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Research Article

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Normal ordering associated with λ-Stirling numbers in λ-shift algebra

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Abstract: It is known that the Stirling numbers of the second kind are related to normal ordering in the Weyl algebra, while the unsigned Stirling numbers of the first kind are related to normal ordering in the shift algebra. Recently, Kim-Kim introduced a λ-analogue of the unsigned Stirling numbers of the first kind and that of the r-Stirling numbers of the first kind. In this article, we introduce a λ-analogue of the shift algebra (called λ-shift algebra) and investigate normal ordering in the λ-shift algebra. From the normal ordering in the λ-shift algebra, we derive some identities about the λ-analogue of the unsigned Stirling numbers of the first kind.

Keywords: λ-shift algebra, normal ordering, unsigned λ-Stirling numbers of the first kind, λ-r-Stirling numbers of the first kind

MSC 2020: 11B73, 11B83

1 Introduction

The Stirling number of the first kind $S_1(n, k)$ is defined in such a way that the unsigned Stirling number of the first kind $\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{n-k} S_1(n, k)$ enumerates the number of permutations of the set $[n] = \{1, 2, 3, \ldots, n\}$, which are the products of k disjoint cycles.

The unsigned r-Stirling number of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_r$ is the number of permutations of $[n]$ with exactly k disjoint cycles in such a way that the numbers $1, 2, \ldots, r$ are in distinct cycles.

In [1], Kim and Kim introduced a λ-analogue of the unsigned Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_\lambda$ and that of the unsigned r-Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_{r,\lambda}$, respectively, as a λ-analogue of $\begin{bmatrix} n \\ k \end{bmatrix}$ and that of $\begin{bmatrix} n \\ k \end{bmatrix}_r$ (see (8) and (9)).

The Stirling numbers of the second kind appear as the coefficients in the normal ordering of the Weyl algebra (see (10) and (11)), while the unsigned Stirling numbers of the first kind appear as those of the shift algebra S (see (12) and (13)).

The aim of this article is to introduce the λ-shift algebra S_λ (for any $\lambda \in \mathbb{C}$), which is a λ-analogue of S (see (14)), and to investigate the normal ordering of the λ-shift algebra. In addition, from the normal ordering of the λ-shift algebra S_λ, we derive some identities about the unsigned λ-Stirling numbers of the first kind.

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The outline of this article is as follows. In Section 1, we recall the λ-falling factorial numbers, the falling factorial numbers, the λ-rising factorial numbers, and the rising factorial numbers. We remind the reader of the unsigned λ-Stirling numbers of the first kind and the λ-r-Stirling numbers of the first kind. We recall the Weyl algebra and the normal ordering result in that algebra. We remind the reader of the shift algebra and the normal ordering result in that algebra. Finally, we define the λ-shift algebra as a λ-analogue of the shift algebra. Section 2 is the main result of this article. We derive normal ordering results in S_λ in Theorems 1 and 2, where $\begin{bmatrix} n \\ k \end{bmatrix}_\lambda$ and $\begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda}$ appear, respectively, as their coefficients. We obtain three other normal ordering results in Theorem 3. In Theorem 4, we obtain a recurrence relation for the unsigned λ-Stirling numbers of the first kind. In Theorem 6, we obtain another expression of the defining equation in (8) in terms of the λ-shift operator (see (30)). In Theorem 7, we show a λ- analogue of the dual to Spivey's identity (see Remark 8). Finally, we conclude this article in Section 3. For the rest of section, we recall what are needed throughout this article.

For any $\lambda \in \mathbb{C}$, the λ-falling factorial sequence is defined by:

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda), \quad (n \geq 1) \text{ (see [1,2])}. \tag{1}$$

In particular, the falling factorial sequence is given by:

$$(x)_0 = 1, \quad (x)_n = x(x-1)\cdots(x-(n-1)), \quad (n \geq 1). \tag{2}$$

Note that $\lim_{\lambda\to 1}(x)_{n,\lambda} = (x)_n$.

For any $\lambda \in \mathbb{C}$, the λ-rising factorial sequence is defined by:

$$\langle x\rangle_{0,\lambda} = 1, \quad \langle x\rangle_{n,\lambda} = x(x+\lambda)\cdots(x+(n-1)\lambda), \quad (n \geq 1) \quad \text{(see [1,2])}. \tag{3}$$

Especially, the rising factorial sequence is given by:

$$\langle x\rangle_0 = 1, \quad \langle x\rangle_n = x(x+1)\cdots(x+(n-1)), \quad (n \geq 1). \tag{4}$$

Observe that $\lim_{\lambda\to 1}\langle x\rangle_{n,\lambda} = \langle x\rangle_n$.

With the notation in (2), the Stirling numbers of the first kind are defined by:

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k, \quad (n \geq 0) \text{ (see [3–6])}. \tag{5}$$

In addition, the unsigned Stirling numbers of the first kind are given by $\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{n-k}S_1(n,k)$, $(n,k \geq 0)$.

The Stirling numbers of the second kind are defined by:

$$x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (x)_k, \quad (n \geq 0). \tag{6}$$

Recently, with the notation in (1), the λ-Stirling numbers of the first kind, which are λ-analogues of the Stirling numbers of the first kind, are defined by:

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}(n,k)x^k, \quad (n \geq 0) \text{ (see [1])}. \tag{7}$$

In addition, with the notation in (3) the unsigned λ-Stirling numbers of the first kind are defined by:

$$\langle x\rangle_{n,\lambda} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda x^k, \quad (n \geq 0) \quad \text{(see [1])}. \tag{8}$$

Note that $\lim_{\lambda\to 1}S_{1,\lambda}(n,k) = S_1(n,k)$ (see (5)); $\lim_{\lambda\to 1}\begin{bmatrix} n \\ k \end{bmatrix}_\lambda = \begin{bmatrix} n \\ k \end{bmatrix}$.

For $r \in \mathbb{N}\cup\{0\}$, the λ-r-Stirling numbers of the first kind, which are λ-analogues of the r-Stirling numbers of the first kind, are defined by:

$$\langle x+r\rangle_{n,\lambda} = \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} x^k, \quad (n \geq 0) \quad \text{(see [1])}. \tag{9}$$

Note that $\lim_{\lambda\to 1}\left[\begin{smallmatrix} n+r \\ k+r \end{smallmatrix}\right]_{r,\lambda} = \left[\begin{smallmatrix} n+r \\ k+r \end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} n+r \\ k+r \end{smallmatrix}\right]$ are the r-Stirling numbers of the first kind, which are introduced by Broder (see [7]) and given by (see (4))

$$\langle x+r\rangle_n = \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r x^k, \quad (n \geq 0).$$

The Weyl algebra is the unital algebra generated by letters a and $a^\dagger$ satisfying the following commutation relation:

$$aa^\dagger - a^\dagger a = 1 \quad (\text{see } [1,5,8\text{–}23]). \tag{10}$$

Katriel proved that the normal ordering in Weyl algebra is given by (see (6))

$$(a^\dagger a)^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (a^\dagger)^k a^k \quad (\text{see } [10,11,12]). \tag{11}$$

From the definition of the Stirling numbers of the second kind and (11), we note that

$$(a^\dagger)^n a^n = (a^\dagger a)_n = a^\dagger a(a^\dagger a - 1)\cdots(a^\dagger a - n + 1), \quad (n \geq 1).$$

The shift algebra S is defined as the complex unital algebra generated by a and $a^\dagger$ satisfying the following commutation relation:

$$aa^\dagger - a^\dagger a = a \quad (\text{see } [22]). \tag{12}$$

A word in S is said to be in normal-ordered form if all letters a stand to the right of all letters $a^\dagger$.

From (12), we note that the normal ordering in the shift algebra S is given by (see (11))

$$(a^\dagger a)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a^\dagger)^k a^n \quad (\text{see } [22]). \tag{13}$$

For any $\lambda \in \mathbb{C}$, we consider a λ-analogue of the shift algebra S, which is defined as the complex unital algebra generated by a and $a^\dagger$ satisfying the following commutation relation (see (12))

$$aa^\dagger - a^\dagger a = \lambda a. \tag{14}$$

The λ-analogue of the shift algebra S is called the λ-shift algebra and denoted by S_λ.

2 λ-analogues of normal ordering in the λ-shift algebra

Let S_λ be the λ-shift algebra defined in (14). A word in S_λ is said to be in normal-ordered form if all letters a stand to the right of all letters $a^\dagger$.

In S_λ, by (14), we obtain

$$(a^\dagger a)^2 = (a^\dagger a)(a^\dagger a) = a^\dagger(aa^\dagger)a = a^\dagger(\lambda a + a^\dagger a)a = a^\dagger(\lambda + a^\dagger)a^2 = \langle a^\dagger\rangle_{2,\lambda} a^2,$$

$$\begin{aligned}(a^\dagger a)^3 &= (a^\dagger a)(a^\dagger a)(a^\dagger a) = a^\dagger(aa^\dagger)(aa^\dagger)a \\ &= a^\dagger(\lambda + a^\dagger)a(\lambda + a^\dagger)a^2 = a^\dagger(\lambda + a^\dagger)(a\lambda + aa^\dagger)a^2 \\ &= a^\dagger(\lambda + a^\dagger)(2\lambda a + a^\dagger a)a^2 = a^\dagger(a^\dagger + \lambda)(a^\dagger + 2\lambda)a^3 \\ &= \langle a^\dagger\rangle_{3,\lambda} a^3.\end{aligned}$$

Continuing this process, we have

$$(a^\dagger a)^n = \langle a^\dagger\rangle_{n,\lambda} a^n, \quad (n \geq 1). \tag{15}$$

Thus, by (8) and (15), we obtain (see (7), (11), (13))

$$(a^{\dagger}a)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\lambda} (a^{\dagger})^k a^n. \tag{16}$$

Therefore, by (16), we obtain the following theorem.

Theorem 1. *In S_λ, the unsigned λ-Stirling numbers of the first kind appear as the coefficients of $(a^{\dagger}a)^n$ in normal-ordered form, as it is given by*:

$$(a^{\dagger}a)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\lambda} (a^{\dagger})^k a^n.$$

For $r \geq 0$, by (14), we obtain

$$\begin{aligned}((a^{\dagger}+r)a)^2 &= ((a^{\dagger}+r)a)((a^{\dagger}+r)a) = (a^{\dagger}+r)(aa^{\dagger}+ra)a \\ &= (a^{\dagger}+r)(a^{\dagger}a+\lambda a+ra)a = (a^{\dagger}+r)(a^{\dagger}+r+\lambda)a^2 \\ &= \langle a^{\dagger}+r\rangle_{2,\lambda}a^2\end{aligned}$$

and

$$\begin{aligned}((a^{\dagger}+r)a)^3 &= ((a^{\dagger}+r)a)((a^{\dagger}+r)a)((a^{\dagger}+r)a) \\ &= (a^{\dagger}+r)a(a^{\dagger}+r)(\lambda+r+a^{\dagger})a^2 \\ &= (a^{\dagger}+r)(\lambda+r+a^{\dagger})a(\lambda+r+a^{\dagger})a^2 \\ &= (a^{\dagger}+r)(a^{\dagger}+r+\lambda)(\lambda a+ra+aa^{\dagger})a^2 \\ &= (a^{\dagger}+r)(a^{\dagger}+r+\lambda)(a^{\dagger}+r+2\lambda)a^3 = \langle a^{\dagger}+r\rangle_{3,\lambda}a^3.\end{aligned}$$

Continuing this process, we have

$$((a^{\dagger}+r))a^n = \langle a^{\dagger}+r\rangle_{n,\lambda}a^n, \quad (n \geq 1). \tag{17}$$

From (9) and (17), we obtain

$$((a^{\dagger}+r)a)^n = \langle a^{\dagger}+r\rangle_{n,\lambda}a^n = \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} (a^{\dagger})^k a^n. \tag{18}$$

Therefore, by (18), we obtain the following theorem.

Theorem 2. *Let r be a non-negative integer. In S_λ, the λ-r-Stirling numbers of the first kind appear as the coefficients of $((a^{\dagger}+r)a)^n$ in the normal-ordered form, as it is given by*:

$$((a^{\dagger}+r)a)^n = \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} (a^{\dagger})^k a^n.$$

From (14), we note that

$$\begin{aligned}a^m a^{\dagger} &= a^{m-1}(aa^{\dagger}) = a^{m-1}(a^{\dagger}+\lambda)a \\ &= (a^{m-1}a^{\dagger})a + \lambda a^m = a^{m-2}(a^{\dagger}a+\lambda a)a + \lambda a^m \\ &= (a^{m-2}a^{\dagger})a^2 + 2\lambda a^m = \cdots = (a^{\dagger}+m\lambda)a^m\end{aligned} \tag{19}$$

and

$$\begin{aligned}a(a^{\dagger})^n &= (aa^{\dagger})(a^{\dagger})^{n-1} = (\lambda+a^{\dagger})a(a^{\dagger})^{n-1} \\ &= (\lambda+a^{\dagger})(aa^{\dagger})(a^{\dagger})^{n-2} = (\lambda+a^{\dagger})^2a(a^{\dagger})^{n-2} = \cdots \\ &= (\lambda+a^{\dagger})^{n-1}aa^{\dagger} = (\lambda+a^{\dagger})^n a.\end{aligned} \tag{20}$$

By (19), we obtain

$$\begin{aligned} a^m(a^\dagger)^n &= (a^m a^\dagger)(a^\dagger)^{n-1} = (a^\dagger + m\lambda)a^m(a^\dagger)^{n-1} \\ &= (a^\dagger + m\lambda)(a^m a^\dagger)(a^\dagger)^{n-2} = (a^\dagger + m\lambda)(a^\dagger + m\lambda)a^m(a^\dagger)^{n-2} \\ &= \cdots = (a^\dagger + m\lambda)^n a^m. \end{aligned} \tag{21}$$

Therefore, by (19), (20), and (21), we obtain the following theorem.

Theorem 3. *For $m, n \in \mathbb{N}$ and $\lambda \neq 0$, we have in S_λ the normal orderings given by:*

$$a^m a^\dagger = (a^\dagger + m\lambda)a^m, \quad a(a^\dagger)^n = (a^\dagger + \lambda)^n a, \quad a^m(a^\dagger)^n = (a^\dagger + m\lambda)^n a^m.$$

Now, we observe from Theorem 3 that

$$\begin{aligned} (a^\dagger a)^{n+1} &= (a^\dagger a)(a^\dagger a)^n = a^\dagger a \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda (a^\dagger)^k a^n \\ &= a^\dagger \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda a(a^\dagger)^k a^n = a^\dagger \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda (\lambda + a^\dagger)^k a a^n \\ &= a^\dagger \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda \sum_{j=0}^{k} \binom{k}{j} \lambda^{k-j}(a^\dagger)^j a^{n+1} \\ &= \sum_{j=0}^{n}\sum_{k=j}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda \binom{k}{j} \lambda^{k-j}(a^\dagger)^{j+1} a^{n+1} \\ &= \sum_{j=1}^{n+1}\left(\sum_{k=j-1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda \binom{k}{j-1} \lambda^{k+1-j}\right)(a^\dagger)^j a^{n+1}. \end{aligned} \tag{22}$$

On the other hand, by Theorem 1, we obtain

$$(a^\dagger a)^{n+1} = \sum_{j=0}^{n+1} \begin{bmatrix} n+1 \\ j \end{bmatrix}_\lambda (a^\dagger)^j a^{n+1} = \sum_{j=1}^{n+1} \begin{bmatrix} n+1 \\ j \end{bmatrix}_\lambda (a^\dagger)^j a^{n+1}. \tag{23}$$

Therefore, by (22) and (23), we obtain the following theorem.

Theorem 4. *Let $n, j \in \mathbb{Z}$ with $n \geq 0$ and $j \geq 1$. In S_λ, the unsigned λ-Stirling numbers of the first kind satisfy the following recurrence relation:*

$$\begin{bmatrix} n+1 \\ j \end{bmatrix}_\lambda = \sum_{k=j-1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda \binom{k}{j-1} \lambda^{k+1-j} = \begin{bmatrix} n \\ j-1 \end{bmatrix}_\lambda + \sum_{k=j}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda \binom{k}{j-1} \lambda^{k+1-j}.$$

For $n \geq 1$, by (15) and (17), we have the λ-analogues of Boole's relations in the λ-shift algebra given by:

$$(a^\dagger a)^n = \langle a^\dagger \rangle_{n,\lambda} a^n, \quad ((a^\dagger + r)a)^n = \langle a^\dagger + r \rangle_{n,\lambda} a^n.$$

Now, we define the λ-analogues of $n!$ as (see (8))

$$(0)_\lambda! = 1, \quad (n)_\lambda! = \langle 1 \rangle_{n,\lambda} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda, \quad (n \geq 1). \tag{24}$$

Note that $\lim_{\lambda \to 1}(n)_\lambda! = n!$.

From (8) and (9), we note that

$$\frac{1}{k!}\left(-\frac{\log(1-\lambda t)}{\lambda}\right)^k = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda \frac{t^n}{n!} \tag{25}$$

and

$$\frac{1}{k!}\left(-\frac{\log(1-\lambda t)}{\lambda}\right)^k\left(\frac{1}{1-\lambda t}\right)^{\frac{r}{\lambda}}=\sum_{n=k}^{\infty}\begin{bmatrix}n+r\\k+r\end{bmatrix}_{r,\lambda}\frac{t^n}{n!},\tag{26}$$

where k is a non-negative integer.

Thus, by (25) and (26), we obtain

$$\begin{aligned}\sum_{n=k}^{\infty}\begin{bmatrix}n+1\\k+1\end{bmatrix}_{1,\lambda}\frac{t^n}{n!}&=\left(\frac{1}{1-\lambda t}\right)^{\frac{1}{\lambda}}\frac{1}{k!}\left(-\frac{1}{\lambda}\log(1-\lambda t)\right)^k\\&=\sum_{l=0}^{\infty}\frac{\langle 1\rangle_{l,\lambda}}{l!}t^l\sum_{m=k}^{\infty}\begin{bmatrix}m\\k\end{bmatrix}_{\lambda}\frac{t^m}{m!}\\&=\sum_{n=k}^{\infty}\left(\sum_{l=0}^{n-k}\binom{n}{l}\langle 1\rangle_{l,\lambda}\begin{bmatrix}n-l\\k\end{bmatrix}_{\lambda}\right)\frac{t^n}{n!}.\end{aligned}\tag{27}$$

Comparing the coefficients on both sides of (27), we obtain

$$\begin{bmatrix}n+1\\k+1\end{bmatrix}_{1,\lambda}=\sum_{l=0}^{n-k}\binom{n}{l}\langle 1\rangle_{l,\lambda}\begin{bmatrix}n-l\\k\end{bmatrix}_{\lambda}.\tag{28}$$

From (9) and (28), we note that

$$\begin{aligned}\langle 1+m\lambda\rangle_{n,\lambda}&=\sum_{k=0}^{n}\begin{bmatrix}n+1\\k+1\end{bmatrix}_{1,\lambda}m^k\lambda^k=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{l}\langle 1\rangle_{l,\lambda}\begin{bmatrix}n-l\\k\end{bmatrix}_{\lambda}m^k\lambda^k\\&=\sum_{l=0}^{n}\left(\sum_{k=0}^{n-l}\begin{bmatrix}n-l\\k\end{bmatrix}_{\lambda}m^k\lambda^k\right)\binom{n}{l}\langle 1\rangle_{l,\lambda}=\sum_{l=0}^{n}\binom{n}{l}\langle 1\rangle_{l,\lambda}\langle m\lambda\rangle_{n-l,\lambda}.\end{aligned}\tag{29}$$

Therefore, by (29), we obtain the following theorem.

Theorem 5. *For $n\geq 0$, we have*

$$\langle 1+m\lambda\rangle_{n,\lambda}=\sum_{l=0}^{n}\binom{n}{l}\langle 1\rangle_{l,\lambda}\langle m\lambda\rangle_{n-l,\lambda}.$$

We note that

$$\langle 1+m\rangle_n=\lim_{\lambda\to 1}\langle 1+m\lambda\rangle_{n,\lambda}=\sum_{l=0}^{n}\binom{n}{l}l!\langle m\rangle_{n-l}.$$

For any $\alpha\in\mathbb{R}$, we define the λ-shift operator δ_λ^α by

$$\delta_\lambda^\alpha f(x)=f(x+\lambda\alpha).\tag{30}$$

Then, we see that

$$\delta_\lambda x-x\delta_\lambda=\lambda\delta_\lambda,$$

where $\delta_\lambda=\delta_\lambda^1$, and x denotes the “multiplication by x” operator.

In the λ-shift algebra S_λ, a concrete representation is given by the operators $a^\dagger\mapsto x$ and $a\mapsto\delta_\lambda$. From Theorems 1 and 3, we note that

$$\delta_\lambda^r x^s=(x+r\lambda)^s\delta_\lambda^r$$

and

$$(x\delta_\lambda)^n=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{\lambda}x^k\delta_\lambda^n,\quad(n\geq 0).\tag{31}$$

Now, we observe from (30) and (31) that

$$(x\delta_\lambda)^n e^x = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda x^k \delta_\lambda^n e^x = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda x^k e^{(n\lambda+x)}. \tag{32}$$

By (8) and (32), we obtain

$$e^{-(n\lambda+x)}(x\delta_\lambda)^n e^x = \langle x\rangle_{n,\lambda} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda x^k. \tag{33}$$

In particular, for $x = 1$, we have (see (24))

$$e^{-(n\lambda+x)}(x\delta_\lambda)^n e^x|_{x=1} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda = (n)_\lambda!. \tag{34}$$

Therefore, by (33) and (34), we obtain the following theorem.

Theorem 6. *For $n \in \mathbb{N}$, in S_λ, we have*

$$e^{-(x+n\lambda)}(x\delta_\lambda)^n e^x = \langle x\rangle_{n,\lambda} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda x^k.$$

In particular, for $x = 1$, we obtain

$$e^{-(x+n\lambda)}(x\delta_\lambda)^n e^x|_{x=1} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda = (n)_\lambda!.$$

From Theorem 6, we note that

$$e^{-(x+(m+n)\lambda)}(x\delta_\lambda)^{m+n} e^x\,|_{x=1} = (m+n)_\lambda!. \tag{35}$$

On the other hand, by (31), we obtain

$$\begin{aligned}(x\delta_\lambda)^{m+n} = (x\delta_\lambda)^m (x\delta_\lambda)^n &= \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_\lambda \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda x^j \delta_\lambda^m x^k \delta_\lambda^n \\ &= \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_\lambda \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda x^j (x+m\lambda)^k \delta_\lambda^{m+n}.\end{aligned} \tag{36}$$

From (36), we have

$$\begin{aligned}(x\delta_\lambda)^{m+n} e^x &= \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_\lambda \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda x^j (x+m\lambda)^k \delta_\lambda^{m+n} e^x \\ &= \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_\lambda \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda x^j (x+m\lambda)^k e^{x+(m+n)\lambda}.\end{aligned} \tag{37}$$

Thus, by (37), we obtain

$$e^{-(x+(m+n)\lambda)}(x\delta_\lambda)^{m+n} e^x = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_\lambda \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda x^j (x+m\lambda)^k. \tag{38}$$

From (35) and (38), we have

$$\begin{aligned}(m+n)_\lambda! &= e^{-(x+(m+n)\lambda)}(x\delta_\lambda)^{n+m} e^x|_{x=1} \\ &= \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_\lambda \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda (1+m\lambda)^k \\ &= \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_\lambda \langle 1+m\lambda\rangle_{n,\lambda}.\end{aligned} \tag{39}$$

By Theorem 5 and (39), we obtain

$$(m+n)_\lambda! = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_\lambda \langle 1+m\lambda\rangle_{n,\lambda} = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_\lambda \sum_{k=0}^{n} \binom{n}{k} \langle m\lambda\rangle_{n-k,\lambda} \langle 1\rangle_{k,\lambda}. \tag{40}$$

Therefore, by (40), we obtain the following theorem.

Theorem 7. *For* $m, n \geq 0$, *we have*

$$(m+n)_\lambda! = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_\lambda \langle 1+m\lambda\rangle_{n,\lambda} = \sum_{j=0}^{m} \sum_{k=0}^{n} \begin{bmatrix} m \\ j \end{bmatrix}_\lambda \binom{n}{k} \langle m\lambda\rangle_{n-k,\lambda} \langle 1\rangle_{k,\lambda}.$$

Remark 8. (a) Taking the limit as $\lambda \to 1$, we see from Theorem 7 that

$$(m+n)! = \lim_{\lambda\to 1}(m+n)_\lambda! = \sum_{j=0}^{m} \sum_{k=0}^{n} \begin{bmatrix} m \\ j \end{bmatrix} \binom{n}{k} \langle m\rangle_{n-k} k! \quad (\text{see [19]}).$$

This was discovered by Mező in [19], which is dual to Spivey's identity (see [22,24,25]).

(b) As the identities in Theorem 7 are obviously symmetric in m and n, we obtain the following symmetric identities:

$$\sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_\lambda \langle 1+m\lambda\rangle_{n,\lambda} = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_\lambda \langle 1+n\lambda\rangle_{m,\lambda},$$

$$\sum_{j=0}^{m} \sum_{k=0}^{n} \begin{bmatrix} m \\ j \end{bmatrix}_\lambda \binom{n}{k} \langle m\lambda\rangle_{n-k,\lambda} \langle 1\rangle_{k,\lambda} = \sum_{j=0}^{n} \sum_{k=0}^{m} \begin{bmatrix} n \\ j \end{bmatrix}_\lambda \binom{m}{k} \langle n\lambda\rangle_{m-k,\lambda} \langle 1\rangle_{k,\lambda}.$$

3 Conclusion

In this article, as a λ-analogue of the shift algebra S, we introduced the λ-shift algebra S_λ, which is defined as the complex unital algebra generated by a and $a^\dagger$ satisfying the followig commutation relation:

$$aa^\dagger - a^\dagger a = \lambda a.$$

The unsigned λ-Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_\lambda$ and the λ-r-Stirling numbers of the first kind $\begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda}$ were introduced, respectively, as a λ-analogue of the unsigned Stirling numbers of the first kind and a λ-analogue of the r-Stirling numbers of the first kind. We showed that those numbers appear as the coefficients in the following normal ordering results in S_λ:

$$(a^\dagger a)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda (a^\dagger)^k a^n, \quad ((a^\dagger + r)a)^n = \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} (a^\dagger)^k a^n.$$

In addition, from those normal ordering results, we derived some properties about the unsigned λ-Stirling numbers of the first kind.

There are various methods that can be used to find some results on special numbers and polynomials. These include generating functions, combinatorial methods, umbral calculus, p-adic analysis, differential equations, analytic number theory, probability, statistics, operator theory, special functions, and mathematical physics.

It is one of our future projects to continue to explore various λ-analogues and degenerate versions of many special numbers and polynomials with these tools.

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