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Research Article

Yanlin Li*, Fatemah Mofarreh, and Rashad A. Abdel-Baky

Kinematic-geometry of a line trajectory and the invariants of the axodes

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Abstract: In this article, we investigate the relationships between the instantaneous invariants of a one-parameter spatial movement and the local invariants of the axodes. Specifically, we provide new proofs for the Euler-Savary and Disteli formulas using the E. Study map in spatial kinematics, showcasing its elegance and efficiency. In addition, we introduce two line congruences and thoroughly analyze their spatial equivalence. Our findings contribute to a deeper understanding of the interplay between spatial movements and axodes, with potential applications in fields such as robotics and mechanical engineering.

Keywords: E. Study map, axodes, line congruence, Disteli's formula

MSC 2020: 53A04, 53A05, 53A17

1 Introduction

The study of invariants of a one-parameter spatial movement and axodes is of great importance in various fields, including mathematics, physics, and engineering. In the context of spatial movements, invariants provide valuable information about the behavior of objects as they move through space. One of the main reasons for studying the invariants of one-parameter spatial movements and axodes is to understand the geometric and kinematic properties of moving objects. Invariants can be used to describe the trajectory, velocity, and acceleration of an object as it moves through space, providing insights into its physical behavior. Moreover, it can be used to construct mathematical models of moving systems, which can be used to design and optimize complex engineering systems. The invariants of a one-parameter spatial movement can be used to characterize the geometry of ruled surfaces, such as their curvature and torsion. This relationship between invariants and ruled surfaces has important applications in fields such as computer graphics and architecture. Throughout the spatial movement of two rigid bodies, the instantaneous screw axis ($\mathbb{ISA}$) changes its position and ownerships and traces two various but connected ruled surfaces named the fixed axode in the fixed body and the moveable axode in the moveable body. Throughout the movement, the axodes roll and slide relative to each other in a specific way such that tangential contact between the axodes is constantly preserved through the entire length of the two mating rulings (one in each axode), which together locate the $\mathbb{ISA}$ at any instant. It is great that not only does a special movement give height to a unique set of axodes, but that the converse also applies. This shows that if the axodes of any movement are known, the given movement can be reconstructed without studying the physical elements of the mechanism, their organization, specific dimensions, or the procedure by which they are constructed [1–3].

*** Corresponding author: Yanlin Li,** School of Mathematics, Hangzhou Normal University, Hangzhou 311120, China, e-mail: liyl@hznu.edu.cn
Fatemah Mofarreh: Mathematical Science Department, Faculty of Sciences, Princess Nourah Bint Abdulrahman, Riyadh 11546, KSA, e-mail: fyalmofarrah@pnu.edu.sa
Rashad A. Abdel-Baky: Department of Mathematics, Faculty of Science, University of Assiut, Assiut 71516, Egypt, e-mail: rbaky@live.com

The utility of axodes is well-known from the work of Garnier [4] who extensively studies the kinematic ownerships of the relative instantaneous movement generated by the axodes up to the second order. Expressions for the velocity and acceleration of points in the moveable body are gained, and special cases of movement are investigated. In addition, he gave the Euler–Savary formulae for the spherical movement in Euclidean 3-space. A method for locating $\mathbb{ISA}$ and determining its properties has been developed by Phillips and Hunt [5]. Skreiner [6] modeled the average of change of the $\mathbb{ISA}$ by intermediary of a screw movement and researched several special cases. Dizioglu [7] generalized the method of Euler-Savary for the construction of the Disteli axis based on the local properties of the axodes. Abdel-Baky and Al-Solamy [8] provided a new geometric and kinematic approach to one-parameter spatial movement based on information specifying the movement of the axodes. There are various recent works in the literature dealing with $\mathbb{ISA}$ and the invariants of the axodes [9–11]. Dual numbers were first introduced by W. Clifford, and after him, E. Study utilized them as an instrument for his research on differential line geometry and kinematics. He gave special care to the impersonation of directed lines by dual unit vectors and defined the mapping that is known by his name. The E. Study map states that: The set of all directed lines in Euclidean 3-space is directly linked to the set of points on the dual unit sphere in the dual 3-space [1–3]. It allows a perfect generalization of mathematical statement for spherical point geometry to spatial line geometry by means of dual number extension, i.e., replacing all ordinary quantities with the corresponding dual numbers quantities. There exists a vast literature on the E. Study map including several monographs, e.g., [8–19].

In this article, based on E. Study map, the instantaneous invariants of the relative movement between two dual unit spheres are utilized for deriving the velocity and the acceleration of point trajectories (dual curve). The curvature possessions of line trajectories are obtained in terms of the invariants that describe the kinematics of the relative movements of two rigid bodies. The invariants of a line trajectory and their monarchies are derived from that of the axodes for a new proof of the Disteli formulae. Then by using E. Study map, two line congruences that are the locus of dual points having specific trajectories, as well as their special cases, examined in detail.

2 Elements of screw calculus

We begin with requisite concepts on dual numbers, dual vectors, and E. Study map (see [1–3]): An oriented line in Euclidean 3-space can be realized by a point $\boldsymbol{\alpha} \in L$ and a normalized direction vector $\mathbf{x}$ of L, i.e., $\langle \mathbf{x}, \mathbf{x} \rangle = 1$. To acquire components for L, one forms the moment vector $\mathbf{x}^* = \boldsymbol{\alpha} \times \mathbf{x}$ with respect to the origin point in $\mathbb{E}^3$. If $\boldsymbol{\alpha}$ is substituted by any point $\boldsymbol{\beta} = \boldsymbol{\alpha} + t\mathbf{x}$, $t \in \mathbb{R}$ on L, this shows that $\mathbf{x}^*$ is independent of $\boldsymbol{\alpha}$ on L. The two vectors $\mathbf{x}$ and $\mathbf{x}^*$ are not independent of one another and fulfill the following conditions:

$$\langle \mathbf{x}, \mathbf{x} \rangle = 1, \ \langle \mathbf{x}^*, \mathbf{x} \rangle = 0.$$

The six components x_i, x_i^* $(i = 1, 2, 3)$ of $\mathbf{x}$ and $\mathbf{x}^*$ are named the normalized Plücker coordinates of the line L. Thus, the two vectors $\mathbf{x}$ and $\mathbf{x}^*$ realize the oriented line L.

A dual number $\hat{x}$ is a number $x + \varepsilon x^*$, where x and x^* in $\mathbb{R}$ and ε is a dual unit with assets that $\varepsilon \neq 0$, and $\varepsilon^2 = 0$. Then, the set:

$$\mathbb{D}^3 = \{\hat{\mathbf{x}} = \mathbf{x} + \varepsilon \mathbf{x}^* = (\hat{x}_1, \hat{x}_2, \hat{x}_3)\},$$

together with the inner product

$$\langle \hat{\mathbf{x}}, \hat{\mathbf{y}} \rangle = \hat{x}_1\hat{y}_1 + \hat{x}_2\hat{y}_2 + \hat{x}_3\hat{y}_3,$$

forms the dual 3-space $\mathbb{D}^3$. The norm of $\hat{\mathbf{x}}$ is defined as follows:

$$\|\hat{\mathbf{x}}\| = \|\mathbf{x}\| + \varepsilon \frac{\langle \mathbf{x}^*, \mathbf{x} \rangle}{\|\mathbf{x}\|}, \ \|\mathbf{x}\| \neq 0.$$

Hence, we write the dual vector $\hat{\mathbf{x}}$ as a dual multiplier of a dual vector in the form

$$\hat{\mathbf{x}} = \|\hat{\mathbf{x}}\|\hat{\mathbf{e}},$$

where $\hat{\mathbf{e}}$ is referred to as the axis. The ratio

$$h = \frac{\langle \mathbf{x}^*, \mathbf{x}\rangle}{\|\mathbf{x}\|^2}$$

is called the pitch along the axis $\hat{\mathbf{e}}$. If $h = 0$ and $\|\mathbf{x}\| = 1$, $\hat{\mathbf{x}}$ is a directed line; when h is finite, $\hat{\mathbf{x}}$ is a proper screw; and when h is infinite, $\hat{\mathbf{x}}$ is called a couple. A dual vector with a norm equal to a unit is called a dual unit vector. Hence, every directed line $L = (\mathbf{x}, \mathbf{x}^*) \in \mathbb{E}^3 \times \mathbb{E}^3$ is represented by a dual unit vector

$$\hat{\mathbf{x}} = \mathbf{x} + \varepsilon\mathbf{x}^*(\langle \mathbf{x}, \mathbf{x}\rangle = 1,\ \langle \mathbf{x}^*, \mathbf{x}\rangle = 0).$$

The dual unit sphere in $\mathbb{D}^3$ is

$$\mathbb{K} = \{\hat{\mathbf{x}} \in \mathbb{D}^3 | \|\hat{\mathbf{x}}\|^2 = \hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 = 1\}.$$

Then, we have the E. Study map: The set of all points of dual unit sphere in dual 3-space is in one-to-one correspondence with the set of all directed lines in Euclidean 3-space.

2.1 One-parameter dual spherical movements

Let $\mathbb{K}_m$ and $\mathbb{K}_f$ be two dual unit spheres with $\mathbf{O}$ as a common center in $\mathbb{D}^3$. We assume that $\{\mathbf{O};\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$, and $\{\mathbf{O};\hat{\mathbf{f}}_1, \hat{\mathbf{f}}_2, \hat{\mathbf{f}}_3\}$ be two orthonormal dual frames rigidly linked to $\mathbb{K}_m$ and $\mathbb{K}_f$, respectively. If we let $\{\mathbf{O};\hat{\mathbf{f}}_1, \hat{\mathbf{f}}_2, \hat{\mathbf{f}}_3\}$ be fixed, whereas the elements of the set $\{\mathbf{O};\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ are functions of a real parameter $t \in \mathbb{R}$ (say the time). Then, we may say that $\mathbb{K}_m$ moves with respect to $\mathbb{K}_f$. Such movement is called a one-parameter dual spherical movement and denoted by $\mathbb{K}_m/\mathbb{K}_f$. If $\mathbb{K}_m$ and $\mathbb{K}_f$ correspond to the line spaces $\mathbb{L}_m$ and $\mathbb{L}_f$, respectively, then $\mathbb{K}_m/\mathbb{K}_f$ represents the one-parameter spatial movement $\mathbb{L}_m/\mathbb{L}_f$. Therefore, $\mathbb{L}_m$ is the moveable space with respect to the fixed space $\mathbb{L}_f$ in $\mathbb{E}^3$. We shall also define a further dual unit sphere $\mathbb{K}_r$ represented by the set $\{\mathbf{O};\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{r}}_3\}$, by the first-order instantaneous properties of the movement, which is defined below.

As we known, there is an instantaneous screw axis ($\mathbb{ISA}$) for the one-parameter spatial movement $\mathbb{L}_m/\mathbb{L}_f$. At any instant $t \in \mathbb{R}$, the $\mathbb{ISA}$ traces the moveable axode π_m in $\mathbb{L}_m$, and the fixed axode π_f in $\mathbb{L}_f$. We take $\hat{\mathbf{r}}_1(t) = \mathbf{r}_1(t) + \varepsilon\mathbf{r}_1^*(t)$ as the $\mathbb{ISA}$ of the movement $\mathbb{L}_m/\mathbb{L}_f$ and

$$\hat{\mathbf{r}}_2(t) = \mathbf{r}_2(t) + \varepsilon\mathbf{r}_2^*(t) = \frac{\mathrm{d}\hat{\mathbf{r}}_1}{\mathrm{d}t}\left\|\frac{\mathrm{d}\hat{\mathbf{r}}_1}{\mathrm{d}t}\right\|^{-1}$$

as the mutual central normal of two separated screw axes. A third dual unit vector is defined as $\hat{\mathbf{r}}_3(t) = \hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2$. This frame is named the relative Blaschke frame, and the congruous lines intersect at the common striction (central) point $\mathbf{S}$ of the axodes π_i $(i = m, f)$. The dual arc length $\mathrm{d}\hat{s}_i = \mathrm{d}s_i + \varepsilon\mathrm{d}s_i^*$ of $\hat{\mathbf{r}}_1(t)$ is $\mathrm{d}\hat{s}_i = \left\|\frac{\mathrm{d}\hat{\mathbf{r}}_1}{\mathrm{d}t}\right\|\mathrm{d}t = \hat{p}(t)\mathrm{d}t$. Since $\hat{p} = p + \varepsilon p^*$ contains only first derivatives of $\hat{\mathbf{r}}_1(t)$, it is a first-order property of the movement $\mathbb{K}_m/\mathbb{K}_f$, in particular its dual speed. We set $\mathrm{d}\hat{s} = \mathrm{d}s + \varepsilon\mathrm{d}s^*$ to indicate $\mathrm{d}\hat{s}_i$, since they are equal to each other. The distribution parameter of the axodes is

$$\mu(s) = \frac{p^*}{p} = \frac{\mathrm{d}s^*}{\mathrm{d}s}. \tag{1}$$

Remark 2.1. In each position of the movement, the axodes have the $\mathbb{ISA}$ of the position in common, i.e., the moveable axode contact with the fixed axode along the $\mathbb{ISA}$ in the first order at any instant t.

Furthermore, the motion $\mathbb{K}_r/\mathbb{K}_i$ is

$$\begin{pmatrix}\hat{\mathbf{r}}_1'\\ \hat{\mathbf{r}}_2'\\ \hat{\mathbf{r}}_3'\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ -1 & 0 & \hat{\gamma}_i\\ 0 & -\hat{\gamma}_i & 0\end{pmatrix} = \hat{\boldsymbol{\omega}}_i(\hat{s}) \times \begin{pmatrix}\hat{\mathbf{r}}_1\\ \hat{\mathbf{r}}_2\\ \hat{\mathbf{r}}_3\end{pmatrix}, \quad \left(' = \frac{\mathrm{d}}{\mathrm{d}\hat{s}}\right), \tag{2}$$

where $\hat{\boldsymbol{\omega}}_i(\hat{s}) \coloneqq \boldsymbol{\omega}_i + \varepsilon\boldsymbol{\omega}_i^* = \hat{\gamma}_i\hat{\mathbf{r}}_1 + \hat{\mathbf{r}}$ is the Darboux vector and $\hat{\gamma}(\hat{s}) \coloneqq \gamma + \varepsilon\gamma^*$ is the dual geodesic curvature of the axodes π_i. The tangent of $\mathbf{c}(s)$ is given as follows:

$$\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}s} = \Gamma_i(s)\mathbf{r}_1(s) + \mu(s)\mathbf{r}_3(s). \tag{3}$$

The dual geodesic curvature $\hat{\gamma}_i = \gamma_i + \varepsilon\gamma_i^*$ is defined in terms of γ_i, μ, and Γ_i as follows:

$$\hat{\gamma}_i = \gamma_i + \varepsilon(\Gamma_i - \mu\gamma_i) = \det(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_1', \hat{\mathbf{r}}_1''). \tag{4}$$

The Disteli-axis (curvature axis or evolute) of the axodes π_i is

$$\hat{\mathbf{b}}_i = \mathbf{b}_i + \varepsilon\mathbf{b}_i^* = \frac{\hat{\boldsymbol{\omega}}_i}{\|\hat{\boldsymbol{\omega}}_i\|} = \frac{\hat{\gamma}_i\hat{\mathbf{r}}_1 + \hat{\mathbf{r}}_3}{\sqrt{\hat{\gamma}_i^2 + 1}}. \tag{5}$$

Let $\hat{\phi}_i = \phi_i + \varepsilon\phi_i^*$ be the radius of curvature between $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{b}}_i$. Then,

$$\hat{\mathbf{b}}_i = \frac{\hat{\gamma}_i}{\sqrt{\hat{\gamma}_i^2 + 1}}\hat{\mathbf{r}}_1 + \frac{1}{\sqrt{\hat{\gamma}_i^2 + 1}}\hat{\mathbf{r}}_3 = \cos\hat{\phi}_i\hat{\mathbf{r}}_1 + \sin\hat{\phi}_i\hat{\mathbf{r}}_3, \tag{6}$$

where

$$\hat{\gamma}_i = \gamma_i + \varepsilon(\Gamma_i - \mu\gamma_i) = \cot\hat{\phi}_i. \tag{7}$$

Thus, we conclude the equality:

$$\hat{\gamma}_f - \hat{\gamma}_m = \cot\hat{\varphi}_f - \cot\hat{\varphi}_m. \tag{8}$$

It is a dual counterpart of a well-known formula of Euler-Savary from ordinary spherical kinematics [1–3]. This dual version provides a relationship among the two axodes in direct contact and the kinematic geometry corresponding to the instantaneous invariants of the one-parameter spatial movement $\mathbb{L}_m/\mathbb{L}_f$. From the real and the dual parts of equation (8), respectively, we obtain:

$$\cot\varphi_f - \cot\varphi_m = \gamma_f - \gamma_m, \tag{9}$$

and

$$\frac{\phi_m^*}{\sin^2\phi_m} - \frac{\phi_f^*}{\sin^2\phi_f} = \Gamma_m - \Gamma_f - \mu(\gamma_f - \gamma_m). \tag{10}$$

Equations (9) and (10) are new Disteli's formulae for the axodes of the movement $\mathbb{L}_m/\mathbb{L}_f$.

Now let us assume that the relative Blaschke frame is fixed in $\mathbb{K}_m$. Then,

$$\mathbb{K}_m/\mathbb{K}_f : \begin{pmatrix}\hat{\mathbf{r}}_1'\\ \hat{\mathbf{r}}_2'\\ \hat{\mathbf{r}}_3'\end{pmatrix} = \hat{\boldsymbol{\omega}} \times \begin{pmatrix}\hat{\mathbf{r}}_1\\ \hat{\mathbf{r}}_2\\ \hat{\mathbf{r}}_3\end{pmatrix}, \tag{11}$$

where

$$\hat{\boldsymbol{\omega}} \coloneqq \hat{\boldsymbol{\omega}}_f - \hat{\boldsymbol{\omega}}_m = \hat{\omega}\hat{\mathbf{r}}_1, \tag{12}$$

is the relative Darboux vector $\|\hat{\boldsymbol{\omega}}\| = \hat{\omega} = \hat{\omega} + \varepsilon\hat{\omega}^* = \gamma_r + \varepsilon(\Gamma_r + \mu\gamma_r)$ is the relative dual geodesic curvature. It follows that $\omega = \gamma_f - \gamma_m$ and $\omega^* = \Gamma_f - \Gamma_m - \mu(\gamma_f - \gamma_m)$ are the rotational angular speed and translational angular speed of the movement $\mathbb{L}_m/\mathbb{L}_f$, respectively, and they are both invariants in kinematics. Hence, the following corollary can be given:

Corollary 2.1. *For the one-parameter spatial movement $\mathbb{L}_m/\mathbb{L}_f$, at any instant $t \in \mathbb{R}$, the pitch of the movement is given as follows*:

$$h(s) = \frac{\omega^*}{\omega} = \frac{\Gamma_f - \Gamma_m}{\gamma_f - \gamma_m} - \mu. \tag{13}$$

In this study, we neglect the pure translational movements, i.e., $\omega^* \neq 0$. Moreover, we eliminate zero divisors $\omega = 0$. Therefore, we shall study only non-torsional movements, so that the axodes are non-developable ruled surfaces ($\mu \neq 0$).

3 Kinematic geometry of a line trajectory

Through the movement $\mathbb{L}_m/\mathbb{L}_f$ that, any fixed line $\hat{\mathbf{x}} \in \mathbb{L}_m$, generally, forms a ruled surface in the space $\mathbb{L}_f$ will be denoted by $(\hat{x})$ and its generator by $\hat{\mathbf{x}}$. Then, we can write

$$\hat{\mathbf{x}}(\hat{s}) = \hat{x}^t\hat{\mathbf{r}}, \hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} = \begin{pmatrix} x_1 + \varepsilon x_1^* \\ x_2 + \varepsilon x_2^* \\ x_3 + \varepsilon x_3^* \end{pmatrix}, \hat{\mathbf{r}} = \begin{pmatrix} \hat{\mathbf{r}}_1 \\ \hat{\mathbf{r}}_2 \\ \hat{\mathbf{r}}_3 \end{pmatrix}, \tag{14}$$

where

$$\begin{gathered} x_1^2 + x_2^2 + x_3^2 = 1, \\ x_1x_1^* + x_2x_2^* + x_3x_3^* = 0. \end{gathered} \tag{15}$$

The velocity $\hat{\mathbf{x}}'$ and the acceleration $\hat{\mathbf{x}}''$ of $\hat{\mathbf{x}}$ fixed in $\mathbb{K}_m$, respectively, are

$$\hat{\mathbf{x}}' = \hat{\boldsymbol{\omega}} \times \hat{\mathbf{x}} = \hat{\omega}(-x_3\hat{\mathbf{r}}_2 + x_2\hat{\mathbf{r}}_3) \tag{16}$$

and

$$\hat{\mathbf{x}}'' = \hat{x}_3\hat{\omega}\hat{\mathbf{r}}_1 - (\hat{x}_2\hat{\omega}^2 + x_3\hat{\omega}')\hat{\mathbf{r}}_2 + (\hat{x}_2\hat{\omega}' - \hat{x}_1\hat{\omega} - \hat{x}_3\hat{\omega}^2)\hat{\mathbf{r}}_3. \tag{17}$$

Then,

$$\hat{\mathbf{x}}' \times \hat{\mathbf{x}}'' = \hat{\boldsymbol{\omega}}^2[(1 - \hat{x}_1^2)\hat{\omega}\hat{\mathbf{r}}_1 + \hat{x}_3\hat{\mathbf{x}}]. \tag{18}$$

The dual arc length $\mathrm{d}\hat{u} = \mathrm{d}u + \varepsilon \mathrm{d}u^*$ of $\hat{\mathbf{x}}(\hat{s})$ is

$$\mathrm{d}\hat{u} = \|\hat{\mathbf{x}}'\|\mathrm{d}\hat{s} = \hat{\omega}\sqrt{1 - \hat{x}_1^2}\,\mathrm{d}\hat{s}. \tag{19}$$

The distribution parameter of $(\hat{x})$ is

$$\lambda(u) \coloneqq \frac{\mathrm{d}u^*}{\mathrm{d}u} = h - \frac{x_1x_1^*}{1 - x_1^2}. \tag{20}$$

The Blaschke frame $\{\mathbf{O}; \hat{\mathbf{x}}(\hat{s}), \hat{\mathbf{t}}(\hat{s}), \hat{\mathbf{g}}(\hat{s})\}$ is derived as follows:

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}(\hat{s}), \hat{\mathbf{t}}(\hat{s}) = \hat{\mathbf{x}}'\|\hat{\mathbf{x}}'\|^{-1}, \hat{\mathbf{g}}(\hat{s}) = \hat{\mathbf{x}} \times \hat{\mathbf{t}}. \tag{21}$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}\hat{u}}\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{g}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \hat{\chi} \\ 0 & -\hat{\chi} & 0 \end{pmatrix}\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{g}} \end{pmatrix} = \hat{\boldsymbol{\varpi}}(\hat{u}) \times \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{g}} \end{pmatrix}, \tag{22}$$

where $\widehat{\boldsymbol{\varpi}} = \boldsymbol{\varpi} + \varepsilon\boldsymbol{\varpi}^* = \widehat{\chi}\,\widehat{\mathbf{x}} + \widehat{\mathbf{g}}$ is the Darboux vector, and

$$\widehat{\chi}(\widehat{u}) = \gamma + \varepsilon(\Gamma - \lambda\gamma) = \det\left(\widehat{\mathbf{x}}, \frac{d\widehat{\mathbf{x}}}{d\widehat{u}}, \frac{d^2\widehat{\mathbf{x}}}{d\widehat{u}^2}\right) = \frac{\widehat{x}_1\widehat{\omega}(1 - \widehat{x}_1^2) + \widehat{x}_3}{\widehat{\omega}(1 - \widehat{x}_1^2)^{\frac{3}{2}}} \tag{23}$$

is the dual geodesic curvature of $\widehat{\mathbf{x}}(\widehat{u})$. The dual unit vectors $\widehat{\mathbf{x}}$, $\widehat{\mathbf{t}}$, and $\widehat{\mathbf{g}}$ match to three concurrent mutually orthogonal lines in $\mathbb{E}^3$. Their point of crossing is the central point $\mathbf{c}$ on the ruling $\widehat{\mathbf{x}}$. $\widehat{\mathbf{g}}(\widehat{s})$ is mutually perpendicular to $\widehat{\mathbf{x}}(\widehat{u})$ and $\widehat{\mathbf{x}}(\widehat{u} + d\widehat{u})$ and is named the central tangent of $(\widehat{x})$ at the central point. The traces of the central points is the striction curve. The line $\widehat{\mathbf{t}}$ is the central normal of $(\widehat{x})$ at the central point. The tangent vector of $\mathbf{c}(u)$ is

$$\frac{d\mathbf{c}}{du} = \Gamma\mathbf{x} + \lambda\mathbf{g}. \tag{24}$$

Here, $\gamma(u)$, $\Gamma(u)$, and $\lambda(u)$ are the curvature (construction) functions of $(\widehat{x})$. Furthermore, the Disteli-axis of $(\widehat{x})$ is as follows:

$$\widehat{\mathbf{b}}(\widehat{u}) = \frac{\widehat{\boldsymbol{\omega}}}{\|\widehat{\boldsymbol{\omega}}\|} = \frac{\widehat{\chi}\,\widehat{\mathbf{x}} + \widehat{\mathbf{g}}}{\sqrt{1 + \widehat{\chi}^2}} = \cos\widehat{\phi}\,\widehat{\mathbf{x}} + \sin\widehat{\phi}\,\widehat{\mathbf{g}}, \tag{25}$$

where $\widehat{\phi} = \phi + \varepsilon\phi^*$ is a dual angle (radius of curvature) between $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{b}}$. Hence, we conclude that

$$\cot\widehat{\phi} := \cot\phi - \varepsilon\phi^*(1 + \cot^2\phi) = \widehat{\chi}(\widehat{u}). \tag{26}$$

Thus, we have

$$\left.\begin{aligned} \widehat{\kappa}(\widehat{u}) &= \kappa + \varepsilon\kappa^* = \sqrt{1 + \widehat{\chi}^2} = \frac{1}{\sin\widehat{\phi}}, \\ \widehat{\tau}(\widehat{u}) &= \tau + \varepsilon\tau^* = \pm\frac{d\widehat{\psi}}{d\widehat{u}} = \pm\frac{1}{1 + \widehat{\chi}^2}\frac{d\widehat{\chi}}{d\widehat{u}}, \end{aligned}\right\} \tag{27}$$

where $\widehat{\kappa}(\widehat{u})$ is the dual curvature, and $\widehat{\tau}(\widehat{u})$ is the dual torsion of the dual curve $\widehat{\mathbf{x}}(\widehat{u})$.

3.1 Disteli formulae of a line trajectory

It is readily seen from equation (16) that $\widehat{\mathbf{x}}'$ is orthogonal to both $\widehat{\boldsymbol{\omega}}$ and $\widehat{\mathbf{x}}$. If $\widehat{\vartheta} = \vartheta + \varepsilon\vartheta^*$ is the dual angle between $\widehat{\boldsymbol{\omega}}$ and $\widehat{\mathbf{x}}$, then we can write

$$\|\widehat{\mathbf{x}}'\| = \widehat{\omega}\sin\widehat{\vartheta}. \tag{28}$$

Then, there exists a mutual perpendicular $\widehat{\mathbf{m}}$ of $\widehat{\mathbf{r}}_1$ and $\widehat{\mathbf{t}}$ (see Figure 1). Therefore, with the chosen sense of rotation angles, one can write

$$\widehat{\mathbf{x}} = \cos\widehat{\vartheta}\,\widehat{\mathbf{r}}_1 + \sin\widehat{\vartheta}\,\widehat{\mathbf{m}}. \tag{29}$$

The dual angle $\widehat{\varphi} = \varphi + \varepsilon\varphi^*$ among the dual unit vectors $\widehat{\mathbf{r}}_3$ and $\widehat{\mathbf{t}}$ leads to

$$\widehat{\mathbf{m}} = \cos\widehat{\varphi}\,\widehat{\mathbf{r}}_2 + \sin\widehat{\varphi}\,\widehat{\mathbf{r}}_3. \tag{30}$$

Hence, we can write

$$\widehat{\mathbf{x}} = \cos\widehat{\vartheta}\,\widehat{\mathbf{r}}_1 + \sin\widehat{\vartheta}(\cos\widehat{\varphi}\,\widehat{\mathbf{r}}_2 + \sin\widehat{\varphi}\,\widehat{\mathbf{r}}_3), \tag{31}$$

where the dual angles $\widehat{\varphi} = \varphi + \varepsilon\varphi^*$, $\widehat{\vartheta} = \vartheta + \varepsilon\vartheta^*$; $0 \le \varphi \le 2\pi$, $0 \le \vartheta \le \pi$, and $\varphi^*, \vartheta^* \in \mathbb{R}$. This choice of coordinates is such that a screw movement of angle φ about the $\mathbb{ISA}$ and distance φ^* along it caries $\widehat{\mathbf{r}}_3$ to be the central normal $\widehat{\mathbf{t}}$ of $\widehat{\mathbf{x}}$. Substituting equation (31) into equation (23), we have

$$\widehat{\chi}(\widehat{u}) = \cot\widehat{\vartheta} + \frac{\sin\widehat{\varphi}}{\widehat{\omega}\sin^2\widehat{\vartheta}}, \tag{32}$$

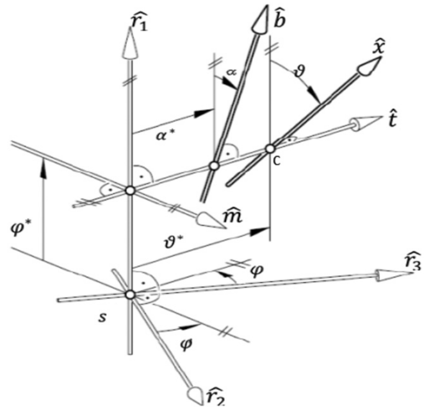

Figure 1: The line $\hat{\mathbf{x}}$ and its Disteli-axis $\hat{\mathbf{b}}$.

or by combining it with equation (26), we obtain

$$\cot\hat{\phi} - \cot\hat{\vartheta} = \frac{\sin\hat{\varphi}}{\hat{\omega}\sin^2\hat{\vartheta}}. \tag{33}$$

Equation (33) shows the relationship between the dual spherical curve $\hat{\mathbf{x}}(\hat{u})$ on $\mathbb{K}_f$, which is corresponding to a line trajectory, and its osculating dual cone, which is corresponding to its Disteli-axis at any instant (see Figure 1). From the real and the dual parts, respectively, we obtain:

$$\cot\phi - \cot\vartheta = \frac{\sin\varphi}{\omega\sin^2\vartheta}, \tag{34}$$

and

$$\phi^* = \frac{\sin^2\phi}{\omega}[(h + 2\vartheta^*\cot\vartheta)\sin\varphi - \varphi^*\cos\varphi] + \frac{\vartheta^*\sin^2\phi}{\sin^2\vartheta}. \tag{35}$$

Equations (34) and (35) are new Disteli formulae for the one-parameter spatial movement $\mathbb{L}_m/\mathbb{L}_f$; the first equation reveals the connection among the positions of the line $\hat{\mathbf{x}}$ in the space $\mathbb{L}_m$ and the Disteli-axis $\hat{\mathbf{b}}$. The second one describes the distance from the line $\hat{\mathbf{x}}$ to the Disteli-axis $\hat{\mathbf{b}}$. The striction point $\mathbf{c}(\varphi^*, \vartheta^*)$ may be on the $\mathbb{ISA}$ if $\vartheta^* = 0$, and on the Disteli-axis $\hat{\mathbf{b}}$ if $\phi^* = 0$. Note that the striction point is the origin of the relative Blaschke frame, i.e., $\mathbf{S} = \mathbf{0}$, see Figure 1. This means that $\vartheta = c_1$ (real const.) and $\vartheta^* = c_2$ (real const.).

4 Inflection and torsion line congruences

In this section, a method for determination of a line congruence is given by using dual vector calculus. Then, analogous to both planar and spherical movements, the well-known inflection, and torsion curves are calculated on $\mathbb{K}_f$. Consequently, two line congruences are introduced, and their geometrical-kinematical properties are examined in detail.

Since $\varepsilon^2 = \varepsilon^3 = \ldots = 0$, the Plücker coordinates of $\hat{\mathbf{x}}$ are:

$$\left.\begin{array}{l} x_1 = \cos\vartheta,\ x_1^* = -\vartheta^*\sin\vartheta, \\ x_2 = \sin\vartheta\cos\varphi,\ x_2^* = \vartheta^*\cos\vartheta\cos\varphi - \varphi^*\sin\varphi\sin\vartheta, \\ x_3 = \sin\vartheta\sin\varphi,\ x_3^* = \varphi^*\cos\vartheta\sin\varphi + \varphi^*\cos\varphi\sin\vartheta. \end{array}\right\} \tag{36}$$

Let $\boldsymbol{\beta}(\beta_1, \beta_2, \beta_3)$ indicate a point on $\hat{\mathbf{x}}$. Since $\mathbf{x}^* = \boldsymbol{\beta} \times \mathbf{x}$, we have the system of linear equations in β_i for i=1, 2, 3 (β_i are the coordinates of $\boldsymbol{\beta}$):

$$\beta_2 \sin\vartheta \sin\varphi - \beta_3 \sin\vartheta \cos\varphi = x_1^*, \quad -\beta_1 \sin\vartheta \sin\varphi + \beta_3 \cos\vartheta = x_2^*, \quad \beta_1 \sin\vartheta \cos\varphi - \beta_2 \cos\vartheta = x_3^*. \tag{37}$$

The matrix of coefficients of unknowns β_1, β_2, and β_3 is the skew symmetric matrix

$$\begin{pmatrix} 0 & \sin\vartheta \sin\varphi & -\sin\vartheta \cos\varphi \\ -\sin\vartheta \sin\varphi & 0 & \cos\vartheta \\ \sin\vartheta \cos\varphi & -\cos\vartheta & 0 \end{pmatrix},$$

and thus its rank is 2 with $\vartheta \neq 2\pi k$ (k is an integer). The rank of the augmented matrix

$$\begin{pmatrix} 0 & \sin\vartheta \sin\varphi & -\sin\vartheta \cos\varphi & x_1^* \\ -\sin\vartheta \sin\varphi & 0 & \cos\vartheta & x_2^* \\ \sin\vartheta \cos\varphi & -\cos\vartheta & 0 & x_2^* \end{pmatrix}$$

is also 2. Thereby, this system has infinite solutions given as follows:

$$\beta_2 \sin\varphi - \beta_3 \cos\varphi = -\vartheta^* \quad \beta_2 = (\beta_1 - \varphi^*) \tan\vartheta \cos\varphi - \vartheta^* \sin\varphi, \quad \beta_3 = (\beta_1 - \varphi^*) \tan\vartheta \sin\varphi + \vartheta^* \cos\varphi. \tag{38}$$

Since β_1 can be arbitrarily, then we may set $\beta_1 = \varphi^*$. In this case, equation (38) reduces to

$$\beta_1 = \varphi^*, \beta_2 = -\vartheta^* \sin\varphi, \beta_3 = \vartheta^* \cos\varphi. \tag{39}$$

Let $\boldsymbol{y}$ (y_1, y_2, y_3) denote a point on the oriented line $\hat{\mathbf{x}}$. We can write:

$$\mathbf{y}(\varphi, \varphi^* v) = \begin{pmatrix} \varphi^* + v \cos\vartheta \\ -\vartheta^* \sin\varphi + v \sin\vartheta \cos\varphi \\ \vartheta^* \cos\varphi + \sin\vartheta \sin\varphi \end{pmatrix}, \quad v \in \mathbb{R}, \tag{40}$$

which represents two-parametric family of oriented lines or a line congruence.

Definition 4.1. Throughout the motion $\mathbb{K}_m/\mathbb{K}_f$, the locus of dual points, whose trajectories have a vanishing dual geodesic curvature in $\mathbb{K}_f$, is called the inflection dual curve.

By definition, from equation (26), we have

$$\widehat{\chi}(\hat{u}) = 0 \Leftrightarrow \cot\widehat{\phi} = 0 \Leftrightarrow \phi = \frac{\pi}{2}, \quad \text{and} \quad \phi^* = 0 \Leftrightarrow \gamma = 0, \quad \text{and} \quad \Gamma = 0. \tag{41}$$

In this instant, the oriented lines $\hat{\mathbf{x}}$, $\hat{\mathbf{t}}$, and $\hat{\mathbf{b}}$ are the Blaschke frame of $(\widehat{x})$. Further, from equation (33) the dual points satisfying $\widehat{\chi}(\hat{u}) = 0$, we have

$$\hat{c} : \widehat{\omega} \sin 2\widehat{\vartheta} + 2 \sin\widehat{\varphi} = 0. \tag{42}$$

The real part of equation (42) identifies the inflection cone for the spherical part of the movement $\mathbb{L}_m/\mathbb{L}_f$ and is given as follows:

$$c : \omega \sin 2\vartheta + 2 \sin\varphi = 0. \tag{43}$$

The intersection of the inflection cone with a real unit sphere centered at the apex of the cone defines a spherical curve. There is a plane for every line direction of a line of the inflection cone, defined by the dual part of equation (42):

$$\pi : \omega^* \sin 2\vartheta + 2\omega\vartheta^* \cos 2\vartheta + 2\varphi^* \cos\varphi = 0. \tag{44}$$

If equation (43) is solved with respect to ϑ, we obtain

$$\sin 2\vartheta = -\left(\frac{2 \sin\varphi}{\omega}\right), \quad \text{and} \quad \cos 2\vartheta = \pm\frac{1}{\omega}\sqrt{\omega^2 - 4 \sin^2\varphi}. \tag{45}$$

By substituting equation (45) into equation (44), we find

$$\pi : h\sin\varphi \mp \sqrt{\omega^2 - 4\sin^2\varphi}\,\vartheta^* - \varphi^*\cos\varphi = 0. \tag{46}$$

Equation (46) is linear in the position coordinates φ^* and ϑ^* of the oriented line $\hat{\mathbf{x}}$. Therefore, for a one-parameter spatial movement $\mathbb{L}_m/\mathbb{L}_f$, the lines in a given fixed direction in $\mathbb{L}_m$-space lie on a plane. As shown in Figure 2, the angle φ identifies the central normal $\hat{\mathbf{t}}$, thus equation (46) defines two lines L^+ and L^- in the plane spanned by $\hat{\mathbf{t}}$ and the $\mathbb{ISA}$ (L^+ and L^- are corresponding to the inflection circle in planar kinematics). If the distance ϑ^* along the central normal $\hat{\mathbf{t}}$ from the $\mathbb{ISA}$ is taking as the independent parameter, then equation (46) becomes

$$\pi : \varphi^* = \mp\frac{\sqrt{\omega^2 - 4\sin^2\varphi}}{\cos\varphi}\vartheta^* + h\tan\varphi. \tag{47}$$

We remark that L^+ (or L^-) will change its location if the parameter ϑ is defined as a various value, but φ = constant. Meanwhile, the location of the plane π is changed if the parameter φ of L^+ (or L^-) has various value, but ϑ = consent. Therefore, the set of all directed lines L^+ and L^- defined by equation (47) is inflection line congruence for all values of (φ^*, ϑ^*) (Figure 2).

Now it is simple to demonstrate a parametric representation of the inflection line congruence. For this objective, from equation (43), we obtain

$$\varphi = \sin^{-1}\left(-\frac{\omega\sin 2\vartheta}{2}\right), \tag{48}$$

and by combining it with the real part of equation (36), we obtain

$$x(\vartheta) = \left(\cos\vartheta, \sqrt{1 - \left(\frac{\omega}{2}\sin^2 2\vartheta\right)^2}\,\sin\vartheta, -\frac{1}{2}\omega\sin 2\vartheta\sin\vartheta\right), \tag{49}$$

which is the inflection curve of the spherical part of movement $\mathbb{L}_m/\mathbb{L}_f$, where $0 \le \vartheta \le \pi$, with $\omega = 1$ (see Figure 3). Furthermore, from equations (48), (49), and (40), we know that

$$\mathbf{y}(\vartheta, \varphi^*, v) = \begin{pmatrix} \varphi^* + v\cos\vartheta \\ \vartheta^*\dfrac{\omega\sin 2\vartheta}{2} + v\sqrt{1 - \left(\dfrac{\omega}{2}\sin^2 2\vartheta\right)^2}\,\sin\vartheta \\ \vartheta^*\sqrt{1 - \left(\dfrac{\omega}{2}\sin^2 2\vartheta\right)^2} - \dfrac{v}{2}\omega\sin 2\vartheta\sin\vartheta \end{pmatrix}, \quad v \in \mathbb{R}. \tag{50}$$

This inflection line congruence composes of the ruled surfaces $\mathbf{y}(\vartheta, \varphi^*, v)$, $\vartheta = \varphi^* = t \in \mathbb{R}$, $\mathbf{y}(\vartheta, \varphi_0^*, v)$, φ_0^* = constant, and $\mathbf{y}(\vartheta_0, \varphi^*, v)$, ϑ_0 = constant. If we choose $\omega = 1$, $\vartheta_0^* = 1$, $\varphi_0^* = 0$ for example, then equation (50)

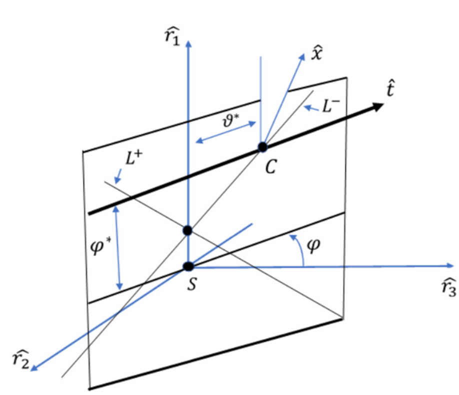

Figure 2: Lines of the inflection line congruence.

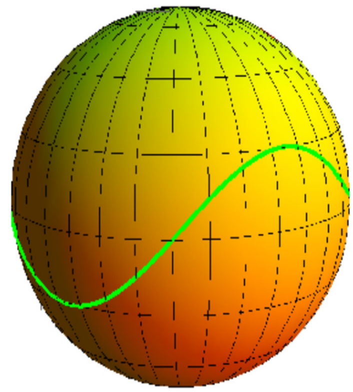

Figure 3: Inflection curve.

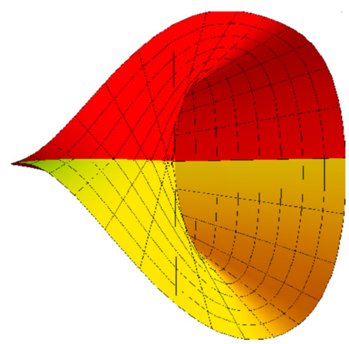

Figure 4: Ruled surface.

represents a ruled surface in the inflection line congruence. The graph of the ruled surface is shown in Figure 4, $0 \leq \vartheta \leq \pi$, and, $-2 \leq v \leq 2$.

Definition 4.2. Throughout the movement $\mathbb{K}_m/\mathbb{K}_f$, the locus of dual points, whose trajectories have a vanishing dual torsion on $\mathbb{K}_f$, is called the torsion dual curve or circling point dual curve.

By Definition 4.2 and equation (27), we obtain the condition

$$\hat{\tau}(\hat{u}) = 0 \Leftrightarrow \frac{d\hat{\chi}}{d\hat{u}} = 0 \Leftrightarrow \gamma = \text{const. and} \quad \Gamma - \lambda\gamma = \text{const.} \tag{51}$$

In this instance, the line trajectories maintain the same dual angle $\hat{\phi}$ with respect to the Disteli-axis $\hat{\mathbf{b}}$ up to the third order. Thus, locally, the ruled surface $(\hat{x})$ is traced. Throughout a one-parameter helical movement of constant pitch h along the constant Disteli-axis $\hat{\mathbf{b}}$, by the line $\hat{\mathbf{x}}$ existing at a constant distance ϕ^* and constant angle ϕ relative to $\hat{\mathbf{b}}$. This implies that the striction curve of $(\hat{x})$ is a cylindrical helix. Then, the torsion dual curve can be described by the following:

Theorem 4.1. *Throughout the one-parameter dual spherical movement* $\mathbb{K}_m/\mathbb{K}_f$, *the dual curve* $\hat{\mathbf{x}}(\hat{u}) \in \mathbb{K}_f$ *has a vanishing dual torsion iff* $\hat{\chi}(\hat{u})$ *is constant.*

Corollary 4.1. *The ruled surface* $(\hat{x})$ *has a constant Disteli-axis iff* (*a*) γ= *constant, and* (*b*) $\Gamma - \lambda\gamma$ = *constant.*

Furthermore, from equations (27) and (32), we have:

$$\hat{\tau}(\hat{u}) = \pm\frac{\hat{\omega}[3\cos\hat{\vartheta}\,\sin\hat{\varphi} + (\hat{\omega} - \hat{\gamma})\sin\hat{\vartheta}]\cos\hat{\varphi} - \hat{\omega}'\sin\hat{\vartheta}\,\sin\hat{\varphi}}{\hat{\omega}^2\hat{\kappa}^2\sin^3\hat{\vartheta}},$$

or by combining it with equation (51), we obtain

$$\hat{\tau}(\hat{u}) = 0 \Leftrightarrow \cot\hat{\vartheta} = \hat{a}\csc\hat{\varphi} + \hat{b}\sec\hat{\varphi}, \tag{52}$$

where

$$\hat{a} = a + \varepsilon a^* = \frac{\hat{\gamma} - \hat{\omega}}{3}, \quad \text{and} \quad \hat{b}(u) = b + \varepsilon b^* = \frac{\hat{\omega}'}{3\hat{\omega}}. \tag{53}$$

If equation (53) is solved with respect to the angle $\hat{\vartheta}$, then we obtain

$$\hat{\vartheta} = \cot^{-1}(\hat{a}\csc\hat{\varphi} + \hat{b}\sec\hat{\varphi}). \tag{54}$$

By a similar procedure, the spherical curve $\mathbf{x}(\varphi)$ can be expressed as follows:

$$\mathbf{x}(\varphi) = \frac{1}{\sqrt{1 + (a\csc\varphi + b\sec\varphi)^2}}(a\csc\varphi + b\sec\varphi, \cos\varphi, \sin\varphi). \tag{55}$$

This parametrization defines the spherical part of the movement $\mathbb{L}_m/\mathbb{L}_f$. For $a = 1, b = 0.3$, and $0 \le \varphi \le 2\pi$, the curve is shown in Figure 5. Similarly, there are associated plane of lines with each direction of the torsion cone, defined by

$$a^*\csc\varphi + b^*\sec\varphi + \varphi^*(b\sec\varphi\tan\varphi - a\csc\varphi\cot\varphi) + \vartheta^* c\sec^2\vartheta = 0. \tag{56}$$

Then, from equations (40), (55), and (56), we know that

$$\mathbf{y}(\varphi, \varphi^* v) = \begin{pmatrix} \varphi^* + \dfrac{v(a\csc\varphi + b\sec\varphi)}{\sqrt{(a\csc\varphi + b\sec\varphi)^2 + 1}} \\ -\vartheta^*\sin\varphi + \dfrac{v\cos\varphi}{\sqrt{(a\csc\varphi + b\sec\varphi)^2 + 1}} \\ \vartheta^*\cos\varphi + \dfrac{v\sin\varphi}{\sqrt{(a\csc\varphi + b\sec\varphi)^2 + 1}} \end{pmatrix}, \quad v \in \mathbb{R}. \tag{57}$$

If we take $a = \varphi^* = \vartheta^* = 1, b = 0, 0 \le \varphi \le 2\pi$, and $-4 \le v \le 4$, then we immediately obtain a ruled surface in the torsion line congruence (Figure 6).

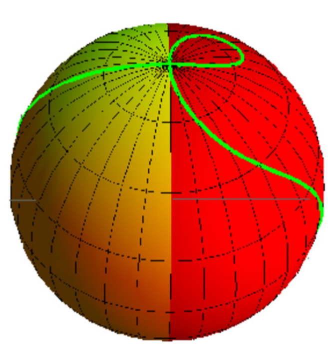

Figure 5: Torsion curve.

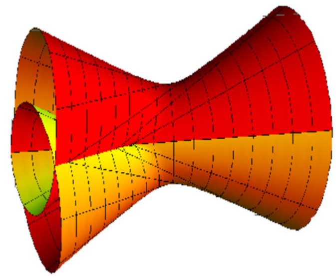

Figure 6: Ruled surface.

5 Conclusion

In this study, we have utilized E. Study map as a direct method for analyzing the kinematic-geometry of one-parameter spatial movement by exploring the properties of axodes and their similarity to spherical kinematics. We have also examined the invariants of a line trajectory and derived dual versions of the planar Euler-Savary equation, resulting in several expressions that depend on the axodes. In addition, we have investigated the theoretical expressions for the direction of a line congruence. These results have the potential to broaden the use of geometric properties of ruled surfaces traced by lines embedded in spatial mechanisms. Our results in this article can contribute to the field of spatial kinematics and have practical applications in mechanical mathematics and engineering. Our future research will focus on exploring some applications of our main discoveries. We plan to integrate concepts from singularity theory, submanifold theory, and other relevant results (referenced in [20–36]) to investigate promising avenues within this article.

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