Research Article

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Generalized Stević-Sharma operators from the minimal Möbius invariant space into Bloch-type spaces

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Abstract: The aim of this study is to investigate the boundedness, essential norm, and compactness of generalized Stević-Sharma operator from the minimal Möbius invariant space into Bloch-type space.

Keywords: generalized Stević-Sharma operator, minimal Möbius invariant space, Bloch-type space, essential norm

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1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{N}$ the set of positive integers. Denote by $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ the family of all analytic self-maps of $\mathbb{D}$.

The set of all conformal automorphisms of $\mathbb{D}$ forms a group, called the Möbius group, and is denoted by $\text{Aut}(\mathbb{D})$. It is well known from complex analysis that every element of $\text{Aut}(\mathbb{D})$ has the form $e^{i\theta}\sigma_w(z)$, where θ is a real number and

$$\sigma_w(z) = \frac{w - z}{1 - \overline{w}z}, \quad w \in \mathbb{D},$$

is a special automorphism of $\mathbb{D}$ exchanging the points w and 0. Let X be a linear space of analytic functions on $\mathbb{D}$. Then, X is said to be Möbius invariant if for all $f \in X$ and $\nu \in \text{Aut}(\mathbb{D})$, $f \circ \nu \in X$ and satisfies that $\|f \circ \nu\|_X = \|f\|_X$ (see [1]). A typical example of Möbius invariant space is the analytic Besov space B_p. Recall that for $1 < p < \infty$, a function $f \in H(\mathbb{D})$ belongs to B_p if

$$\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} \mathrm{d}A(z) < \infty,$$

where $\mathrm{d}A$ is the normalized Lebesgue area measure on $\mathbb{D}$. Note that when $p = 2$, B_2 is known as the Dirichlet space, which is the only Möbius invariant Hilbert space (see [2]).

The analytic Besov space B_1 consists of all $f \in H(\mathbb{D})$, which have a representation as:

$$f(z) = \sum_{n=1}^{\infty} a_n \sigma_{\lambda_n}(z),$$

for some sequences $\{a_n\}_{n\in\mathbb{N}} \in l^1$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ in $\mathbb{D}$. The norm in B_1 is defined by:

$$\|f\|_{B_1} = \inf\left\{\sum_{n=1}^{\infty} |a_n| : f(z) = \sum_{n=1}^{\infty} a_n \sigma_{\lambda_n}(z)\right\}.$$

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By [1], we know that the space B_1 is the minimal Möbius invariant space, as it is contained in any Möbius invariant space. Furthermore, B_1 is identical with the set of $f \in H(\mathbb{D})$ for which $f'' \in L^1(\mathbb{D}, dA)$, and there exist constants C_1 and C_2 such that

$$C_1\|f\|_{B_1} \le |f(0)| + |f'(0)| + \int_{\mathbb{D}} |f''(z)| dA(z) \le C_2\|f\|_{B_1}.$$

For more studies of B_1 space, see also [3–8].

Suppose that μ is a weight, namely, a strictly positive continuous function on $\mathbb{D}$. We also assume that μ is radial: $\mu(z) = \mu(|z|)$ for any $z \in \mathbb{D}$. An $f \in H(\mathbb{D})$ is said to belong to the Bloch-type space $\mathcal{B}_\mu$, if

$$\sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty.$$

$\mathcal{B}_\mu$ is a Banach space under the norm $\|f\|_{\mathcal{B}_\mu} = |f(0)| + \sup_{z\in\mathbb{D}}\mu(z)|f'(z)|$. When $\mu(z) = 1 - |z|^2$, the induced space $\mathcal{B}_\mu$ reduces to the classical Bloch space, which is the maximal Möbius invariant space [9]. For some results on the Bloch-type spaces and operators on them, see, for instance, [4,10–14].

Suppose that $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$, the composition and multiplication operators on $H(\mathbb{D})$ are defined, respectively, by:

$$C_\varphi f(z) = f(\varphi(z)) \quad \text{and} \quad M_u f(z) = u(z)f(z),$$

where $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. The product of these two operators is known as the weighted composition operator $W_{u,\varphi} = u(z)f(\varphi(z))$. It is important to provide function theoretic characterizations when φ and u induce a bounded or compact weighted composition operator on various function spaces. See [7,15] for more research about the (weighted) composition operators acting on several spaces of analytic functions. The differentiation operator D, which is defined by $Df(z) = f'(z)$ for $f \in H(\mathbb{D})$, plays an important role in operator theory and dynamical system.

The first papers on product-type operators including the differentiation operator dealt with the operators DC_φ and $C_\varphi D$ (see, for example, [11,16–19]). In [20,21], Stević and co-workers introduced the so-called Stević-Sharma operator as follows:

$$T_{u,v,\varphi}f(z) = u(z)f(\varphi(z)) + v(z)f'(\varphi(z)), \quad f \in H(\mathbb{D}),$$

where $u, v \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. By taking some specific choices of the involving symbols, we can easily obtain the general product-type operators:

$$\begin{gathered} M_uC_\varphi = T_{u,0,\varphi}, \quad C_\varphi M_u = T_{u\circ\varphi,0,\varphi}, \quad M_uD = T_{0,u,id}, \quad DM_u = T_{u',u,id}, \quad C_\varphi D = T_{0,1,\varphi}, \\ DC_\varphi = T_{0,\varphi',\varphi}, \quad M_uC_\varphi D = T_{0,u,\varphi}, \quad M_uDC_\varphi = T_{0,u\varphi',\varphi}, \quad C_\varphi M_uD = T_{0,u\circ\varphi,\varphi}, \\ DM_uC_\varphi = T_{u',u\varphi',\varphi}, \quad C_\varphi DM_u = T_{u'\circ\varphi,u\circ\varphi,\varphi}, \quad DC_\varphi M_u = T_{\varphi'(u'\circ\varphi),\varphi'(u\circ\varphi),\varphi}. \end{gathered}$$

Recently, there has been an increasing interest in studying the Stević-Sharma operator between various spaces of analytic function. For instance, the boundedness, compactness, and essential norm of $T_{u,v,\varphi}$ on the weighted Bergman space were characterized by Stević et al. in [20,21]. Wang et al. in [22] considered the difference of two Stević-Sharma operators and investigated its boundedness, compactness, and order boundedness between Banach spaces of analytic functions. Zhu et al. in [14] provided some necessary and sufficient conditions for $T_{u,v,\varphi}$ to be bounded or compact when considered as an operator from the analytic Besov space B_p into Bloch space. Abbasi et al. in [23] generalized the Stević-Sharma operator as follows:

$$T^m_{u,v,\varphi}f(z) = u(z)f(\varphi(z)) + v(z)f^{(m)}(\varphi(z)), \quad m \in \mathbb{N},$$

and studied its boundedness, compactness, and essential norm from Hardy space into the nth weighted-type space, which was introduced by Stević in [24] (see also [25]). Note that when $m = 1$, we obtain the Stević-Sharma operator $T_{u,v,\varphi}$. Some more related results can be found (see, e.g., [4, 5,8,10–14,26–32] and references therein).

Motivated by the aforementioned studies, here we investigate the boundedness and essential norm of the generalized Stević-Sharma operator $T^m_{u,v,\varphi}$ from the minimal Möbius invariant space B_1 into the Bloch-type space $\mathcal{B}_\mu$. As a corollary, we give the characterizations of its compactness.

Recall that the essential norm of a bounded linear operator $T : X \to Y$ is the distance from T to the compact operators $K : X \to Y$, that is,

$$||T||_{e,X\to Y} = \inf\{||T - K||_{X\to Y} : K \text{ is compact}\},$$

where X and Y are the Banach spaces. Note that $||T||_{e,X\to Y} = 0$ if and only if $T : X \to Y$ is compact.

Throughout this article, for nonnegative quantities X and Y, we use the abbreviation $X \lesssim Y$ or $Y \gtrsim X$ if there exists a positive constant C independent of X and Y such that $X \le CY$. Moreover, we write $X \approx Y$ if $X \lesssim Y \lesssim X$.

2 Auxiliary results

In this section, we state several auxiliary results that are needed in the proofs of our main results. The following lemma can be found, for example, in [8] (see also [33]).

Lemma 1. *Let $k \in \mathbb{N}$, then*

$$||f||_\infty \lesssim ||f||_{B_1} \quad \text{and} \quad (1 - |z|^2)^k |f^{(k)}(z)| \lesssim ||f||_{B_1}$$

for each $f \in B_1$.

For any $w \in \mathbb{D}$ and $j \in \mathbb{N}$, set

$$f_{j,w}(z) = \frac{(1 - |w|^2)^j}{(1 - \overline{w}z)^j}, \quad z \in \mathbb{D}. \tag{1}$$

It is easily seen that $f_{j,w} \in B_1$ and $\sup_{w\in\mathbb{D}} ||f_{j,w}||_{B_1} \lesssim 1$ for each $j \in \mathbb{N}$. Moreover, $f_{j,w}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $|w| \to 1$.

Lemma 2. *Let $m \in \mathbb{N}$ and $m > 1$. For any $w \in \mathbb{D}\backslash\{0\}$ and $i, k \in \{0, 1, m, m + 1\}$, there exists a function $g_{i,w} \in B_1$ such that*

$$g_{i,w}^{(k)}(w) = \frac{\overline{w}^k \delta_{ik}}{(1 - |w|^2)^k},$$

where δ_{ik} is the Kronecker delta.

Proof. For any $w \in \mathbb{D}\backslash\{0\}$ and constants c_1, c_2, c_3, and c_4, let

$$g_w(z) = \sum_{j=1}^{4} c_j f_{j,w}(z),$$

where $f_{j,w}$ is defined in (1). For each $i \in \{0, 1, m, m + 1\}$, the system of linear equations

$$\begin{cases} g_w(w) = c_1 + c_2 + c_3 + c_4 = \delta_{i0}, \\ g_w'(w) = (c_1 + 2c_2 + 3c_3 + 4c_4)\dfrac{\overline{w}}{1 - |w|^2} = \dfrac{\overline{w}\delta_{i1}}{1 - |w|^2}, \\ g_w^{(m)}(w) = \left(m!c_1 + (m + 1)!c_2 + \dfrac{(m + 2)!}{2}c_3 + \dfrac{(m + 3)!}{6}c_4\right)\dfrac{\overline{w}^m}{(1 - |w|^2)^m} = \dfrac{\overline{w}^m \delta_{im}}{(1 - |w|^2)^m}, \\ g_w^{(m+1)}(w) = \left((m + 1)!c_1 + (m + 2)!c_2 + \dfrac{(m + 3)!}{2}c_3 + \dfrac{(m + 4)!}{6}c_4\right)\dfrac{\overline{w}^{m+1}}{(1 - |w|^2)^{m+1}} = \dfrac{\overline{w}^{m+1}\delta_{i(m+1)}}{(1 - |w|^2)^{m+1}}, \end{cases}$$

has a unique solution c_1^i, c_2^i, c_3^i, and c_4^i, which is independent of w, since the determinant of the system

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ m! & (m+1)! & \frac{(m+2)!}{2} & \frac{(m+3)!}{6} \\ (m+1)! & (m+2)! & \frac{(m+3)!}{2} & \frac{(m+4)!}{6} \end{vmatrix} = \frac{1}{12}m!(m+1)!m^2(m-1)(m+1) \neq 0.$$

For such c_j^i, $j \in \{1, 2, 3, 4\}$, the function

$$g_{i,w}(z) \coloneqq \sum_{j=1}^{4} c_j^i f_{j,w}(z)$$

satisfies the desired result. □

By a similar argument, we can obtain the following lemma.

Lemma 3. *For any $w \in \mathbb{D}\backslash\{0\}$ and $i, k \in \{0, 1, 2\}$, there exists a function $h_{i,w} \in B_1$ such that*

$$h_{i,w}^{(k)}(z) = \frac{\overline{w}^k \delta_{ik}}{(1-|w|^2)^k},$$

where δ_{ik} is the Kronecker delta.

In order to estimate the essential norm of $T_{u,v,\varphi}^m : B_1 \to \mathcal{B}_\mu$, we need the following two lemmas. The first one characterizes the compactness in terms of sequential convergence, whose proof is similar to that of [15, Proposition 3.11], so we omit the details.

Lemma 4. *Let $m \in \mathbb{N}$, $u, v \in H(\mathbb{D})$, and $\varphi \in S(\mathbb{D})$. Then, the operator $T_{u,v,\varphi}^m : B_1 \to \mathcal{B}_\mu$ is compact if and only if for each bounded sequence, $\{f_n\}_{n\in\mathbb{N}}$ in B_1 converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, we have $\|T_{u,v,\varphi}^m f_n\|_{\mathcal{B}_\mu} \to 0$ as $n \to \infty$.*

Lemma 5. [8] *Every bounded sequence in B_1 has a subsequence that converges uniformly in $\overline{\mathbb{D}}$ to a function in B_1.*

3 Main results

In this section, we formulate our main results. For simplicity of the expressions, we write

$$A_1(z) = |u(z)\varphi'(z)|,$$
$$A_m(z) = |v'(z)|,$$
$$A_{m+1}(z) = |v(z)\varphi'(z)|.$$

We first give several characterizations of the generalized Stević-Sharma operator $T_{u,v,\varphi}^m : B_1 \to \mathcal{B}_\mu$ to be bounded.

Theorem 1. *Let $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m \in \mathbb{N}$, $m > 1$, and μ be a radial weight. Then, the following statements are equivalent.*

(*i*) *The operator $T_{u,v,\varphi}^m : B_1 \to \mathcal{B}_\mu$ is bounded.*

(*ii*) *$u \in \mathcal{B}_\mu$,*

$$\sum_{j=1}^{4} \sup_{w\in\mathbb{D}} \|T_{u,v,\varphi}^m f_{j,w}\|_{\mathcal{B}_\mu} < \infty,$$

and

$$\sum_{i\in\{1,m,m+1\}} \sup_{z\in\mathbb{D}} \mu(z)A_i(z) < \infty,$$

where $f_{j,w}$ *are defined in* (1).

(*iii*) $u \in \mathcal{B}_\mu$, *and*

$$\sum_{i\in\{1,m,m+1\}} \sup_{z\in\mathbb{D}} \frac{\mu(z)A_i(z)}{(1-|\varphi(z)|^2)^i} < \infty.$$

Proof. (i) ⇒ (ii). Suppose that $T^m_{u,v,\varphi} : B_1 \to \mathcal{B}_\mu$ is bounded. Taking $f_0(z) = 1 \in B_1$ we obtain, $T^m_{u,v,\varphi} f_0 = u \in \mathcal{B}_\mu$, that is,

$$\sup_{z\in\mathbb{D}} \mu(z)|u'(z)| < \infty. \tag{2}$$

For each $w \in \mathbb{D}$ and $j \in \{1,2,3,4\}$, $\|f_{j,w}\|_{B_1} \lesssim 1$ and hence by the boundedness of $T^m_{u,v,\varphi}$ we have $\|T^m_{u,v,\varphi} f_{j,w}\|_{\mathcal{B}_\mu} < \infty$. Therefore,

$$\sum_{j=1}^{4} \sup_{w\in\mathbb{D}} \|T^m_{u,v,\varphi} f_{j,w}\|_{\mathcal{B}_\mu} < \infty.$$

Taking $f_1(z) = z \in B_1$ and using the boundedness of $T^m_{u,v,\varphi} : B_1 \to \mathcal{B}_\mu$, we obtain

$$\begin{aligned}\infty > \|T^m_{u,v,\varphi} f_1\|_{\mathcal{B}_\mu} &\geq \sup_{z\in\mathbb{D}} \mu(z)|(T^m_{u,v,\varphi} f_1)'(z)|\\ &= \sup_{z\in\mathbb{D}} \mu(z)|u'(z)\varphi(z) + u(z)\varphi'(z)|\\ &\geq \sup_{z\in\mathbb{D}} \mu(z)|u(z)\varphi'(z)| - \sup_{z\in\mathbb{D}} \mu(z)|u'(z)\varphi(z)|,\end{aligned}$$

which along with (2) and the fact that $|\varphi(z)| < 1$, it follows that

$$\sup_{z\in\mathbb{D}} \mu(z)|u(z)\varphi'(z)| \leq \|T^m_{u,v,\varphi} f_1\|_{\mathcal{B}_\mu} + \sup_{z\in\mathbb{D}} \mu(z)|u'(z)| < \infty. \tag{3}$$

Applying the operator $T^m_{u,v,\varphi}$ for $f_m(z) = z^m \in B_1$ yields

$$\infty > \|T^m_{u,v,\varphi} f_m\|_{\mathcal{B}_\mu} \geq \sup_{z\in\mathbb{D}} \mu(z)|(T^m_{u,v,\varphi} f_m)'(z)| = \sup_{z\in\mathbb{D}} \mu(z)|u'(z)\varphi(z)^m + mu(z)\varphi'(z)\varphi(z)^{m-1} + m!v'(z)|.$$

Using (2), (3), the fact that $|\varphi(z)| < 1$, and the triangle inequality, we obtain

$$\sup_{z\in\mathbb{D}} \mu(z)|v'(z)| < \infty. \tag{4}$$

By choosing $f_{m+1}(z) = z^{m+1} \in B_1$, we conclude that

$$\begin{aligned}\infty > \|T^m_{u,v,\varphi} f_{m+1}\|_{\mathcal{B}_\mu} &\geq \sup_{z\in\mathbb{D}} \mu(z)|(T^m_{u,v,\varphi} f_{m+1})'(z)|\\ &= \sup_{z\in\mathbb{D}} \mu(z)|u'(z)\varphi(z)^{m+1} + (m+1)u(z)\varphi'(z)\varphi(z)^m + (m+1)!v'(z)\varphi(z) + (m+1)!v(z)\varphi'(z)|.\end{aligned}$$

By using (2), (3), and (4), in the same manner, we obtain

$$\sup_{z\in\mathbb{D}} \mu(z)|v(z)\varphi'(z)| < \infty. \tag{5}$$

Combining (3), (4), and (5), we deduce that

$$\sum_{i\in\{1,m,m+1\}} \sup_{z\in\mathbb{D}} \mu(z)A_i(z) < \infty.$$

(ii) ⇒ (iii). Assume that (ii) holds. By Lemma 2, for each $i \in \{1, m, m+1\}$ and $\varphi(w) \neq 0$, there exist constants c^i_1, c^i_2, c^i_3, and c^i_4 such that

$$g_{i,\varphi(w)}(z) = \sum_{j=1}^{4} c_j^i f_{j,\varphi(w)}(z) \in B_1, \tag{6}$$

and

$$g_{i,\varphi(w)}^{(k)}(w) = \frac{\overline{\varphi(w)}^i \delta_{ik}}{(1 - |\varphi(w)|^2)^k},$$

where $f_{j,w}$ are defined in (1) and $k \in \{0, 1, m, m+1\}$. Then,

$$\begin{aligned} \infty > \sum_{j=1}^{4} \sup_{w \in \mathbb{D}} \|T_{u,v,\varphi}^m f_{j,\varphi(w)}\|_{\mathcal{B}_\mu} &\gtrsim \sup_{w \in \mathbb{D}} \|T_{u,v,\varphi}^m g_{i,\varphi(w)}\|_{\mathcal{B}_\mu} \\ &\geq \mu(w)|(T_{u,v,\varphi}^m g_{i,\varphi(w)})'(w)| = \frac{\mu(w)A_i(w)|\varphi(w)|^i}{(1 - |\varphi(w)|^2)^i}. \end{aligned} \tag{7}$$

From (7) and (ii), for each $i \in \{1, m, m+1\}$, we have

$$\sup_{|\varphi(w)| > \frac{1}{2}} \frac{\mu(w)A_i(w)}{(1 - |\varphi(w)|^2)^i} < \infty$$

and

$$\sup_{|\varphi(w)| \leq \frac{1}{2}} \frac{\mu(w)A_i(w)}{(1 - |\varphi(w)|^2)^i} \lesssim \sup_{w \in \mathbb{D}} \mu(w)A_i(w) < \infty.$$

Therefore,

$$\sum_{i \in \{1, m, m+1\}} \sup_{z \in \mathbb{D}} \frac{\mu(z)A_i(z)}{(1 - |\varphi(z)|^2)^i} < \infty.$$

(iii) ⇒ (i). Suppose that (iii) holds. For any $f \in B_1$, by Lemma 1, we have

$$\begin{aligned} \mu(z)|(T_{u,v,\varphi}^m f)'(z)| &\leq \mu(z)|u'(z)||f(\varphi(z))| + \sum_{i \in \{1, m, m+1\}} \mu(z)A_i(z)|f^{(i)}(\varphi(z))| \\ &\lesssim \left(\|u\|_{\mathcal{B}_\mu} + \sum_{i \in \{1, m, m+1\}} \frac{\mu(z)A_i(z)}{(1 - |\varphi(z)|^2)^i} \right) \|f\|_{B_1}. \end{aligned}$$

Moreover,

$$|(T_{u,v,\varphi}^m f)(0)| = |u(0)f(\varphi(0)) + v(0)f^m(\varphi(0))| \lesssim \left(|u(0)| + \frac{|v(0)|}{(1 - |\varphi(0)|^2)^m} \right) \|f\|_{B_1}.$$

Thus, $T_{u,v,\varphi}^m : B_1 \to \mathcal{B}_\mu$ is bounded. The proof is completed. □

By using Lemma 3 instead of Lemma 2, the following result may be proved in much the same way as Theorem 1.

Theorem 2. *Let $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, and μ be a radial weight. Then, the following statements are equivalent.*

(*i*) *The operator $T_{u,v,\varphi} : B_1 \to \mathcal{B}_\mu$ is bounded.*

(*ii*) *$u \in \mathcal{B}_\mu$,*

$$\sum_{j=1}^{3} \sup_{w \in \mathbb{D}} \|T_{u,v,\varphi} f_{j,w}\|_{\mathcal{B}_\mu} < \infty,$$

and

$$\sup_{z\in\mathbb{D}}\mu(z)|u(z)\varphi'(z)+v'(z)|+\sup_{z\in\mathbb{D}}\mu(z)|v(z)\varphi'(z)|<\infty.$$

(iii) $u\in\mathcal{B}_\mu$, *and*

$$\sup_{z\in\mathbb{D}}\frac{\mu(z)|u(z)\varphi'(z)+v'(z)|}{1-|\varphi(z)|^2}+\sup_{z\in\mathbb{D}}\frac{\mu(z)|v(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2}<\infty.$$

Now, we estimate the essential norm of $T^m_{u,v,\varphi}$ acting from the minimal Möbius invariant space to the Bloch-type space. Then, we obtain some equivalence conditions for compactness of $T^m_{u,v,\varphi}$.

Theorem 3. *Let $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, $m\in\mathbb{N}$, $m>1$, and μ be a radial weight such that $T^m_{u,v,\varphi}:B_1\to\mathcal{B}_\mu$ is bounded. Then,*

$$\|T^m_{u,v,\varphi}\|_{e,B_1\to\mathcal{B}_\mu}\approx\sum_{j=1}^{4}\limsup_{|w|\to1}\|T^m_{u,v,\varphi}f_{j,w}\|_{\mathcal{B}_\mu}\approx\sum_{i\in\{1,m,m+1\}}\limsup_{|\varphi(z)|\to1}\frac{\mu(z)A_i(z)}{(1-|\varphi(z)|^2)^i},$$

where $f_{j,w}$ are defined in (1).

Proof. We first show that

$$\|T^m_{u,v,\varphi}\|_{e,B_1\to\mathcal{B}_\mu}\gtrsim\sum_{j=1}^{4}\limsup_{|w|\to1}\|T^m_{u,v,\varphi}f_{j,w}\|_{\mathcal{B}_\mu}.$$

It is obvious that for each $j\in\{1,2,3,4\}$ and $w\in\mathbb{D}$, $\|f_{j,w}\|_{B_1}\lesssim1$. Moreover, $f_{j,w}$ converge to zero uniformly on compact subsets of $\mathbb{D}$. For any compact operator K from B_1 into $\mathcal{B}_\mu$, by using some standard arguments (see, e.g., [34,35]), we obtain

$$\lim_{|w|\to1}\|Kf_{j,w}\|_{\mathcal{B}_\mu}=0.$$

It follows that

$$\begin{aligned}\|T^m_{u,v,\varphi}-K\|_{B_1\to\mathcal{B}_\mu}&\gtrsim\limsup_{|w|\to1}\|(T^m_{u,v,\varphi}-K)f_{j,w}\|_{\mathcal{B}_\mu}\\&\geq\limsup_{|w|\to1}\|T^m_{u,v,\varphi}f_{j,w}\|_{\mathcal{B}_\mu}-\limsup_{|w|\to1}\|Kf_{j,w}\|_{\mathcal{B}_\mu}.\end{aligned}$$

Therefore,

$$\|T^m_{u,v,\varphi}\|_{e,B_1\to\mathcal{B}_\mu}=\inf_K\|T^m_{u,v,\varphi}-K\|_{B_1\to\mathcal{B}_\mu}\gtrsim\sum_{j=1}^{4}\limsup_{|w|\to1}\|T^m_{u,v,\varphi}f_{j,w}\|_{\mathcal{B}_\mu}.\tag{8}$$

Next, we prove that

$$\|T^m_{u,v,\varphi}\|_{e,B_1\to\mathcal{B}_\mu}\gtrsim\sum_{i\in\{1,m,m+1\}}\limsup_{|\varphi(z)|\to1}\frac{\mu(z)A_i(z)}{(1-|\varphi(z)|^2)^i}.$$

Let $\{z_j\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)|\to1$ as $j\to\infty$. Since $T^m_{u,v,\varphi}:B_1\to\mathcal{B}_\mu$ is bounded, for any compact operator $K:B_1\to\mathcal{B}_\mu$ and $i\in\{1,m,m+1\}$, applying Lemma 4 and (7), we obtain

$$\begin{aligned}\|T^m_{u,v,\varphi}-K\|_{B_1\to\mathcal{B}_\mu}&\gtrsim\limsup_{j\to\infty}\|T^m_{u,v,\varphi}g_{i,\varphi(z_j)}\|_{\mathcal{B}_\mu}-\limsup_{j\to\infty}\|Kg_{i,\varphi(z_j)}\|_{\mathcal{B}_\mu}\\&\gtrsim\limsup_{j\to\infty}\frac{\mu(z_j)A_i(z_j)|\varphi(z_j)|^i}{(1-|\varphi(z_j)|^2)^i},\end{aligned}$$

where $g_{i,\varphi(z_j)}$ are defined in (6). Therefore,

$$\|T^m_{u,v,\varphi}\|_{e,B_1\to\mathcal{B}_\mu}\gtrsim\limsup_{j\to\infty}\frac{\mu(z_j)A_i(z_j)|\varphi(z_j)|^i}{(1-|\varphi(z_j)|^2)^i}=\limsup_{|\varphi(z)|\to1}\frac{\mu(z)A_i(z)}{(1-|\varphi(z)|^2)^i},$$

from which we have

$$\|T^m_{u,v,\varphi}\|_{e,B_1\to\mathcal{B}_\mu} \gtrsim \sum_{i\in\{1,m,m+1\}} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)A_i(z)}{(1-|\varphi(z)|^2)^i}. \tag{9}$$

Combining (8) and (9) yields

$$\|T^m_{u,v,\varphi}\|_{e,B_1\to\mathcal{B}_\mu} \gtrsim \min\left\{\sum_{j=1}^4 \limsup_{|w|\to 1}\|T^m_{u,v,\varphi}f_{j,w}\|_{\mathcal{B}_\mu}, \sum_{i\in\{1,m,m+1\}} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)A_i(z)}{(1-|\varphi(z)|^2)^i}\right\}.$$

It is sufficient to show that

$$\|T^m_{u,v,\varphi}\|_{e,B_1\to\mathcal{B}_\mu} \lesssim \min\left\{\sum_{j=1}^4 \limsup_{|w|\to 1}\|T^m_{u,v,\varphi}f_{j,w}\|_{\mathcal{B}_\mu}, \sum_{i\in\{1,m,m+1\}} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)A_i(z)}{(1-|\varphi(z)|^2)^i}\right\}.$$

Define $K_r f(z) = f_r(z) = f(rz)$, where $0 \le r < 1$. Then, $K_r : B_1 \to B_1$ is a compact operator with $\|K_r\| \le 1$ and $f_r \to f$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$ clearly. Let $\{r_j\} \subset (0,1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Then, for each $j \in \mathbb{N}$, $T^m_{u,v,\varphi}K_{r_j} : B_1 \to \mathcal{B}_\mu$ is compact, and so

$$\|T^m_{u,v,\varphi}\|_{e,B_1\to\mathcal{B}_\mu} \le \limsup_{j\to\infty}\|T^m_{u,v,\varphi} - T^m_{u,v,\varphi}K_{r_j}\|_{B_1\to\mathcal{B}_\mu}.$$

Therefore, we only need to show that

$$\begin{aligned}&\limsup_{j\to\infty}\|T^m_{u,v,\varphi} - T^m_{u,v,\varphi}K_{r_j}\|_{B_1\to\mathcal{B}_\mu}\\ &\quad\lesssim \min\left\{\sum_{j=1}^4 \limsup_{|w|\to 1}\|T^m_{u,v,\varphi}f_{j,w}\|_{\mathcal{B}_\mu}, \sum_{i\in\{1,m,m+1\}} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)A_i(z)}{(1-|\varphi(z)|^2)^i}\right\}.\end{aligned} \tag{10}$$

For every $f \in B_1$ such that $\|f\|_{B_1} \le 1$, we have

$$\begin{aligned}\|(T^m_{u,v,\varphi} - T^m_{u,v,\varphi}K_{r_j})f\|_{\mathcal{B}_\mu} &= |T^m_{u,v,\varphi}f(0) - T^m_{u,v,\varphi}f_{r_j}(0)| + \sup_{z\in\mathbb{D}}\mu(z)|(T^m_{u,v,\varphi}f - T^m_{u,v,\varphi}f_{r_j})'(z)|\\ &\le \underbrace{|(f-f_{r_j})(\varphi(0))u(0)| + |(f-f_{r_j})^{(m)}(\varphi(0))v(0)|}_{E_0} + \underbrace{\sup_{z\in\mathbb{D}}\mu(z)|(f-f_{r_j})(\varphi(z))u'(z)|}_{E_1}\\ &\quad+ \underbrace{\sup_{|\varphi(z)|\le r_N}\mu(z)\sum_{i\in\{1,m,m+1\}}|(f-f_{r_j})^{(i)}(\varphi(z))|A_i(z)}_{E_2}\\ &\quad+ \underbrace{\sup_{|\varphi(z)|> r_N}\mu(z)\sum_{i\in\{1,m,m+1\}}|(f-f_{r_j})^{(i)}(\varphi(z))|A_i(z)}_{E_3},\end{aligned} \tag{11}$$

where $N \in \mathbb{N}$ such that $r_j \ge \frac{2}{3}$ for all $j \ge N$. Furthermore, we have $(f - f_{r_j})^{(t)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$ for any nonnegative integer t. Now, Theorem 1 implies

$$\limsup_{j\to\infty} E_0 = \limsup_{j\to\infty} E_2 = 0. \tag{12}$$

From Lemma 5,

$$\lim_{j\to\infty} E_1 \le \|u\|_{\mathcal{B}_\mu}\lim_{j\to\infty}\sup_{z\in\mathbb{D}}|(f-f_{r_j})(z)| = 0. \tag{13}$$

Finally, we estimate E_3.

$$E_3 \le \sum_{i\in\{1,m,m+1\}}\underbrace{\sup_{|\varphi(z)|>r_N}\mu(z)|f^{(i)}(\varphi(z))|A_i(z)}_{F_i} + \sum_{i\in\{1,m,m+1\}}\underbrace{\sup_{|\varphi(z)|>r_N}\mu(z)|r_j^i f^{(i)}(r_j\varphi(z))|A_i(z)}_{G_i}. \tag{14}$$

For each $i \in \{1, m, m+1\}$, using Lemma 1, (6), and (7), we obtain

$$\begin{aligned}F_i &= \sup_{|\varphi(z)|>r_N} \frac{(1-|\varphi(z)|^2)^i|f^{(i)}(\varphi(z))|}{|\varphi(z)|^i}\frac{\mu(z)A_i(z)|\varphi(z)|^i}{(1-|\varphi(z)|^2)^i}\\ &\lesssim \|f\|_{B_1}\sup_{|\varphi(z)|>r_N}\|T^m_{u,v,\varphi}g_{i,\varphi(z)}\|_{\mathcal{B}_\mu}\\ &\lesssim \sum_{j=1}^{4}\sup_{|w|>r_N}\|T^m_{u,v,\varphi}f_{j,w}\|_{\mathcal{B}_\mu},\end{aligned} \tag{15}$$

and

$$F_i = \sup_{|\varphi(z)|>r_N}(1-|\varphi(z)|^2)^i|f^{(i)}(\varphi(z))|\frac{\mu(z)A_i(z)}{(1-|\varphi(z)|^2)^i} \lesssim \|f\|_{B_1}\sup_{|\varphi(z)|>r_N}\frac{\mu(z)A_i(z)}{(1-|\varphi(z)|^2)^i}. \tag{16}$$

Taking the limits as $N\to\infty$ in (15) and (16), we obtain

$$\limsup_{j\to\infty}F_i \lesssim \sum_{j=1}^{4}\limsup_{|w|\to1}\|T^m_{u,v,\varphi}f_{j,w}\|_{\mathcal{B}_\mu} \tag{17}$$

and

$$\limsup_{j\to\infty}F_i \lesssim \limsup_{|\varphi(z)|\to1}\frac{\mu(z)A_i(z)}{(1-|\varphi(z)|^2)^i}. \tag{18}$$

Similarly, we have

$$\limsup_{j\to\infty}G_i \lesssim \sum_{j=1}^{4}\limsup_{|w|\to1}\|T^m_{u,v,\varphi}f_{j,w}\|_{\mathcal{B}_\mu} \quad\text{and}\quad \limsup_{j\to\infty}G_i \lesssim \limsup_{|\varphi(z)|\to1}\frac{\mu(z)A_i(z)}{(1-|\varphi(z)|^2)^i}. \tag{19}$$

Therefore, by (11)–(14) and (17)–(19), we obtain

$$\begin{aligned}\limsup_{j\to\infty}\|T^m_{u,v,\varphi}-T^m_{u,v,\varphi}K_{r_j}\|_{B_1\to\mathcal{B}_\mu} &= \limsup_{j\to\infty}\sup_{\|f\|_{B_1}\le1}\|(T^m_{u,v,\varphi}-T^m_{u,v,\varphi}K_{r_j})f\|_{\mathcal{B}_\mu}\\ &\lesssim \sum_{j=1}^{4}\limsup_{|w|\to1}\|T^m_{u,v,\varphi}f_{j,w}\|_{\mathcal{B}_\mu},\end{aligned}$$

and

$$\limsup_{j\to\infty}\|T^m_{u,v,\varphi}-T^m_{u,v,\varphi}K_{r_j}\|_{B_1\to\mathcal{B}_\mu} \lesssim \sum_{i\in\{1,m,m+1\}}\limsup_{|\varphi(z)|\to1}\frac{\mu(z)A_i(z)}{(1-|\varphi(z)|^2)^i}.$$

From the last two inequalities, we obtain (10) and the proof is completed. □

Corollary 1. *Let $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m \in \mathbb{N}$, $m > 1$, and μ be a radial weight. Suppose that $T^m_{u,v,\varphi}: B_1 \to \mathcal{B}_\mu$ is bounded, then the following statements are equivalent.*

(*i*) *The operator $T^m_{u,v,\varphi}: B_1 \to \mathcal{B}_\mu$ is compact.*

(*ii*)

$$\sum_{j=1}^{4}\limsup_{|w|\to1}\|T^m_{u,v,\varphi}f_{j,w}\|_{\mathcal{B}_\mu} = 0.$$

(*iii*)

$$\sum_{i\in\{1,m,m+1\}}\limsup_{|\varphi(z)|\to1}\frac{\mu(z)A_i(z)}{(1-|\varphi(z)|^2)^i} = 0.$$

By the same method as in the proof of Theorem 3, we can obtain the following results for the case $m = 1$, namely, the Stević-Sharma operator.

Theorem 4. *Let $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, and μ be a radial weight such that $T_{u,v,\varphi} : B_1 \to \mathcal{B}_\mu$ is bounded. Then,*

$$\|T_{u,v,\varphi}\|_{e,B_1\to\mathcal{B}_\mu} \approx \sum_{j=1}^{3} \limsup_{|w|\to 1} \|T_{u,v,\varphi} f_{j,w}\|_{\mathcal{B}_\mu}$$
$$\approx \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)\varphi'(z) + v'(z)|}{1 - |\varphi(z)|^2} + \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|v(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^2}.$$

Corollary 2. *Let $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, and μ be a radial weight. Suppose that $T_{u,v,\varphi} : B_1 \to \mathcal{B}_\mu$ is bounded, then the following statements are equivalent.*

(*i*) *The operator $T_{u,v,\varphi} : B_1 \to \mathcal{B}_\mu$ is compact.*

(*ii*)

$$\sum_{j=1}^{3} \limsup_{|w|\to 1} \|T_{u,v,\varphi} f_{j,w}\|_{\mathcal{B}_\mu} = 0.$$

(*iii*)

$$\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)\varphi'(z) + v'(z)|}{1 - |\varphi(z)|^2} + \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|v(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^2} = 0.$$

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