_n$ for all $n \in \mathbb{N}$. Hence, we obtain a decreasing sequence $\{(G_n, H_n)\}$ of nonempty, closed and convex pairs in $P_0 \times Q_0$. Also, $T(H_n) \subseteq T(H_{n-1}) \subseteq \overline{\mathrm{con}}(T(H_{n-1})) = H_n$ and $T(G_n) \subseteq T(G_{n-1}) \subseteq \overline{\mathrm{con}}(T(G_{n-1})) = G_n$. Therefore, for all $n \in \mathbb{N}$, the pair $(G_n, H_n)$ is $T$-invariant. From the proof of Theorem 2.1, we have $(G_n, H_n)$ is a proximinal pair such that $\mathrm{dist}(G_n, H_n) = \mathrm{dist}(P, Q)$ for all $n \in \mathbb{N} \cup \{0\}$.

Now since $\{\mu(G_n \cup H_n)\}$ is a positive nonincreasing sequence and following the proof of Theorem 2.1, we can prove that $\{\mu(G_n \cup H_n)\}$ converges to 0.

Therefore, $\mu(G_n \cup H_n) \to 0$ as $n \to \infty$. That is, $\lim_{n\to\infty}\mu(G_n \cup H_n) = \max\{\lim_{n\to\infty}\mu(G_n), \lim_{n\to\infty}\mu(H_n)\} = 0$. Now let $G_\infty = \cap_{n=0}^{\infty}G_n$ and $H_\infty = \cap_{n=0}^{\infty}H_n$. By property $(d)$ of MNC, $(G_\infty, H_\infty)$ is nonempty, convex, compact and $T$-invariant pair with $\text{dist}(G_\infty, H_\infty) = \text{dist}(P, Q)$. Thus, application of Theorem 1.3 ensures that $T$ admits a best proximity pair. $\qquad\square$

Now, we give some consequences of the aforementioned theorems as corollaries.

**Corollary 2.3.** *Let $(P, Q)$ be a nonempty and convex pair in a (strictly convex) Banach space $E$ with $P_0$ nonempty and $\mu$ an MNC on $E$. A relatively nonexpansive cyclic (noncyclic) mapping $T : P \cup Q \to P \cup Q$ has at least one best proximity point (pair) if for every $\mathcal{NBCC}$, proximinal and $T$-invariant pair $(M_1, M_2) \subseteq (P, Q)$ with $\text{dist}(M_1, M_2) = \text{dist}(P, Q)$ and for continuous mappings $\Delta_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i \in \{1, 2, 3\}$ such that for $t > 0$, $\Delta_1(t) > \Delta_2(t) - \Delta_3(t)$, $T$ satisfies*

$$\Delta_1(\mu(T(M_1) \cup T(M_2)) + \varphi(\mu(T(M_1) \cup T(M_2))))$$
$$\leq \Delta_2(\mu(M_1 \cup M_2) + \varphi(\mu(M_1 \cup M_2))) - \Delta_3(\mu(M_1 \cup M_2) + \varphi(\mu(M_1 \cup M_2))),$$

*where $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing and continuous function.*

**Proof.** Taking $\kappa(a, b) = a + b$ in Theorem 2.1 (Theorem 2.2), we obtain the desired result. $\qquad\square$

**Corollary 2.4.** *Let $(P, Q)$ be a nonempty and convex pair in a (strictly convex) Banach space $E$ with $P_0$ nonempty and $\mu$ an MNC on $E$. A relatively nonexpansive cyclic (noncyclic) mapping $T : P \cup Q \to P \cup Q$ has at least one best proximity point (pair) if for every $\mathcal{NBCC}$, proximinal and $T$-invariant pair $(M_1, M_2) \subseteq (P, Q)$ with $\text{dist}(M_1, M_2) = \text{dist}(P, Q)$ and for continuous mappings $\Delta_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i \in \{1, 2, 3\}$ such that for $t > 0$, $\Delta_1(t) > \Delta_2(t) - \Delta_3(t)$, $T$ satisfies*

$$\Delta_1(\mu(T(M_1) \cup T(M_2))) \leq \Delta_2(\mu(M_1 \cup M_2)) - \Delta_3(\mu(M_1 \cup M_2)).$$

**Proof.** If we take $\varphi$ as a constant zero function in Corollary 2.3, we obtain the desired result. $\qquad\square$

The following corollary is the main result of [10].

**Corollary 2.5.** *Let $(P, Q)$ be a nonempty and convex pair in a (strictly convex) Banach space $E$ with $P_0$ is nonempty and $\mu$ an MNC on $E$. A relatively nonexpansive cyclic (noncyclic) mapping $T : P \cup Q \to P \cup Q$ has at least one best proximity point (pair) if for every $\mathcal{NBCC}$, proximinal and $T$-invariant pair $(M_1, M_2) \subseteq (P, Q)$ with $\text{dist}(M_1, M_2) = \text{dist}(P, Q)$, $T$ satisfies*

$$\mu(T(M_1) \cup T(M_2)) \leq \lambda\mu(M_1 \cup M_2),$$

*where $\lambda \in [0, 1)$.*

**Proof.** If we take $\Delta_1(t) = t + \frac{1}{2}$, $\Delta_2(t) = 2\lambda t + 1$, $\Delta_3(t) = \lambda t + \frac{1}{2}$ ($\lambda \in [0, 1)$) and $\varphi$ as a constant zero function in Corollary 2.3, we obtain the required result. $\qquad\square$

# 3 Application

In the last decade, from the view point of the numerous applications, the study of FDEs gained significant importance. The number of characteristics of physical events arising in the area of biology, medicines, and branches of engineering such as mechanics and electrical engineering are expressible in the form of mathematical model using impulsive FDEs. These applications lead to the development of theory of solutions of

impulsive FDEs in various aspects. The interested readers can refer to the articles [17–23] for more details on FDEs and their applications.

In this section, we survey an application of the best proximity point results proved in Section 2 of this article. The existence of an optimal solution of systems of FDEs involving $(k, \psi)$- Hilfer fractional derivative using the said result is established.

Let $a, b$ and $r$ be positive real numbers, $\mathcal{I} = [a, b]$ and $(E, \|\cdot\|)$ be a Banach space. Let $B_1 = B(\alpha_a, r)$ and $B_2 = B(\beta_a, r)$ be closed balls in $E$, where $\alpha_a, \beta_a \in E$.

We present some concepts and outcomes from fractional calculus which will be used in this section of the article. Let $-\infty < a < b < \infty$. Let $C[a, b]$ denote the space of all continuous functions on $[a, b]$. We denote by $L^m(a, b), m \geq 1$, the spaces of Lebesgue integrable functions on $(a, b)$. See [24] for more details. Let $(a, b)$ be a finite or infinite interval of the real line $\mathbb{R}$. Let $\psi(x)$ be an increasing and positive monotone function on $[a, b]$, having a continuous derivative $\psi'(t) \neq 0$ on $(a, b)$.

**Definition 3.1.** [25] For $k > 0 (k \in \mathbb{R})$, the $k$-gamma function of a complex number $z \in \mathbb{C}$ with $\text{Re}(z) > 0$ is defined by

$$\Gamma_k(z) = \int_0^\infty s^{z-1} e^{-\frac{s^k}{k}} ds.$$

**Definition 3.2.** [26] Let $f \in L^1(a, b)$ and $k \in \mathbb{R}_+$. The integral

$$^k I_{a^+}^{\eta; \psi} f(x) = \frac{1}{k \Gamma_k(\eta)} \int_a^x \psi'(s)(\psi(x) - \psi(s))^{\frac{\eta}{k}-1} f(s) ds, \, x > a$$

is called a $(k, \psi)$-Riemann-Liouville fractional integral of function $f$ on $[a, b]$ of order $\eta \in \mathbb{R}_+$.

**Definition 3.3.** [27] Let $\eta, k \in \mathbb{R}_+, \nu \in [0, 1]$, and $m \in \mathbb{N}$. Let $f, \psi \in \mathbb{C}^m([a, b], \mathbb{R})$ be two functions such that $\psi(x)$ is increasing and $\psi'(t) \neq 0$ for all $t \in [a, b]$. The $(k, \psi)$-Hilfer fractional derivative of order $\eta$ and type $\nu$ of function $f$ is defined as the following expression:

$$^{k,H} D_{a^+}^{\eta, \nu; \psi} f(x) = {}^k I_{a^+}^{(mk-\eta); \psi} \left( \frac{k}{\psi'(x)} \frac{d}{dx} \right)^m {}^k I_{a^+}^{(1-\nu)(mk-\eta); \psi} f(x), \, x > a, m = \left\lceil \frac{\eta}{k} \right\rceil,$$

provided the right-hand side exists.

**Remark 3.4.** The $(k, \psi)$-Hilfer fractional derivative is considered to be the most general and unified definition of fractional derivative. In fact, by choosing different values of $\psi(x), k, a$ and taking limits on parameters $\eta, \nu$ in definition of $(k, \psi)$-Hilfer fractional derivative, we obtain a wide variety of fractional derivatives in the literature. See [27–29] for more information related to this.

We have the following results for the fractional derivatives.

**Lemma 3.5.** [27] *Let* $\mu, k \in \mathbb{R}_+$ *and let* $\xi \in \mathbb{R}$ *such that* $\frac{\xi}{k} > -1$. *Then, we have*

$$^k I_{a^+}^{\mu; \psi} (\psi(t) - \psi(a))^{\frac{\xi}{k}} = \frac{\Gamma_k(\xi + k)}{\Gamma_k(\xi + k + \mu)} (\psi(t) - \psi(a))^{\frac{\xi+\mu}{k}}.$$

**Lemma 3.6.** [27] *For* $p, q, k \in \mathbb{R}_+$, *and* $f \in L^1(a, b)$, *we have*

$$^k I_{a^+}^{p; \psi} {}^k I_{a^+}^{q; \psi} f(x) = {}^k I_{a^+}^{p+q; \psi} f(x), \quad a.e., \, x \in [a, b].$$

We consider the following system of nonlinear FDEs involving $(k, \psi)$-Hilfer fractional derivatives of arbitrary order with initial conditions of the form

$$\begin{cases} {}^{k,H}D_{a^+}^{p,q;\,\psi}x(t) = f(t, x(t)), \, t \in (a, b], \, 0 < p < k, \, q \in [0, 1] \\ {}^{k}I_{a^+}^{(k-\xi_k);\,\psi}x(a) = \alpha_a, \, \xi_k = p + q(k - p), \end{cases} \tag{2}$$

$$\begin{cases} {}^{k,H}D_{a^+}^{p,q;\,\psi}y(t) = g(t, y(t)), \, t \in (a, b], \, 0 < p < k, \, q \in [0, 1] \\ {}^{k}I_{a^+}^{(k-\xi_k);\,\psi}y(a) = \beta_a, \, \xi_k = p + q(k - p), \end{cases} \tag{3}$$

where ${}^{k,H}D_{a^+}^{p,q;\,\psi}$ is the $(k, \psi)$-Hilfer fractional differential operator of order $p$ and type $q$; ${}^{k}I_{a^+}^{(k-\xi_k);\,\psi}$ is the $(k, \psi)$-Riemann-Liouville fractional integral of order $(k - \xi_k)$; the state $x(\cdot)$ takes the values from Banach space $E$; $f : \mathcal{I} \times B_1 \to E$ and $g : \mathcal{I} \times B_2 \to E$ are given mappings satisfying some assumptions.

The following result establishes the equivalence of (2) with the fractional integral equation. Let $\xi_k = p + q(k - p)$.

**Lemma 3.7.** [27] *The initial value problem* (2) *is equivalent to the following integral equation:*

$$x(t) = \frac{(\psi(t) - \psi(a))^{\frac{\xi_k}{k} - 1}}{\Gamma_k(\xi_k)} \alpha_a + \frac{1}{k\Gamma_k(p)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\frac{p}{k} - 1} f(s, x(s)) \mathrm{d}s, \quad t \in \mathcal{I}.$$

Let $\mathcal{J} \subseteq \mathcal{I}$ and $S = C(\mathcal{J}, E)$ be a Banach space of continuous mappings from $\mathcal{J}$ into $E$ endowed with supremum norm. Let

$$S_1 = \{x \in C(\mathcal{J}, B_1) : {}^{k}I^{(k-\xi_k);\,\psi}x(a) = \alpha_a\}, \quad S_2 = \{y \in C(\mathcal{J}, B_2) : {}^{k}I^{(k-\xi_k);\,\psi}x(a) = \beta_a\}.$$

So $(S_1, S_2)$ is an NBCC pair in $S \times S$. Now for every $u \in S_1$ and $v \in S_2$, we have $\|u - v\| = \sup\|u(s) - v(s)\| \geq \|\alpha_a - \beta_a\|$. Therefore, $\mathrm{dist}(S_1, S_2) = \|\alpha_a - \beta_a\|$ which ensures that $(S_1)_0$ is nonempty.

Now let us define the operator $T : S_1 \cup S_2 \to S$ as follows:

$$Tx(t) = \begin{cases} \dfrac{(\psi(t) - \psi(a))^{\frac{\xi_k}{k} - 1}}{\Gamma(\xi_k)} \beta_a + \dfrac{1}{k\Gamma_k(p)} \displaystyle\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\frac{p}{k} - 1} f(s, x(s)) \mathrm{d}s & x \in S_1, \\[4mm] \dfrac{(\psi(t) - \psi(a))^{\frac{\xi_k}{k} - 1}}{\Gamma(\xi_k)} \alpha_a + \dfrac{1}{k\Gamma_k(p)} \displaystyle\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\frac{p}{k} - 1} g(s, x(s)) \mathrm{d}s, & x \in S_2. \end{cases} \tag{4}$$

**Lemma 3.8.** *The operator* $T : S_1 \cup S_2 \to S$ *defined by* (4) *is cyclic if* $f$ *and* $g$ *are bounded and continuous such that* $f, g \in L^1(a, b)$.

**Proof.** For $x \in S_1$ and set $\xi_k = p + q(k - p)$, we have

$$Tx(t) = \frac{(\psi(t) - \psi(a))^{\frac{\xi_k}{k} - 1}}{\Gamma_k(\xi_k)} \beta_a + \frac{1}{k\Gamma_k(p)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\frac{p}{k} - 1} f(s, x(s)) \mathrm{d}s.$$

Applying ${}^{k}I_{a^+}^{k-\xi_k;\,\psi}$ on both sides and applying Lemmas 3.5 and 3.6, we obtain

$$\begin{aligned} {}^{k}I_{a^+}^{k-\xi_k;\,\psi}Tx(t) &= \frac{\beta_a}{\Gamma_k(\xi_k)}{}^{k}I_{a^+}^{k-\xi_k;\,\psi}(\psi(t) - \psi(a))^{\frac{\xi_k}{k} - 1} + {}^{k}I_{a^+}^{k-\xi_k;\,\psi k}I_{a^+}^{p;\,\psi}f(s, x(s))(t) \\[2mm] &= \frac{\beta_a}{\Gamma_k(k)} + [{}^{k}I_{a^+}^{k-q(k-p);\,\psi}f(s, x(s))](t). \end{aligned}$$

Using the fact that $\Gamma_k(k) = 1$ and $[I_a^{1-\theta_2(1-\theta_1);\,\psi}f(s, x(s))](t) \to 0$ as $t \to a$ (see the proof of Theorem 7.1 in [27]). Therefore, ${}^{k}I_{a^+}^{k-\xi_k;\,\psi}Tx(a) = \beta_a$ which means that $Tx(t) \in S_2$. Similarly, one can show that $Tx(t) \in S_1$ if $x \in S_2$. Thus, $T$ is a cyclic operator. □

We say that $z \in S_1 \cup S_2$ is an optimal solution for systems (2) and (3) provided that $\|z - Tz\| = \mathrm{dist}(S_1, S_2)$, that is $z$ is a best proximity point of the operator $T$ defined in (4).

**Assumptions:** We consider the following hypotheses to prove existence of optimal solutions.

(A1) Let $\mu$ be an MNC on $E$ and $\Lambda > 0$ such that for any bounded pair $(N_1, N_2) \subseteq (B_1, B_2)$,

$$\mu(f(\mathcal{J} \times N_1) \cup g(\mathcal{J} \times N_2)) \leq \Lambda\mu(N_1 \cup N_2),$$

(A2) $\|f(t, x(t)) - g(t, y(t))\| \leq \dfrac{k\Gamma_k(p)}{(\psi(b) - \psi(a))^{\frac{p}{k}}}(\|x(t) - y(t)\| - \dfrac{(\psi(b) - \psi(a))^{\frac{\xi_k}{k}-1}}{\Gamma_k(\xi_k)}\|\beta_a - \alpha_a\|)$, for all $(x, y) \in S_1 \times S_2$.

Following result is the Mean-Value Theorem for FDEs.

**Theorem 3.9.** [30,31] *Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $\alpha > 0$. Moreover, let $g \in L^1([a, b])$ be a function which does not change its sign on its domain. Then, for almost every $t \in [a, b]$, there exists some $\zeta \in (a, x)$ such that*

$$I_a^\alpha(fg)(t) = f(\zeta)I_a^\alpha g(t). \tag{5}$$

Then, we give the following result.

**Theorem 3.10.** *Under notations defined above, the hypotheses of Lemma 3.8 and assumptions $(A_1)$ and $(A_2)$, the system of Hilfer FDEs (2) and (3) has an optimal solution.*

**Proof.** It is clear that systems (2) and (3) have an optimal solution if the operator $T$ defined in (4) has a best proximity point.

From Lemma 3.8, $T$ is a cyclic operator. It follows trivially that $T(S_1)$ is a bounded subset of $S_2$. We prove that $T(S_1)$ is also an equicontinuous subset of $S_2$. For $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in S_1$, we observe that

$$\begin{aligned}
\|Tx(t_2) - Tx(t_1)\| &= \left\| \frac{(\psi(t_2) - \psi(a))^{\frac{\xi_k}{k}-1}}{\Gamma_k(\xi_k)}\beta_a + \frac{1}{k\Gamma_k(p)}\int_a^{t_2}\psi'(s)(\psi(t_2) - \psi(s))^{\frac{p}{k}-1}f(s, x(s))ds \right. \\
&\quad \left. - \frac{(\psi(t_1) - \psi(a))^{\frac{\xi_k}{k}-1}}{\Gamma_k(\xi_k)}\beta_a - \frac{1}{k\Gamma_k(p)}\int_a^{t_1}\psi'(t_1)(\psi(t_1) - \psi(s))^{\frac{p}{k}-1}f(s, x(s))ds \right\| \\
&= \left| \frac{\beta_a}{\Gamma_k(\xi_k)}((\psi(t_2) - \psi(a))^{\frac{\xi_k}{k}-1} - (\psi(t_1) - \psi(a))^{\frac{\xi_k}{k}-1}) \right. \\
&\quad + \frac{1}{k\Gamma_k(p)}\int_a^{t_1}[\psi'(s)(\psi(t_2) - \psi(s))^{\frac{p}{k}-1} - \psi'(s)(\psi(t_1) - \psi(s))^{\frac{p}{k}-1}]f(s, x(s))ds \\
&\quad \left. + \frac{1}{k\Gamma_k(p)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2) - \psi(s))^{\frac{p}{k}-1}f(s, x(s))ds \right| \\
&\leq \frac{\beta_a}{\Gamma_k(\xi_k)}|((\psi(t_2) - \psi(a))^{\frac{\xi_k}{k}-1} - (\psi(t_1) - \psi(a))^{\frac{\xi_k}{k}-1})| \\
&\quad + \frac{M}{k\Gamma_k(p)}\left|\int_a^{t_1}\left[\psi'(s)(\psi(t_2) - \psi(s))^{\frac{p}{k}-1} - \psi'(s)(\psi(t_1) - \psi(s))^{\frac{p}{k}-1}\right]ds\right| \\
&\quad + \frac{M}{k\Gamma_k(p)}\left|\int_{t_1}^{t_2}\psi'(s)(\psi(t_2) - \psi(s))^{\frac{p}{k}-1}ds\right|.
\end{aligned}$$

As $t_2 \to t_1$, right-hand side tends to 0. Thus, $\|Tx(t_2) - Tx(t_1)\| \to 0$ as $t_2 \to t_1$. Thus, $T(S_1)$ is equicontinuous. With the similar argument we can prove that $T(S_2)$ is bounded and equicontinuous subset of $S_1$. Thus, with the application of Arzela-Ascoli theorem we can conclude that $(S_1, S_2)$ is relatively compact.

Next we show that $T$ is relatively nonexpansive. For any $(x, y) \in S_1 \times S_2$, with assumption $(A_2)$, we have

$$\|Tx(t) - Ty(t)\| = \left\| \frac{\beta_a}{\Gamma_k(\xi_k)}(\psi(t) - \psi(a))^{\frac{\xi_k}{k}-1} + \frac{1}{k\Gamma_k(p)}\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\frac{p}{k}-1}f(s, x(s))\mathrm{d}s \right.$$

$$\left. - \left[\frac{\alpha_a}{\Gamma_k(\xi_k)}(\psi(t) - \psi(a))^{\frac{\xi_k}{k}-1} + \frac{1}{k\Gamma_k(p)}\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\frac{p}{k}-1}g(s, x(s))\mathrm{d}s\right] \right\|$$

$$= \frac{(\psi(t) - \psi(a))^{\frac{\xi_k}{k}-1}}{\Gamma_k(\xi_k)}|\beta_a - \alpha_a| + \left| \frac{1}{k\Gamma_k(p)}\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\frac{p}{k}-1}[f(s, x(s)) - g(s, y(s))]\mathrm{d}s \right|$$

$$\leq \frac{(\psi(b) - \psi(a))^{\frac{\xi_k}{k}-1}}{\Gamma_k(\xi_k)}\|\beta_a - \alpha_a\| + \frac{1}{k\Gamma_k(p)}\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\frac{p}{k}-1}\frac{k\Gamma_k(p)}{(\psi(b) - \psi(a))^{\frac{p}{k}}}\left[\|x(s) - y(s)\|\right.$$

$$\left. - \frac{(\psi(b) - \psi(a))^{\frac{\xi_k}{k}-1}}{\Gamma_k(\xi_k)}\|\beta_a - \alpha_a\|\right]\mathrm{d}s$$

$$= \|x - y\|,$$

and thereby, $\|Tx - Ty\| \leq \|x - y\|$. Therefore, $T$ is relatively nonexpansive.

At last, let $(K_1, K_2) \subseteq (S_1, S_2)$ be nonempty, closed, convex and proximinal pair which is $T$-invariant and such that $\mathrm{dist}(K_1, K_2) = \mathrm{dist}(S_1, S_2)(=\|\alpha_a - \beta_a\|)$. By using a generalized version of Arzela-Ascoli theorem (see Ambrosetti [32]) and assumption $(A_1)$, we obtain

$$\mu(T(K_1) \cup T(K_2)) = \max\{\mu(T(K_1)), \mu(T(K_2))\}$$

$$= \max\left\{\sup_{t\in\mathcal{J}}\{\mu(\{Tx(t) : x \in K_1\})\}, \sup_{t\in\mathcal{J}}\{\mu(\{Ty(t) : y \in K_2\})\}\right\}$$

$$= \max\left\{\sup_{t\in\mathcal{J}}\left\{\mu\left(\left\{\frac{\beta_a(\psi(t) - \psi(a))^{\frac{\xi_k}{k}-1}}{\Gamma_k(\xi_k)} + \frac{1}{k\Gamma_k(p)}\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\frac{p}{k}-1}f(s, x(s))\mathrm{d}s : x \in K_1\right\}\right)\right\},\right.$$

$$\left.\sup_{t\in\mathcal{J}}\left\{\mu\left(\left\{\frac{\alpha_a\ (\psi(t) - \psi(a))^{\frac{\xi_k}{k}-1}}{\Gamma_k(\xi_k)} + \frac{1}{k\Gamma_k(p)}\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\frac{p}{k}-1}g(s, y(s))\mathrm{d}s : y \in K_2\right\}\right)\right\}\right\}.$$

So, in view of Theorem (3.9), it follows that

$$\mu(T(K_1) \cup T(K_2)) \leq \max\left\{\sup_{t\in\mathcal{J}}\left\{\mu\left(\left\{\frac{\beta_a(\psi(t) - \psi(a))^{\frac{\xi_k}{k}-1}}{\Gamma_k(\xi_k)} + \frac{(\psi(t) - \psi(a))^{\frac{p}{k}}}{k\Gamma_k(p)}\overline{\mathrm{con}}(\{f(\sigma, x(\sigma)) : \sigma \in J\})\right\}\right)\right\},\right.$$

$$\left.\sup_{t\in\mathcal{J}}\left\{\mu\left(\left\{\frac{\alpha_a\ (\psi(t) - \psi(a))^{\frac{\xi_k}{k}-1}}{\Gamma_k(\xi_k)} + \frac{(\psi(t) - \psi(a))^{\frac{p}{k}}}{k\Gamma_k(p)}\overline{\mathrm{con}}(\{g(\sigma, x(\sigma)) : \sigma \in J\})\right\}\right)\right\}\right\}$$

$$= \max\left\{\frac{(\psi(t) - \psi(a))^{\frac{p}{k}}}{k\Gamma_k(p)}\mu(f(\mathcal{J} \times K_1)), \frac{(\psi(t) - \psi(a))^{\frac{p}{k}}}{k\Gamma_k(p)}\mu(g(\mathcal{J} \times K_2))\right\}$$

$$= \frac{(\psi(t) - \psi(a))^{\frac{p}{k}}}{k\Gamma_k(p)}\mu(f(\mathcal{J} \times K_1)) \cup \mu(g(\mathcal{J} \times K_2))$$

$$< \frac{\Lambda(\psi(t) - \psi(a))^{\frac{p}{k}}}{k\Gamma_k(p)}\mu(K_1 \cup K_2).$$

Since, $\psi$ is continuous, there exists a $t \in J$ such that $w = \frac{\Lambda(\psi(t) - \psi(a))^{\frac{p}{k}}}{k\Gamma_k(p)} < 1$. Thus, we obtain

$\mu(T(K_1) \cup T(K_2)) + \frac{1}{2} \leq 2w\mu(K_1 \cup K_2) + 1 - (w\mu(K_1 \cup K_2) + \frac{1}{2})$. Choosing $\Delta_1(t) = t + \frac{1}{2}$, $\Delta_2(t) = 2wt + 1$, and $\Delta_3 = wt + \frac{1}{2}$, we obtain

$$\Delta_1(\mu(T(K_1) \cup T(K_2))) \leq \Delta_2(\mu(K_1 \cup K_2)) - \Delta_3(\mu(K_1 \cup K_2)).$$

Therefore, we conclude that $T$ satisfies all the hypotheses of Corollary 2.4 and so the operator $T$ has a best proximity point $z \in S_1 \cup S_2$, which is an optimal solution for systems (2) and (3). □

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