

## Research Article

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# Properties of a subclass of analytic functions defined by Riemann-Liouville fractional integral applied to convolution product of multiplier transformation and Ruscheweyh derivative

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**Abstract:** The contribution of fractional calculus in the development of different areas of research is well known. This article presents investigations involving fractional calculus in the study of analytic functions. Riemann-Liouville fractional integral is known for its extensive applications in geometric function theory. New contributions were previously obtained by applying the Riemann-Liouville fractional integral to the convolution product of multiplier transformation and Ruscheweyh derivative. For the study presented in this article, the resulting operator is used following the line of research that concerns the study of certain new subclasses of analytic functions using fractional operators. Riemann-Liouville fractional integral of the convolution product of multiplier transformation and Ruscheweyh derivative is applied here for introducing a new class of analytic functions. Investigations regarding this newly introduced class concern the usual aspects considered by researchers in geometric function theory targeting the conditions that a function must meet to be part of this class and the properties that characterize the functions that fulfil these conditions. Theorems and corollaries regarding neighborhoods and their inclusion relation involving the newly defined class are stated, closure and distortion theorems are proved, and coefficient estimates are obtained involving the functions belonging to this class. Geometrical properties such as radii of convexity, starlikeness, and close-to-convexity are also obtained for this new class of functions.

**Keywords:** analytic functions, fractional integral, multiplier transformation, radii of convexity and starlikeness, neighborhood property

**MSC 2020:** 30C45, 30A10, 33C15

## 1 Introduction

Riemann-Liouville fractional integral is a classical fractional calculus operator, which is intensely used in theoretical studies, while certain operators defined involving this operator can be applied in numerous fields of science and engineering. Scientific and real-world problems can be modeled and analyzed using such

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fractional integral operators. A comprehensive overview of the theory and applications of the fractional-calculus operators is given in the very recent review papers [1,2].

Fractional calculus operators provided interesting applications in the theory of analytic functions, and examples can be given as classical results, which are still inspiring for researchers nowadays. Riemann-Liouville fractional integral was first used in [3], further investigations were conducted combining it with the generalized hypergeometric function in [4], and a general class of fractional integral operators involving the Gauss hypergeometric function is introduced in [5] and used for obtaining distortion results involving certain subclasses of univalent functions analytic in the unit disk. After many more other published results, a unified approach on special functions and fractional calculus operators is presented in [6], where a series of articles are listed that can be read for following the development of the topic. The investigations continued using the familiar Dziok-Srivastava convolution operator associated with fractional derivative for defining and studying new classes of analytic functions [7] or for using the means of differential subordination and superordinations theory on those newly defined classes as in [8]. Different integral operators were also associated with Riemann-Liouville fractional integral for studying starlikeness of certain classes of analytic functions in [9] and for obtaining new differential subordinations and superordinations in [10]. Geometric properties of classes of analytic functions were studied using a fractional integral operator in [11], and multi-index Mittag-Leffler function was involved in defining a new fractional operator in [12]. Recent studies involve fractional integral connected to Bessel functions [13], and important special cases connected with fractional operators in fractional calculus are highlighted. Mittag-Leffler confluent hypergeometric function is used for extending properties of fractional integral in [14] and for highlighting certain implementations in fractional calculus in [15].

An important line of research in geometric function theory concerns finding methods of constructing different operators that preserve classes of the univalent functions and using them to define certain relevant subclasses. Very recent interesting results on this topic can be listed such as classes of functions introduced and studied involving generalized differential operators [16,17], considering bi-univalent functions [18,19], or using integral operators [20,21].

The classical definition of fractional calculus operators and generalizations involving such operators have been applied for introducing and studying new classes of functions. In [22], a generalized fractional operator is studied regarding its geometric properties and is used for defining classes of univalent functions.

This article follows the line of research concerned with introducing and studying new classes of functions involving fractional calculus aspects. In this regard, Riemann-Liouville fractional integral applied to convolution product between multiplier transformation and Ruscheweyh derivative introduced in [23] is used for defining a new subclass of analytic functions. The newly introduced class is interesting due to its geometrical properties since aspects regarding convexity are known to be of interest for investigations using fractional integral inequalities [24,25]. Radii of convexity, starlikeness, and close-to-convexity of order  $\delta$  for  $0 \leq \delta < 1$  are given for this new class. The characteristics of the class can be used in further studies concerning special classes of functions, which have particular geometric properties. Other useful results are obtained such as neighborhood, closure, and distortion theorems, and coefficient studies reveal interesting inequalities, which can be used further for conducting studies with a focus on Fekete-Szegő problem as seen for example in [26] or estimates regarding Henkel determinants of certain order [27,28].

Some notations and special classes are needed for the investigation.

$\mathcal{H}(U)$ , denotes the class of analytic functions defined on the open unit disc  $U = \{z \in \mathbb{C} : |z| < 1\}$  of the complex plane and  $\mathcal{H}(a, n)$  denotes the class that contains functions of the form  $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$ . The subclass of the functions of the form  $f(z) = z + a_{n+1} z^{n+1} + \dots$  is denoted by  $\mathcal{A}_n$ , with  $\mathcal{A} = \mathcal{A}_1$ .

**Definition 1.1.** [23] The Riemann-Liouville fractional integral applied to the convolution product of multiplier transformation and Ruscheweyh derivative is defined by:

$$\begin{aligned}
D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z) &= \frac{1}{\Gamma(\lambda)} \int_0^z \frac{\text{IR}_{a,l}^{m,n} f(t)}{(z-t)^{1-\lambda}} dt \\
&= \frac{1}{\Gamma(\lambda)} \int_0^z \frac{t}{(z-t)^{1-\lambda}} dt \frac{1}{\Gamma(\lambda)\Gamma(n+1)} \sum_{k=2}^{\infty} \frac{\Gamma(n+k)}{\Gamma(k)} \left( \frac{1+\alpha(k-1)+l}{l+1} \right)^m a_k^2 \int_0^z \frac{t^k}{(z-t)^{1-\lambda}} dt,
\end{aligned}$$

which can be written making a simple calculation as follows:

$$D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z) = \frac{1}{\Gamma(\lambda+2)} z^{\lambda+1} + \frac{1}{\Gamma(n+1)} \sum_{k=2}^{\infty} \frac{k\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left( \frac{1+\alpha(k-1)+l}{l+1} \right)^m a_k^2 z^{k+\lambda},$$

considering the function  $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$ . We observe that  $D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z) \in \mathcal{H}(\lambda+1, 1)$ .

We remind that for two analytic functions  $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$  and  $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{A}$ , the convolution product of  $f$  and  $g$ , written as  $f * g$ , is defined by:

$$(f * g)(z) = f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

The operators used in the convolution product are defined in the following.

**Definition 1.2.** [29] The multiplier transformation  $I(m, \alpha, l)f(z)$  is defined by the following infinite series:

$$I(m, \alpha, l)f(z) = z + \sum_{k=2}^{\infty} \left( \frac{1+\alpha(k-1)+l}{1+l} \right)^m a_k z^k,$$

for  $f \in \mathcal{A}$ ,  $m \in \mathbb{N} \cup \{0\}$ ,  $\alpha, l \geq 0$ .

**Definition 1.3.** (Ruscheweyh [30]) The Ruscheweyh derivative  $R^n$  is defined by  $R^n : \mathcal{A} \rightarrow \mathcal{A}$ ,

$$\begin{aligned}
R^0 f(z) &= f(z), \\
R^1 f(z) &= z f'(z), \\
&\dots \\
(n+1)R^{n+1} f(z) &= z(R^n f(z))' + nR^n f(z), \quad z \in U,
\end{aligned}$$

for  $f \in \mathcal{A}$  and  $n \in \mathbb{N}$ .

**Remark 1.1.** We observe that  $R^n f(z) = z + \frac{1}{\Gamma(n+1)} \sum_{k=2}^{\infty} \frac{\Gamma(n+k)}{\Gamma(k)} a_k z^k$  for  $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$ ,  $z \in U$ .

**Definition 1.4.** [31] The convolution product  $\text{IR}_{a,l}^{m,n} : \mathcal{A} \rightarrow \mathcal{A}$  of the multiplier transformation  $I(m, \alpha, l)$  and the Ruscheweyh derivative  $R^n$  is defined by:

$$\text{IR}_{a,l}^{m,n} f(z) = (I(m, \alpha, l) * R^n) f(z),$$

for  $n, m \in \mathbb{N}$  and  $\alpha, l \geq 0$ .

**Remark 1.2.** We observe that  $\text{IR}_{a,l}^{m,n} f(z) = z + \frac{1}{\Gamma(n+1)} \sum_{k=2}^{\infty} \left( \frac{1+\alpha(k-1)+l}{l+1} \right)^m \frac{\Gamma(n+k)}{\Gamma(k)} a_k^2 z^k$ , for  $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$ ,  $z \in U$ .

We also remind the definition of Riemann-Liouville fractional integral.

**Definition 1.5.** [3,32] The fractional integral of order  $\lambda$  ( $\lambda > 0$ ) for an analytic function  $f$  is defined by:

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt. \quad (1)$$

Inspired by the results obtained in [33], we define a new subclass of analytic functions using the operator introduced in [23].

**Definition 1.6.** The class  $\mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$  consists of the function  $f \in \mathcal{A}$ , which satisfies the following relation:

$$\left| \frac{1}{d} \left( \frac{z(D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z))' + \gamma z^2 (D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z))''}{(1 - \gamma) D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z) + \gamma z (D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z))'} - 1 \right) \right| < \beta, \quad (2)$$

with  $m, n \in \mathbb{N}$ ,  $\lambda > 0$ ,  $a, l \geq 0$ ,  $d \in \mathbb{C} - \{0\}$ ,  $0 < \beta \leq 1$ ,  $0 \leq \gamma \leq 1$ , and  $z \in U$ .

Considering the definition of the newly introduced class, in Section 2, a necessary and sufficient condition for a function  $f \in \mathcal{A}$  to belong to the class is stated. Coefficient-related studies establish relevant inequalities in Section 2, and distortion properties for functions from class  $\mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$  and for their derivatives are described in Section 3. Partial sums of functions belonging to the class  $\mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$  are presented in Section 4, and the extreme points are obtained for the class. Certain inclusion properties are proved in Section 5 for the class  $\mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$ , and, as a closure of this study, the important aspects regarding convexity and starlikeness properties of the class are discussed in Section 6 of this article.

## 2 Properties related to coefficient inequality

**Theorem 2.1.** The function  $f \in \mathcal{A}$  is contained in the class  $\mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$  if and only if

$$\begin{aligned} & \sum_{k=2}^{\infty} k \{ \gamma k^2 + [1 + \gamma(2\lambda - 2 + \beta|d|)]k + [1 + \gamma(\lambda - 1)](\lambda - 1 + \beta|d|) \} \\ & \left( \frac{1 + a(k-1) + l}{l+1} \right)^m \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} a_k^2 \leq (\gamma\lambda + 1)(\beta|d| - \lambda) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}, \end{aligned} \quad (3)$$

with  $m, n \in \mathbb{N}$ ,  $\lambda > 0$ ,  $a, l \geq 0$ ,  $d \in \mathbb{C} - \{0\}$ ,  $0 < \beta \leq 1$ ,  $0 \leq \gamma \leq 1$ , and  $z \in U$ .

**Proof.** Let  $f \in \mathcal{A}$ . Considering Inequality (3) and making an easy calculus, we obtain

$$\begin{aligned} & \left| \frac{z(D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z))' + \gamma z^2 (D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z))''}{(1 - \gamma) D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z) + \gamma z (D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z))'} - 1 \right| \\ & = \left| \frac{\frac{\lambda(\gamma\lambda+1)\Gamma(n+1)}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{k=2}^{\infty} k \{ \gamma k^2 + [1 + 2\gamma(\lambda - 1)]k + (\lambda - 1)[1 + \gamma(\lambda - 1)] \} \left( \frac{1 + a(k-1) + l}{l+1} \right)^m a_k^2 z^{k+\lambda}}{\frac{(\gamma\lambda+1)\Gamma(n+1)}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{k=2}^{\infty} k [1 + \gamma(k + \lambda - 1)] \left( \frac{1 + a(k-1) + l}{l+1} \right)^m a_k^2 z^{k+\lambda}} \right| \leq, \end{aligned}$$

and applying properties of the modulus function, we obtain

$$\begin{aligned} & \frac{\frac{\lambda(\gamma\lambda+1)\Gamma(n+1)}{\Gamma(\lambda+2)} + \sum_{k=2}^{\infty} k \{ \gamma k^2 + [1 + 2\gamma(\lambda - 1)]k + (\lambda - 1)[1 + \gamma(\lambda - 1)] \} \left( \frac{1 + a(k-1) + l}{l+1} \right)^m \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} a_k^2 |z|^{k-1}}{\frac{(a\lambda+1)\Gamma(n+1)}{\Gamma(\lambda+2)} - \sum_{k=2}^{\infty} k [1 + \gamma(k + \lambda - 1)] \left( \frac{1 + a(k-1) + l}{l+1} \right)^m \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} a_k^2 |z|^{k-1}}. \end{aligned}$$

Considering on real axis values of  $z$  and taking  $z \rightarrow 1^-$ , we obtain using Relation (3)

$$\begin{aligned} & \sum_{k=2}^{\infty} k \{ \gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1) \} \\ & \left( \frac{1 + a(k-1) + l}{l+1} \right)^m \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} a_k^2 \leq (\gamma\lambda + 1)(\beta|d| - \lambda) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}, \end{aligned}$$

that

$$\left| \frac{z(D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z))' + \gamma z^2 (D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z))''}{(1-\gamma)D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z) + \gamma z (D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z))'} - 1 \right| \leq \beta |d|,$$

equivalently with  $f \in \mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$ .

Conversely, suppose that  $f \in \mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$ . In this condition, we obtain the following equivalently inequalities:

$$\begin{aligned} \text{Re} \left\{ \frac{z(D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z))' + \gamma z^2 (D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z))''}{(1-\gamma)D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z) + \gamma z (D_z^{-\lambda} \text{IR}_{a,l}^{m,n} f(z))'} - 1 \right\} > -\beta |d|, \\ \text{Re} \left\{ \frac{\frac{\lambda(\gamma\lambda+1)\Gamma(n+1)}{\Gamma(\lambda+2)}z^{\lambda+1} + \sum_{k=2}^{\infty} k\{\gamma k^2 + [2\gamma(\lambda-1)+1]k + (\lambda-1)[\gamma(\lambda-1)+1]\} \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left( \frac{1+\alpha(k-1)+l}{l+1} \right)^m a_k^2 z^{k+\lambda}}{\frac{(\gamma\lambda+1)\Gamma(n+1)}{\Gamma(\lambda+2)}z^{\lambda+1} + \sum_{k=2}^{\infty} k[1 + \gamma(k+\lambda-1)] \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left( \frac{1+\alpha(k-1)+l}{l+1} \right)^m a_k^2 z^{k+\lambda}} - 1 \right. \\ \left. + \beta |d| \right\} > 0, \\ \text{Re} \frac{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)\Gamma(n+1)}{\Gamma(\lambda+2)}r^{\lambda+1} - \sum_{k=2}^{\infty} k\{\gamma k^2 + [1 + (2\lambda + \beta|d| - 2) + 1]k + [\gamma(\lambda-1) + 1](\lambda + \beta|d| - 1)\} \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left( \frac{1+\alpha(k-1)+l}{l+1} \right)^m a_k^2 r^{k+\lambda}}{\frac{(\gamma\lambda+1)\Gamma(n+1)}{\Gamma(\lambda+2)}r^{\lambda+1} - \sum_{k=2}^{\infty} k[1 + \gamma(k+\lambda-1)] \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left( \frac{1+\alpha(k-1)+l}{l+1} \right)^m a_k^2 r^{k+\lambda}} > 0. \end{aligned}$$

Because  $\text{Re}(e^{i\theta}) = r$  and  $\text{Re}(-e^{i\theta}) \geq -|e^{i\theta}| = -1$ , the inequality becomes

$$\frac{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)\Gamma(n+1)}{\Gamma(\lambda+2)}r^{\lambda+1} - \sum_{k=2}^{\infty} k\{\gamma k^2 + [1 + (2\lambda + \beta|d| - 2) + 1]k + [\gamma(\lambda-1) + 1](\lambda + \beta|d| - 1)\} \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left( \frac{1+\alpha(k-1)+l}{l+1} \right)^m a_k^2 r^{k+\lambda}}{\frac{(\gamma\lambda+1)\Gamma(n+1)}{\Gamma(\lambda+2)}r^{\lambda+1} - \sum_{k=2}^{\infty} k[1 + \gamma(k+\lambda-1)] \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left( \frac{1+\alpha(k-1)+l}{l+1} \right)^m a_k^2 r^{k+\lambda}} > 0.$$

Letting  $r \rightarrow 1^-$  and by using the mean value theorem, we obtain Inequality (3), which finalizes the proof of Theorem 2.1.  $\square$

**Corollary 2.2.** *We obtain for the function  $f \in \mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$ , the following coefficient inequality:*

$$a_k \leq \sqrt{\frac{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [\gamma(\lambda-1) + 1](\lambda + \beta|d| - 1)\} \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left( \frac{1+\alpha(k-1)+l}{l+1} \right)^m}}, \quad k \geq 2.$$

### 3 Properties related to distortion

**Theorem 3.1.** *The function  $f \in \mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$  has the property*

$$\begin{aligned} r - \sqrt{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)(\lambda+2)}{2(n+1)\left(\frac{1+\alpha+l}{l+1}\right)^m \{\beta|d| + (\lambda+1)[1 + \gamma(\lambda + \beta|d| + 1)]\}}} r^2 &\leq |f(z)| \\ \leq r + \sqrt{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)(\lambda+2)}{2(n+1)\left(\frac{1+\alpha+l}{l+1}\right)^m \{\beta|d| + (\lambda+1)[1 + \gamma(\lambda + \beta|d| + 1)]\}}} r^2, \end{aligned}$$

with  $|z| = r < 1$ .

The equality holds for

$$f(z) = z + \sqrt{\frac{(\lambda + 2)(\beta|d| - \lambda)(\gamma\lambda + 1)}{2(n + 1)\left(\frac{1+\alpha+l}{l+1}\right)^m \{\beta|d| + (\lambda + 1)[1 + \gamma(\lambda + 1 + \beta|d|)]\}}} z^2,$$

with  $z \in U$ .

**Proof.** Considering  $f \in \mathcal{IR}_{m,n}(\lambda, \alpha, l, d, \beta, \gamma)$ , from Relation (3) and taking into account that the sequence

$$k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\} \\ \left(\frac{1 + \alpha(k - 1) + l}{l + 1}\right)^m \frac{\Gamma(n + k)}{\Gamma(k + \lambda + 1)}$$

is increasing and positive for  $k \geq 2$ , in these conditions, we obtain

$$\sqrt{2(n + 1)\left(\frac{1+\alpha+l}{l+1}\right)^m \{\beta|d| + (\lambda + 1)[1 + \gamma(\lambda + \beta|d| + 1)]\}} \sum_{k=2}^{\infty} a_k \\ \leq \sum_{k=2}^{\infty} \sqrt{k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\}} \cdot \sqrt{\left(\frac{1 + \alpha(k - 1) + l}{l + 1}\right)^m \frac{\Gamma(n + k)}{\Gamma(k + \lambda + 1)}} a_k \\ \leq \sqrt{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n + 1)}{\Gamma(\lambda + 2)}},$$

equivalently with

$$\sum_{k=2}^{\infty} a_k \leq \sqrt{\frac{(\lambda + 2)(\beta|d| - \lambda)(\gamma\lambda + 1)}{2(n + 1)\left(\frac{1+\alpha+l}{l+1}\right)^m \{\beta|d| + (\lambda + 1)[1 + \gamma(\lambda + \beta|d| + 1)]\}}}. \quad (4)$$

Using the properties of the modulus function for

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

we obtain

$$r - r^2 \sum_{k=2}^{\infty} a_k \leq r - \sum_{k=2}^{\infty} a_k r^k \leq |z| - \sum_{k=2}^{\infty} a_k |z|^k \leq |f(z)| \leq |z| + \sum_{k=2}^{\infty} a_k |z|^k \leq r + \sum_{k=2}^{\infty} a_k r^k \leq r + r^2 \sum_{k=2}^{\infty} a_k,$$

and applying Relation (4), we obtain

$$r - \sqrt{\frac{(\lambda + 2)(\beta|d| - \lambda)(\gamma\lambda + 1)}{2(n + 1)\left(\frac{1+\alpha+l}{l+1}\right)^m \{\beta|d| + (\lambda + 1)[1 + \gamma(\lambda + \beta|d| + 1)]\}}} r^2 \leq |f(z)| \\ \leq r + \sqrt{\frac{(\lambda + 2)(\beta|d| - \lambda)(\gamma\lambda + 1)}{2(n + 1)\left(\frac{1+\alpha+l}{l+1}\right)^m \{\beta|d| + (\lambda + 1)[\gamma(\lambda + \beta|d| + 1) + 1]\}}} r^2,$$

which completes the proof of Theorem 3.1.  $\square$

**Theorem 3.2.** The function  $f \in \mathcal{IR}_{\lambda,\alpha,l}^{m,n}(d, \beta, \gamma)$  has the property

$$-\sqrt{\frac{2(\lambda + 2)(\beta|d| - \lambda)(\gamma\lambda + 1)}{(n + 1)\left(\frac{1+\alpha+l}{l+1}\right)^m \{\beta|d| + (\lambda + 1)[1 + \beta|d| + \gamma(\lambda + 1)]\}}} r \leq |f'(z)|$$

$$\leq \sqrt{\frac{2(\lambda+2)(\beta|d| - \lambda)(\gamma\lambda + 1)}{(n+1)\left(\frac{1+\alpha+l}{l+1}\right)^m \{\beta|d| + (\lambda+1)[1 + \beta|d| + \gamma(\lambda+1)]\}}} r,$$

with  $|z| = r < 1$ .

The equality holds for

$$f(z) = z + \sqrt{\frac{(\lambda+2)(\beta|d| - \lambda)(\gamma\lambda + 1)}{2(n+1)\left(\frac{1+\alpha+l}{l+1}\right)^m \{\beta|d| + (\lambda+1)[1 + \beta|d| + \gamma(\lambda+1)]\}}} z^2, \quad z \in U.$$

**Proof.** The following result is obtained by applying the properties of the modulus function for

$$f'(z) = 1 + \sum_{k=2}^{\infty} k a_k z^{k-1},$$

obtaining

$$1 - \sum_{k=2}^{\infty} k a_k |z| \leq 1 - \sum_{k=2}^{\infty} k a_k |z|^{k-1} \leq |f'(z)| \leq 1 + \sum_{k=2}^{\infty} k a_k |z|^{k-1} \leq 1 + \sum_{k=2}^{\infty} k a_k |z|.$$

Now, using Relation (4), we obtain

$$\begin{aligned} 1 - \sqrt{\frac{2(\lambda+2)(\beta|d| - \lambda)(\gamma\lambda + 1)}{(n+1)\left(\frac{1+\alpha+l}{l+1}\right)^m \{\beta|d| + (\lambda+1)[1 + \gamma(\lambda + \beta|d| + 1)]\}}} r &\leq |f'(z)| \\ &\leq 1 + \sqrt{\frac{2(\lambda+2)(\beta|d| - \lambda)(\gamma\lambda + 1)}{(n+1)\left(\frac{1+\alpha+l}{l+1}\right)^m \{\beta|d| + (\lambda+1)[1 + \gamma(\lambda + \beta|d| + 1)]\}}} r, \end{aligned}$$

which finalize the proof of Theorem 3.2.  $\square$

## 4 Properties related to closure

**Theorem 4.1.** Consider the functions  $f_p$ ,  $p = 1, 2, \dots, q$ , given by:

$$f_p(z) = z + \sum_{k=2}^{\infty} a_{k,p} z^k, \quad a_{k,p} \geq 0, \quad z \in U, \quad (5)$$

from the class  $\mathcal{IR}_{\lambda,\alpha,l}^{m,n}(d, \beta, \gamma)$ .

The function

$$h(z) = \sum_{p=1}^q \mu_p f_p(z), \quad z \in U, \quad \mu_p \geq 0,$$

is in the class  $\mathcal{IR}_{\lambda,\alpha,l}^{m,n}(d, \beta, \gamma)$ , when

$$\sum_{p=1}^q \mu_p = 1.$$

**Proof.** For the function  $f$ , we can write

$$h(z) = \sum_{p=1}^q \mu_p z + \sum_{p=1}^q \sum_{k=2}^{\infty} \mu_p a_{k,p} z^k = z + \sum_{k=2}^{\infty} \sum_{p=1}^q \mu_p a_{k,p} z^k,$$

and since the functions  $f_p$ ,  $p = 1, 2, \dots, q$  belong to the class  $\mathcal{IR}_{\lambda, a, l}^{m, n}(d, \beta, \gamma)$ , applying Corollary 2.2, we obtain

$$\begin{aligned} & \sum_{k=2}^{\infty} \sqrt{k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\}} \cdot \sqrt{\frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left(\frac{1+a(k-1)+l}{l+1}\right)^m} a_k \\ & \leq \sqrt{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}. \end{aligned}$$

In this condition, it is enough to demonstrate that

$$\begin{aligned} & \sum_{k=2}^{\infty} \sqrt{k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\}} \sqrt{\frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left(\frac{1+a(k-1)+l}{l+1}\right)^m} \left( \sum_{p=1}^q \mu_p a_{k,p} \right) \\ & = \sum_{p=1}^q \mu_p \sum_{k=2}^{\infty} \sqrt{k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\}} \\ & \quad \cdot \sqrt{\frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left(\frac{1+a(k-1)+l}{l+1}\right)^m} a_{k,p} \\ & \leq \sum_{p=1}^q \mu_p \sqrt{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}} = \sqrt{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}. \end{aligned}$$

Hence, the proof is finalized.  $\square$

**Corollary 4.2.** *Considering the functions  $f_p$ ,  $p = 1, 2$ , given by (5) from the class  $\mathcal{IR}_{\lambda, a, l}^{m, n}(d, \beta, \gamma)$ , the function*

$$h(z) = (1 - \xi)f_1(z) + \xi f_2(z), \quad 0 \leq \xi \leq 1, \quad z \in U,$$

*belongs to the class  $\mathcal{IR}_{\lambda, a, l}^{m, n}(d, \beta, \gamma)$ .*

**Theorem 4.3.** *Let*

$$f_1(z) = z,$$

*and*

$$f_k(z) = z + \sqrt{\frac{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\}} \left(\frac{1+a(k-1)+l}{l+1}\right)^m} z^k,$$

$$k \geq 2, \quad z \in U.$$

*The function  $f$  is contained in the class  $\mathcal{IR}_{\lambda, a, l}^{m, n}(d, \beta, \gamma)$  if and only if it has the following form:*

$$f(z) = \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z), \quad z \in U,$$

*with  $\mu_k \geq 0$ ,  $k \geq 1$ , and  $\sum_{k=1}^{\infty} \mu_k = 1$ .*

**Proof.** Considering the function

$$\begin{aligned} f(z) &= \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z) \\ &= z + \sum_{k=2}^{\infty} \sqrt{\frac{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\}} \left(\frac{1+a(k-1)+l}{l+1}\right)^m} \mu_k z^k, \end{aligned}$$

we obtain

$$\begin{aligned}
& \sum_{k=2}^{\infty} \sqrt{\frac{k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\} \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m}{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}} \\
& \cdot \sqrt{\frac{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\} \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m} \mu_k} \\
& = \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \leq 1.
\end{aligned}$$

Hence,  $f \in \mathcal{IR}_{\lambda,\alpha,l}^{m,n}(d, \beta, \gamma)$ .

Conversely, we assume that  $f \in \mathcal{IR}_{\lambda,\alpha,l}^{m,n}(d, \beta, \gamma)$ .

Putting

$$\mu_k = \sqrt{\frac{k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\} \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m}{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}} a_k,$$

since

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k.$$

Thus,

$$f(z) = \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z).$$

Hence, the proof is finalized.  $\square$

**Corollary 4.4.** *The functions*

$$f_1(z) = z$$

and

$$f_k(z) = z + \sqrt{\frac{(\gamma\lambda + 1)(\beta|d| - \lambda) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{k\{\gamma k^2 + [1 + \gamma(2\lambda - 2 + \beta|d|)]k + [1 + \gamma(\lambda - 1)](\lambda - 1 + \beta|d|)\} \left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)}}} z^k,$$

$k \geq 2$  and  $z \in U$  are the extreme points for the class  $\mathcal{IR}_{\lambda,\alpha,l}^{m,n}(d, \beta, \gamma)$ .

## 5 Properties related to inclusion and neighborhood

We define the  $\delta$ -neighborhood of a function  $f \in \mathcal{A}$  by:

$$N_{\delta}(f) = \{g \in \mathcal{A} : g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=2}^{\infty} k|a_k - b_k| \leq \delta\}, \quad (6)$$

and for a particular function  $e(z) = z$ , we obtain

$$N_{\delta}(e) = \left\{ g \in \mathcal{A} : g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=2}^{\infty} k|b_k| \leq \delta \right\}. \quad (7)$$

A function  $f \in \mathcal{A}$  is contained in the class  $\mathcal{IR}_{\lambda, \alpha, l}^{m, n}(d, \beta, \gamma, \zeta)$  if there exists a function  $h \in \mathcal{IR}_{\lambda, \alpha, l}^{m, n}(d, \beta, \gamma)$  such that

$$\left| \frac{f(z)}{h(z)} - 1 \right| < 1 - \zeta, \quad 0 \leq \zeta < 1, \quad z \in U. \quad (8)$$

**Theorem 5.1.** *If*

$$\delta = \sqrt{\frac{2(\lambda + 2)(\beta|d| - \lambda)(\gamma\lambda + 1)}{(n + 1)\{\beta|d| + (\lambda + 1)[1 + \gamma(\lambda + \beta|d| + 1)]\}\left(\frac{1+\alpha+l}{l+1}\right)^m}},$$

then

$$\mathcal{IR}_{\lambda, \alpha, l}^{m, n}(d, \beta, \gamma) \subset N_\delta(e).$$

**Proof.** Consider  $f \in \mathcal{IR}_{\lambda, \alpha, l}^{m, n}(d, \beta, \gamma)$ . By applying Corollary 2.2, taking into account that

$$\begin{aligned} & k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\} \cdot \frac{\Gamma(n + k)}{\Gamma(k + \lambda + 1)} \left(\frac{1 + \alpha(k - 1) + l}{l + 1}\right)^m \\ & \geq \{\beta|d| + (\lambda + 1)[1 + \gamma(\lambda + \beta|d| + 1)]\} \frac{\Gamma(n + 2)}{4\Gamma(\lambda + 3)} \left(\frac{1 + \alpha + l}{l + 1}\right)^m, \end{aligned}$$

for  $k \geq 2$ , we obtain

$$\begin{aligned} & \sqrt{\{\beta|d| + (\lambda + 1)[1 + \gamma(\lambda + \beta|d| + 1)]\} \frac{\Gamma(n + 2)}{4\Gamma(\lambda + 3)} \left(\frac{1 + \alpha + l}{l + 1}\right)^m} \sum_{k=2}^{\infty} a_k \\ & \leq \sum_{k=2}^{\infty} \sqrt{k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\}} \\ & \quad \cdot \sqrt{\frac{\Gamma(n + k)}{\Gamma(k + \lambda + 1)} \left(\frac{1 + \alpha(k - 1) + l}{l + 1}\right)^m} a_k \\ & \leq \sqrt{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n + 1)}{\Gamma(\lambda + 2)}}, \end{aligned}$$

therefore

$$\sum_{k=2}^{\infty} a_k \leq \sqrt{\frac{(\lambda + 2)(\beta|d| - \lambda)(\gamma\lambda + 1)}{2(n + 1)\{\beta|d| + (\lambda + 1)[1 + \gamma(\lambda + \beta|d| + 1)]\}\left(\frac{1+\alpha+l}{l+1}\right)^m}}. \quad (9)$$

Applying Corollary 2.2 and Relation (9), we obtain

$$\sum_{k=2}^{\infty} k a_k \leq \sqrt{\frac{2(\lambda + 2)(\beta|d| - \lambda)(\gamma\lambda + 1)}{(n + 1)\{\beta|d| + (\lambda + 1)[1 + \gamma(\lambda + \beta|d| + 1)]\}\left(\frac{1+\alpha+l}{l+1}\right)^m}} = \delta$$

By virtue of (6), we obtain  $f \in N_\delta(e)$ , which finalizes the proof of Theorem 5.1.  $\square$

**Theorem 5.2.** *If  $h \in \mathcal{IR}_{\lambda, \alpha, l}^{m, n}(d, \beta, \gamma)$  and*

$$\zeta = 1 + \frac{\delta}{2} \sqrt{\frac{(\lambda + 2)(\beta|d| - \lambda)(\gamma\lambda + 1)}{2(n + 1)\{\beta|d| + (\lambda + 1)[1 + \gamma(\lambda + \beta|d| + 1)]\}\left(\frac{1+\alpha+l}{l+1}\right)^m}}, \quad (10)$$

then

$$N_\delta(h) \subset \mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma, \zeta).$$

**Proof.** Considering  $f \in N_\delta(h)$ , from Relation (6), we deduce that

$$\sum_{k=2}^{\infty} k|a_k - b_k| \leq \delta,$$

which gives the coefficient inequality

$$\sum_{k=2}^{\infty} |a_k - b_k| \leq \frac{\delta}{2}. \quad (11)$$

Using Relation (9) and taking into account that  $h \in \mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$ , we obtain

$$\sum_{k=2}^{\infty} b_k \leq \sqrt{\frac{(\lambda + 2)(\beta|d| - \lambda)(\gamma\lambda + 1)}{2(n + 1)\{\beta|d| + (\lambda + 1)[1 + \gamma(\lambda + \beta|d| + 1)]\}\left(\frac{1+a+l}{l+1}\right)^m}}. \quad (12)$$

Using (11) and (12), we have

$$\left| \frac{f(z)}{h(z)} - 1 \right| \leq \frac{\sum_{k=2}^{\infty} |a_k - b_k|}{1 - \sum_{k=2}^{\infty} b_k} \leq \frac{\delta}{2 \left( 1 - \sqrt{\frac{(\lambda + 2)(\beta|d| - \lambda)(\gamma\lambda + 1)}{2(n + 1)\{\beta|d| + (\lambda + 1)[1 + \gamma(\lambda + \beta|d| + 1)]\}\left(\frac{1+a+l}{l+1}\right)^m}} \right)} = 1 - \zeta.$$

Thus, by Condition (8),  $f \in \mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma, \zeta)$ , and  $\zeta$  is given by (10).  $\square$

## 6 Properties related to radii of convexity, starlikeness, and close-to-convexity

**Theorem 6.1.** *The function  $f \in \mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$  is analytic starlike of order  $\delta$ , with  $|z| < r_1$ , for  $0 \leq \delta < 1$  and*

$$r_1 = \inf_k \left\{ \frac{k\{yk^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\}(1 - \delta)^2 \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left(\frac{1+a(k-1)+l}{l+1}\right)^m}{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)} (k - \delta)^2} \right\}^{\frac{1}{2(k-1)}}.$$

*The function*

$$f_k(z) = z + \sqrt{\frac{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{k\{yk^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\} \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left(\frac{1+a(k-1)+l}{l+1}\right)^m}} z^k, \quad k \geq 2,$$

*gives the sharp result.*

**Proof.** We have to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \delta, \quad \text{for } |z| < r_1.$$

Taking into account that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{\sum_{k=2}^{\infty} (k-1)a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} a_k z^{k-1}} \right| \leq \frac{\sum_{k=2}^{\infty} (k-1)a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k |z|^{k-1}},$$

we will demonstrate that

$$\frac{\sum_{k=2}^{\infty} (k-1)a_k|z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k|z|^{k-1}} \leq 1 - \delta,$$

equivalently to

$$\sum_{k=2}^{\infty} (k - \delta)a_k|z|^{k-1} \leq 1 - \delta.$$

Using Theorem 2.1, we obtain

$$|z| \leq \frac{k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\}(1 - \delta)^2 \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left(\frac{1 + \alpha(k-1) + l}{l+1}\right)^m}{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)} (k - \delta)^2}^{\frac{1}{2(k-1)}},$$

and the proof is finalized.  $\square$

**Theorem 6.2.** *The function  $f \in \mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$  is analytic convex of order  $\delta$ , with  $|z| < r_2$ , for  $0 \leq \delta \leq 1$  and*

$$r_2 = \inf_k \frac{k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\}(1 - \delta)^2 \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left(\frac{1 + \alpha(k-1) + l}{l+1}\right)^m}{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)} (k - \delta)^2}^{\frac{1}{2(k-1)}}.$$

The function

$$f_k(z) = z + \sqrt{\frac{(\gamma\lambda + 1)(\beta|d| - \lambda) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{k\{\gamma k^2 + [1 + \gamma(2\lambda - 2 + \beta|d|)]k + [1 + \gamma(\lambda - 1)](\lambda - 1 + \beta|d|)\} \left(\frac{1 + \alpha(k-1) + l}{l+1}\right)^m \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)}}} z^k, \quad k \geq 2,$$
(13)

gives the sharp result.

**Proof.** It is enough to demonstrate that

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - \delta, \quad |z| < r_2.$$

Taking into account that

$$\left| \frac{zf''(z)}{f'(z)} \right| = \left| \frac{\sum_{k=2}^{\infty} k(k-1)a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} ka_k z^{k-1}} \right| \leq \frac{\sum_{k=2}^{\infty} k(k-1)a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} ka_k |z|^{k-1}},$$

we have to show that

$$\frac{\sum_{k=2}^{\infty} k(k-1)a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} ka_k |z|^{k-1}} \leq 1 - \delta,$$

$$\sum_{k=2}^{\infty} k(k - \delta)a_k |z|^{k-1} \leq 1 - \delta,$$

and using Theorem 2.1, we have the following result:

$$|z|^{k-1} \leq \frac{(1 - \delta)}{k(k - \delta)} \sqrt{\frac{k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\} \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left(\frac{1 + \alpha(k-1) + l}{l+1}\right)^m}{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}},$$

written as:

$$|z| \leq \left| \frac{k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\}(1 - \delta)^2 \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left( \frac{1 + a(k-1) + l}{l+1} \right)^m}{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)} (k - \delta)^2} \right|^{\frac{1}{2(k-1)}},$$

and the proof is finalized.  $\square$

**Theorem 6.3.** *The function  $f \in \mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$  is analytic close-to-convex of order  $\delta$ , with  $|z| < r_3$ , for  $0 \leq \delta < 1$  and*

$$r_3 = \inf_k \left| \frac{\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\}(1 - \delta)^2 \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left( \frac{1 + a(k-1) + l}{l+1} \right)^m}{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}} \right|^{\frac{1}{2(k-1)}}.$$

For the function  $f$  defined by (13), the result is sharp.

**Proof.** It is enough to demonstrate that

$$|f'(z) - 1| \leq 1 - \delta, \quad \text{where } |z| < r_3.$$

Then,

$$|f'(z) - 1| = \left| \sum_{k=2}^{\infty} k a_k z^{k-1} \right| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1},$$

and we can write  $|f'(z) - 1| \leq 1 - \delta$ , if we have  $\sum_{k=2}^{\infty} \frac{k a_k}{1 - \delta} |z|^{k-1} \leq 1$ . Using Theorem 2.1, the aforementioned inequality gives the result:

$$|z|^{k-1} \leq \frac{(1 - \delta)}{k} \sqrt{\frac{k\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\} \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left( \frac{1 + a(k-1) + l}{l+1} \right)^m}{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}},$$

written as:

$$|z| \leq \left| \frac{\{\gamma k^2 + [1 + \gamma(2\lambda + \beta|d| - 2)]k + [1 + \gamma(\lambda - 1)](\lambda + \beta|d| - 1)\}(1 - \delta)^2 \frac{\Gamma(n+k)}{\Gamma(k+\lambda+1)} \left( \frac{1 + a(k-1) + l}{l+1} \right)^m}{(\beta|d| - \lambda)(\gamma\lambda + 1) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}} \right|^{\frac{1}{2(k-1)}},$$

which finalizes the proof.  $\square$

## 7 Conclusion

The topic of introducing and studying new classes of univalent functions by applying fractional calculus operators is further developed with the results presented in this article. A new class of analytic functions  $\mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$  given in Definition 1.6 is introduced by using the Riemann-Liouville fractional integral for the convolution product of multiplier transformation and Ruscheweyh operator shown in Definition 1.1. This operator was previously introduced in [31], and the research exhibited in this article comes as a complement to those previous studies and to the ones presented in [34] where a new class is introduced and studied using

the fractional integral associated with the convolution product of generalized Sălăgean operator and Ruscheweyh derivative. Just as in [31] and [34], comprehensive investigations are done on the newly introduced class  $\mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$ . The first theorem proved here in Section 2 gives the necessary conditions for a function  $f \in \mathcal{A}$  to belong to the class  $\mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$ , and a coefficient inequality for such functions follows as a corollary. Bounds for the modulus of the functions belonging to class  $\mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$  and for the modulus of the derivative of the functions belonging to class  $\mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$  are obtained in the distortion results presented in Section 3. Closure theorems are proved in Section 4 for the functions of the class  $\mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$ , and extreme points of the class are obtained as a corollary. In Section 5, neighborhood results are obtained and the inclusion relations involving class  $\mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$  are proved. Section 6 concerns geometric properties of the class  $\mathcal{IR}_{\lambda,a,l}^{m,n}(d, \beta, \gamma)$ , radii of starlikeness, convexity, and close-to-convexity being provided for this class.

The results obtained in this article could inspire future investigations including quantum calculus, which is another topic of interest at the moment proved by recent results contained in [35]. Fractional integral was also associated with quantum calculus aspects in [36] and [37]. Differential subordinations and superordinations could be obtained using the newly introduced class, and also, fuzzyfication of the results could be considered as seen in [38]. Differential subordination theory can also be applied to the Riemann-Liouville fractional integral for the convolution product of multiplier transformation and Ruscheweyh operator shown in Definition 1.1 as nice results have been obtained recently involving fractional integral and Libera integral operator [39].

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