

Research Article

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Integral transforms involving a generalized k -Bessel function

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Abstract: The main goal of this study was to look into some new integral transformations that are associated with a generalized k -Bessel function. Integral formulas for the generalized k -Bessel function have been established using the Laplace transform, Euler transform, Whittaker transform, and k -transforms. The results presented here have the potential to be helpful, and some special cases of corollaries are explicitly demonstrated.

Keywords: k -gamma function, generalized k -Bessel function, integral transforms

MSC 2020: 33C10, 33C60, 33B15, 65R10

1 Introduction

The theory of special functions plays a discernible role in a variety of appropriate fields of mathematical analysis, applied science, and engineering. Bessel functions and their extensions with their various generalizations are among the most important in the special functions. In recent years, the k -calculus started with Diaz and Parigun's [1] definition (see also Diaz et al. [2], Diaz and Teruel [3]). Following Diaz and Parigaun's [1] concept, several researchers later introduced different types of k -special functions; for example, the properties of k -gamma, k -beta, and k -zeta have been studied by Kokologiannaki [4]. Krasniqi [5] studied the limits to k -gamma and k -beta functions.

In addition, the hypergeometric function and its generalized extensions have been introduced and extensively investigated mainly due to their applications in diverse areas of mathematics [6–8]. Similarly, the hypergeometric k -function has become important in recent years; for example, in [9,11], Mubeen and Habiullah defined the integral notation of generalized confluent hypergeometric k -functions and hypergeometric k -functions and k -fractional integral with applications based on Pochhammer k -symbols, k -gamma, and k -beta functions. Mansour [12] defined the k -generalized gamma function using a functional equation. The power product constraints for the k -gamma function were provided by Merovci [13].

Suthar et al. [14] have studied and presented the interesting Pochhammer k -symbol (rising factorial), fractional integral representations of k -gamma with some other related functions, such as k -beta and k -digamma functions, and k -Bessel function (which is one of the valuable speculations of the Bessel function). Also, for more details regarding the developed and applied new class of k -Bessel function with their applied

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properties, several inequalities, and integrals in different fields have been studied by various researchers and mathematicians [15–21].

Recently, Ali et al. [22], Ghayasuddin and Khan [23], Khan et al. [24–26], and Nisar et al. [27,28] gave certain interesting new class of integral formulas involving some types of the generalized Bessel functions, which are expressed in terms of the generalized (right) hypergeometric function (for more details, see [6–8]). Motivated by these recent studies, we start by mentioning some notations with some results and definitions used in this study.

The k -Bessel function of the first kind has been established and discussed by Romero et al. [16] given as follows:

$$J_{k,\nu}^{\lambda}(z) = \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^r}{r! \Gamma_k(\lambda r + \nu + 1)}, \quad (1.1)$$

where $k \in \mathfrak{R}$; $\lambda, \gamma, \alpha, \vartheta \in \mathbb{C}$; $\Re(\lambda) > 0$; and $\Re(\vartheta) > 0$.

The generalized k -Bessel function of the first kind introduced by [27] has the form:

$$J_{k,\nu}^{b,c,\gamma,\lambda}(z) = \sum_{r=0}^{\infty} \frac{(c)^r (\gamma)_{r,k} (z/2)^{\nu+2r}}{(r!)^2 \Gamma_k\left(\lambda r + \nu + \frac{b+1}{2}\right)}, \quad (1.2)$$

where $k \in \mathfrak{R}$; $\lambda, \gamma, \nu, b, c \in \mathbb{C}$; $\Re(\nu) > 0$, and $\Re(\lambda) > 0$, and $(\gamma)_{r,k}$ is the k -Pochhammer symbol [1] defined by:

$$(\gamma)_{\nu,k} = \frac{\Gamma_k(\gamma + \nu k)}{\Gamma_k(\gamma)}, \quad (\gamma \in \mathbb{C} \setminus \{0\}). \quad (1.3)$$

The k -gamma functions $\Gamma_k(x)$, where $\Gamma_k(x) \rightarrow \Gamma(x)$ if $k \rightarrow 1$, are given by the following relationship:

$$\Gamma_k(x) = k^{\frac{x}{k}-1} \Gamma\left(\frac{x}{k}\right). \quad (1.4)$$

To obtain our main results, we use the following definitions of integral transforms:

(1) Euler transform [29] of the function $f(z)$ is defined by:

$$B\{f(z); \lambda, \mu\} = \int_0^1 z^{\lambda-1} (1-z)^{\mu-1} f(z) dz, \quad \lambda, \mu \in \mathbb{C} \quad \text{and} \quad \Re(\lambda), \Re(\mu) > 0. \quad (1.5)$$

(2) Laplace transform [29] of the function $f(z)$ is defined by:

$$F(s) = L\{f(z); s\} = \int_0^{\infty} e^{-sz} f(z) dz, \quad \Re(s) > 0. \quad (1.6)$$

(3) We also recall the following known Whittaker transform results [30]:

$$\int_0^{\infty} t^{\vartheta-1} e^{-\frac{t}{z}} W_{\lambda,\mu}(t) dt = \frac{\Gamma\left(\frac{1}{2} + \mu + \vartheta\right) \Gamma\left(\frac{1}{2} - \mu + \vartheta\right)}{\Gamma(1 - \lambda + \vartheta)}, \quad \Re(\vartheta \pm \mu) > -\frac{1}{2}, \quad (1.7)$$

where $W_{\lambda,\mu}$ is the Whittaker function [31] (see, e.g., [32])

$$W_{\lambda,\mu}(t) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \lambda - \mu\right)} M_{\lambda,\mu}(t) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} + \lambda + \mu\right)} M_{\lambda,-\mu}(t),$$

where $M_{\lambda,\mu}(t)$ is the confluent hypergeometric function written as:

$$M_{\lambda,\mu}(t) = z^{\frac{1}{2}+\mu} e^{-\frac{1}{2}t} {}_1F_1\left(\frac{1}{2} + \mu - \lambda; 2\mu + 1; t\right).$$

Also, we have the transformation formula

$$\int_0^{\infty} t^{\vartheta-1} e^{-\frac{t}{2}} M_{\lambda, \mu}(t) dt = \frac{\Gamma(2\mu+1) \Gamma\left(\frac{1}{2} + \mu + \vartheta\right) \Gamma(\lambda - \vartheta)}{\Gamma\left(\frac{1}{2} + \mu - \vartheta\right) \Gamma\left(\frac{1}{2} + \mu + \lambda\right)}. \quad (1.8)$$

(4) The K -transform [31] with a complex parameter a is defined by:

$$R_{\vartheta}\{f(z); a\} = g(a; \vartheta) = \int_0^{\infty} (az)^{\frac{1}{2}} K_{\vartheta}(az) f(z) dz, \quad (1.9)$$

where $\Re(a) > 0$ and $K_{\vartheta}(z)$ is the Bessel function of the second kind defined in ([31], p. 32) as:

$$K_{\vartheta}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} W_{0, \vartheta}(2z),$$

where $W_{0, \vartheta}(\cdot)$ is the Whittaker function defined in (1.7).

The following relationship [32, p. 54] is required in our results:

$$\int_0^{\infty} t^{\rho-1} K_{\vartheta}(at) dt = 2^{\rho-1} a^{-\rho} \Gamma\left(\frac{\rho \pm \vartheta}{2}\right), \quad \Re(a) > 0; \quad \Re(\rho \pm \vartheta) > 0. \quad (1.10)$$

The generalized hypergeometric function is defined in [33] as:

$${}_pF_q \left[\begin{matrix} (A)_p; \\ (B)_q; \end{matrix} x \right] = \sum_{r=0}^{\infty} \frac{\prod_{i=1}^p (A_i)_r x^r}{\prod_{j=1}^q (B_j)_r r!}, \quad (1.11)$$

provided $p \leq q$, $p = q + 1$ and $|x| < 1$, where $(A)_r$ is known as the Pochhammer symbol.

Also, the Fox-Wright generalization ${}_p\Omega_q$ of the hypergeometric function ${}_pF_q$ [34–36] is

$$\begin{aligned} {}_pF_q \left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{matrix} x \right] &= {}_p\Omega_q((\alpha_i, A_i)_{1,p}; (\beta_j, B_j)_{1,q}; x) \\ &= \sum_{r=0}^{\infty} \frac{\Gamma((\alpha_1 + A_1 r) \dots (\alpha_p + A_p r)) x^r}{\Gamma((\beta_1 + B_1 r) \dots (\beta_q + B_q r)) r!}, \end{aligned} \quad (1.12)$$

where $A_i > 0$ ($i = 1, 2, 3, \dots, p$); $B_i > 0$ ($i = 1, 2, 3, \dots, q$); and $1 + \sum_{i=1}^q B_i - \sum_{i=1}^p A_i \geq 0$ for a suitably bounded value of $|x|$.

In [20], the generalized k -Wright function defined for $k \in \Re^+$; $z \in \mathbb{C}$, $\alpha_i, \beta_j \in \Re(\alpha_i, \beta_j \neq 0; i = 1, 2, 3, \dots, p; j = 1, 2, 3, \dots, q)$ and $(s_i + \alpha_i r), (d_j + \beta_j r) \in \mathbb{C} \setminus k\mathbb{Z}^-$ is written as:

$${}_p\Psi_q^k(x) = {}_p\Psi_q^k \left[\begin{matrix} (s_i, \alpha_i)_{1,p}; \\ (d_j, \beta_j)_{1,q}; \end{matrix} x \right] = \sum_{r=0}^{\infty} \frac{\prod_{i=1}^p \Gamma_k(s_i + \alpha_i r) x^r}{\prod_{j=1}^q \Gamma_k(d_j + \beta_j r) r!}. \quad (1.13)$$

In this article, we initiate a generalized k -Bessel function. We also establish some integral formulas for the generalized k -Bessel function by applying the Laplace transform, Euler transform, Whittaker transform, and k -transform. Moreover, as consequences, we present some special cases of our obtained results.

2 Main results

First, we introduce the generalized k -Bessel function of the first kind in the following form:

$$J_{k,\vartheta,\delta}^{b,c,\gamma,\lambda}(z) = \sum_{r=1}^{\infty} \frac{(c)^r (\gamma)_{n,k} (z/2)^{\vartheta+2r}}{(\delta)_{r,k} \Gamma_k \left(\lambda r + \vartheta + \frac{b+1}{2} \right) r!}, \quad (2.1)$$

for $k \in \Re$; $\lambda, \gamma, \alpha, \vartheta \in C$; $\Re(\lambda) > 0$; and $\Re(\vartheta) > 0$, where $(\gamma)_{n,k}$ and $(\delta)_{n,k}$ are the k -Pochhammer symbol.

Note that

$$\lim_{b,\lambda,\gamma,k,\delta \rightarrow 1, c \rightarrow -1} J_{k,\vartheta,\delta}^{b,c,\gamma,\lambda}(z) = \sum_{r=1}^{\infty} \frac{(-1)^r (z/2)^{\vartheta+2r}}{\Gamma(r + \vartheta + 1) r!} = J_{\vartheta}(z). \quad (2.2)$$

The integral transforms for generalized k -Bessel functions such as Euler, Laplace, and Whittaker are listed in the following theorems:

Theorem 2.1. If $k \in \Re$; $\alpha, \beta, \gamma, b, c, \vartheta, \delta \in C$; and $\min\{\Re(\sigma\vartheta + \alpha), \Re(\sigma\vartheta + \beta), \Re(\beta), \Re(\sigma), \Re(\lambda), \Re(\vartheta), \Re(\frac{\vartheta}{k}), \Re(\frac{\delta}{k}), \text{ and } \Re(\frac{\vartheta}{k} + \frac{b+1}{k})\} > 0$, then

$$\begin{aligned} & \int_0^1 z^{\alpha-1} (1-z)^{\beta-1} J_{k,\vartheta,\delta}^{b,c,\gamma,\lambda}(xz^{\sigma}) dz \\ &= \frac{\Gamma(\beta) \left(\frac{x}{z} \right)^{\vartheta} \Gamma \left(\frac{\delta}{k} \right) k^{1-\frac{\vartheta}{k}-\frac{b+1}{2k}}}{\Gamma \left(\frac{\gamma}{k} \right)} \\ & \quad \times {}_2\Psi_3 \left[\left(\frac{\gamma}{k}, 1 \right), (\alpha + \sigma\vartheta, 2\sigma); \right. \\ & \quad \left. (\alpha + \sigma\vartheta + \beta, 2\sigma), \left(\frac{\vartheta}{k} + \frac{b+1}{2k}, \frac{\lambda}{k} \right), \left(\frac{\delta}{k}, 1 \right); \frac{cx^2}{4k^{\frac{\lambda}{k}}} \right], \end{aligned} \quad (2.3)$$

where ${}_2\Psi_3$ is the Wright hypergeometric function (1.2).

Proof. For convenience, let R_1 be the left-hand side (L.H.S.) of (2.3):

$$R_1 = \int_0^1 z^{\alpha-1} (1-z)^{\beta-1} J_{k,\vartheta,\delta}^{b,c,\gamma,\lambda}(xz^{\sigma}) dz.$$

By applying (2.1) in the integrand of the L.H.S. of (2.3), we have

$$R_1 = \int_0^1 z^{\alpha-1} (1-z)^{\beta-1} \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_{n,k} \left(\frac{xz^{\sigma}}{2} \right)^{\vartheta+2n}}{\Gamma_k \left(\lambda n + \vartheta + \frac{b+1}{2} \right) (\delta)_{n,k} n!} dz.$$

By interchanging the integration and summation and then from (1.3) and (1.4), we obtain

$$R_1 = \frac{\Gamma \left(\frac{\delta}{k} \right) \Gamma(\beta) \left(\frac{x}{z} \right)^{\vartheta} k^{1-\frac{\vartheta}{k}-\frac{b+1}{2k}}}{\Gamma \left(\frac{\gamma}{k} \right)} \sum_{n=0}^{\infty} \frac{\Gamma \left(\frac{\gamma}{k} + n \right) \Gamma(\alpha + \sigma\vartheta + 2\sigma n) \left(\frac{cx^2}{4k^{\frac{\lambda}{k}}} \right)^n}{n! \Gamma \left(\frac{\lambda n}{k} + \frac{\vartheta}{k} + \frac{b+1}{2k} \right) \Gamma(\alpha + \sigma\vartheta + 2\sigma n + \beta) \Gamma \left(\frac{\delta}{k} + n \right)}.$$

In view of the definition of the Wright hypergeometric functions (1.12), we obtain the desired result. $\square$

If we let $k = 1$ in Theorem 2.1, we state the following:

Corollary 2.1. If $k \in \Re$; $\alpha, \beta, \gamma, \vartheta, \delta \in \mathbb{C}$; and $\min\{\Re(\sigma\vartheta + \alpha), \Re(\sigma\vartheta + \beta), \Re(\beta), \Re(\sigma), \Re(\lambda)\} > 0$, then

$$\int_0^1 z^{\alpha-1} (1-z)^{\beta-1} J_{1,\vartheta,\delta}^{b,c,\gamma,\lambda}(xz^\sigma) dz = \frac{\Gamma(\delta)\Gamma(\beta)\left(\frac{x}{2}\right)^\vartheta}{\Gamma(\gamma)} {}_2\Psi_3 \left[\begin{matrix} (y, 1), (\alpha + \sigma\vartheta, 2\sigma); \\ (\alpha + \sigma\vartheta + \beta, 2\sigma), \left(\gamma + \frac{b+1}{2}, \lambda\right), (\delta, 1); \end{matrix} ; cx^2 \right]. \quad (2.4)$$

Setting $\gamma = \delta, b = 1$, and $c = -1$ in Theorem 2.1, we obtain

Corollary 2.2. If $k \in \Re$; $\alpha, \beta, \vartheta \in \mathbb{C}$; and $\min\{\Re(\sigma\vartheta + \alpha), \Re(\sigma\vartheta + \beta), \Re(\beta), \Re(\sigma), \Re(\lambda), \Re(\vartheta), \Re(\frac{\vartheta}{k}), \Re(\frac{\vartheta}{k} + \frac{1}{k})\} > 0$, then

$$\int_0^1 z^{\alpha-1} (1-z)^{\beta-1} J_{k,\vartheta}^\lambda(xz^\sigma) dz = \Gamma(\beta) \left(\frac{x}{2}\right)^\vartheta k^{1-\frac{\vartheta}{k}-\frac{1}{k}} \times {}_1\Psi_2 \left[\begin{matrix} (\alpha + \sigma\vartheta, 2\sigma); \\ (\alpha + \sigma\vartheta + \beta, 2\sigma), \left(\frac{\vartheta}{k} + \frac{1}{k}, \frac{\lambda}{k}\right); \end{matrix} ; -\frac{x^2}{4k^{\frac{1}{k}}} \right], \quad (2.5)$$

where $J_{k,\vartheta}^\lambda$ is the k-Bessel function defined by (1.1) and ${}_1\Psi_2$ is the Wright hypergeometric function (1.12).

Corollary 2.3.

- (i) Taking $\delta = 1$ in Relation (2.3) corresponds to the result given by Nisar et al. [28, Theorem 2.1],
- (ii) Taking $k = 1$ and $\delta = 1$ in Relation (2.3), we obtain the result given by Nisar et al. [28, Corollary 2.1].

Theorem 2.2. If $k \in \Re$; $b, \lambda, \vartheta, \gamma, a, c, \sigma, \delta \in \mathbb{C}$; $\Re\left(\frac{\vartheta}{k} + \frac{b+1}{k}\right) > 0$, $\min\{\Re(\lambda), \Re(\vartheta), \Re(s), \Re\left(\frac{\vartheta}{k}\right), \Re\left(\frac{\delta}{k}\right), \Re(\sigma\vartheta + a)\} > 0$ and $|\frac{x}{s^\sigma}| < 1$ in Theorem 2.1, then we have

$$\begin{aligned} & \int_0^\infty z^{a-1} e^{-sz} J_{k,\vartheta,\delta}^{b,c,\gamma,\lambda}(xz^\sigma) dz \\ &= \frac{\left(\frac{x}{2}\right)^\vartheta s^{-(a+\sigma\vartheta)} \Gamma\left(\frac{\delta}{k}\right) k^{1-\frac{\vartheta}{k}-\frac{b+1}{2k}}}{\Gamma\left(\frac{\vartheta}{k}\right)} {}_2\Psi_2 \left[\begin{matrix} \left(\frac{\vartheta}{k}, 1\right), (\alpha + \sigma\vartheta, 2\sigma); \\ \left(\frac{\vartheta}{k} + \frac{b+1}{2k}, \frac{\lambda}{k}\right), \left(\frac{\delta}{k}, 1\right); \end{matrix} ; \frac{cx^2}{4s^{2\sigma}k^{\frac{1}{k}}} \right]. \end{aligned} \quad (2.6)$$

Proof. Let R_1 denote the L.H.S. of (2.6). Then, applying (2.1) on the L.H.S. of (2.6), we obtain

$$R_1 = \int_0^\infty z^{a-1} e^{-sz} \sum_{n=0}^\infty \frac{(c)^n (\gamma)_{n,k} \left(\frac{xz^\sigma}{2}\right)^{\vartheta+2n}}{\Gamma_k\left(\lambda n + \vartheta + \frac{b+1}{2}\right) (\delta)_{n,k} n!} dz.$$

Now, by interchanging the integration and summation allows us to write

$$R_1 = \sum_{n=0}^\infty \frac{(c)^n (\gamma)_{n,k} \left(\frac{x}{2}\right)^{\vartheta+2n}}{\Gamma_k\left(\lambda n + \vartheta + \frac{b+1}{2}\right) (\delta)_{n,k} n!} \int_0^\infty z^{a+\sigma\vartheta+2n\sigma-1} e^{-sz} dz.$$

Using the Laplace transform definition, we obtain

$$R_1 = \sum_{n=0}^\infty \frac{(c)^n (\gamma)_{n,k} \Gamma(a + \sigma\vartheta + 2n\sigma) \left(\frac{x}{2}\right)^{\vartheta+2n}}{\Gamma_k\left(\lambda n + \vartheta + \frac{b+1}{2}\right) (\delta)_{n,k} s^{a+\sigma\vartheta+2n\sigma} n!}.$$

Using the relationship defined in equations (1.3) and (1.4), we obtain

$$R_1 = \frac{\Gamma\left(\frac{\delta}{k}\right)\left(\frac{x}{2}\right)^{\vartheta} s^{-(a+\sigma\vartheta)} k^{1-\frac{\vartheta}{k}-\frac{b+1}{2k}}}{\Gamma\left(\frac{\gamma}{k}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma}{k} + n\right) \Gamma(a + \sigma\vartheta + 2\sigma n) \left(\frac{cx^2}{4s^{2n}k^{\frac{\lambda}{k}}}\right)^n}{n! \Gamma\left(\frac{\lambda n + \vartheta}{k} + \frac{b+1}{2k}\right) \Gamma\left(\frac{\delta}{k} + n\right)}.$$

In view of the definition of the Wright hypergeometric functions (1.12), we obtain the desired result. $\square$

If we take $k = 1$ in Theorem 2.2, then we have

Corollary 2.4. *If $b, \lambda, \vartheta, \gamma, a, c, \sigma, \delta \in C$; $\min\{\Re(\lambda), \Re(\vartheta), \Re(s), \Re(\sigma\vartheta + a)\} > 0$, and $|\frac{x}{s\sigma}| < 1$, then we have*

$$\int_0^{\infty} z^{a-1} e^{-sz} J_{1,\vartheta,\delta}^{b,c,\gamma,\lambda}(xz^{\sigma}) dz = \frac{\left(\frac{x}{2}\right)^{\vartheta} s^{-(a+\sigma\vartheta)} \Gamma(\delta)}{\Gamma(\gamma)} {}_2\Psi_2\left[\left(\gamma, 1\right), (a + \sigma\vartheta, 2\sigma); \left[\vartheta + \frac{b+1}{2}, \lambda\right], (\delta, 1); \frac{cx^2}{4s^{2\sigma}}\right]. \quad (2.7)$$

Moreover, if we set $\gamma = \delta$ in the aforementioned corollary, we obtain the following result:

Corollary 2.5. *If $b, \lambda, \vartheta, a, c, \sigma \in C$; $\min\{\Re(\lambda), \Re(\vartheta), \Re(s), \Re(\sigma\vartheta + a)\} > 0$, and $|\frac{x}{s\sigma}| < 1$, then we have*

$$\int_0^{\infty} z^{a-1} e^{-sz} J_{1,\vartheta}^{b,c,\lambda}(xz^{\sigma}) dz = \left(\frac{x}{2}\right)^{\vartheta} s^{-(a+\sigma\vartheta)} {}_1\Psi_1\left[\left(a + \sigma\vartheta, 2\sigma\right); \left[\vartheta + \frac{b+1}{2}, \lambda\right]; \frac{cx^2}{4s^{2\sigma}}\right]. \quad (2.8)$$

Theorem 2.3. *If $k \in \Re, b, \lambda, \vartheta, \gamma, a, c, \sigma, \delta \in C$; $\Re\left(\frac{\rho + \sigma\vartheta \pm \mu}{2}\right) > 0$; and $\min\{\Re(\lambda), \Re(\rho), \Re(\frac{\delta}{k}), \Re(\frac{\gamma}{k}), \Re(\sigma\vartheta + a)\} > 0$, then we have*

$$\begin{aligned} & \int_0^{\infty} z^{\rho-1} K_{\mu}(az) J_{k,\vartheta,\delta}^{b,c,\gamma,\lambda}(xz^{\sigma}) dz \\ &= \frac{\left(\frac{x}{2}\right)^{\vartheta} 2^{(\sigma\vartheta+\rho-2)} a^{-(\sigma\vartheta+\rho)} \Gamma\left(\frac{\delta}{k}\right) k^{1-\frac{\vartheta}{k}-\frac{b+1}{2k}}}{\Gamma\left(\frac{\gamma}{k}\right)} {}_2\Psi_2\left[\left(\frac{\gamma}{k}, 1\right), \left(\frac{\rho + \sigma\vartheta \pm \mu}{2}, \sigma\right); \left[\frac{\vartheta}{k} + \frac{b+1}{2k}, \frac{\lambda}{k}\right], \left(\frac{\delta}{k}, 1\right); \frac{c4^{\sigma}x^2}{4a^{2\sigma}k^{\frac{\lambda}{k}}}\right]. \end{aligned} \quad (2.9)$$

Proof. Let R_1 denote the L.H.S. of (2.9). Then, by applying (2.1) on the L.H.S. of (2.9) with interchanging the integration and summation, we obtain

$$R_1 = \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_{n,k} \left(\frac{x}{2}\right)^{\vartheta+2n}}{\Gamma_k\left(\lambda n + \vartheta + \frac{b+1}{2}\right) (\delta)_{n,k} n!} \int_0^{\infty} z^{\rho+\sigma\vartheta+2n\sigma-1} K_{\mu}(az) dz.$$

Using (1.10) in the aforementioned expression, we obtain

$$R_1 = \left(\frac{x}{2}\right)^{\vartheta} 2^{(\rho+\sigma\vartheta-2)} a^{-(\sigma\vartheta+\rho)} \sum_{n=0}^{\infty} \frac{2^{2\sigma n} a^{-2\sigma n} (\gamma)_{n,k} \Gamma\left(\frac{\rho + 2\sigma n + \sigma\vartheta \pm \mu}{2}\right) \left(\frac{cx^2}{4}\right)^n}{n! \Gamma_k(\lambda n + \vartheta + \frac{b+1}{2}) (\delta)_{n,k}}.$$

Now, using the same method as previous theories' proof by applying Relationships (1.3) and (1.4), then in view of definition (1.12), we obtain the desired result. $\square$

In Theorem 2.3, by setting $k = 1$ we obtain

Corollary 2.6. *If $b, \lambda, \vartheta, \gamma, a, c, \sigma, \delta \in \mathbb{C}; \Re\left(\frac{\rho + \sigma\vartheta \pm \mu}{2}\right) > 0$, and $\min\{\Re(\lambda), \Re(\rho), \Re(\sigma\vartheta + a) > 0\} > 0$, then we have*

$$\begin{aligned} & \int_0^\infty z^{\rho-1} K_\mu(az) J_{1,\vartheta,\delta}^{b,c,\gamma,\lambda}(xz^\sigma) dz \\ &= \frac{\left(\frac{x}{2}\right)^\vartheta 2^{(\sigma\vartheta+\rho-2)} a^{-(\sigma\vartheta+\rho)} \Gamma(\delta) k^{1-\frac{\vartheta}{k}-\frac{b+1}{2k}}}{\Gamma(\gamma)} {}_2\Psi_2 \left[\begin{matrix} (\gamma, 1), \left(\frac{\rho + \sigma\vartheta \pm \mu}{2}, \sigma\right); \\ \left(\vartheta + \frac{b+1}{2}, \lambda\right), (\delta, 1); \end{matrix} \right] \frac{c4^\sigma x^2}{4a^{2\sigma}}. \end{aligned} \quad (2.10)$$

Theorem 2.4. *If $k \in \mathbb{R}^+, b, c, \lambda, \gamma, \mu, \eta, \delta \in \mathbb{C}; \Re(\rho \pm \vartheta) > -\frac{1}{2}; \Re(\mu) > -\frac{1}{2}, \Re\left(\frac{b+1}{2k} + \frac{\nu}{k}\right) > 0$, $\min\{\Re(u), \Re(v), \Re\left(\frac{\gamma}{k}\right), \Re\left(\frac{\delta}{k}\right), \Re(\lambda)\} > 0$; and $|\frac{w}{p^u}| < 1$, then we have*

$$\begin{aligned} & \int_0^\infty t^{\rho-1} e^{-\frac{pt}{2}} M_{\eta,\vartheta}(pt) J_{k,\sigma,\delta}^{b,c,\gamma,\lambda}(wt^u) dt \\ &= \frac{\Gamma\left(\frac{\delta}{k}\right) \left(\frac{w}{2}\right)^\nu k^{1-\frac{\nu}{k}-\frac{b+1}{2k}}}{p^{\beta+u\nu} \Gamma\left(\frac{\gamma}{k}\right)} {}_3\Psi_3 \left[\begin{matrix} \left(\frac{\gamma}{k}, 1\right), \left(\frac{1}{2} + \vartheta + u\sigma + \rho, 2u\right), \left(\frac{1}{2} - \vartheta + u\sigma + \rho, 2u\right); \\ \left(\frac{\sigma}{k} + \frac{b+1}{2k}, \frac{\lambda}{k}\right), (1 - \eta + u\sigma + \rho, 2u), \left(\frac{\delta}{k}, 1\right); \end{matrix} \right] \frac{cw^2}{4p^{2u} k^{\frac{\lambda}{k}-1}}. \end{aligned} \quad (2.11)$$

Proof. Let R_1 denote the L.H.S. of (2.11). By setting $pt = v$ in the L.H.S. of (2.11), interchanging the integration and summation of (2.11) and using (2.1), then we obtain

$$R_1 = \frac{1}{p^{\rho+u\sigma}} \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_{n,k} \left(\frac{w}{2}\right)^{\sigma+2n}}{p^{2un} \Gamma_k\left(\lambda n + \sigma + \frac{b+1}{2}\right) (\delta)_{n,k} n!} \int_0^\infty v^{u\sigma+2un+\rho-1} e^{-\frac{v}{2}} W_{\eta,\vartheta}(v) dv.$$

Using (1.7) in the aforementioned expression, we obtain

$$\begin{aligned} R_1 &= \frac{1}{p^{\rho+u\sigma}} \sum_{n=0}^{\infty} \frac{(c)^n \Gamma_k(\gamma + nk) \Gamma_k(\delta) \left(\frac{w}{2}\right)^{\sigma+2n}}{\Gamma_k(\gamma) p^{2un} \Gamma_k\left(\lambda n + \sigma + \frac{b+1}{2}\right) \Gamma_k(\delta + nk) n!} \\ &\quad \times \frac{\Gamma\left(\frac{1}{2} + \vartheta + u\sigma + 2un + \rho\right) \Gamma\left(\frac{1}{2} - \vartheta + u\sigma + 2un + \rho\right)}{\Gamma(1 - \eta + u\sigma + 2un + \rho)}. \end{aligned}$$

Using the same technique as previous theories' proof by applying Relations (1.3) and (1.4), we obtain the desired result (2.9) in light of Definition (1.12). $\square$

If we set $k = 1$ in Theorem 2.4, we obtain the following result:

Corollary 2.7. *If $b, c, \lambda, \gamma, \mu, \eta, \delta \in \mathbb{C}; \Re(\rho \pm \vartheta) > -\frac{1}{2}, \Re(\mu) > -\frac{1}{2}; \Re\left(\frac{b+1}{2} + \nu\right) > 0; \min\{\Re(u), \Re(v), \Re(\lambda)\} > 0$; and $|\frac{w}{p^u}| < 1$, then we have*

$$\begin{aligned}
& \int_0^\infty t^{\rho-1} e^{-\frac{pt}{2}} W_{\eta,\vartheta}(\rho t) J_{1,\sigma,\delta}^{b,c,\gamma,\lambda}(wt^u) dt \\
&= \frac{\left(\frac{w}{2}\right)^\sigma \Gamma(\delta)}{p^{\rho+u\sigma} \Gamma(\gamma)} {}_3\Psi_3 \left[\begin{matrix} (\gamma, 1), \left(\frac{1}{2} + \vartheta + u\sigma + \rho, 2u\right), \left(\frac{1}{2} - \vartheta + u\sigma + \rho, 2u\right) \\ \left(\sigma + \frac{b+1}{2}, \lambda\right), (1 - \eta + \rho + \sigma u, 2u)(\delta, 1) \end{matrix} ; \frac{cw^2}{4p^{2u}} \right].
\end{aligned} \quad (2.12)$$

Now, we list another theorem that can be proved in steps similar to the proof of Theorem 2.4, using the integral transform given in equation (1.8).

Theorem 2.5. If $k \in \Re^+$, $b, c, \lambda, \gamma, \eta, \delta \in \mathbb{C}$; $\Re(m \pm \beta) > -\frac{1}{2}$, $\Re(m) > -\frac{1}{2}$; $\Re\left(\frac{b+1}{2k} + \frac{\gamma}{k}\right) > 0$; $\min\{\Re(u), \Re(\eta), \Re(v), \Re(\lambda)\} > 0$; and $|\frac{w}{p^u}| < 1$, then we have

$$\begin{aligned}
& \int_0^\infty t^{\beta-1} e^{-\frac{pt}{2}} M_{\eta,m}(pt) J_{k,v,\delta}^{b,c,\gamma,\lambda}(wt^u) dt \\
&= \frac{\left(\frac{w}{2}\right)^\gamma k^{1-\frac{v}{k}-\frac{b+1}{2k}} \Gamma(2m+1) \Gamma\left(\frac{\delta}{k}\right)}{p^{\beta+uv} \Gamma\left(\frac{\gamma}{k}\right) \Gamma\left(\frac{1}{2} + m + \eta\right)} \\
&\quad \times {}_3\Psi_3 \left[\begin{matrix} \left(\frac{\gamma}{k}, 1\right), \left(\frac{1}{2} + \beta + uv + m, 2u\right), (\mu - \beta - vu, -2u); \\ \left(\frac{v}{k} + \frac{b+1}{2k}, \frac{\lambda}{k}\right), (m - \beta - vu + 1/2, -2u), \left(\frac{\delta}{k}, 1\right) \end{matrix} ; \frac{cw^2}{4p^{2u} k^{\frac{\lambda}{k}-1}} \right].
\end{aligned} \quad (2.13)$$

By setting $k = 1$ in Theorem 2.5, we have the following relationship:

Corollary 2.8. If $b, c, \lambda, \gamma, \eta, \delta \in \mathbb{C}$; $\Re(m \pm \beta) > -\frac{1}{2}$, $\Re(m) > -\frac{1}{2}$; $\Re\left(\frac{b+1}{2} + \gamma\right) > 0$; $\min\{\Re(u), \Re(\eta), \Re(v), \Re(\lambda)\} > 0$; and $|\frac{w}{p^u}| < 1$, then we have

$$\begin{aligned}
& \int_0^\infty t^{\beta-1} e^{-\frac{pt}{2}} M_{\eta,m}(pt) J_{1,v,\delta}^{b,c,\gamma,\lambda}(wt^u) dt = \frac{\left(\frac{w}{2}\right)^\gamma \Gamma(2m+1) \Gamma(\delta)}{p^{\beta+uv} \Gamma(\gamma) \Gamma\left(\frac{1}{2} + m + \eta\right)} \\
&\quad \times {}_3\Psi_3 \left[\begin{matrix} (\gamma, 1), \left(\frac{1}{2} + \beta + uv + m, 2u\right), (\eta - \beta - vu, -2u); \\ \left(v + \frac{b+1}{2}, \lambda\right), (m - \beta - vu + 1/2, -2u), (\delta, 1) \end{matrix} ; \frac{cw^2}{4p^{2u}} \right].
\end{aligned} \quad (2.14)$$

Remark 2.1. Note that $J_{k,\vartheta,1}^{1,1,\gamma,\lambda}(z)$ denotes the k -Bessel function and $J_{k,\vartheta,1}^{1,-1,\gamma,\lambda}(z)$ denotes the modified k -Bessel function given by [16], (see also [27]). Similarly, for $J_{k,\vartheta,1}^{b,c,\gamma,\lambda}(z)$, the results presented in this study can be in fact reduced to the well-known results of Nisar et al. [28].

3 Conclusion

In this research work, we have established new integral transformations associated with a generalized k -Bessel function. We have also pointed out several integral transforms, such as Laplace transform, Euler transform, Whittaker transform, and k -transform. Special cases of our obtained results are presented. Due to the unified nature of the generalized k -Bessel function with the general class of polynomials, our main results

can also be used to derive a number of new integrals involving different types of integral and fractional special functions, some other integral transformations involving various (generalized) k -Bessel functions such as the relationships between the first-kind k -Bessel function, the k -Wright function, the k -Mittag-Leffler function, and also fractional kinetic equations involving generalized k -Bessel function via the Sumudu transform, which will be discussed in a forthcoming article.

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