Research Article

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Some results on fractional Hahn difference boundary value problems

https://doi.org/10.1515/dema-2022-0247
received September 17, 2022; accepted May 18, 2023

Abstract: Fractional Hahn boundary value problems are significant tools to describe mathematical and physical phenomena depending on non-differentiable functions. In this work, we develop certain aspects of the theory of fractional Hahn boundary value problems involving fractional Hahn derivatives of the Caputo type. First, we construct the Green function for an αth-order fractional boundary value problem, with $1 < \alpha < 2$, and discuss some important properties of the Green function. The solutions to the proposed problems are obtained in terms of the Green function. The uniqueness of the solutions is proved by various fixed point theorems. The Banach's contraction mapping theorem, the Schauder's theorem, and the Browder's theorem are used.

Keywords: fractional Hahn difference operator, fractional Hahn integral, boundary value problem, Green function, existence theorem, uniqueness theorem

MSC 2020: 39A10, 39A13, 39A27, 39A70

1 Introduction

Recently, the Hahn calculus and fractional Hahn difference equations have gained much attention. The Hahn calculus (also called q, ω-calculus) can be dated back to 1949, Hahn's work [1]. Based on the fractional Hahn calculus, the fractional Hahn difference equations were established that can describe some physical processes appearing in quantum dynamics, discrete dynamical systems, discrete stochastic processes, and many others. Here, one should point out that the Hahn difference equations are usually defined on a time scale set $I_{q,\omega}$, with $\sigma_{q,\omega}(t) = qt + \omega$ being the scale index. The Hahn difference operator is defined by [1]:

$$D_{q,\omega} f(t) \coloneqq \frac{f(qt + \omega) - f(t)}{qt + \omega - t}, \quad t \neq \omega_0.$$

We note that this operator is combined from the well-known operators: the forward difference operator and the Jackson q-difference operator.

$$\begin{aligned} D_{q,\omega} f(t) &= \Delta_\omega f(t) \quad \text{for } q = 1, \\ D_{q,\omega} f(t) &= D_q f(t) \quad \text{for } \omega = 0, \\ D_{q,\omega} f(t) &= f'(t) \quad \text{for } q = 1, \omega \to 0. \end{aligned}$$

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With the development of the Hahn calculus theory, some related concepts and results have also been introduced and studied, such as the theory of linear Hahn difference equations, Leibniz's rule and Fubini's theorem associated with Hahn difference operator, and q, ω-Taylor expansion [2–8] (see [9–14] for more details on Hahn and fractional Hahn difference equations). Up to now, compared with the classical fractional differential equations, the study of the fractional Hahn difference equations is still immature. At present, the literature have many studies on the existence and uniqueness of solutions of fractional Hahn difference boundary value problems. In [10–13], the Banach's fixed point theorem and the Schauder's fixed-point theorem are used to prove the existence and uniqueness results of Caputo fractional Hahn difference boundary value problems for fractional Hahn integro-difference equations. In [11], the authors studied a nonlocal Robin boundary value problem for the fractional Hahn integro-difference equation. The existence and uniqueness results were proved by using the Banach's fixed point theorem and the Schauder's fixed point theorem. More recently, nonlocal fractional symmetric Hahn integral boundary value problems for the fractional symmetric Hahn integro-difference equation were studied in [12].

In this work, we are going to gain further insight into the theorem of fractional Hahn difference boundary value problems. Mainly, we consider the following Caputo fractional Hahn difference boundary value problem:

$$\begin{cases} {}^{C}D^{\alpha}_{q,\omega}y(t) = h(t, y(t)), \;\; t \in I^{T}_{q,w}, \\ y(\omega_0) = \phi(y), \;\; y(T) = \psi(y), \end{cases} \tag{1.1}$$

where $\alpha \in (1, 2]$, $q \in (0, 1)$, $\omega > 0$, $I^{T}_{q,w} = \{q^k T + \omega[k]_q : k \in \mathbb{N}_0\} \cup \{\omega_0\}$, with $\omega_0 = \frac{\omega}{1-q}$ and $\phi, \psi : C(I^{T}_{q,\omega}, \mathbb{R}) \to \mathbb{R}$ being given functionals. We aim to study the existence and uniqueness of the solution to Problem (1.1) by using the Banach's fixed point theorem and the existence of at least one solution by using the Browder's and the Schauder's fixed point theorems. This work is organized as follows: Section 2 provides some basic definitions and relevant results on the Hahn and fractional Hahn calculi. In Section 3, we study the existence of solutions of Caputo fractional Hahn difference boundary value problems. Section 4 is devoted to the uniqueness of the solutions by using various fixed point theorems. Finally, illustrative examples are given in Section 5.

2 Preliminaries

In this section, we present some basic definitions and notations for the q, ω-calculus (see [1–3,5–8,13]). Let $q \in (0, 1)$ and $\omega > 0$ and define the q-analogue of both the integer n and the factorial

$$[n]_q := \frac{1 - q^n}{1 - q} = q^{n-1} + \cdots + q + 1, \quad [n]_q! := \prod_{k=1}^{n-1} \frac{1 - q^k}{1 - q}, \quad n \in \mathbb{N}$$

(see [15,16]). The q, ω-forward jump operator and the q, ω-backward jump operator are defined by:

$$\sigma_{q,\omega}(t) := qt + \omega$$

and

$$\rho_{q,\omega}(t) := \frac{t - \omega}{q}, \;\; t \in \mathbb{R},$$

respectively. The kth-order iteration of $\sigma_{q,\omega}(t)$ is given by:

$$\sigma^{k}_{q,\omega}(t) = q^k t + \omega[k]_q, \;\; t \in \mathbb{R}.$$

For $a, b \in \mathbb{R}$, the q-analogue of the power function $(a - b)^{n}_{q}$ with $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ is defined by:

$$(a-b)_q^{\underline{0}} := 1, \quad (a-b)_q^{\underline{n}} := \prod_{k=0}^{n-1}(a-bq^k), \quad n \in \mathbb{N}.$$

The q, ω-analogue of the power function $(a-b)_{q,\omega}^{\underline{n}}$ with $n \in \mathbb{N}_0$ is defined by: (see [9,10,13])

$$(a-b)_{q,\omega}^{\underline{0}} = 1, \quad (a-b)_{q,\omega}^{\underline{n}} = \prod_{k=0}^{n-1}[a-\sigma_{q,\omega}^k(b)] \quad n \in \mathbb{N}.$$

In general, for $\alpha \in \mathbb{R}$, we have

$$(a-b)_q^{\underline{\alpha}} = a^\alpha \prod_{n=0}^{\infty} \frac{1-(b/a)q^n}{1-(b/a)q^{n+\alpha}}, \; a \neq 0,$$

$$(a-b)_{q,\omega}^{\underline{\alpha}} = (a-\omega_0)^\alpha \prod_{n=0}^{\infty} \frac{1-[(b-\omega_0)/(a-\omega_0)]q^n}{1-[(b-\omega_0)/(a-\omega_0)]q^{n+\alpha}}, \; a \neq \omega_0, \; \omega_0 = \frac{\omega}{1-q}.$$

Note that $a_q^{\underline{\alpha}} = a^\alpha$ and $(a-\omega_0)_{q,\omega}^{\underline{\alpha}} = (a-\omega_0)^\alpha$. We use the notation $(0)_q^{\underline{\alpha}} = (0)_{q,\omega}^{\underline{\alpha}} = 0$ for $\alpha > 0$. The q-gamma function is defined as [16]:

$$\Gamma_q(x) = \frac{(1-q)_q^{\underline{x-1}}}{(1-q)^{x-1}}, \; x \in \mathbb{R}\setminus\{0,-1,-2,\ldots\}.$$

For $a, b \in \mathbb{R}$ with $a < \omega_0 < b$, we define the q, ω-interval by:

$$I_{q,\omega}^{a,b} = [a,b]_{q,\omega} := \{q^k a + \omega[k]_q : k \in \mathbb{N}_0\} \cup \{q^k b + \omega[k]_q : k \in \mathbb{N}_0\} \cup \{\omega_0\} = [a,\omega_0]_{q,\omega} \cup [\omega_0, b]_{q,\omega}.$$

It is clear that for $t \in [a,b]_{q,\omega}$, the sequence $\{\sigma_{q,\omega}^n(t)\}_{n=0}^{\infty}$ uniformly converges to ω_0.

Also, we consider

$$I_{q,\omega}^T = I_{q,\omega}^{\omega_0,T} := [\omega_0, T]_{q,\omega}.$$

From now, I is a closed interval of $\mathbb{R}$ containing ω_0.

Definition 2.1. [1,3,8] For any function $f: I \to \mathbb{R}$, the Hahn difference operator is defined by:

$$D_{q,\omega} f(t) := \frac{f(qt+\omega)-f(t)}{qt+\omega-t}, \; t \neq \omega_0, \tag{2.1}$$

$D_{q,\omega} f(\omega_0) = f'(\omega_0)$ provided that f is differentiable at ω_0 in the usual sense. We call $D_{q,\omega} f$ the q, ω-derivative of f and say that f is q, ω-differentiable on I.

The nth q, ω-derivative, $n \in \mathbb{N}$ of a function $f: I \to \mathbb{R}$ is given by:

$$D_{q,\omega}^n f(t) := D_{q,\omega}(D_{q,\omega}^{n-1} f(t)),$$

provided that $D_{q,\omega}^{n-1} f(t)$ is q, ω-differentiable on I and $D_{q,\omega}^0 f(t) = f(t)$.

Lemma 2.1. [6] *Let $f, g : I_{q,\omega}^T \to \mathbb{R}$ be q, ω-differentiable at $t \in I_{q,\omega}^T$. Then,*

(i) $D_{q,\omega}(f+g)(t) = D_{q,\omega} f(t) + D_{q,\omega} g(t)$,
(ii) $D_{q,\omega}(fg)(t) = D_{q,\omega}(f(t))g(t) + f(\sigma_{q,\omega}(t))D_{q,\omega} g(t)$,
(iii) *For any constant* $c \in \mathbb{R}$, $D_{q,\omega}(cf)(t) = cD_{q,\omega}(f(t))$,
(iv) $D_{q,\omega}\left(\frac{f}{g}\right)(t) = \frac{D_{q,\omega}(f(t))g(t) - f(t)D_{q,\omega} g(t)}{g(t)g(\sigma_{q,\omega}(t))}$ *provided that* $g(t)g(\sigma_{q,\omega}(t)) \neq 0$.

Lemma 2.2. [3] *For $n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$*

$$(a) \quad D_{q,\omega}(\alpha t+\beta)^n = \alpha \sum_{k=0}^{n-1} (\alpha(qt+\omega)+\beta)^k (\alpha t+\beta)^{n-k-1},$$

$$(b) \quad D_{q,\omega}(\alpha t+\beta)^{-n} = -\alpha \sum_{k=0}^{n-1} (\alpha(qt+\omega)+\beta)^{-n+k} (\alpha t+\beta)^{-k-1},$$

provided that $(\alpha(qt+\omega)+\beta)(\alpha t+\beta) \neq 0$.

Lemma 2.3. [9] *Let $t \in I_{q,\omega}^T$ and $\alpha, \beta \in \mathbb{R}$. Then,*

$$(a) \quad D_{q,\omega}(t-\beta)_{q,\omega}^{\underline{\alpha}} = [\alpha]_q(\rho_{q,\omega}(t)-\beta)_{q,\omega}^{\underline{\alpha-1}},$$

$$(b) \quad D_{q,\omega}(\beta-t)_{q,\omega}^{\underline{\alpha}} = -[\alpha]_q(\beta-t)_{q,\omega}^{\underline{\alpha-1}}.$$

Definition 2.2. [3,8, 9] Let I be a closed interval of $\mathbb{R}$ containing a, b, and ω_0. If $f : I \to \mathbb{R}$ is a function, we define the q, ω-integral of f from a to b by

$$\int_a^b f(t) d_{q,\omega}t = \int_{\omega_0}^b f(t) d_{q,\omega}t - \int_{\omega_0}^a f(t) d_{q,\omega}t,$$

where

$$\int_{\omega_0}^x f(t) d_{q,\omega}t = (x(1-q)-\omega) \sum_{k=0}^{\infty} q^k f(\sigma_{q,\omega}^k(x)), \quad x \in I,$$

provided that the series converges at $x = a$ and $x = b$.

Lemma 2.4. [8] *Let $f, g : I \to \mathbb{R}$ be q, ω-integrable functions on I, $k \in \mathbb{R}$, and $a, b, c \in I$ with $a < c < b$. Then,*

$$(i) \quad \int_a^a f(t) d_{q,\omega}t = 0,$$

$$(ii) \quad \int_a^b k f(t) d_{q,\omega}t = k \int_a^b f(t) d_{q,\omega},$$

$$(iii) \quad \int_a^b f(t) d_{q,\omega}t = -\int_b^a f(t) d_{q,\omega}t,$$

$$(iv) \quad \int_a^b f(t) d_{q,\omega}t = \int_a^c f(t) d_{q,\omega}t + \int_c^b f(t) d_{q,\omega}t,$$

$$(v) \quad \int_a^b (f(t)+g(t)) d_{q,\omega}t = \int_a^b f(t) d_{q,\omega}t + \int_a^b g(t) d_{q,\omega}t.$$

Lemma 2.5. [3,8,9] *Let $f : I \to \mathbb{R}$ be a continuous function at ω_0.* Define

$$F(x) := \int_{\omega_0}^x f(t) d_{q,\omega}t.$$

Then, F is continuous at ω_0. Furthermore, $D_{q,\omega}F(x)$ exists for every $x \in I$ and

$$D_{q,\omega}F(x) = f(x), \ \ x \in I.$$

Conversely,

$$\int_a^b D_{q,\omega} f(t) d_{q,\omega} t = f(b) - f(a), \quad \textit{for all } a, b \in I.$$

Lemma 2.6. [3,8] *If $f, g : I \to \mathbb{R}$ are continuous at ω_0, then*

$$\int_a^b f(t) D_{q,\omega} g(t) d_{q,\omega} t = f(t)g(t)|_a^b - \int_a^b D_{q,\omega}(f(t)) g(qt + \omega) d_{q,\omega} t, \ \ a, b \in I.$$

Next, we are going to present some basic notations for the fractional Hahn calculus. For more details, one can refer to [9–14].

Definition 2.3. For $\alpha, \omega > 0$, $q \in (0, 1)$, and $f : I_{q,\omega}^T \to \mathbb{R}$, the fractional Hahn integral is defined by:

$$\begin{aligned} \mathcal{I}_{q,\omega}^{\alpha} f(t) &:= \frac{1}{\Gamma_q(\alpha)} \int_{\omega_0}^{t} (t - \sigma_{q,\omega}(s))_{\underline{q,\omega}}^{\underline{\alpha-1}} f(s) d_{q,\omega} s \\ &= \frac{[t(1-q) - \omega]}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} q^n (t - \sigma_{q,\omega}^{n+1}(t))_{\underline{q,\omega}}^{\underline{\alpha-1}} f(\sigma_{q,\omega}^n(t)) \end{aligned}$$

and $(\mathcal{I}_{q,\omega}^0 f)(t) = f(t)$.

Lemma 2.7. *For $\alpha, \beta > 0$, $p \in \mathbb{N}$, $a \in I_{q,\omega}^T$, and $f : I_{q,\omega}^T \to \mathbb{R}$, we have*

$$\begin{aligned} &(i) \quad \mathcal{I}_{q,\omega}^{\alpha}[\mathcal{I}_{q,\omega}^{\beta} f(t)] = \mathcal{I}_{q,\omega}^{\beta}[\mathcal{I}_{q,\omega}^{\alpha} f(t)] = \mathcal{I}_{q,\omega}^{\alpha+\beta} f(t), \\ &(ii) \quad \mathcal{I}_{q,\omega}^{\alpha}[D_{q,\omega}^{p} f(t)] = D_{q,\omega}^{p}[\mathcal{I}_{q,\omega}^{\alpha} f(t)] - \sum_{k=0}^{p-1} \frac{(t-\omega_0)^{\alpha-p+k}}{\Gamma_q(\alpha - p + k + 1)} [D_{q,\omega}^{k} f(\omega_0)], \\ &(iii) \quad \int_{\omega_0}^{a} (t - \sigma_{q,\omega}(s))_{\underline{q,\omega}}^{\underline{\beta-1}} \mathcal{I}_{q,\omega}^{\alpha} f(s) d_{q,\omega} s = 0. \end{aligned}$$

Definition 2.4. For $\alpha, \omega > 0$, $q \in (0, 1)$, and $f : I_{q,\omega}^T \to \mathbb{R}$, the fractional Hahn difference operator of Caputo type of order α is defined as:

$${}^{C}D_{q,\omega}^{\alpha} f(t) := (\mathcal{I}_{q,\omega}^{N-\alpha} D_{q,\omega}^{N} f)(t) = \frac{1}{\Gamma_q(N-\alpha)} \int_{\omega_0}^{t} (t - \sigma_{q,\omega}(s))_{\underline{q,\omega}}^{\underline{N-\alpha-1}} D_{q,\omega}^{N} f(s) d_{q,\omega} s$$

and ${}^{C}D_{q,\omega}^{0} f(t) = f(t)$, where $N - 1 < \alpha < N$, $N \in \mathbb{N}$.

Lemma 2.8. *For $\alpha, \omega > 0$, $q \in (0, 1)$, and $f : I_{q,\omega}^T \to \mathbb{R}$,*

$${}^{C}D_{q,\omega}^{\alpha} \mathcal{I}_{q,\omega}^{\alpha} f(t) = f(t).$$

Lemma 2.9. *For $\alpha, \omega > 0$, $q \in (0, 1)$, and $f : I_{q,\omega}^T \to \mathbb{R}$,*

$$\mathcal{I}_{q,\omega}^{\alpha} \, {}^{C}D_{q,\omega}^{\alpha} f(t) = f(t) - \sum_{k=0}^{N-1} \frac{(t - \omega_0)^k}{\Gamma_q(k+1)} D_{q,\omega}^{k} f(\omega_0),$$

where $N - 1 < \alpha < N$, $N \in \mathbb{N}$.

Thus, the following results can be proved.

Lemma 2.10. *Let $\alpha, \omega > 0$, $q \in (0,1)$, and f be a function on $I_{q,\omega}^{T}$. Then, the function*

$$y(t) = \mathcal{I}_{q,\omega}^{\alpha} f(t)$$

is a solution of the following Caputo-type fractional Hahn initial value problem

$$\begin{cases} {}^{C}D_{q,\omega}^{\alpha} y(t) = f(t), & t \in I_{q,\omega}^{T}, \\ D_{q,\omega}^{k} y(\omega_0) = 0, & k = 0, 1, 2, \ldots, N-1. \end{cases} \tag{2.2}$$

Proof. Setting

$$z(t) = \mathcal{I}_{q,\omega}^{\alpha} f(t). \tag{2.3}$$

By using Definition 2.3 and Lemma 2.7, we obtain

$$D_{q,\omega}^{k} z(t)|_{t=\omega_0} = D_{q,\omega}^{k}[\mathcal{I}_{q,\omega}^{\alpha} f(t)]|_{t=\omega_0} = 0, \quad k = 0, 1, 2, \ldots, N-1.$$

Therefore, $z(t)$ satisfies the initial conditions in (2.2). Consequently, from Lemma 2.8 $z(t) = \mathcal{I}_{q,\omega}^{\alpha} f(t)$ solves the initial value problem (2.2). □

In the following lemmas, we present the solvability of linear forms of Problem (1.1). These results play a serious role in sequel investigations.

Lemma 2.11. *Let $\alpha \in (1,2)$, $t \in I_{q,\omega}^{T}$. The function*

$$y(t) = \phi(y) + \frac{t-\omega_0}{T-\omega_0}[\psi(y) - \phi(y)] \tag{2.4}$$

satisfies the following fractional boundary value problem (FBVP):

$$\begin{cases} {}^{C}D_{q,\omega}^{\alpha} y(t) = 0, \\ y(\omega_0) = \phi(y), \quad y(T) = \psi(y). \end{cases} \tag{2.5}$$

Proof. From [9, Corollary 5.1], the solution of equation (2.5) can be written as:

$$y(t) = c_0 + c_1(t-\omega_0). \tag{2.6}$$

From the first condition $y(\omega_0) = \phi(y)$, we have $c_0 = \phi(y)$. From the second condition $y(T) = \psi(y)$, we obtain

$$c_1 = \frac{1}{T-\omega_0}[\psi(y) - \phi(y)].$$

By substituting the values of c_0 and c_1 in equation (2.6), we obtain (2.4). □

Lemma 2.12. *Let $\alpha \in (1,2)$, $t \in I_{q,\omega}^{T}$. The function*

$$\begin{aligned} y(t) = \phi(y) + \frac{t-\omega_0}{T-\omega_0}\left[\psi(y) - \varphi(y) + \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{T}(T-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}} h(s)\, d_{q,\omega}s\right] \\ - \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{t}(t-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}} h(s)\, d_{q,\omega}s \end{aligned} \tag{2.7}$$

satisfies the following FBVP:

$$\begin{cases} {}^{C}D^{\alpha}_{q,\omega}y(t) = -h(t), \quad 1 < \alpha < 2, \\ y(\omega_0) = \phi(y), \quad y(T) = \psi(y). \end{cases} \tag{2.8}$$

Proof. Equation (2.8) has a solution

$$y(t) = c_0 + c_1(t - \omega_0) - \mathcal{I}^{\alpha}_{q,\omega}h(t). \tag{2.9}$$

From the first boundary condition $y(\omega_0) = \phi(y)$, we have $c_0 = \phi(y)$. The second boundary condition $y(T) = \psi(y)$ yields

$$c_1 = \frac{1}{(T-\omega_0)}\left[\psi(y) - \phi(y) + \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{T}(T - \sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}h(s)d_{q,\omega}s\right].$$

By substituting the values of c_0 and c_1 in equation (2.9), we directly arrive at Conclusion (2.7). □

From (2.7), we have

$$\begin{aligned} &\frac{t-\omega_0}{(T-\omega_0)\Gamma_q(\alpha)}\int_{\omega_0}^{T}(T - \sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}h(s)d_{q,\omega}s - \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{t}(t - \sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}h(s)d_{q,\omega}s \\ &= \frac{t-\omega_0}{(T-\omega_0)\Gamma_q(\alpha)}\int_{\omega_0}^{t}(T - \sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}h(s)d_{q,\omega}s + \frac{t-\omega_0}{(T-\omega_0)\Gamma_q(\alpha)}\int_{t}^{T}(T - \sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}h(s)d_{q,\omega}s \\ &\quad - \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{t}(t - \sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}h(s)d_{q,\omega}s \\ &= \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{t}\left(\frac{(t-\omega_0)(T - \sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}}{T-\omega_0} - (t - \sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}\right)h(s)d_{q,\omega}s \\ &\quad + \frac{t-\omega_0}{(T-\omega_0)\Gamma_q(\alpha)}\int_{t}^{T}(T - \sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}h(s)d_{q,\omega}s. \end{aligned} \tag{2.10}$$

Now, we define the Green function as follows.

Lemma 2.13. *The Green function for FBVP*

$$\begin{cases} {}^{C}D^{\alpha}_{q,\omega}x(t) = -h(t), \quad 1 < \alpha < 2, \quad t \in I^{T}_{q,\omega}, \\ x(\omega_0) = 0, \quad x(T) = 0, \end{cases} \tag{2.11}$$

is given by:

$$G_{q,\omega}(t,s) = \begin{cases} \dfrac{(t-\omega_0)(T - \sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}}{(T-\omega_0)\Gamma_q(\alpha)} - \dfrac{(t - \sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}}{\Gamma_q(\alpha)}, & \omega_0 \leqslant s \leqslant t, \\ \dfrac{(t-\omega_0)(T - \sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}}{(T-\omega_0)\Gamma_q(\alpha)}, & t \leqslant s \leqslant T. \end{cases} \tag{2.12}$$

Lemma 2.14. *Let $\alpha \in (1,2)$, $t \in I^{T}_{q,\omega}$, and $h : I^{T}_{q,\omega} \to \mathbb{R}$. The function*

$$y(t) = z(t) + \int_{\omega_0}^{T} G_{q,\omega}(t,s)h(s)d_{q,\omega}s \tag{2.13}$$

satisfies the following FBVP:

$$\begin{cases} {}^{C}D^{\alpha}_{q,\omega}y(t) = -h(t), \\ y(\omega_0) = \phi(y), \quad y(T) = \psi(y), \end{cases} \tag{2.14}$$

where $G_{q,\omega}(t,s)$ is defined by (2.12) *and $z(t)$ is a solution to FBVP*

$$\begin{cases} {}^{C}D^{\alpha}_{q,\omega}z(t) = 0, \\ z(\omega_0) = \phi(z), \quad z(T) = \psi(z). \end{cases} \tag{2.15}$$

Proof. Applying Lemmas 2.11 and 2.12 to obtain the desired result. □

3 Properties of a Green's function

In this section, we study some properties of the Green function $G_{q,\omega}(t,s)$ defined by (2.12).

Proposition 3.1. *Let $G_{q,\omega}(t,s)$ be Green's function given in* (2.12). *Then, $G_{q,\omega}(t,s) \geqslant 0$ for each $s \in I^{T}_{q,\omega}$ and α is closed to* 2.

Proof. From (2.12), we define the functions

$$g_1(t,s) = \frac{(t-\omega_0)(T-\sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}}{\Gamma_q(\alpha)(T-\omega_0)} - \frac{(t-\sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}}{\Gamma_q(\alpha)}, \quad \omega_0 \leqslant s \leqslant t,$$
$$g_2(t,s) = \frac{(t-\omega_0)(T-\sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}}{\Gamma_q(\alpha)(T-\omega_0)}, \quad t \leqslant s \leqslant T.$$

(i) First for $\omega_0 \leqslant s \leqslant t$,

$$\begin{aligned} g_1(t,s) &= \frac{(t-\omega_0)(T-\sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}}{\Gamma_q(\alpha)(T-\omega_0)} - \frac{(t-\sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}}{\Gamma_q(\alpha)} \\ &= \frac{1}{\Gamma_q(\alpha)} \frac{(t-\omega_0)(T-\sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega} - (T-\omega_0)(t-\sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}}{(T-\omega_0)}. \end{aligned}$$

It is sufficient to show that

$$\frac{(t-\omega_0)(T-\sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}}{(T-\omega_0)(t-\sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}} > 1.$$

Since

$$\begin{aligned} (T-\sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega} &= (T-\omega_0)^{\alpha-1} \prod_{n\geqslant 0} \frac{1-\left(\frac{\sigma_{q,\omega}(s)-\omega_0}{T-\omega_0}\right)q^{n}}{1-\left(\frac{\sigma_{q,\omega}(s)-\omega_0}{T-\omega_0}\right)q^{\alpha-1+n}} \\ &= (T-\omega_0)^{\alpha-1} \prod_{n\geqslant 0} \frac{1-\left(\frac{s-\omega_0}{T-\omega_0}\right)q^{n+1}}{1-\left(\frac{s-\omega_0}{T-\omega_0}\right)q^{\alpha+n}} \end{aligned}$$

and

$$(t-\sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega} = (t-\omega_0)^{\alpha-1} \prod_{n\geqslant 0} \frac{1-\left(\frac{s-\omega_0}{t-\omega_0}\right)q^{n+1}}{1-\left(\frac{s-\omega_0}{t-\omega_0}\right)q^{\alpha+n}},$$

then

$$\frac{(t-\omega_0)(T-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}}}{(T-\omega_0)(t-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}}} = \left(\frac{T-\omega_0}{t-\omega_0}\right)^{\alpha-2}\prod_{n\geqslant 0}\frac{1-\left(\frac{s-\omega_0}{T-\omega_0}\right)q^{n+1}}{1-\left(\frac{s-\omega_0}{T-\omega_0}\right)q^{\alpha+n}}\cdot\frac{1-\left(\frac{s-\omega_0}{t-\omega_0}\right)q^{\alpha+n}}{1-\left(\frac{s-\omega_0}{t-\omega_0}\right)q^{n+1}}$$

$$= \left(\frac{T-\omega_0}{t-\omega_0}\right)^{\alpha-2}\prod_{n\geqslant 0}\frac{(T-\omega_0)-(s-\omega_0)q^{n+1}}{(T-\omega_0)-(s-\omega_0)q^{\alpha+n}}\cdot\frac{(t-\omega_0)-(s-\omega_0)q^{\alpha+n}}{(t-\omega_0)-(s-\omega_0)q^{n+1}}.$$

We have two cases: Case (1) $\alpha \to 2$

$$\frac{(t-\omega_0)(T-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}}}{(T-\omega_0)(t-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}}} \to [(T-\omega_0)-(s-\omega_0)q]\frac{1}{(t-\omega_0)-(s-\omega_0)q} \geqslant 1.$$

Then, $g_1(t,s) \geqslant 0$, and hence, $G_{q,\omega}(t,s) \geqslant 0$, for $\omega_0 \leqslant s \leqslant t \leqslant T$, and α is closed to 2.

Case (2) $\alpha \to 1$

$$0 < \frac{(t-\omega_0)(T-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}}}{(T-\omega_0)(t-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}}} \to \frac{t-\omega_0}{T-\omega_0} < 1,$$

i.e.,

$$g_1(t,s) < 0.$$

(ii) Second for $t \leqslant s \leqslant T$,

$$g_2(t,s) = \frac{(t-\omega_0)(T-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}}}{\Gamma_q(\alpha)(T-\omega_0)} = \frac{(t-\omega_0)}{\Gamma_q(\alpha)(T-\omega_0)}(T-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}}.$$

Since

$$(T-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}} = (T-\omega_0)^{\alpha-1}\prod_{n\geqslant 0}\frac{1-\left(\frac{\sigma_{q,\omega}(s)-\omega_0}{T-\omega_0}\right)q^{n}}{1-\left(\frac{\sigma_{q,\omega}(s)-\omega_0}{T-\omega_0}\right)q^{\alpha-1+n}}$$

$$= (T-\omega_0)^{\alpha-1}\prod_{n\geqslant 0}\frac{1-\left(\frac{s-\omega_0}{T-\omega_0}\right)q^{n+1}}{1-\left(\frac{s-\omega_0}{T-\omega_0}\right)q^{\alpha+n}}$$

$$= (T-\omega_0)^{\alpha-1}\prod_{n\geqslant 0}\frac{(T-\omega_0)-(s-\omega_0)q^{n+1}}{(T-\omega_0)-(s-\omega_0)q^{\alpha+n}} > 0,$$

then $g_2(t,s) > 0$. Therefore, $G_{q,\omega}(t,s) > 0$, for $t \leqslant s \leqslant T$. □

Proposition 3.2. *For the function* $G_{q,\omega}(t,s)$ *defined by* (2.12), *the following relation holds true*:

$$\int_{\omega_0}^{T}|G_{q,\omega}(t,s)|d_{q,\omega}s = \frac{(t-\omega_0)}{\Gamma_q(\alpha+1)}[(T-\omega_0)^{\alpha-1}+(t-\omega_0)^{\alpha-1}]. \tag{3.1}$$

Proof. From (2.12), we find

$$\int_{\omega_0}^{T}|G_{q,\omega}(t,s)|d_{q,\omega}s = \frac{(t-\omega_0)}{(T-\omega_0)}\frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{T}(T-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}}d_{q,\omega}s + \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{t}(t-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}}d_{q,\omega}s. \tag{3.2}$$

By [9, Lemma 3.1], we obtain

$$\frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{T}(T-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}}d_{q,\omega}s=\frac{(T-\omega_0)^\alpha}{\alpha\Gamma_q(\alpha)}=\frac{(T-\omega_0)^\alpha}{\Gamma_q(\alpha+1)} \tag{3.3}$$

and

$$\frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{t}(t-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}}d_{q,\omega}s=\frac{(t-\omega_0)^\alpha}{\alpha\Gamma_q(\alpha)}=\frac{(t-\omega_0)^\alpha}{\Gamma_q(\alpha+1)}. \tag{3.4}$$

Substituting from (3.3) and (3.4) into (3.2), we have

$$\begin{aligned}\int_{\omega_0}^{T}|G_{q,\omega}(t,s)|d_{q,\omega}s&=\frac{(t-\omega_0)}{(T-\omega_0)}\frac{(T-\omega_0)^\alpha}{\Gamma_q(\alpha+1)}+\frac{(t-\omega_0)^\alpha}{\Gamma_q(\alpha+1)}\\&=\frac{1}{\Gamma_q(\alpha+1)}[(t-\omega_0)(T-\omega_0)^{\alpha-1}+(t-\omega_0)^\alpha]\\&=\frac{(t-\omega_0)}{\Gamma_q(\alpha+1)}[(T-\omega_0)^{\alpha-1}+(t-\omega_0)^{\alpha-1}],\end{aligned}$$

which completes the proof. □

In the following result, we find the maximum of $\int_{\omega_0}^{T}|G_{q,\omega}(t,s)|d_{q,\omega}s$.

Corollary 3.1. *For $q\in(0,1)$ and $1<\alpha\leqslant 2$, we have*

$$\max_{t\in I_{q,\omega}^T}\int_{\omega_0}^{T}|G_{q,\omega}(t,s)|d_{q,\omega}s\leqslant 2\frac{(T-\omega_0)^\alpha}{\Gamma_q(\alpha+1)}, \tag{3.5}$$

where $G_{q,\omega}(t,s)$ is defined by (2.12).

4 Existence and uniqueness results

4.1 Existence of at least one solution

In this section, we prove the existence of at least one solution to Problem (1.1) by applying the Schauder's fixed point theorem and a special case of Browder's fixed point theorem.

Lemma 4.1. (Schauder's fixed-point theorem) [17] *Let M be a nonempty, closed, bounded, convex, subset of a Banach space X, and suppose that $T: M\to M$ is a continuous operator. Then, T has a fixed point.*

The following theorem is an application of the Schauder's fixed point theorem.

Theorem 4.1. *Let $h: I_{q,\omega}^T\times\mathbb{R}\to\mathbb{R}$ be a continuous function in the second variable,* $1<\alpha\leqslant 2$, *and* $\max_{t\in I_{q,\omega}^T}|z(t)|\leqslant M$ *for some* $M>0$, *where z is the unique solution to the FBVP*

$$\begin{cases}{}^{C}D_{q,\omega}^{\alpha}z(t)=0,\\ z(\omega_0)=\phi(z),\quad z(T)=\psi(z).\end{cases}$$

Then, the nonlinear FBVP (1.1) *has a solution provided that*

$$\frac{(t-\omega_0)}{\Gamma_q(\alpha+1)}[(T-\omega_0)^{\alpha-1}-(t-\omega_0)^{\alpha-1}]\leqslant\frac{M}{c},$$

where $c=\max\{|h(t,u)|:\omega_0\leqslant t\leqslant T, u\in\mathbb{R}, |u|\leqslant 2M\}$.

Proof. Let S be the space of all real-valued functions defined on $I_{q,\omega}^{T}$. Then, we define a norm $\|.\|$ on S by $\|y\| = \max_{t\in I_{q,\omega}^{T}}|y(t)|$. So that the pair $(S, \|.\|)$ is a Banach space. Let $K = \{y \in S : \|y\| \leqslant 2M\}$. Then, K is a compact convex subset of S. Next, define the map $T : S \to S$ by

$$Ty(t) := z(t) + \int_{\omega_0}^{T} G_{q,\omega}(t,s)h(s,y(s))d_{q,\omega}s, \quad t \in I_{q,\omega}^{T}. \tag{4.1}$$

Lemma 2.14 shows that (4.1) is equivalent to (1.1). Therefore, Problem (1.1) has a solution if and only if the operator T defined by (4.1) has a fixed point. To do so, we need only to show that T is a continuous operator from K into itself. First, we show that T maps K into itself. For $t \in I_{q,\omega}^{T}$ and $y \in K$,

$$\begin{aligned}
|Ty(t)| &= \left| z(t) + \int_{\omega_0}^{T} G_{q,\omega}(t,s)h(s,y(s))d_{q,\omega}s \right| \\
&\leqslant |z(t)| + \left| \int_{\omega_0}^{T} G_{q,\omega}(t,s)h(s,y(s))d_{q,\omega}s \right| \\
&\leqslant M + \int_{\omega_0}^{T} |G_{q,\omega}(t,s)||h(s,y(s))|d_{q,\omega}s \\
&\leqslant M + c\int_{\omega_0}^{T} |G_{q,\omega}(t,s)|d_{q,\omega}s \\
&\leqslant M + c\frac{(t-\omega_0)}{\Gamma_q(\alpha+1)}[(T-\omega_0)^{\alpha-1} - (t-\omega_0)^{\alpha-1}] \\
&\leqslant M + c\frac{M}{c} \\
&\leqslant 2M.
\end{aligned}$$

Now, we will show that T is continuous on K. Let $\varepsilon > 0$ be given and assume that

$$l := \max_{t\in I_{q,\omega}^{T}} \int_{\omega_0}^{T} |G_{q,\omega}(t,s)|d_{q,\omega}s.$$

Since h is continuous in its second variable on $\mathbb{R}$, h is uniformly continuous in its second variable on $[-2M, 2M]$. Therefore, there exists $\delta > 0$ such that for all $t \in I_{q,\omega}^{T}$ and for all $u, v \in [-2M, 2M]$ with $|(t,u) - (t,v)| < \delta$, we have $|h(t,u) - h(t,v)| < \frac{\varepsilon}{l}$. Thus, for all $t \in I_{q,\omega}^{T}$, we have

$$\begin{aligned}
|Ty(t) - Tx(t)| &= \left| \int_{\omega_0}^{T} G_{q,\omega}(t,s)h(s,y(s))d_{q,\omega}s - \int_{\omega_0}^{T} G_{q,\omega}(t,s)h(s,x(s))d_{q,\omega}s \right| \\
&\leqslant \int_{\omega_0}^{T} |G_{q,\omega}(t,s)||h(s,y(s)) - h(s,x(s))|d_{q,\omega}s \\
&< \frac{\varepsilon}{l}\int_{\omega_0}^{T} |G_{q,\omega}(t,s)|d_{q,\omega}s < \varepsilon.
\end{aligned}$$

Then, for $t \in [\omega_0, T]_{q,\omega}$, we have $\|Ty(t) - Tx(t)\| < \varepsilon$. This shows the continuity of T on K. Hence, T is a continuous map from K into itself. Then, T has a fixed point in K. □

Lemma 4.2. (A specific case of Browder's theorem). [17] *Let S be a Banach space and $T : S \to S$ be a compact operator. If $T(S)$ is bounded, then T has a fixed point.*

As an application of this lemma, we will prove the following result on the existence of a solution to FBVP (1.1) under a strong assumption on h.

Theorem 4.2. *Let $h : I_{q,\omega}^T \times \mathbb{R} \to \mathbb{R}$ be a continuous function in second variable and be bounded, and let $\max_{t\in I_{q,\omega}^T}|z(t)| \leqslant M$ for some $M > 0$. Then, the nonlinear FBVP* (1.1) *has a solution.*

Proof. Let S be the space of all real-valued functions defined on $I_{q,\omega}^T$. Then, the pair $(S, \|.\|)$ is a Banach space. Define the operator T as in (4.1). It is easy to see that the operator T is compact. Next, we shall show that $T(S)$ is bounded. Since h is bounded, then there exists $m > 0$ such that for all $t \in I_{q,\omega}^T$ and for all $u \in \mathbb{R}$, $|h(t, u)| \leqslant m$. Thus, for any $y \in S$ and $t \in I_{q,\omega}^T$, we have

$$\begin{aligned}
|Ty(t)| &= \left| z(t) + \int_{\omega_0}^{T} G_{q,\omega}(t, s)h(s, y(s))d_{q,\omega}s \right| \\
&\leqslant |z(t)| + \int_{\omega_0}^{T} |G_{q,\omega}(t, s)||h(s, y(s))|d_{q,\omega}s \\
&\leqslant |z(t)| + m\int_{\omega_0}^{T} |G_{q,\omega}(t, s)|d_{q,\omega}s \\
&\leqslant \max_{t\in I_{q,\omega}^T}|z(t)| + m \max_{t\in I_{q,\omega}^T} \int_{\omega_0}^{T} |G_{q,\omega}(t, s)|d_{q,\omega}s \\
&\leqslant M + m\frac{(T - \omega_0)^\alpha}{\Gamma_q(\alpha + 1)}.
\end{aligned}$$

Therefore, T is bounded on S. Then, T has a fixed point. □

In the next section, we prove the uniqueness result for Problem (1.1) by applying the Banach's fixed point theorem.

4.2 Existence of a unique solution

Lemma 4.3. (Contraction mapping theorem) [17] *Let $(X, \|.\|)$ be a Banach space and $T : X \to X$ be a contraction mapping. Then, T has a unique fixed point in X.*

Applying the aforementioned lemma helps to prove the following theorems.

Theorem 4.3. *Let $h : I_{q,\omega}^T \times \mathbb{R} \to \mathbb{R}$ satisfy a uniform Lipschitz condition with respect to its second variable, i.e., there exists $K \geqslant 0$ such that for all $t \in I_{q,\omega}^T$ and $u, v \in \mathbb{R}$,*

$$|h(t, u) - h(t, v)| \leqslant K|u - v|.$$

If $(T - \omega_0)^\alpha < \frac{\Gamma_q(\alpha+1)}{K}$, then the nonlinear fractional Hahn boundary value problem (1.1) *has a unique solution.*

Proof. Let S be the space of all real-valued functions defined on $I_{q,\omega}^T$. Then, we define a norm $\|.\|$ on S by $\|y\| = \max_{t\in I_{q,\omega}^T}|y(t)|$. So that the pair $(S, \|.\|)$ is a Banach space. Now, we define the map $T : S \to S$ as in (4.1). From the equivalence between (4.1) and (1.1), it is enough to prove that T is a contractive mapping. Observe for all $t \in I_{q,\omega}^T$ and for all $x, y \in S$ that

$$\begin{aligned}\|Ty(t)-Tx(t)\| &= \max_{t\in I_{q,\omega}^T}\left|\int_{\omega_0}^{T} G_{q,\omega}(t,s)[h(s,y(s))-h(s,x(s))]d_{q,\omega}s\right|\\ &\leqslant \max_{t\in I_{q,\omega}^T}\int_{\omega_0}^{T}|G_{q,\omega}(t,s)||h(s,y(s))-h(s,x(s))|d_{q,\omega}s\\ &\leqslant K\|y-x\|\max_{t\in I_{q,\omega}^T}\int_{\omega_0}^{T}|G_{q,\omega}(t,s)|d_{q,\omega}s\\ &\leqslant K\|y-x\|\frac{(T-\omega_0)^\alpha}{\Gamma_q(\alpha+1)}\\ &\leqslant \gamma\|y-x\|,\end{aligned}$$

where $\gamma := \frac{K(T-\omega_0)^\alpha}{\Gamma_q(\alpha+1)} < 1$ by the assumption. Therefore, T is a contraction mapping on S. Therefore, T has a unique fixed point in S. □

Theorem 4.4. *Let $h : I_{q,\omega}^T \times \mathbb{R} \to \mathbb{R}$ satisfy a uniform Lipschitz condition with respect to its second variable, i.e., there exists $k \geqslant 0$ such that for all $t \in I_{q,\omega}^T$ and $u, v \in \mathbb{R}$,*

$$|h(t,u)-h(t,v)| \leqslant k|u-v|,$$

and the equation

$$^{C}D_{q,\omega}^{\alpha}y(t)+ky(t)=0$$

has a positive solution u. Then, the nonlinear fractional boundary value problem (1.1) *has a unique solution.*

Proof. If the equation $^{C}D_{q,\omega}^{\alpha}y(t)+ky(t)=0$ has a positive solution, it follows that $u(t)$ is a solution to FBVP

$$\begin{cases} ^{C}D_{q,\omega}^{\alpha}y(t) = -ku(t),\\ y(\omega_0)=\phi(u), \quad y(T)=\psi(u),\end{cases}$$

where $\phi(u)=u(\omega_0)$ and $\psi(u)=u(T)$. Then, according to Lemma 2.14, we have

$$u(t)=z(t)+k\int_{\omega_0}^{T}G_{q,\omega}(t,s)u(s)d_{q,\omega}s,$$

where $G_{q,\omega}(t,s)$ is the Green function defined by (2.12) and $z(t)$ is the unique solution to FBVP

$$\begin{cases} ^{C}D_{q,\omega}^{\alpha}z(t) = 0,\\ z(\omega_0)=\phi(z), \quad z(T)=\psi(z),\end{cases}$$

which is of the form

$$z(t)=\phi(z)+\frac{t-\omega_0}{T-\omega_0}[\psi(z)-\phi(z)].$$

Now, since ϕ and ψ are positive,

$$z(t)=\frac{T-t}{T-\omega_0}\phi(z)+\frac{t-\omega_0}{T-\omega_0}\psi(z)>0, \quad t\in I_{q,\omega}^T.$$

Then,

$$u(t)>k\int_{\omega_0}^{T}G_{q,\omega}(t,s)u(s)d_{q,\omega}s.$$

Therefore,

$$\gamma := \max_{t \in I_{q,\omega}^T} \frac{k}{u(t)} \int_{\omega_0}^{T} G_{q,\omega}(t,s)u(s)d_{q,\omega}s < 1.$$

Consider the weighted norm $\|.\|$ defined by:

$$\|x\| := \max_{t \in I_{q,\omega}^T} \frac{|x(t)|}{u(t)}.$$

Let S be the space of all real-valued functions defined on $I_{q,\omega}^T$. Then, the pair $(S, \|.\|)$ is a Banach space. The operator

$$Ty(t) = z(t) + \int_{\omega_0}^{T} G_{q,\omega}(t,s)h(s,y(s))d_{q,\omega}s,$$

for all $t \in I_{q,\omega}^T$, satisfies

$$\begin{aligned}
\left|\frac{Ty(t) - Tx(t)}{u(t)}\right| &= \frac{1}{u(t)}\left|\int_{\omega_0}^{T} G_{q,\omega}(t,s)[h(s,y(s)) - h(s,x(s))]d_{q,\omega}s\right| \\
&\leqslant \frac{1}{u(t)}\int_{\omega_0}^{T} |G_{q,\omega}(t,s)||h(s,y(s)) - h(s,x(s))|d_{q,\omega}s \\
&\leqslant \frac{k}{u(t)}\int_{\omega_0}^{T} |G_{q,\omega}(t,s)||y(s) - x(s)|d_{q,\omega}s \\
&\leqslant \frac{1}{u(t)}\|y - x\|\int_{\omega_0}^{T} k|G_{q,\omega}(t,s)|u(s)d_{q,\omega}s \\
&\leqslant \|y - x\| \max_{t \in I_{q,\omega}^T} \frac{k}{u(t)}\int_{\omega_0}^{T} G_{q,\omega}(t,s)u(s)d_{q,\omega}s \\
&\leqslant \gamma\|y - x\|.
\end{aligned}$$

Since $\gamma < 1$, T is a contraction mapping on S. Therefore, T has a unique fixed on S (by the contraction mapping theorem). This shows the existence of the unique solution to the nonlinear FBVP (1.1). □

5 Examples

5.1 Example 1

Consider the following fractional Hahn boundary value problem:

$$\begin{cases} {}^{C}D_{\frac{1}{2},\frac{2}{3}}^{\frac{4}{3}}u(t) = \dfrac{e^{-3t+1}(u^2 + |u(t)|)}{(100\pi^2 + t^3)(1 + |u(t)|)}, & t \in I_{\frac{1}{2},\frac{2}{3}}^{10}, \\ u\left(\dfrac{4}{3}\right) = 10e, \quad u(10) = \dfrac{1}{10\pi}. \end{cases} \tag{5.1}$$

Here, $\alpha = \frac{4}{3}$, $\omega = \frac{2}{3}$, $q = \frac{1}{2}$, $\omega_0 = \frac{4}{3}$, $T = 10$, and

$$h(t,u(t)) = \frac{e^{-3t+1}(u^2 + |u(t)|)}{(100\pi^2 + t^3)(1 + |u(t)|)}.$$

For $t \in I^{10}_{\frac{1}{2},\frac{2}{3}}$ and $u, v \in \mathbb{R}$, we obtain

$$|h(t,u(t)) - h(t,v(t))| \leqslant \frac{1}{e^3(100\pi^2 + (\frac{4}{3})^3)}\|u - v\|.$$

Hence, the first condition

$$|h(t,u(t)) - h(t,v(t))| \leqslant K\|u - v\|$$

of Theorem 4.3 holds true, with

$$K = \frac{1}{e^3(100\pi^2 + (\frac{4}{3})^3)}.$$

On the other hand, we have

$$(T - \omega_0)^\alpha K = \left(10 - \frac{4}{3}\right)^{\frac{4}{3}} \cdot \frac{1}{e^3\left(100\pi^2 + \left(\frac{4}{3}\right)^3\right)} = 0.000895869.$$

However,

$$\Gamma_q(\alpha + 1) = \Gamma_{\frac{1}{2}}\left(\frac{7}{3}\right) = \frac{\left(1 - \frac{1}{2}\right)^{1/3}_{\frac{1}{2}}}{\left(1 - \frac{1}{2}\right)^{1/3}} = 0.893616.$$

So, the second condition

$$(T - \omega_0)^\alpha < \frac{\Gamma_q(\alpha + 1)}{K}$$

of Theorem 4.3 is satisfied. Hence, by Theorem 4.3, Problem (5.1) has a unique solution.

5.2 Example 2

For the following fractional Hahn boundary value problem

$$\begin{cases} {}^{C}D^{\frac{5}{3}}_{\frac{1}{2},2}u(t) = \dfrac{e^{-\sin^2(\pi t)}u^2(t)}{t^2(100 + e^{\cos^2(\pi t/2)})(1 + |u(t)|)}, & t \in I^{10}_{\frac{1}{2},2}, \\ u(4) = \dfrac{1}{125e^3}, \quad u(10) = \dfrac{1}{100\pi^2}, \end{cases} \tag{5.2}$$

we have $\alpha = \frac{5}{3}$, $\omega = 2$, $q = \frac{1}{2}$, $\omega_0 = 4$, $T = 10$, and

$$h(t,u(t)) = \frac{e^{-\sin^2(\pi t)}}{t^2(100 + e^{\cos^2(\pi/t)})} \cdot \frac{u^2(t)}{1 + |u(t)|}.$$

Then, for $t \in I^{10}_{\frac{1}{2},2}$ and $u, v \in \mathbb{R}$, we obtain

$$|h(t,u(t)) - h(t,v(t))| \leqslant \frac{1}{1{,}616}\|u - v\|.$$

Hence, the first condition

$$|h(t,u(t)) - h(t,v(t))| \leqslant K\|u - v\|$$

of Theorem 4.3 holds true, with $K = \frac{1}{1{,}616}$. On the other hand, we have

$$(T-\omega_0)^\alpha K = (10-4)^{\frac{5}{3}} \cdot \frac{1}{1{,}616} = 0.0122596.$$

However,

$$\Gamma_q(\alpha+1) = \Gamma_{\frac{1}{2}}\left(\frac{8}{3}\right) = \frac{\left(1-\frac{1}{2}\right)^{2/3}_{\frac{1}{2}}}{\left(1-\frac{1}{2}\right)^{2/3}} = 1.07789.$$

So, the second condition

$$(T-\omega_0)^\alpha < \frac{\Gamma_q(\alpha+1)}{K}$$

of Theorem 4.3 is satisfied. Hence, by Theorem 4.3, Problem (5.2) has a unique solution.

6 Conclusion

In this study, we have considered a boundary value problem for a fractional Hahn difference equation subject to two boundary conditions. Our results extend and generalize the results obtained in [18–20]. After proving the existence and uniqueness results concerning linear variants of the main nonlinear problem, it is transformed into a fixed point problem. A Green's function is constructed and used to express the solutions to the considered boundary value problems. Banach's, Schauder's, and Browder's fixed point theorems are used to prove the existence and uniqueness results. The main results are illustrated by numerical examples. Some properties of the fractional Hahn calculus needed in our work are also presented. The results of this article are new and enrich the theory of boundary value problems for Hahn difference equations. In future works, we will study the Laplace transform and Baskakov basis functions associated with the fractional Hahn difference operators. Also, the q, ω-analogs of degenerate derivatives and their applications and the Boson operator can be investigated. These potential results will generalize the recent works [21–24].

Acknowledgement: The authors thank the editor and the referee for their comments and suggestions to improve the manuscript.

Funding information: No funding sources to be declared.

Author contributions: All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Conflict of interest: The authors declare that they have no competing interest.

Data availability statement: Data sharing does not apply to the article as no datasets were generated or analyzed during the current study.

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