

Research Article

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Increasing property and logarithmic convexity of functions involving Dirichlet lambda function[#]

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Abstract: In this article, with the help of an integral representation of the Dirichlet lambda function, by means of a monotonicity rule for the ratio of two integrals with a parameter, and by virtue of complete monotonicity and another property of an elementary function involving the exponential function, the authors find increasing property and logarithmic convexity of two functions containing the gamma function and the Dirichlet lambda function.

Keywords: Dirichlet lambda function, increasing property, logarithmic convexity, integral representation, complete monotonicity, monotonicity rule, gamma function, exponential function, zeta function

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1 Motivations and main results

In this article, we use the following notation:

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}, \quad \mathbb{N} = \{1, 2, \dots\}, \quad \mathbb{N}_0 = \{0, 1, 2, \dots\}, \quad \mathbb{N}_- = \{-1, -2, \dots\}.$$

According to [1, Fact 13.3], for complex number $z \in \mathbb{C}$ such that $\Re(z) > 1$, the Riemann zeta function $\zeta(z)$ can be defined by

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z} = \frac{1}{1-2^{-z}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^z} = \frac{1}{1-2^{1-z}} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^z}. \quad (1.1)$$

In [2, Section 3.5], the analytic continuation of the Riemann zeta function $\zeta(z)$ into the punctured complex plane $\mathbb{C} \setminus \{1\}$ is discussed: the only singularity $z = 1$ is a simple pole with residue 1.

Viewing from a subseries of the first definition in (1.1) of the Riemann zeta function $\zeta(z)$, or basing on the second definition in (1.1), one considers the Dirichlet lambda function

$$\lambda(z) = \left(1 - \frac{1}{2^z}\right) \zeta(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^z}, \quad \Re(z) > 1.$$

This lambda function $\lambda(z)$ has an integral representation

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$$\lambda(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^t - e^{-t}} dt, \quad \Re(z) > 1 \quad (1.2)$$

in [3, p. 1046, 9.513.2] and [4, p. 604, 25.5.8], where the classical Euler gamma function $\Gamma(z)$ can be defined by

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{\prod_{k=0}^n (z + k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

For more information and recent developments of the gamma function $\Gamma(z)$ and its logarithmic derivatives $\psi^{(n)}(z)$ for $n \in \mathbb{N}_0$, please refer to [2, Chapter 3] and [5, Chapter 6].

In 2009, Cerone and Dragomir [6] established many inequalities and properties for the Riemann eta function $\zeta(x)$ and the Dirichlet lambda function $\lambda(x)$. In 2010, Zhu and Hua [7] presented that the sequence $\lambda(n)$ for $n \in \mathbb{N}$ is decreasing. In [8–10], Qi used this decreasing property of $\lambda(n)$ to establish a double inequality and monotonicity of the ratio $\frac{|B_{2(n+1)}|}{|B_{2n}|}$ for $n \in \mathbb{N}$, where the Bernoulli numbers B_{2n} for $n \in \mathbb{N}_0$ are generated by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} B_{2n} \frac{z^{2n}}{(2n)!}, \quad |z| < 2\pi.$$

In 2020, Yang and Tian [11] used some properties of the Riemann zeta function $\zeta(z)$ and Zhu [12] used once the monotonicity of the sequence $\lambda(n)$ to extend and sharpen the double inequality discovered in [8,9]. In 2019, Hu and Kim [13] obtained a number of infinite families of linear recurrence relations and convolution identities for the Dirichlet lambda function $\lambda(2n)$ for $n \in \mathbb{N}$.

In this article, we consider

(1) the function

$$x \mapsto \left(x + \alpha + \ell \right) \frac{\lambda(x + \alpha)}{\lambda(x)} \quad (1.3)$$

and its monotonicity on $(1, \infty)$, where $\alpha > 0$ is a constant, $\ell \in \mathbb{N}_0$,

$$\binom{z}{w} = \begin{cases} \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}, & z \notin \mathbb{N}_-, \quad w, z-w \notin \mathbb{N}_- \\ 0, & z \notin \mathbb{N}_-, \quad w \in \mathbb{N}_- \quad \text{or} \quad z-w \in \mathbb{N}_- \\ \frac{\langle z \rangle_w}{w!}, & z \in \mathbb{N}_-, \quad w \in \mathbb{N}_0 \\ \frac{\langle z \rangle_{z-w}}{(z-w)!}, & z, w \in \mathbb{N}_-, \quad z-w \in \mathbb{N}_0 \\ 0, & z, w \in \mathbb{N}_-, \quad z-w \in \mathbb{N}_- \\ \infty, & z \in \mathbb{N}_-, \quad w \notin \mathbb{Z} \end{cases} \quad (1.4)$$

for $z, w \in \mathbb{C}$ denotes the extended binomial coefficient, and

$$\langle \beta \rangle_n = \prod_{k=0}^{n-1} (\beta - k) = \begin{cases} \beta(\beta-1)\cdots(\beta-n+1), & n \in \mathbb{N} \\ 1, & n = 0 \end{cases}$$

is called the falling factorial of $\beta \in \mathbb{C}$;

(2) the function $\Gamma(x + \ell)\lambda(x)$ for $\ell \in \mathbb{N}$ and its logarithmic convexity on $(1, \infty)$.

2 Lemmas

For proving our main results in this article, we need the following lemmas.

Lemma 2.1. ([14, Theorem 2.1], [15, Theorem 2.1], and [16, Theorem 3.1]) *Let $\vartheta \neq 0$ and $\theta \neq 0$ be real constants and $k \in \mathbb{N}$. When $\vartheta > 0$ and $t \neq -\frac{\ln \vartheta}{\theta}$ or when $\vartheta < 0$ and $t \in \mathbb{R}$, we have*

$$\frac{d^k}{dt^k} \left(\frac{1}{\partial e^{\theta t} - 1} \right) = (-1)^k \theta^k \sum_{p=1}^{k+1} (p-1)! S(k+1, p) \left(\frac{1}{\partial e^{\theta t} - 1} \right)^p, \quad (2.1)$$

where

$$S(k, p) = \frac{1}{p!} \sum_{q=1}^p (-1)^{p-q} \binom{p}{q} q^k, \quad 1 \leq p \leq k$$

denotes the Stirling numbers of the second kind.

For detailed information on the Stirling numbers of the second kind $S(k, m)$ for $1 \leq m \leq k$, please refer to [2, pp. 18–21, Section 1.3], [5, pp. 824–825, 24.1.4], the articles [17,18], or the monograph [19] and closely related references therein.

Recall from [20, Chapter XIII], [21, Chapter 1], and [22, Chapter IV] that

- (1) a function $q(x)$ is said to be completely monotonic on an interval I if it is infinitely differentiable and $(-1)^n q^{(n)}(x) \geq 0$ for $n \in \mathbb{N}_0$ on I .
- (2) a positive function $q(x)$ is said to be logarithmically completely monotonic on an interval $I \subseteq \mathbb{R}$ if it is infinitely differentiable and its logarithm $\ln f(x)$ satisfies $(-1)^k [\ln q(x)]^{(k)} \geq 0$ for $k \in \mathbb{N}$ on I .

Lemma 2.2. ([23, p. 98] and [24, p. 395]) *If a function $q(x)$ is non-identically zero and completely monotonic on $(0, \infty)$, then $q(x)$ and its derivatives $q^{(k)}(x)$ for $k \in \mathbb{N}$ are impossibly equal to 0 on $(0, \infty)$.*

Lemma 2.3. ([25, Lemma 2.8 and Remark 6.3] and [26, Remark 7.2]) *Let $U(t)$, $V(t) > 0$, and $W(t, x) > 0$ be integrable in $t \in (a, b)$,*

- (1) *if the ratios $\frac{\partial W(t, x) / \partial x}{W(t, x)}$ and $\frac{U(t)}{V(t)}$ are both increasing or both decreasing in $t \in (a, b)$, then the ratio*

$$R(x) = \frac{\int_a^b W(t, x) U(t) dt}{\int_a^b W(t, x) V(t) dt}$$

is increasing in x ;

- (2) *if one of the ratios $\frac{\partial W(t, x) / \partial x}{W(t, x)}$ and $\frac{U(t)}{V(t)}$ is increasing and another one of them is decreasing in $t \in (a, b)$, then the ratio $R(x)$ is decreasing in x .*

We call Lemma 2.3 the monotonicity rule for the ratio of two integrals with a parameter.

3 Increasing property and logarithmic convexity

We are now in a position to state and prove our main results in this article.

Theorem 3.1. *Let $\alpha > 0$ be a constant and let $\ell \in \mathbb{N}_0$. Then the function defined in (1.3) is increasing from $(1, \infty)$ onto $(0, \infty)$. Consequently, for fixed $\ell \in \mathbb{N}$, the function $\Gamma(x + \ell)\lambda(x)$ is logarithmically convex in $x \in (1, \infty)$.*

Proof. From (2.1) in Lemma 2.1 for $\theta = \pm \vartheta = 1$, we obtain

$$\frac{d^k}{dt^k} \left(\frac{1}{e^t \mp 1} \right) = (-1)^k \sum_{p=1}^{k+1} (\pm 1)^{p+1} (p-1)! S(k+1, p) \left(\frac{1}{e^t \mp 1} \right)^p.$$

Hence, it follows that

$$\frac{d^k}{dt^k} \left(\frac{1}{e^t - e^{-t}} \right) = \frac{1}{2} \frac{d^k}{dt^k} \left(\frac{1}{e^t - 1} + \frac{1}{e^t + 1} \right) = \frac{(-1)^k}{2} \sum_{p=1}^{k+1} (p-1)! S(k+1, p) \left[\left(\frac{1}{e^t - 1} \right)^p - (-1)^p \left(\frac{1}{e^t + 1} \right)^p \right].$$

This implies that

$$t^{k+1} \frac{d^k}{dt^k} \left(\frac{1}{e^t - e^{-t}} \right) \rightarrow \begin{cases} (-1)^k \frac{k!}{2}, & t \rightarrow 0^+ \\ 0, & t \rightarrow \infty \end{cases}$$

for $k \in \mathbb{N}_0$, where we used $S(k+1, k+1) = 1$ for $k \in \mathbb{N}_0$.

It is immediate that

$$\frac{1}{e^t - e^{-t}} = \frac{e^{-t}}{1 - e^{-2t}} = \sum_{k=0}^{\infty} e^{-(2k+1)t},$$

which implies that the function $\frac{1}{e^t - e^{-t}}$ is completely monotonic on $(0, \infty)$. Further considering Lemma 2.2, we conclude that

$$(-1)^k \frac{d^k}{dt^k} \left(\frac{1}{e^t - e^{-t}} \right) > 0, \quad t \in (0, \infty), \quad k \in \mathbb{N}_0.$$

Making use of the recurrence relation $\Gamma(z+1) = z\Gamma(z)$, using the integral representation (1.2), and integrating by parts yield

$$\begin{aligned} \frac{\Gamma(x+\alpha+1)}{\Gamma(x+1)} \frac{\lambda(x+\alpha)}{\lambda(x)} &= \frac{\Gamma(x+\alpha+1)}{\Gamma(x+1)} \frac{\frac{1}{\Gamma(x+\alpha)} \int_0^{\infty} \frac{t^{x+\alpha-1}}{e^t - e^{-t}} dt}{\frac{1}{\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{e^t - e^{-t}} dt} \\ &= \frac{(x+\alpha) \int_0^{\infty} \frac{t^{x+\alpha-1}}{e^t - e^{-t}} dt}{x \int_0^{\infty} \frac{t^{x-1}}{e^t - e^{-t}} dt} = \frac{\int_0^{\infty} \frac{1}{e^t - e^{-t}} (t^{x+\alpha})'_t dt}{\int_0^{\infty} \frac{1}{e^t - e^{-t}} (t^x)'_t dt} \\ &= \frac{\left(\frac{1}{e^t - e^{-t}} t^{x+\alpha} \right) \Big|_{t \rightarrow 0^+}^{t \rightarrow \infty} - \int_0^{\infty} \left(\frac{1}{e^t - e^{-t}} \right)' t^{x+\alpha} dt}{\left(\frac{1}{e^t - e^{-t}} t^x \right) \Big|_{t \rightarrow 0^+}^{t \rightarrow \infty} - \int_0^{\infty} \left(\frac{1}{e^t - e^{-t}} \right)' t^x dt} = \frac{\int_0^{\infty} \left[\left(\frac{1}{e^t - e^{-t}} \right)' t^a \right] t^x dt}{\int_0^{\infty} \left(\frac{1}{e^t - e^{-t}} \right)' t^x dt}. \end{aligned}$$

Consecutively and inductively, we obtain

$$\frac{\Gamma(x+\alpha+\ell)}{\Gamma(x+\ell)} \frac{\lambda(x+\alpha)}{\lambda(x)} = \frac{\int_0^{\infty} \left[\left(\frac{1}{e^t - e^{-t}} \right)^{(\ell)} t^a \right] t^{x+\ell-1} dt}{\int_0^{\infty} \left(\frac{1}{e^t - e^{-t}} \right)^{(\ell)} t^{x+\ell-1} dt}, \quad \ell \in \mathbb{N}.$$

Applying Lemma 2.3 to

$$U(t) = \left(\frac{1}{e^t - e^{-t}} \right)^{(\ell)} t^a, \quad V(t) = \left(\frac{1}{e^t - e^{-t}} \right)^{(\ell)} > 0, \quad W(t, x) = t^{x+\ell-1} > 0,$$

and $(a, b) = (0, \infty)$, since $\frac{U(t)}{V(t)} = t^a$ and $\frac{\partial W(t, x) / \partial x}{W(t, x)} = \ln t$ are both increasing on $(0, \infty)$, we conclude that the function

$$\frac{\int_0^{\infty} \left[\left(\frac{1}{e^t - e^{-t}} \right)^{(\ell)} t^a \right] t^{x+\ell-1} dt}{\int_0^{\infty} \left(\frac{1}{e^t - e^{-t}} \right)^{(\ell)} t^{x+\ell-1} dt} = \frac{\Gamma(x+\alpha+\ell)}{\Gamma(x+\ell)} \frac{\lambda(x+\alpha)}{\lambda(x)} = \Gamma(\alpha+1) \binom{x+\alpha+\ell-1}{\alpha} \frac{\lambda(x+\alpha)}{\lambda(x)}$$

is increasing in $x \in (1, \infty)$ for $\ell \in \mathbb{N}$, where we used Definition (1.4).

Because the function

$$\frac{\Gamma(x+\alpha+\ell)}{\Gamma(x+\ell)} \frac{\lambda(x+\alpha)}{\lambda(x)} = \frac{\Gamma(x+\alpha+\ell)\lambda(x+\alpha)}{\Gamma(x+\ell)\lambda(x)}$$

for $\ell \in \mathbb{N}$ is increasing in $x \in (0, \infty)$, its first derivative

$$\begin{aligned} \left[\frac{\Gamma(x + \alpha + \ell) \lambda(x + \alpha)}{\Gamma(x + \ell) \lambda(x)} \right]' &= \left[\frac{\Gamma(x + \alpha + \ell) \lambda(x + \alpha)}{\Gamma(x + \ell) \lambda(x)} \right]' \\ &= \frac{[\Gamma(x + \alpha + \ell) \lambda(x + \alpha)]' [\Gamma(x + \ell) \lambda(x)] - [\Gamma(x + \alpha + \ell) \lambda(x + \alpha)] [\Gamma(x + \ell) \lambda(x)]'}{[\Gamma(x + \ell) \lambda(x)]^2} \end{aligned}$$

is positive for $x \in (0, \infty)$. Hence, we have

$$\frac{[\Gamma(x + \alpha + \ell) \lambda(x + \alpha)]'}{\Gamma(x + \alpha + \ell) \lambda(x + \alpha)} > \frac{[\Gamma(x + \ell) \lambda(x)]'}{[\Gamma(x + \ell) \lambda(x)]},$$

that is, the logarithmic derivative

$$(\ln[\Gamma(x + \ell) \lambda(x)])' = \frac{[\Gamma(x + \ell) \lambda(x)]'}{[\Gamma(x + \ell) \lambda(x)]}$$

is increasing in $x \in (0, \infty)$. Consequently, for $\ell \in \mathbb{N}$, the function $\Gamma(x + \ell) \lambda(x)$ is logarithmically convex in $(1, \infty)$. The proof of Theorem 3.1 is complete. $\square$

4 Conclusion

Our main results are included in Theorem 3.1. One of our main tools is Lemma 2.3. The ideas, approaches, and techniques used in this article have also been used in the articles [10,27–29]. This means that the ideas, approaches, and techniques used in this article are deep and applicable.

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