Research Article

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On split generalized equilibrium problem with multiple output sets and common fixed points problem

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Abstract: In this article, we introduce and study the notion of split generalized equilibrium problem with multiple output sets (SGEPMOS). We propose a new iterative method that employs viscosity approximation technique for approximating the common solution of the SGEPMOS and common fixed point problem for an infinite family of multivalued demicontractive mappings in real Hilbert spaces. Under mild conditions, we prove a strong convergence theorem for the proposed method. Our method uses self-adaptive step size that does not require prior knowledge of the operator norm. The results presented in this article unify, complement, and extend several existing recent results in the literature.

Keywords: split generalized equilibrium problem, viscosity approximation method, multivalued demicontractive mapping, fixed point problem, strong convergence

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1 Introduction

Let C and Q be nonempty, closed, and convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, with its adjoint $A^* : H_2 \to H_1$. The split feasibility problem (SFP) introduced and studied by Censor and Elfving [1] for modelling inverse problems is to find an element

$$x^* \in C \ \text{ such that } \ Ax^* \in Q. \tag{1}$$

The SFP has been found useful in the study of several problems such as phase retrievals, signal processing, medical image reconstruction, and radiation therapy treatment planning (see, for instance, [2–5] and the references contained therein). Several optimization problems such as split variational inequality problem (SVI_qP), split variational inclusion problem, split equilibrium problem (SEP), split common fixed point problem are all generalizations of the SFP (see, e.g., [6–13] and the references therein).

In 2013, Kazmi and Rizvi [14] introduced and studied the split generalized equilibrium problem (SGEP) in real Hilbert spaces, which is formulated as finding an element $x^* \in C$ such that

$$F_1(x^*, x) + \phi_1(x^*, x) \geq 0 \quad \forall x \in C \tag{2}$$

and

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$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) + \phi_2(y^*, y) \geq 0 \quad \forall y \in Q, \tag{3}$$

where $F_1, \phi_1 : C \times C \to \mathbb{R}$ and $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ are nonlinear bifunctions and $A : H_1 \to H_2$ is a bounded linear operator. We denote the set of solutions of the SGEP (2)–(3) by

$$\text{SGEP}(F_1, \phi_1, F_2, \phi_2) := \{x^* \in C : x^* \in \text{GEP}(F_1, \phi_1) \quad \text{and} \quad Ax^* \in \text{GEP}(F_2, \phi_2)\},$$

If $F_2 = 0$ and $\phi_2 = 0$, the SGEP reduces to the generalized equilibrium problem (GEP) studied by Cianciruso et al. [15] which is defined as finding an element $x^* \in C$ such that

$$F(x^*, x) + \phi(x^*, x) \geq 0 \quad \forall x \in C, \tag{4}$$

where $F : C \times C \to \mathbb{R}$ and $\phi : Q \times Q \to \mathbb{R}$ are two bifunctions. We denote by $\text{GEP}(F, \phi)$ the solution set of the GEP (4). The GEP includes as particular cases, minimization problems, fixed point problems (FPPs), Nash equilibrium problems in noncooperative games, mixed equilibrium problems, variational inequality problems (VIPs) to mention but few, see, e.g., [16–25]. When $\phi \equiv 0$ in equation (4), the GEP reduces to the classical equilibrium problem (EP) as described by Blum and Oettli [26].

Observe also from equations (2) and (3) that when $\phi_1, \phi_2 = 0$, the SGEP reduces to SEP as stated by Suantai et al. [27], which is defined as finding an element $x^* \in C$ such that

$$F_1(x^*, x) \geq 0 \quad \forall x \in C$$

such that

$$y^* = Ax^* \text{ solves } F_2(y^*, y) \geq 0 \quad \forall y \in Q.$$

We denote by $\text{SEP}(F_1, F_2)$, the solution set of the SEP.

Recently, Reich and Tuyen [28] introduced the following generalized split common null point problem (GSCNPP): for $i = 1, 2, \ldots, N$, let H_i be real Hilbert spaces and let $B_i : H_i \to 2^{H_i}$ be maximal monotone operators on H_i, respectively. Furthermore, let $A_i : H_i \to H_{i+1}$ be bounded linear operators for $i = 1, 2, \ldots, N-1$, such that $A_i \neq 0$. The GSCNPP is defined as find an element $x^* \in H_1$ such that

$$0 \in B_1(x^*), \quad 0 \in B_2(A_1x^*), \ldots, \quad 0 \in B_N(A_{N-1}(A_{N-2}, \ldots, A_1(x^*))). \tag{5}$$

Very recently, Reich and Tuyen [29] introduced and studied the split common null point problem with multiple output sets (SCNPPMOS) in real Hilbert spaces as follows: let $H, H_1, \ldots, H_N$ be real Hilbert spaces and let $A_i : H \to H_i$, $i = 1, 2, \ldots N$ be bounded linear operators. Let $B : H \to 2^H$ and $B_i : H_i \to 2^{H_i}$, $i = 1, 2, \ldots, N$ be maximal monotone operators, the SCNPPMOS is to find an element x^* such that

$$x^* \in B^{-1}(0) \cap \left(\bigcap_{i=1}^{N} A_i^{-1}(B_i^{-1}(0))\right) \neq \emptyset. \tag{6}$$

Reich and Tuyen [29] proposed two algorithms for approximating the solutions of the SCNPPMOS. Moreover, the authors established the relationship between equations (5) and (6).

In this study, we introduce and study the notion of split generalized equilibrium problem with multiple output sets. Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let C_i be nonempty closed convex subset of Hilbert spaces H_i and let $A_i : H \to H_i$ be bounded linear operators. Let $F, \phi : C \times C \to \mathbb{R}, F_i, \phi_i : C_i \times C_i \to \mathbb{R}$ be bifunctions. The SGEPMOS is formulated as finding a point $x^* \in C$ such that

$$x^* \in GEP(F, \phi) \cap \left(\bigcap_{i=1}^{N} A_i^{-1}(GEP(F_i, \phi_i))\right) \neq \emptyset. \tag{7}$$

We denote by Ω the solution set of the SGEPMOS (7).

Remark 1.1. The major motivation for extending the study of SGEP (2)–(3) to SGEPMOS (7) lies in its potential application to mathematical models whose constraints can be expressed as equation (7). This arises in real-life problems such as signal processing, network resource allocation, and image recovery (see, e.g., [30, 31] and the references therein).

In this article, we also consider FPP for nonlinear mappings. The FPP finds application in so many real life problems such as optimization problems as well as in proving the existence of solutions of many physical problems arising in differential and integral equations (see, e.g., [32,33]). We denote by $\mathrm{Fix}(S)$ the fixed points set of S, i.e., $\mathrm{Fix}(S) \coloneqq \{x^* \in C : x^* = Sx^*\}$, where $S : C \to C$ is a nonlinear mapping.

If S is a multivalued mapping, i.e., $S : C \to 2^C$, then $x^* \in C$ is called a fixed point of S if

$$x^* \in Sx^*. \tag{8}$$

The fixed point theory for multivalued mappings can be utilized in various areas such as game theory, control theory, and mathematical economics.

In this article, we consider the problem of approximating a common solution of the SGEPMOS (7) and common FPP for an infinite family of multivalued demicontractive mappings. That is, find x^* such that

$$x^* \in \bigcap_{j=1}^{+\infty} \mathrm{Fix}(S_j) \cap GEP(F, \phi) \cap \left(\bigcap_{i=1}^{N} A_i^{-1}(GEP(F_i, \phi_i)) \right), \tag{9}$$

where $S_j : H \to CB(H), \quad j = 1, 2, \ldots$, is an infinite family of multivalued demicontractive mappings. We denote by Γ the solution set of problem (9).

Motivated and inspired by some studies [28,29,34–36], and the ongoing research in this direction, we introduce and study the notion of SGEPMOS. We propose a self-adaptive iterative method for approximating a common solution of the SGEPMOS (7) and common FPP for an infinite family of multivalued demicontractive mappings in real Hilbert spaces. The current study is more general as it includes several other optimization problems as special cases. In simple and clear terms, the proposed method of this article has the following features:

(1) The current literature extends the works of [35,36] from SGEP to the problem of SGEPMOS in Hilbert spaces.
(2) Our method uses simple self-adaptive step size that is generated at each iteration by some simple computation. Thus, the implementation of our method does not depend on the prior knowledge of the norm of the bounded linear operators. This feature is important as algorithms whose implementation depends on the operator norm require the computation of the norm of the bounded linear operator, which in general is very difficult to accomplish.
(3) The sequence generated by our proposed method converges strongly to the solution of the problem (9) in real Hilbert spaces. Moreover, we prove strong convergence theorem for the proposed method without following the conventional "Two Cases Approach" widely employed in several articles to guarantee strong convergence.

The rest of this article is organized as follows: in Section 2, we shall recall some useful definitions and lemmas that are needed in the proof of our main results; in Section 3, we present our proposed algorithm and highlight some of its features, while in Section 4, we analyze the convergence of the proposed method. The application of the proposed method is presented in Section 5. Moreover, some examples and numerical experiments to show the efficiency and implementation of our method are also presented in Section 6. We then conclude with some final remarks in Section 7.

2 Preliminaries

We state some known and useful results that will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$," respectively.

A subset D of H is called *proximal* if for each $x \in H$, there exists $y \in D$ such that

$$\|x - y\| = \mathrm{dist}(x, D),$$

where $\mathrm{dist}(x, D) = \inf\{\|x - y\| : y \in D\}$ is the distance from a point x to D.

Let H be a real Hilbert space. We denote by $CB(H)$, $CC(H)$, and $P(H)$ the collections of all nonempty closed bounded subsets of H, nonempty closed convex subset of H, and nonempty proximal bounded subsets of H, respectively. The Hausdorff metric $\mathcal{H}$ on $CB(H)$ is defined as follows:

$$\mathcal{H}(D_1, D_2) := \max\left\{\sup_{x\in D_1} \operatorname{dist}(x, D_2), \sup_{y\in D_2} \operatorname{dist}(y, D_1)\right\} \quad \forall D_1, D_2 \in CB(H).$$

Let $S : H \to 2^H$ be a multivalued mapping. We say that S satisfies the *endpoint condition* if $Sp = \{p\}$ for all $p \in \operatorname{Fix}(S)$, where $\operatorname{Fix}(S)$ is the fix point of a multivalued mapping S. For multivalued mappings $S_i : H \to 2^H (i \in \mathbb{N})$ with $\cap_{i=1}^{+\infty}\operatorname{Fix}(S_i) \neq \emptyset$, we say S_i satisfies the *common endpoint condition* if $S_i(p) = \{p\}$ for all $i \in \mathbb{N}, p \in \cap_{i=1}^{+\infty}\operatorname{Fix}(S_i)$, where $\operatorname{Fix}(S_i)$ is the fixed point of a multivalued mappings S_i.

Recall that a multivalued mapping $S : H \to 2^H$ is called

(i) L-Lipschitzian if there exists $L > 0$ such that

$$\mathcal{H}(Sx, Sy) \leq L\|x - y\| \quad \forall x, y \in H.$$

If $L \in (0, 1)$, then S is called a contraction, while S is called nonexpansive if $L = 1$.

(ii) quasi-nonexpansive if $\operatorname{Fix}(S) \neq \emptyset$ and

$$\mathcal{H}(Sx, Sp) \leq \|x - p\|, \quad \forall x \in H, p \in \operatorname{Fix}(S),$$

(iii) demicontractive as defined by Isiogugu and Osilike [37] if $\operatorname{Fix}(S) \neq \emptyset$ and

$$\mathcal{H}^2(Sx, Sp) \leq \|x - p\|^2 + k \operatorname{dist}^2(x, Sx) \quad \forall x \in H, p \in \operatorname{Fix}(S), \quad \text{and} \quad k \in [0, 1).$$

Remark 2.1. Clearly, every multivalued quasi-nonexpansive mapping is a multivalued demicontractive mapping. However, the following counter example demonstrates that the converse is not always true.

Example 2.2. [38] Let $H = \mathbb{R}$ (endowed with the usual metric) and $T : \mathbb{R} \to 2^{\mathbb{R}}$ be defined as follows:

$$Tx = \begin{cases} \left[-(\alpha + 1)x, \quad -\frac{2\alpha + 1}{2}x\right], & x \in [0, +\infty) \\ \left[-\frac{2\alpha + 1}{2}x, \quad -(\alpha + 1)x\right], & x \in (-+\infty, 0) \quad \forall \alpha > 0. \end{cases}$$

Then, T is a demicontractive mapping but not quasi-nonexpansive.

Example 2.3. [39] Let $\mathbb{R}$ denote the set of real number with the usual norm and $T : \mathbb{R} \to \mathbb{R}$ be a function defined as follows:

$$Tx = \begin{cases} x, & \text{if } (-\infty, 0); \\ -3x, & \text{if } [0, \infty). \end{cases}$$

Observe that $F(T) = (-\infty, 0]$. Then, T is $\frac{1}{2}$-demicontractive but not quasi-nonexpansive.

Definition 2.4. Let H be a real Hilbert space and $T : H \to CB(H)$ be a multivalued mapping. Then, the mapping $I^H - T$ is said to be demiclosed at the origin if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$, and $\operatorname{dist}(x_n, T(x_n)) \to 0$, we have $x^* \in Tx^*$, where I^H is identity mapping on H.

The following lemmas are useful in establishing our main result.

Lemma 2.5. *In a real Hilbert space H, the following inequalities hold for all* $x, y \in H, \alpha \in \mathbb{R}$:

(*i*) $2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x - y\|^2 - \|x\|^2 - \|y\|^2$;

(*ii*) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;

(*iii*) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$.

Lemma 2.6. [40,41] *Let H be a Hilbert space. Let $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1)$ such that $\alpha + \beta + \gamma = 1$. Then, we have*

$$||\alpha x + \beta y + \gamma z||^2 = \alpha||x||^2 + \beta||y||^2 + \gamma||z||^2 - \alpha\beta||x - y||^2 - \alpha\gamma||x - z||^2 - \beta\gamma||y - z||^2.$$

Lemma 2.7. [42,43] *Let H be a real Hilbert space and $\{x_i\}_{i\geq 1}$ be bounded sequences in H. For $\alpha_i \in (0, 1)$ such that $\sum_{i=1}^{+\infty}\alpha_i = 1$, the following identity holds*:

$$\left\|\sum_{i=1}^{+\infty}\alpha_i x_i\right\|^2 = \sum_{i=1}^{+\infty}\alpha_i||x_i||^2 - \sum_{1\leq i<j<+\infty}\alpha_i\alpha_j||x_i - x_j||^2.$$

Lemma 2.8. [44] *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $(0, 1)$ with the condition: $\sum_{n=1}^{+\infty}\alpha_n = +\infty$, and $\{d_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n d_n, \quad \forall n \geq 0.$$

if $\limsup_{k\to+\infty} d_{n_k} \leq 0$ *for every subsequence* $\{a_{n_k}\}$ of $\{a_n\}$ *satisfying the following condition*:

$$\liminf_{k\to+\infty}\left(a_{n_k+1} - a_{n_k}\right) \geq 0,$$

then $\lim_{n\to+\infty} a_n = 0$.

For solving the SGEP, we assume that the bifunctions $F : C \times C \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$ satisfy the following assumption:

Assumption 2.9. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $F : C \times C \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$ be two bifunctions satisfying the following conditions:*

(B1) $F(x, x) \geq 0$ for all $x \in C$;
(B2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0 \quad \forall x, y \in C$;
(B3) F is upper hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t\to+\infty} F(tz + (1 - t)x, y) \leq F(x, y);$$

(B4) For each $x \in C$ fixed, the function $y \mapsto F(x, y)$ is convex and lower semicontinuous;
(B5) $\phi(x, x) \geq 0$ for all $x \in C$;
(B6) For each $y \in C$ fixed, the function $x \to \phi(x, y)$ is upper semicontinuous;
(B7) For each $x \in C$ fixed, the function $y \to \phi(x, y)$ is convex and lower semicontinuous.

Assume that for fixed $r > 0$ and $z \in C$, there exists a nonempty, compact, convex subset K of H and $x \in C \cap K$ such that

$$F(y, x) + \phi(y, x) + \frac{1}{r}\langle y - x, x - z\rangle < 0, \quad \forall y \in C\backslash K.$$

Lemma 2.10. [45,46] *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$ be bifunctions satisfying the Assumptions* B1–B7 *and ϕ is monotone. For $r > 0$ and for all $x \in H$, define a mapping $T_r^{(F,\phi)} : H \to C$ as follows*:

$$T_r^{(F,\phi)}x = \left\{x^* \in C : F(x^*, y) + \phi(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - x\rangle \geq 0 \quad \forall y \in C\right\}.$$

Then, the following conclusions hold:
(*i*) $T_r^{(F,\phi)}$ *is single-valued*;
(*ii*) $T_r^{(F,\phi)}$ *is firmly nonexpansive, i.e., for any $x, y \in H$*,

$$||T_r^{(F,\phi)}x - T_r^{(F,\phi)}y||^2 \le \langle T_r^{(F,\phi)}x - T_r^{(F,\phi)}y, x - y\rangle; \tag{10}$$

(*iii*) $\mathrm{Fix}(T_r^{(F,\phi)}) = \mathrm{GEP}(F,\phi)$;

(*iv*) $\mathrm{GEP}(F,\phi)$ *is compact and convex.*

Lemma 2.11. [47,48] *Let H be a real Hilbert space. A mapping $T : H \to H$ is firmly nonexpansive if and only if its complement $I - T$ is firmly nonexpansive.*

3 Main results

In this section, we present a modified viscosity-type algorithm for approximating a common element of the set of solution of the SGEPMOS and the common FEP for an infinite family of multivalued demicontractive mappings in real Hilbert spaces.

Let C be a nonempty, closed, convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let C_i be nonempty, closed, convex subset of Hilbert spaces H_i and let $A_i : H \to H_i$ be bounded linear operators. Let $F, \phi : C \times C \to \mathbb{R}$, $F_i, \phi_i : C_i \times C_i \to \mathbb{R}$ be bifunctions satisfying assumptions (B1)-(B7) in Assumption 2.9 and for each $j \in \mathbb{N}$, let $S_j : H \to CB(H)$ be a family of multivalued demicontractive mappings with constant $k_j \in (0,1)$ such that $I - S_j$ is demiclosed at zero and $S_j(p) = \{p\}$ for each $j \in \mathbb{N}$, $p \in \cap_{j=1}^{+\infty}\mathrm{Fix}(S_j)$. Suppose the solution set is denoted by

$$\Gamma \coloneqq \bigcap_{j=1}^{+\infty}\mathrm{Fix}(S_j) \cap GEP(F,\phi) \cap \left(\bigcap_{i=1}^{N} A_i^{-1}(GEP(F_i,\phi_i))\right) \neq \varnothing. \tag{11}$$

Let $g : H \to H$ be a ρ-contraction with constant $\rho \in (0,1)$. Let $\{\alpha_n\}, \{\delta_n\}, \{\mu_n\}, \{\gamma_{n,j}\}, j \in \mathbb{N}$, be sequences in $(0,1)$, and $\{\phi_{n,i}\}$ is a sequence of positive real numbers for each $i = 0, 1, 2, \ldots, N$ and $n \ge 0$. Let $\{x_n\}$ be a sequence generated as follows:

Algorithm 3.1.
Step 0: For any $x_0 \in H$, let $H_0 = H$, $T_0 = I^H$ is the identity operator in Hilbert space H, $F_0 = F$, $\phi_0 = \phi$, and set $n = 0$.
Step 1: Compute

$$v_n = \sum_{i=0}^{N}\beta_{i,n}[x_n - \tau_{i,n}A_i^*(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n]. \tag{12}$$

Step 2: Compute

$$y_n = \gamma_{n,0}v_n + \sum_{j=1}^{+\infty}\gamma_{n,j}z_n^j, \tag{13}$$

where $z_n^j \in S_j v_n, \quad j = 1, 2, \ldots$.
Step 3: Compute

$$x_{n+1} = \alpha_n g(x_n) + \delta_n x_n + \mu_n y_n, \quad n \in \mathbb{N}. \tag{14}$$

Update

$$\tau_{i,n} = \theta_{i,n}\frac{||(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n||^2}{||A_i^*(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n||^2 + \phi_{i,n}}. \tag{15}$$

Set $n = n + 1$ and go to **Step 1**.

The following assumptions are needed in order to establish the strong convergence result for Algorithm 3.1:

(A1) $\lim_{n\to+\infty}\alpha_n = 0$ and $\sum_{n=0}^{+\infty}\alpha_n = +\infty$, and $\alpha_n + \delta_n + \mu_n = 1$, $\mu_n \subset [a,b] \subset (0,1)$;

(A2) $\sum_{j=0}^{+\infty}\gamma_{n,j} = 1$, $\liminf_{n\to+\infty}\gamma_{n,j}(\gamma_{n,0} - k) > 0$ for each $j = 1, 2, \ldots,$ where $k \coloneqq \sup_{j\ge1}\{k_j\} < 1$;

(A3) $\{\beta_{i,n}\} \subset [c, d] \subset (0, 1)$ such that $\sum_{i=0}^{N}\beta_{i,n} = 1$, $\{\theta_{i,n}\} \subset [e, f] \subset (0, 2)$;
(A4) $r_i > 0$ for each $i = 1, 2, \ldots, N$, $\max_{i=0,1,\ldots,N}\{\sup_n\{\phi_{i,n}\}\} = K < +\infty$.

We now highlight some of the features of our proposed method.

Remark 3.2.
- Observe that Algorithm 3.1 can be viewed as a modified viscosity approximation method involving adaptive step size for approximating the solution of a more general problem, SGEPMOS in real Hilbert spaces, in contrast to Jolaoso et al. and Phuengrattana and Klanarong [35, 36] who considered SGEP.
- The step size $\{\tau_{i,n}\}$ given by equation (15) is generated at each iteration by some simple computation. Thus, Algorithm 3.1 is easily implemented without prior knowledge of the operators norm.

4 Convergence analysis

We first establish the following lemmas needed for proving our strong convergence theorem for the proposed algorithm.

Lemma 4.1. *The sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *is bounded.*

Proof. Let $p \in \Gamma$, we obtain $A_i p = T_{r_i}^{(F_i,\phi_i)}A_i p$, for each $i = 0, 1, \ldots, N$. Thus, by the convexity of the function $\|\cdot\|^2$, we obtain

$$\begin{aligned}\|v_n - p\|^2 &= \left\|\sum_{i=0}^{N}\beta_{i,n}(x_n - \tau_{i,n}A_i^*(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n) - p\right\|^2 \\ &\le \sum_{i=0}^{N}\beta_{i,n}\|x_n - \tau_{i,n}A_i^*(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n - p\|^2.\end{aligned} \tag{16}$$

Applying the firmly nonexpansiveness of $I^{H_i} - T_{r_i}^{(F_i,\phi_i)}$ for each $i = 0, 1, \ldots, N$, we obtain

$$\begin{aligned}&\|x_n - \tau_{i,n}A_i^*(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n - p\|^2 \\ &\quad = \|x_n - p\|^2 + \tau_{i,n}^2\|A_i^*(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^2 - 2\tau_{i,n}\langle A_i^*(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n, x_n - p\rangle \\ &\quad = \|x_n - p\|^2 + \tau_{i,n}^2\|A_i^*(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^2 - 2\tau_{i,n}\langle (I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n, A_i x_n - A_i p\rangle \\ &\quad = \|x_n - p\|^2 - 2\tau_{i,n}\langle (I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n - (I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i p, A_i x_n - A_i p\rangle + \tau_{i,n}^2\|A_i^*(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^2 \\ &\quad \le \|x_n - p\|^2 - 2\tau_{i,n}\|(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^2 + \tau_{i,n}^2(\|A_i^*(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^2 + \phi_{i,n}) \\ &\quad = \|x_n - p\|^2 - \theta_{i,n}(2 - \theta_{i,n})\frac{\|(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^4}{\|A_i^*(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^2 + \phi_{i,n}},\end{aligned} \tag{17}$$

which follows from equations (16) and (17) that

$$\|v_n - p\|^2 \le \|x_n - p\|^2 - \sum_{i=0}^{N}\beta_{i,n}\theta_{i,n}(2 - \theta_{i,n})\frac{\|(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^4}{\|A_i^*(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^2 + \phi_{i,n}}. \tag{18}$$

Now, from Algorithm 3.1, Lemma 2.7, and the fact that S_j is demicontractive for each $j \in \mathbb{N}$, we have that

$$
\begin{aligned}
\|y_n - p\|^2 &= \left\| \gamma_{n,0}v_n + \sum_{j=1}^{+\infty} \gamma_{n,j} z_n^j - p \right\|^2 \\
&= \left\| \gamma_{n,0}(v_n - p) + \sum_{j=1}^{+\infty} \gamma_{n,j}(z_n^j - p) \right\|^2 \\
&= \gamma_{n,0}\|v_n - p\|^2 + \sum_{j=1}^{+\infty} \gamma_{n,j}\|z_n^j - p\|^2 - \sum_{j=1}^{+\infty} \gamma_{n,0}\gamma_{n,j}\|v_n - z_n^j\|^2 \\
&\le \gamma_{n,0}\|v_n - p\|^2 + \sum_{j=1}^{+\infty} \gamma_{n,j}\mathcal{H}^2(S_j v_n, S_j p) - \sum_{j=1}^{+\infty} \gamma_{n,0}\gamma_{n,j}\|v_n - z_n^j\|^2 \\
&\le \gamma_{n,0}\|v_n - p\|^2 + \sum_{j=1}^{+\infty} \gamma_{n,j}[\|v_n - p\|^2 + k\,\mathrm{dist}(v_n, S_j v_n)^2] - \sum_{j=1}^{+\infty} \gamma_{n,0}\gamma_{n,j}\|v_n - z_n^j\|^2 \\
&\le \gamma_{n,0}\|v_n - p\|^2 + \sum_{j=1}^{+\infty} \gamma_{n,j}[\|v_n - p\|^2 + k\|v_n - z_n^j\|^2] - \sum_{j=1}^{+\infty} \gamma_{n,0}\gamma_{n,j}\|v_n - z_n^j\|^2 \\
&= \|v_n - p\|^2 - \sum_{j=1}^{+\infty} \gamma_{n,j}(\gamma_{n,0} - k)\|v_n - z_n^j\|^2.
\end{aligned}
$$

It follows from equation (18) that

$$
\|y_n - p\|^2 \le \|x_n - p\|^2 - \sum_{j=1}^{+\infty} \gamma_{n,j}(\gamma_{n,0} - k)\|v_n - z_n^j\|^2 - \sum_{i=0}^{N} \beta_{i,n}\theta_{i,n}(2 - \theta_{i,n})\frac{\|(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^4}{\|A_i^*(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^2 + \phi_{i,n}}, \tag{19}
$$

and it follows from the conditions (A2)–(A4), that

$$
\|y_n - p\|^2 \le \|x_n - p\|^2. \tag{20}
$$

Further, by applying equation (20), we have

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n g(x_n) + \delta_n x_n + \mu_n y_n - p\| \\
&\le \alpha_n\|g(x_n) - p\| + \delta_n\|x_n - p\| + \mu_n\|y_n - p\| \\
&\le \alpha_n(\|g(x_n) - g(p)\| + \|g(p) - p\|) + \delta_n\|x_n - p\| + \mu_n\|y_n - p\| \\
&\le \alpha_n(\rho\|x_n - p\| + \|g(p) - p\|) + \delta_n\|x_n - p\| + \mu_n\|x_n - p\| \\
&= (\rho\alpha_n + \delta_n + \mu_n)\|x_n - p\| + \alpha_n\|g(p) - p\| \\
&= (1 - \alpha_n(1 - \rho))\|x_n - p\| + \frac{\alpha_n(1 - \rho)\|g(p) - p\|}{1 - \rho} \\
&\le \max\left\{\|x_n - p\|, \frac{\|g(p) - p\|}{1 - \rho}\right\}.
\end{aligned}
$$

Therefore, It follows by induction that

$$
\|x_{n+1} - p\| \le \max\left\{\|x_1 - p\|, \frac{\|g(p) - p\|}{1 - \rho}\right\}.
$$

Hence, $\{x_n\}$ is bounded, and consequently, $\{v_n\}$, $\{y_n\}$, and $\{z_n^j\}$ are all bounded. □

It is easy to see that the operator $P_\Gamma \circ g$ is a contraction. Thus, by the Banach contraction principle, there exists a unique point $x^* \in \Gamma$ such that $x^* = P_\Gamma \circ g(x^*)$. It follows from the characterization of the projection mapping that

$$
\langle g(x^*) - x^*, x - x^* \rangle \le 0 \quad \forall x \in \Gamma. \tag{21}
$$

Lemma 4.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 and let $p \in \Gamma$. Then, under conditions (A1)–(A4) and Assumption 2.9 the following inequality holds for all $n \in \mathbb{N}$:*

$$\|x_{n+1} - p\|^2 \le \left(1 - \frac{2\alpha_n[1-\rho]}{(1-\alpha_n\rho)}\right)\|x_n - p\|^2 + \frac{2\alpha_n(1-\rho)}{(1-\alpha\rho)}\left[\frac{\alpha_n M_1}{2(1-\rho)} + \frac{1}{(1-\rho)}\langle g(p) - p, x_{n+1} - p\rangle\right]$$
$$- \frac{\mu_n(1-\alpha_n)}{(1-\alpha_n\rho)}\left[\sum_{j=1}^{+\infty}\gamma_{n,j}(\gamma_{n,0} - k)\|v_n - z_n^j\|^2\right.$$
$$\left. + \sum_{i=0}^{N}\beta_{i,n}\theta_{i,n}(2-\theta_{i,n})\frac{\|(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^4}{\|A_i^*(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^2 + \phi_{i,n}}\right].$$

Proof. Let $p \in \Gamma$. By applying Lemma 2.5(ii), and (19), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n g(x_n) + \delta_n x_n + \mu_n y_n - p\|^2 \\
&\le \|\delta_n(x_n - p) + \mu_n(y_n - p)\|^2 + 2\alpha_n\langle g(x_n) - p, x_{n+1} - p\rangle \\
&\le \delta_n^2\|x_n - p\|^2 + \mu_n^2\|y_n - p\|^2 + 2\delta_n\mu_n\|x_n - p\|\|y_n - p\| + 2\alpha_n\langle g(x_n) - p, x_{n+1} - p\rangle \\
&\le \delta_n^2\|x_n - p\|^2 + \mu_n^2\|y_n - p\|^2 + \delta_n\mu_n(\|x_n - p\|^2 + \|y_n - p\|^2) + 2\alpha_n\langle g(x_n) - g(p), x_{n+1} - p\rangle \\
&\quad + 2\alpha_n\langle g(p) - p, x_{n+1} - p\rangle \\
&= \delta_n(\delta_n + \mu_n)\|x_n - p\|^2 + \mu_n(\mu_n + \delta_n)\|y_n - p\|^2 + 2\alpha_n\langle g(x_n) - g(p), x_{n+1} - p\rangle \\
&\quad + 2\alpha_n\langle g(p) - p, x_{n+1} - p\rangle \\
&\le \delta_n(1-\alpha_n)\|x_n - p\|^2 + \mu_n(1-\alpha_n)\left[\|x_n - p\|^2 - \sum_{j=1}^{+\infty}\gamma_{n,j}(\gamma_{n,0} - k)\|v_n - z_n^j\|^2\right. \\
&\quad \left. - \sum_{i=0}^{N}\beta_{i,n}\theta_{i,n}(2-\theta_{i,n})\frac{\|(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^4}{\|A_i^*(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^2 + \phi_{i,n}}\right] \\
&\quad + 2\alpha_n\rho\|x_n - p\|\|x_{n+1} - p\| + 2\alpha_n\langle g(p) - p, x_{n+1} - p\rangle \\
&\le (1-\alpha_n)^2\|x_n - p\|^2 - \mu_n(1-\alpha_n)\left[\sum_{j=1}^{+\infty}\gamma_{n,j}(\gamma_{n,0} - k)\|v_n - z_n^j\|^2\right. \\
&\quad \left. + \sum_{i=0}^{N}\beta_{i,n}\theta_{i,n}(2-\theta_{i,n})\frac{\|(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^4}{\|A_i^*(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^2 + \phi_{i,n}}\right] \\
&\quad + \alpha_n\rho[\|x_n - p\|^2 + \|x_{n+1} - p\|^2] + 2\alpha_n\langle g(p) - p, x_{n+1} - p\rangle \\
&= ((1-\alpha_n)^2 + \alpha_n\rho)\|x_n - p\|^2 + \alpha_n\rho\|x_{n+1} - p\|^2 + 2\alpha_n\langle g(p) - p, x_{n+1} - p\rangle \\
&\quad - \mu_n(1-\alpha_n)\left[\sum_{j=1}^{+\infty}\gamma_{n,j}(\gamma_{n,0} - k)\|v_n - z_n^j\|^2\right. \\
&\quad \left. + \sum_{i=0}^{N}\beta_{i,n}\theta_{i,n}(2-\theta_{i,n})\frac{\|(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^4}{\|A_i^*(I^{H_i} - T_{r_i}^{(F_i,\phi_i)})A_i x_n\|^2 + \phi_{i,n}}\right].
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \frac{(1-2\alpha_n+\alpha_n^2+\alpha_n\rho)}{(1-\alpha_n\rho)}\|x_n-p\|^2+\frac{2\alpha_n}{(1-\alpha_n\rho)}\langle g(p)-p, x_{n+1}-p\rangle \\
&\quad -\frac{\mu_n(1-\alpha_n)}{(1-\alpha_n\rho)}\Bigg[\sum_{j=1}^{+\infty}\gamma_{n,j}(\gamma_{n,0}-k)\|v_n-z_n^j\|^2 \\
&\quad +\sum_{i=0}^{N}\beta_{i,n}\theta_{i,n}(2-\theta_{i,n})\frac{\|(I^{H_i}-T_{r_i}^{(F_i,\phi_i)})A_ix_n\|^4}{\|A_i^*(I^{H_i}-T_{r_i}^{(F_i,\phi_i)})A_ix_n\|^2+\phi_{i,n}}\Bigg] \\
&= \frac{(1-2\alpha_n+\alpha_n\rho)}{(1-\alpha_n\rho)}\|x_n-p\|^2+\frac{\alpha_n^2}{(1-\alpha_n\rho)}\|x_n-p\|^2+\frac{2\alpha_n}{(1-\alpha_n\rho)}\langle g(p)-p, x_{n+1}-p\rangle \\
&\quad -\frac{\mu_n(1-\alpha_n)}{(1-\alpha_n\rho)}\Bigg[\sum_{j=1}^{+\infty}\gamma_{n,j}(\gamma_{n,0}-k)\|v_n-z_n^j\|^2 \\
&\quad +\sum_{i=0}^{N}\beta_{i,n}\theta_{i,n}(2-\theta_{i,n})\frac{\|(I^{H_i}-T_{r_i}^{(F_i,\phi_i)})A_ix_n\|^4}{\|A_i^*(I^{H_i}-T_{r_i}^{(F_i,\phi_i)})A_ix_n\|^2+\phi_{i,n}}\Bigg] \\
&\le \left(1-\frac{2\alpha_n(1-\rho)}{(1-\alpha_n\rho)}\right)\|x_n-p\|^2+\frac{2\alpha_n(1-\rho)}{(1-\alpha\rho)}\left[\frac{\alpha_nM_1}{2(1-\rho)}+\frac{1}{(1-\rho)}\langle g(p)-p, x_{n+1}-p\rangle\right] \\
&\quad -\frac{\mu_n(1-\alpha_n)}{(1-\alpha_n\rho)}\Bigg[\sum_{j=1}^{+\infty}\gamma_{n,j}(\gamma_{n,0}-k)\|v_n-z_n^j\|^2 \\
&\quad +\sum_{i=0}^{N}\beta_{i,n}\theta_{i,n}(2-\theta_{i,n})\frac{\|(I^{H_i}-T_{r_i}^{(F_i,\phi_i)})A_ix_n\|^4}{\|A_i^*(I^{H_i}-T_{r_i}^{(F_i,\phi_i)})A_ix_n\|^2+\phi_{i,n}}\Bigg],
\end{aligned}$$

where $M_1 := \sup\{\|x_n-p\|^2 : n\in\mathbb{N}\}$. Hence, the proof is complete. □

We now present the strong convergence theorem for the proposed algorithm as follows.

Theorem 4.3. *Suppose that conditions* (A1)–(A4) *and Assumption* 2.9 *hold. Then, the sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *converges strongly to* $\hat{x}\in\Gamma$, *where* $\hat{x}=P_\Gamma\circ g(\hat{x})$.

Proof. Let $\hat{x}=P_\Omega\circ g(\hat{x})$. From Lemma 4.2, we obtain

$$\|x_{n+1}-\hat{x}\|^2 \le \left(1-\frac{2\alpha_n[1-\rho]}{(1-\alpha_n\rho)}\right)\|x_n-\hat{x}\|^2+\frac{2\alpha_n(1-\rho)}{(1-\alpha\rho)}\left[\frac{\alpha_nM_1}{2(1-\rho)}+\frac{1}{(1-\rho)}\langle g(\hat{x})-\hat{x}, x_{n+1}-\hat{x}\rangle\right]. \tag{22}$$

Next, we show that the sequence $\{\|x_n-\hat{x}\|\}$ converges to zero. In order to establish this, by Lemma 2.8, it is enough to show that $\limsup_{k\to+\infty}\langle g(\hat{x})-\hat{x}, x_{n_k+1}-\hat{x}\rangle\le 0$ for every subsequence $\{\|x_{n_k}-\hat{x}\|\}$ of $\{\|x_n-\hat{x}\|\}$ satisfying

$$\liminf_{k\to+\infty}(\|x_{n_k+1}-\hat{x}\|-\|x_{n_k}-\hat{x}\|)\ge 0. \tag{23}$$

Now, suppose that $\{\|x_{n_k}-\hat{x}\|\}$ is a subsequence of $\{\|x_n-\hat{x}\|\}$ such that equation (23) holds. Then,

$$\liminf_{k\to+\infty}(\|x_{n_k+1}-\hat{x}\|^2-\|x_{n_k}-\hat{x}\|^2)=\liminf_{k\to+\infty}[(\|x_{n_k+1}-\hat{x}\|-\|x_{n_k}-\hat{x}\|)(\|x_{n_k+1}-\hat{x}\|+\|x_{n_k}-\hat{x}\|)]\ge 0. \tag{24}$$

Again, from Lemma 4.2, we have

$$\begin{aligned}
&\frac{\mu_{n_k}(1-\alpha_{n_k})}{(1-\alpha_{n_k}\rho)}\sum_{j=1}^{+\infty}\gamma_{n_k,j}(\gamma_{n_k,0}-k)\|v_{n_k}-z_{n_k}^j\|^2 \\
&\le \left(1-\frac{2\alpha_{n_k}[1-\rho]}{(1-\alpha_{n_k}\rho)}\right)\times\|x_{n_k}-\hat{x}\|^2-\|x_{n_k+1}-\hat{x}\|^2+\frac{2\alpha_n(1-\rho)}{(1-\alpha\rho)}\times\left[\frac{\alpha_{n_k}M_1}{2(1-\rho)}+\frac{1}{(1-\rho)}\langle g(\hat{x})-\hat{x}, x_{n_k+1}-\hat{x}\rangle\right].
\end{aligned}$$

By applying equation (24) together with condition (A1), we obtain

$$\frac{\mu_{n_k}(1-\alpha_{n_k})}{(1-\alpha_{n_k}\rho)}\sum_{j=1}^{+\infty}\gamma_{n_k,j}(\gamma_{n_k,0}-k)\|v_{n_k}-z_{n_k}^j\|^2\to 0,\quad k\to+\infty.$$

Consequently, we have

$$\lim_{k\to+\infty}\|v_{n_k}-z_{n_k}^j\|=0,\quad j=1,2,\ldots. \tag{25}$$

It then follows that

$$\lim_{k\to+\infty}\operatorname{dist}(v_{n_k},S_jv_{n_k})\leq\lim_{k\to+\infty}\|v_{n_k}-z_{n_k}^j\|=0,\quad j=1,2,\ldots. \tag{26}$$

By similar argument, we obtain from Lemma 4.2 that

$$\lim_{k\to+\infty}\mu_{n_k}\left[\sum_{i=0}^{N}\beta_{i,n_k}\theta_{i,n_k}(2-\theta_{i,n_k})\frac{\|(I^{H_i}-T_{r_i}^{(F_i,\phi_i)})A_ix_{n_k}\|^4}{\|A_i^*(I^{H_i}-T_{r_i}^{(F_i,\phi_i)})A_ix_{n_k}\|^2+\phi_{i,n_k}}\right]=0\quad\forall i=0,1,2,\ldots,N.$$

By condition (A3), we have

$$\frac{\|(I^{H_i}-T_{r_i}^{(F_i,\phi_i)})A_ix_{n_k}\|^4}{\|A_i^*(I^{H_i}-T_{r_i}^{(F_i,\phi_i)})A_ix_{n_k}\|^2+\phi_{i,n_k}}\to 0,\quad k\to+\infty,\quad\forall i=0,1,2,\ldots,N. \tag{27}$$

It follows from the boundedness of the operators A_i, the nonexpansivity of the mappings $T_{r_i}^{(F_i,\phi_i)}$, and the boundedness of the sequence $\{x_{n_k}\}$ that

$L:=\max_{i=0,1,\ldots,N}\{\sup_n\{\|A_i^*(I^{H_i}-T_{r_i}^{(F_i,\phi_i)})A_ix_{n_k}\|^2\}\}<+\infty$. Therefore, it follows from (A4), that

$$\frac{\|(I^{H_i}-T_{r_i}^{(F_i,\phi_i)})A_ix_{n_k}\|^4}{\|A_i^*(I^{H_i}-T_{r_i}^{(F_i,\phi_i)})A_ix_{n_k}\|^2+\phi_{i,n_k}}\geq\frac{\|(I^{H_i}-T_{r_i}^{(F_i,\phi_i)})A_ix_{n_k}\|^4}{L+K}.$$

Using the last inequality together with (27), we have

$$\lim_{k\to+\infty}\|(I^{H_i}-T_{r_i}^{(F_i,\phi_i)})A_ix_{n_k}\|=0\quad\forall i=0,1,2,\ldots,N. \tag{28}$$

Furthermore, we obtain from equation (12) that

$$\lim_{k\to+\infty}\|v_{n_k}-x_{n_k}\|=\left\|\sum_{i=0}^{N}\beta_{i,n_k}\tau_{i,n_k}A_i^*(I^{H_i}-T_{r_i}^{(F_i,\phi_i)})A_ix_{n_k}\right\|. \tag{29}$$

Applying equation (28) together with (A3), it follows from the last inequality that

$$\lim_{k\to+\infty}\|v_{n_k}-x_{n_k}\|=0. \tag{30}$$

Also, from equations (13) and (25), we have that

$$\begin{aligned}\|y_{n_k}-v_{n_k}\|&=\left\|\gamma_{n_k,0}(v_{n_k}-v_{n_k})+\sum_{j=1}^{+\infty}\gamma_{n_k,j}(z_{n_k}^j-v_{n_k})\right\|\\&\leq\sum_{j=1}^{+\infty}\gamma_{n_k,j}\|v_{n_k}-z_{n_k}^j\|\to 0,\quad k\to+\infty.\end{aligned} \tag{31}$$

Observe that from equations (25) and (31), we have

$$\|y_{n_k}-z_{n_k}^j\|\leq\|y_{n_k}-v_{n_k}\|+\|v_{n_k}-z_{n_k}^j\|\to 0,\quad k\to+\infty\quad\forall j=1,2,\ldots \tag{32}$$

It follows from equations (30) and (31) that

$$\|x_{n_k} - y_{n_k}\| \leq \|x_{n_k} - v_{n_k}\| + \|v_{n_k} - y_{n_k}\| \to 0, \quad k \to +\infty. \tag{33}$$

Consequently, by applying condition (A1), we have

$$\begin{aligned}\|x_{n_{k+1}} - x_{n_k}\| &= \|\alpha_{n_k} g(x_{n_k}) + \delta_{n_k} x_{n_k} + \mu_{n_k} y_{n_k} - x_{n_k}\| \\ &\leq \alpha_{n_k}\|g(x_{n_k}) - x_{n_k}\| + \delta_{n_k}\|x_{n_k} - x_{n_k}\| + \mu_{n_k}\|y_{n_k} - x_{n_k}\| \to 0, \quad k \to +\infty.\end{aligned} \tag{34}$$

To complete the proof, we need to show that $w_\omega(x_n) \subset \Gamma$. If the sequence $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $\bar{x} \in w_\omega(x_n)$ be an arbitrary element. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k \to +\infty$. From equation (30), we have that $v_{n_k} \rightharpoonup \bar{x}$. Now, from the fact that $I - S_j$ is demiclosed at zero for each $j = 1, 2, \ldots,$ and since from equation (25), $\lim_{k\to+\infty}\|v_{n_k} - z^j_{n_k}\| \to 0$ as $k \to +\infty$ for each $j = 1, 2, \ldots,$ we have that $\bar{x} \in \mathrm{Fix}(S_j)$ for all $j = 1, 2, \ldots$ Hence, $\bar{x} \in \bigcap_{j=1}^{+\infty}\mathrm{Fix}(S_j)$. Also, since for each $i = 0, 1, \ldots, N$, A_i is a bounded linear operator, it follows that $A_i x_{n_k} \rightharpoonup A_i\bar{x}$. Thus, by the demiclosedness principle, it follows from equation (28) that $A_i\bar{x} \in \mathrm{Fix}(T_{r_i}^{F_i,\phi_i})$ for all $i = 0, 1, \ldots, N$. Hence, $A_i\bar{x} \in \bigcap_{i=0}^{N}(GEP(F_i, \phi_i))$. Consequently, we have $\bar{x} \in \Gamma$, which implies that $w_\omega(x_n) \subset \Gamma$.

By the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup x^\dagger$ and

$$\lim_{j\to+\infty}\langle g(\hat{x}) - \hat{x}, x_{n_{k_j}} - \hat{x}\rangle = \limsup_{k\to+\infty}\langle g(\hat{x}) - \hat{x}, x_{n_k} - \hat{x}\rangle.$$

Since $\hat{x} = P_\Omega \circ g(\hat{x})$, then from equations (34) and (21), we have

$$\begin{aligned}\limsup_{k\to+\infty}\langle g(\hat{x}) - \hat{x}, x_{n_{k+1}} - \hat{x}\rangle &= \limsup_{k\to+\infty}\langle g(\hat{x}) - \hat{x}, x_{n_{k+1}} - x_{n_k}\rangle + \limsup_{k\to+\infty}\langle g(\hat{x}) - \hat{x}, x_{n_k} - \hat{x}\rangle \\ &= \limsup_{j\to+\infty}\langle g(\hat{x}) - \hat{x}, x_{n_{k_j}} - \hat{x}\rangle \\ &= \langle g(\hat{x}) - \hat{x}, x^\dagger - \hat{x}\rangle \leq 0.\end{aligned} \tag{35}$$

Applying Lemma 2.8 to equation (22), and using equation (35) together with the fact that $\lim_{n\to+\infty}\alpha_n = 0$, we deduce that $\lim_{n\to+\infty}\|x_n - \hat{x}\| = 0$ as required. □

If we take $\phi_i = 0$, $i = 0, 1, 2, \ldots, N$, in Theorem 4.3, we have the following consequent result for approximating a common solution of the set of solutions of SPE with multiple output sets and the common FPP for an infinite family of multivalued demicontractive mappings in real Hilbert spaces.

Corollary 4.4. *Let C be a nonempty, closed, convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let C_i be nonempty, closed, convex subset of Hilbert spaces H_i and let F, F_i, and A_i be as defined in Theorem* 4.3. *For each $j \in \mathbb{N}$, let $S_j : H \to CB(H)$ be a family of multivalued demicontractive mappings with constant $k_j \in (0, 1)$ such that $I - S_j$ is demiclosed at zero for each $j \in \mathbb{N}$. Suppose the solution set denoted by $\Gamma \coloneqq \bigcap_{j=1}^{+\infty}\mathrm{Fix}(S_j) \cap EP(F) \cap (\bigcap_{i=1}^{N}A_i^{-1}(EP(F_i))) \neq \emptyset$, and conditions* (A1)–(A4) *and Assumption* 2.9 *hold. Then, the sequence $\{x_n\}$ generated by the following algorithm converges strongly to $\hat{x} \in \Gamma$, where $\hat{x} = P_\Gamma \circ g(\hat{x})$.*

Algorithm 4.5.

Step 0: For any $x_0 \in H$, let $H_0 = H$, $T_0 = I^H$ is the identity operator in Hilbert space H, $F_0 = F$, and set $n = 0$.

Step 1: Compute

$$v_n = \sum_{i=0}^{N}\beta_{i,n}[x_n - \tau_{i,n}A_i^*(I^{H_i} - T_{r_i}^{F_i})A_i x_n]. \tag{36}$$

Step 2: Compute

$$y_n = \gamma_{n,0}v_n + \sum_{j=1}^{+\infty}\gamma_{n,j}z_n^j, \tag{37}$$

where $z_n^j \in S_j v_n, \quad j = 1, 2, \ldots.$

Step 3: Compute

$$x_{n+1} = \alpha_n g(x_n) + \delta_n x_n + \mu_n y_n, \quad n \in \mathbb{N}. \tag{38}$$

Update:

$$\tau_{i,n} = \theta_{i,n} \frac{\|(I^{H_i} - T_{r_i}^{F_i})A_i x_n\|^2}{\|A_i^*(I^{H_i} - T_{r_i}^{F_i})A_i x_n\|^2 + \phi_{i,n}}.$$

Set $n = n + 1$ and go to **Step 1**.

5 Application

5.1 Split variational inequality problem with multiple output sets

Let C be a nonempty, closed, convex subset of a real Hilbert space H, and $B : H \to H$ be a single-valued mapping. The VIP is defined as follows:

$$\text{Find } x^* \in C \text{ such that } \langle y - x^*, Bx^* \rangle \geq 0, \quad \forall y \in C. \tag{39}$$

The solution set of the VIP is denoted by $VI(C, B)$. Variational inequality was first introduced independently by Fichera [49] and Stampacchia [50]. The VIP is a useful mathematical model, which unifies many important concepts in applied mathematics, such as necessary complementarity problems, network equilibrium

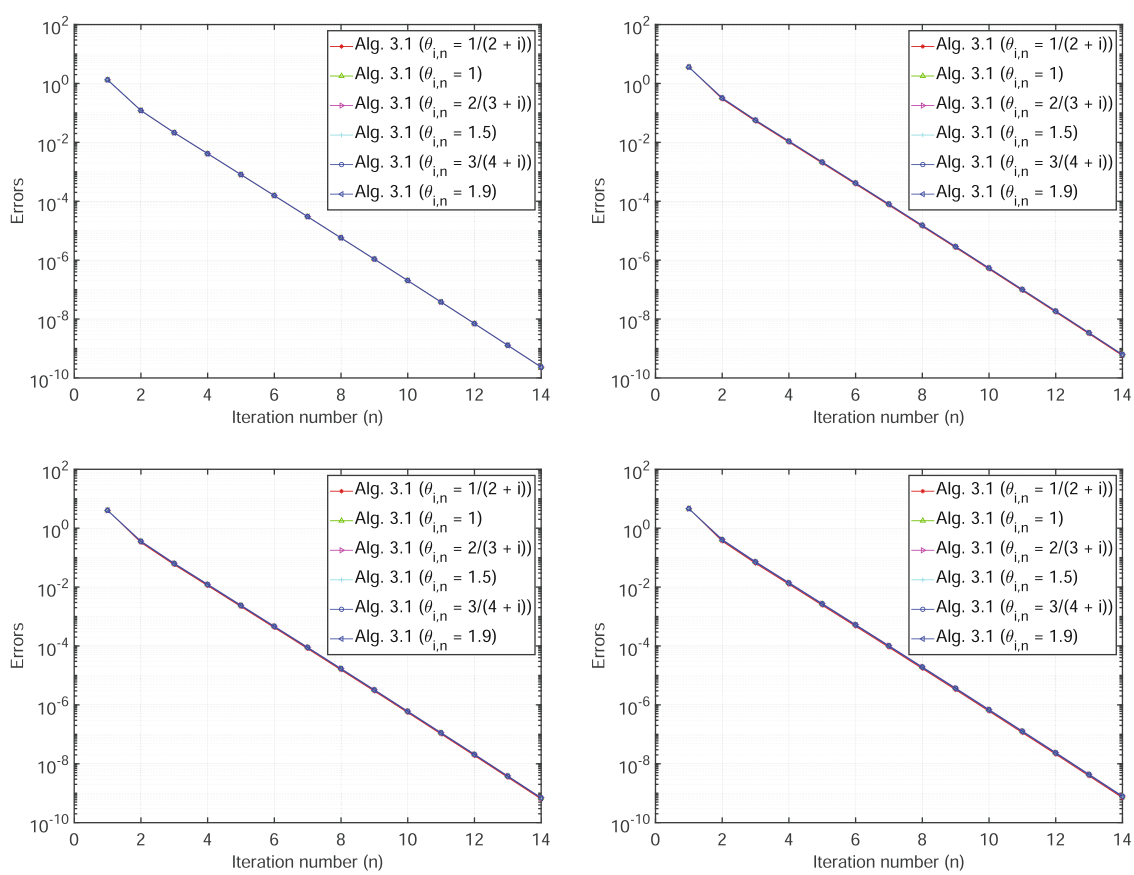

Figure 1: Top left: Case 1; top right: Case 2; bottom left: Case 3; and bottom right: Case 4.

problems, optimality conditions, and systems of nonlinear equations (see [51,52]). Several methods have been proposed and analyzed for approximating the solution of VIP (39) (see [53–56] and references therein).

Let C be a nonempty, closed, convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let C_i be nonempty, closed, convex subset of Hilbert spaces H_i and let $A_i : H \to H_i$ be bounded linear operators. Let $B : C \to H$, $B_i : C_i \to H_i$ be monotone mappings. The SVIPMOS is formulated as finding a point $x^* \in C$ such that

$$x^* \in VI(C, B) \cap \left(\bigcap_{i=1}^{N} A_i^{-1}(VI(C_i, B_i)) \right) \neq \emptyset. \tag{40}$$

We denote the solution set of problem (40) by $\mathcal{F}$. By taking $F_i(x, y) \coloneqq \langle y - x, B_i x \rangle$, $i = 0, 1, 2, \ldots, N$, where $F_0 = F$, $B_0 = B$, then the SVIPMOS (40) becomes the problem of finding a solution of SEP with multiple output sets. Consequently, Corollary 4.4 can be used to approximate the common solution of SVIPMOS (40) and the common FPP for an infinite family of multivalued demicontractive mappings $S_j : H \to CB(H)$, $j \in \mathbb{N}$ in real Hilbert spaces, where the solution set denoted by $\Gamma \coloneqq \bigcap_{j=1}^{+\infty} \mathrm{Fix}(S_j) \cap \mathcal{F}$ is assumed to be nonempty.

6 Numerical examples

This section provides some examples to illustrate the implementation of our proposed methods, Algorithm 3.1 . In our experiment, we let $g(x) = \frac{x}{3}$, $\alpha_n = \frac{1}{140n + 1}$, $\delta_n = \frac{1}{3n + 14}$, and $\mu_n = 1 - \alpha_n - \delta_n$, for $i = 0, 1, 2$, and let $r_i = r = 0.5$, $\beta_{i,n} = \frac{1}{3}$, $\gamma_{n,0} = \frac{1}{2}$, and $\gamma_{n,j} = \frac{1}{2^{j+1}}$, $j = 1, 2, \ldots$. Moreover, we consider the effect of varying values of the following parameters $\theta_{i,n} = \frac{1}{2+i}, 1.0, \frac{2}{3+i}, 1.5, \frac{3}{4+i}, 1.9$, $\phi_{i,n} = 0.5, \frac{3}{4+i}, 1.0, \frac{5}{2+i}, 2.0, \frac{7}{3+i}$ on our method. All

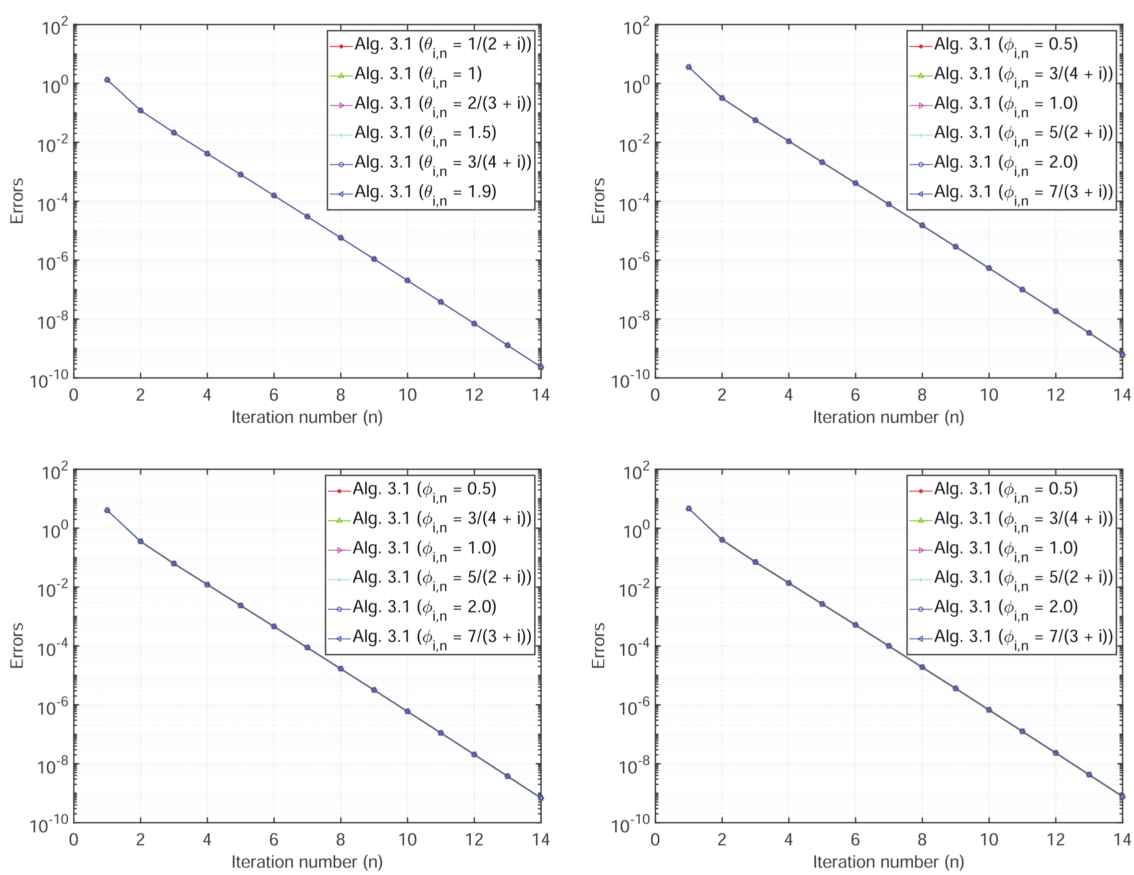

Figure 2: Top left: Case 1; top right: Case 2; bottom left: Case 3; and bottom right: Case 4.

numerical computations were carried out using Matlab version R2019(b). We plot the graphs of errors against the number of iterations in each case. The stopping criterion used for our computation is $\|x_{n+1} - x_n\| < 10^{-9}$. The numerical results are reported in Figures 1–4 and Tables 1–4. In Tables 1–4, "Iter." means the number of iterations and "CPU" means the CPU time in seconds.

Example 6.1. For $i = 0, 1, 2$, let $H_i = \mathbb{R}^2$ ($H = H_0$) with the inner product $\langle.,.\rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

$$\langle x, y\rangle = x \cdot y = x_1 \cdot y_1 + x_2 \cdot y_2, \quad x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.$$

We define the mappings $F = F_0 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $F_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, and $F_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, respectively, by $F(x, y) = -3x^2 + xy + 2y^2$, $F_1(x, y) = -4x^2 + xy + 3y^2$ and $F_2(x, y) = -5y^2 + 2y + 5xy - 5xy^2$ for each $x = (x_1, x_2) \in \mathbb{R}^2$, and $y = (y_1, y_2) \in \mathbb{R}^2$. Also, for $i = 0, 1, 2$, let $\phi_0 = \phi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $\phi_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and $\phi_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined as $\phi(x, y) = x^2 - xy$, $\phi_1(x, y) = 2x(x - y)$ and $\phi_2(x, y) = 5y^2 - 2x$, respectively, for each $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$. For some $r > 0$, we obtain by some simple calculation that

$$v = T_r^{F,\phi} u = \frac{1}{4r+1} u, \quad y = T_r^{F_1,\phi_1} x = \frac{1}{1+5r} x, \quad \text{and} \quad w = T_r^{F_2,\phi_2} z = \frac{z - 2r}{1+5r}.$$

Let $A_i : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $A_i(x) = \frac{x}{i+1}$ where $x = (x_1, x_2) \in \mathbb{R}^2$. Let $S_j : C \to C(B)$ be defined as follows:

$$S_j x \coloneqq \frac{-3}{2j} x, \quad j = 1, 2, \ldots.$$

It is easy to see that S_j is demicontractive for each $j = 1, 2, \ldots$.

The next example is in the framework of an infinite dimensional Hilbert spaces.

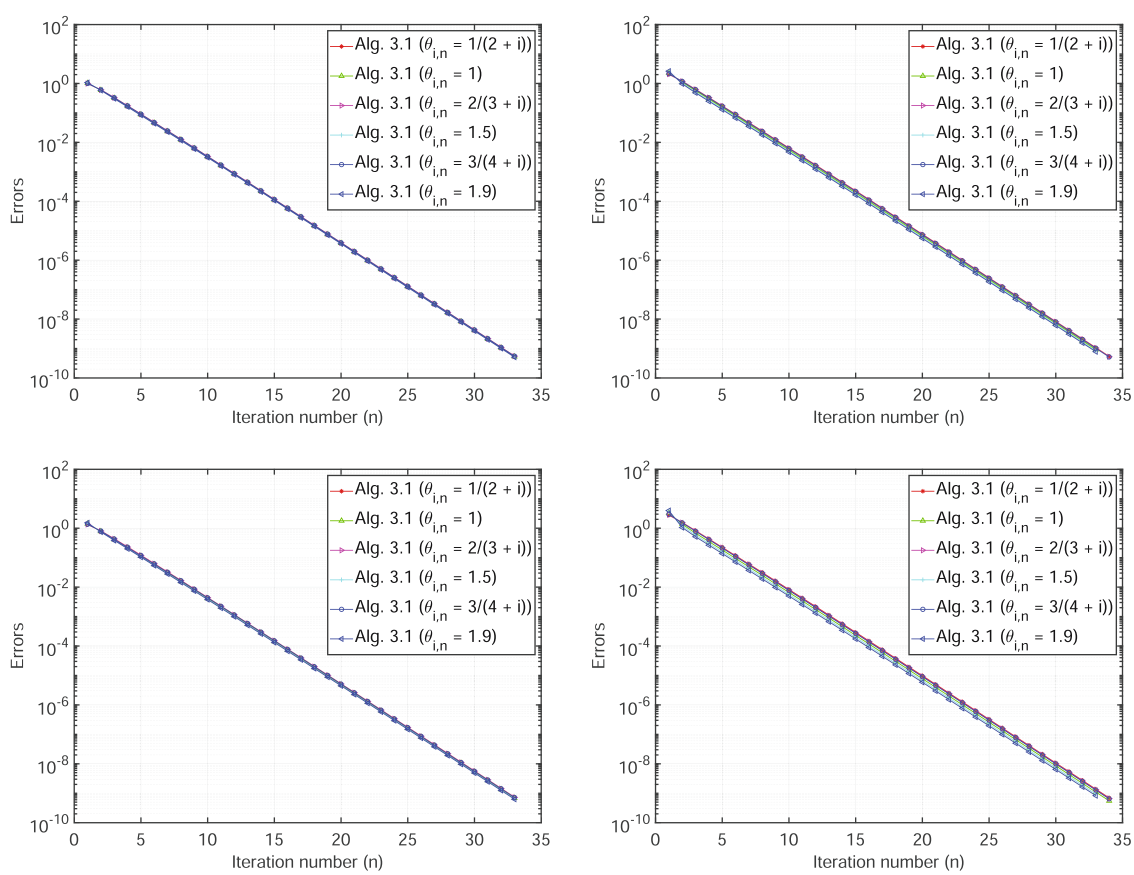

Figure 3: Top left: Case 1; top right: Case 2; bottom left: Case 3; and bottom right: Case 4.

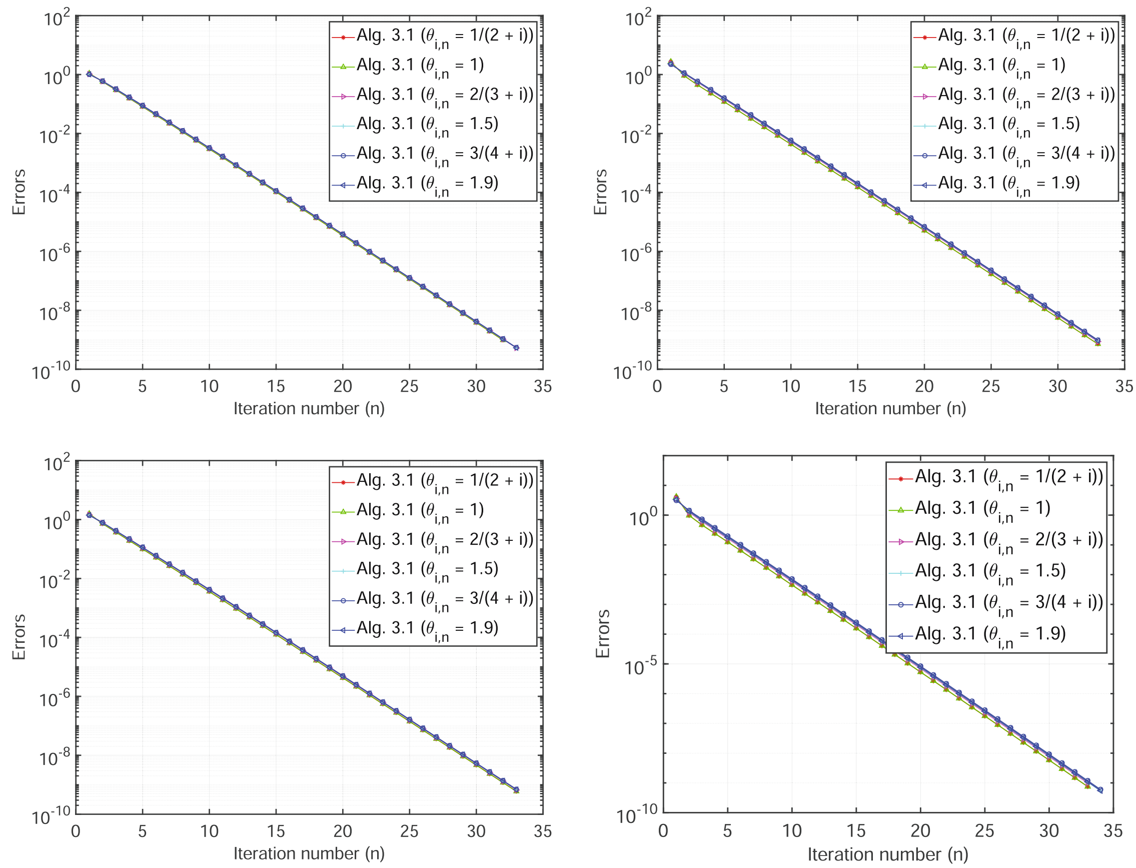

Figure 4: Top left: Case 1; top right: Case 2; bottom left: Case 3; bottom right: Case 4.

Table 1: Numerical results for Example 6.1 (**Experiment 1**)

Cases		$\theta_{i,n} = \frac{1}{2+i}$	$\theta_{i,n} = 1$	$\theta_{i,n} = \frac{2}{3+i}$	$\theta_{i,n} = 1.5$	$\theta_{i,n} = \frac{3}{4+i}$	$\theta_{i,n} = 1.9$
1	CPU time (s)	0.0155	0.0128	0.0096	0.0140	0.0150	0.0144
	No. of Iter.	14	14	14	14	14	14
2	CPU time (s)	0.0136	0.0088	0.0128	0.0139	0.0150	0.0098
	No. of Iter.	14	14	14	14	14	14
3	CPU time (s)	0.0137	0.0089	0.05	0.0162	0.0161	0.0154
	No. of Iter.	14	14	14	14	14	14
4	CPU time (s)	0.0135	0.0123	0.0097	0.0146	0.0151	0.0142
	No. of Iter.	14	14	14	14	14	14

Example 6.2. Let $H,\ H_i = \ell_2$ for $i = 0, 1, 2, \dots$ be the linear spaces whose elements consists of two summable sequences $(x_1, x_2, \dots, x_i, \dots)$ of scalars, i.e.,

$$\ell_2 = \{x : x = (x_1, x_2, \dots, x_i, \dots) \quad \text{and} \quad \sum_{i=1}^{+\infty} |x_i|^2 < +\infty\},$$

with an inner product $\langle .,. \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined as follows:

$$\langle x, y \rangle = x \cdot y = \sum_{i=1}^{+\infty} x_i y_i \quad \text{where } x = \{x_i\}_{i=1}^{+\infty}, \quad y = \{y_i\}_{i=1}^{+\infty} \in \ell_2.$$

Table 2: Numerical results for Example 6.1 (**Experiment 2**)

Cases		$\phi_{i,n} = 0.5$	$\phi_{i,n} = \frac{3}{4+i}$	$\phi_{i,n} = 1.0$	$\phi_{i,n} = \frac{5}{2+i}$	$\phi_{i,n} = 2.0$	$\phi_{i,n} = \frac{7}{3+i}$
1	CPU time (s)	0.0138	0.0090	0.0132	0.0156	0.0092	0.0130
	No. of Iter.	14	14	14	14	14	14
2	CPU time (s)	0.0139	0.0089	0.0128	0.0144	0.0140	0.0171
	No. of Iter.	14	14	14	14	14	14
3	CPU time (s)	0.0139	0.0091	0.0148	0.0151	0.0093	0.0143
	No. of Iter.	14	14	14	14	14	14
4	CPU time (s)	0.0136	0.0088	0.0140	0.0152	0.0092	0.0127
	No. of Iter.	14	14	14	14	14	14

Table 3: Numerical results for Example 6.2 (**Experiment 1**)

Cases		$\theta_{i,n} = \frac{1}{2+i}$	$\theta_{i,n} = 1.0$	$\theta_{i,n} = \frac{2}{3+i}$	$\theta_{i,n} = 1.5$	$\theta_{i,n} = \frac{3}{4+i}$	$\theta_{i,n} = 1.9$
1	CPU time (s)	0.0544	0.0112	0.0144	0.0130	0.0158	0.0242
	No. of Iter.	33	33	33	33	33	33
2	CPU time (s)	0.0129	0.0093	0.0133	0.0133	0.0138	0.0124
	No. of Iter.	34	33	34	33	34	33
3	CPU time (s)	0.0134	0.0124	0.0096	0.0142	0.0137	0.0134
	No. of Iter.	33	33	33	33	33	33
4	CPU time (s)	0.0161	0.0174	0.0164	0.0166	0.0133	0.0124
	No. of Iter.	33	33	33	34	34	34

Table 4: Numerical results for Example 6.2 (**Experiment 2**)

Cases		$\phi_{i,n} = 0.5$	$\phi_{i,n} = \frac{3}{4+i}$	$\phi_{i,n} = 1.0$	$\theta_{i,n} = \frac{5}{2+i}$	$\theta_{i,n} = 2.0$	$\theta_{i,n} = \frac{7}{3+i}.$
1	CPU time (s)	0.0132	0.0100	0.0112	0.0116	0.0118	0.0130
	No. of Iter.	33	33	33	33	33	33
2	CPU time (s)	0.0154	0.0122	0.0102	0.0146	0.0138	0.0138
	No. of Iter.	33	33	33	33	33	33
3	CPU time (s)	0.0141	0.0100	0.0133	0.0141	0.0140	0.0154
	No. of Iter.	33	33	33	33	33	33
4	CPU time (s)	0.0161	0.0169	0.0142	0.0169	0.0160	0.0148
	No. of Iter.	33	33	33	34	34	34

For $i = 0, 1, 2$, let the mapping $A_i : \ell_2 \to \ell_2$ be defined as $A_i x = \left(\frac{x_1}{3}, \frac{x_2}{3}, \ldots, \frac{x_m}{3}, \ldots\right)$ for all $x = \{x_m\}_{m=1}^{+\infty} \in \ell_2$ and $A_i^* : \ell_2 \to \ell_2$ be defined by $A_i^* z = \left(\frac{z_1}{3}, \frac{z_2}{3}, \ldots, \frac{z_m}{3}, \ldots\right)$ for all $z = \{z_m\}_{m=1}^{+\infty} \in \ell_2$. Define the mapping $F_i : \ell_2 \times \ell_2 \to \mathbb{R}$ by F_i, $=F$ such that $F(x, y) = -x^2 + y^2, \quad \forall \quad x = \{x_i\}_{i=1}^{+\infty}, \quad y = \{y_i\}_{i=1}^{+\infty}$. and $\phi_i = 0$, for each $i = 0, 1, 2$. It is easy to see that

$$T_r^{(F,\phi)} x = \frac{1 - r}{5r + 1} x.$$

Also, for $j = 1, 2, \ldots$, we define $S_j : C \to CB(\ell_2)$ by

$$S_j x = \left[0, \frac{x}{5j}\right] \quad \forall j = 1, 2, \ldots.$$

It is easy to see that S_j is zero demicontractive for each $j = 1, 2, \dots$ and $\mathrm{Fix}(S_j) = \{0\}$.

We test these examples under the following experiments:

Experiment 1:

In this experiment, we check the behavior of our method by fixing the other parameters and varying $\theta_{i,n}$. We do this to check the effects of the parameter $\theta_{i,n}$ on our method.

For **Example** 6.1 We consider the following cases for the initial value of x_0:

Case 1 $x_0 = (0.78, 1.25)$;
Case 2 $x_0 = (3.78, 1.25)$;
Case 3 $x_0 = (4, 2)$;
Case 4 $x_0 = (-1, -5)$.

Also, we consider $\theta_{i,n} \in \left\{\frac{1}{2+i}, 1.0, \frac{2}{3+i}, 1.5, \frac{3}{4+i}, 1.9\right\}$, which satisfies Assumption (A3). We use Algorithm 3.1 for the experiment and report the numerical results in Table 1 and Figure 1.

For Example 6.2 We consider the following cases for the initial value of x_0:

Case 1 $x_0 = (2, 1, \frac{1}{2}, \cdots)$;
Case 2 $x_0 = (4, -2, 1, \cdots)$;
Case 3 $x_0 = (-3, \frac{3}{5}, -\frac{3}{25}, \cdots)$;
Case 4 $x_0 = (6, 1, \frac{1}{6}, \cdots)$.

Also, we consider $\theta_{i,n} \in \left\{\frac{1}{2+i}, 1.0, \frac{2}{3+i}, 1.5, \frac{3}{4+i}, 1.9\right\}$, which satisfies Assumption (A3). We use Algorithm 3.1 for the experiment and report the numerical results in Table 3 and Figure 3.

Experiment 2:

In this experiment, we check the behavior of our method by fixing the other parameters and varying $\phi_{i,n}$. We do this to check the effects of the parameter $\phi_{i,n}$ on our method.

For Example 6.1: We consider the following cases for the initial value of x_0:

Case 1 $x_0 = (0.78, 1.25)$;
Case 2 $x_0 = (3.78, 1.25)$;
Case 3 $x_0 = (4, 2)$;
Case 4 $x_0 = (-1, -5)$.

Also, we consider $\phi_{i,n} \in \left\{0.5, \frac{3}{4+i}, 1.0, \frac{5}{2+i}, 2.0, \frac{7}{3+i}\right\}$, which satisfies Assumption (A4). We use Algorithm 3.1 for the experiment and report the numerical results in Table 2 and Figure 2.

For Example 6.2: We consider the following cases for the initial value of x_0:

Case 1 $x_0 = (2, 1, \frac{1}{2}, \cdots)$;
Case 2 $x_0 = (4, -2, 1, \cdots)$;
Case 3 $x_0 = (-3, \frac{3}{5}, -\frac{3}{25}, \cdots)$;
Case 4 $x_0 = (6, 1, \frac{1}{6}, \cdots)$.

Also, we consider $\phi_{i,n} \in \left\{0.5, \frac{3}{4+i}, 1.0, \frac{5}{2+i}, 2.0, \frac{7}{3+i}\right\}$, which satisfies Assumption (A4). We use Algorithm 3.1 for the experiment and report the numerical results in Table 4 and Figure 4.

7 Conclusion

A new efficient algorithm for approximating a common solution of finite family of split common generalized equilibrium problems with multiple output sets and common fixed point of an infinite family of demicontractive mappings in real Hilbert spaces. We proved that the proposed algorithm converges strongly to the solution set without relying on the usual "**Two Cases Approach**" widely used in the literature to guarantee strong convergence. We gave some applications of our result and also presented some numerical examples to illustrate the applicability and efficiency of our method. In all examples, we checked the sensitivity of key parameters for each starting points in order to know if their choices affect the performance of our methods. We can see from the tables and graphs that the number of iterations and CPU times for our proposed method remain consistent and well-behaved for different choices of these key parameters. Furthermore, our method uses a simple self-adaptive step size that is generated at each iteration by some simple computation, which allows easy implementation of our algorithm without prior knowledge of the operator norm. Our method unifies and extends a whole lot of results in the literature in this direction of research.

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