Research Article

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Some results of homogeneous expansions for a class of biholomorphic mappings defined on a Reinhardt domain in $\mathbb{C}^n$

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Abstract: Let $S^*_{\gamma,A,B}(\mathbb{D})$ be the usual class of g-starlike functions of complex order γ in the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, where $g(\zeta) = (1 + A\zeta)/(1 + B\zeta)$, with $\gamma \in \mathbb{C}\backslash\{0\}$, $-1 \le A < B \le 1$, $\zeta \in \mathbb{D}$. First, we obtain the bounds of all the coefficients of homogeneous expansions for the functions $f \in S^*_{\gamma,A,B}(\mathbb{D})$ when $\zeta = 0$ is a zero of order $k + 1$ of $f(\zeta) - \zeta$. Second, we generalize this result to several complex variables by considering the corresponding biholomorphic mappings defined in a bounded complete Reinhardt domain. These main theorems unify and extend many known results.

Keywords: biholomorphic mappings, coefficient problems, Reinhardt domain, g-starlike mappings of complex order gamma

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1 Introduction

Let $\mathbb{C}^n$ be the n-dimensional complex variable space. Let $\mathbb{D}$ be the unit open disk in $\mathbb{C}^1$. Suppose that $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ are two domains. Let $H(\Omega_1, \Omega_2)$ be the family of all holomorphic mappings from Ω_1 into Ω_2. Let $\phi_1, \phi_2 \in H(\mathbb{D}, \mathbb{C}^1)$. We say that ϕ_1 is subordinate to ϕ_2, and write $\phi_1 \prec \phi_2$, if there exists a Schwarz function ω on $\mathbb{D}$ such that $\phi_1 = \phi_2(\omega(z))$ on $\mathbb{D}$ (see, Amini et al. [1]). Let Ω be a domain (connected open set) in $\mathbb{C}^n$, which contains 0. It is said that $z = 0$ is a zero of order k of $f(z)$ if $f(0) = 0, \ldots, D^{k-1}f(0) = 0$, but $D^k f(0) \neq 0$, where $k \in \mathbb{N}$ (see, Lin and Hong [2]). In one complex variable, the following Theorem A concerning starlike functions of order α is classical and well known.

Theorem A. (Boyd [3]) *Let $\alpha \in (0, 1)$ and $k \in \mathbb{N}^+$. If $f(z) = z + \sum_{m=k+1}^{\infty} a_m z^m$ is a starlike function of order α on the unit disk $\mathbb{D}$, then*

$$|a_m| \le \frac{\prod_{\mu=1}^{s}((\mu - 1)k + 2 - 2\alpha)}{s!k^s}, \quad sk + 1 \le m \le (s + 1)k, \quad s = 1, 2, \ldots.$$

These estimates are sharp for $m = sk + 1$, $s = 1, 2, \ldots$. Especially, when $k = 1$,

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$$|a_m| \le \frac{1}{(m-1)!}\prod_{\mu=2}^{m}(\mu-2\alpha), \quad m=2,3,\ldots.$$

It is known that the coefficient inequalities are related to the Bieberbach conjectures [4], which was settled by de Branges [5]. However, Cartan [6] stated that the Bieberbach conjecture does not hold in several complex variables. In fact, only a few complete results are known for the inequalities of homogeneous expansions for subclasses of biholomorphic mappings in $\mathbb{C}^n$ (see, e.g., Długosz and Liczberski [7], Hamada and Honda [8], Liu and Wu [9], Liu and Liu [10], Liu et al. [11], and Xu et al. [12]). Many works are concentrating on the bounds of second- and third-order terms of homogeneous expansions for starlike mappings and the sharp bounds of all homogeneous expansions for the special subclasses of starlike mappings with some restricted conditions (see, e.g., Hamada et al. [13], Liu and Liu [14], Tu and Xiong [15], Xiong [16], Xu et al. [17], and Xu et al. [18]). In Graham et al. [19], the estimate of the second-order coefficients of the first elements of g-Loewner chains in several complex variables was first obtained. In Bracci [20], a sharp estimate for the second-order coefficients for the first elements of g-Loewner chains on the Euclidean unit ball of $\mathbb{C}^2$, where $g(\zeta)=(1+\zeta)/(1-\zeta)$, which gives a support point for the family, was obtained. Generalizations of this result to the unit polydisk in $\mathbb{C}^2$ and to bounded symmetric domains were considered in Graham et al. [21], Hamada and Kohr [22], respectively. In Xu and Liu [23], the Fekete-Szegö inequality for starlike mappings in several complex variables was first obtained. Very recent important results related to the Fekete-Szegö inequality in several complex variables were obtained in other articles (see, e.g., Długosz and Liczberski [24], Elin and Jacobzon [25], Hamada [26], and Lai and Xu [27]). In particular, the Fekete-Szegö inequality for univalent mappings in several complex variables was first obtained in Hamada et al. [28]. Also, the other related results may consult in Długosz and Liczberski [29], Graham and Kohr [30], and Nunokawa and Sokol [31]. Liu et al. [32] considered only the main coefficients that are analogous with the diagonal elements of a square matrix and generalized Theorem A to the case on a Reinhardt domain in $\mathbb{C}^n$ from a new viewpoint. Let

$$D_{p_1,p_2,\ldots,p_n}=\left\{z\in\mathbb{C}^n:\sum_{l=1}^{n}|z_l|^{p_l}<1\right\}, \quad (p_l>1,\ l=1,2,\ \ldots,n)$$

be a bounded complete Reinhardt domain in $\mathbb{C}^n$, its Minkowski function $\rho(z)$ is C^1 except for some lower-dimensional manifolds in $\mathbb{C}^n$, and $\rho:\mathbb{C}^n\to[0,+\infty)$ is defined by:

$$\rho(z)=\inf\left\{t>0:\frac{z}{t}\in D_{p_1,p_2,\ldots,p_n}\right\}, \quad z\in\mathbb{C}^n. \tag{1}$$

Let $\gamma\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}$. Now, we introduce the following classes $\mathcal{S}^*_{g,\gamma}(D_{p_1,p_2,\ldots,p_n})$, which extend the usual class $\mathcal{S}^*(\gamma)$ of starlike functions of complex order γ on $\mathbb{D}$ in $\mathbb{C}$ to the classes of g-starlike mapping of complex order γ on the bounded Reinhardt domain $D_{p_1,p_2,\ldots,p_n}$ in $\mathbb{C}^n$. The function class $\mathcal{S}^*(\gamma)$ was considered earlier by Nasr and Aouf [33] (also, see Srivastava et al. [34]).

Definition 1.1. Suppose that $\gamma\in\mathbb{C}^*$ and $g:\mathbb{D}\to\mathbb{C}$ be a biholomorphic function such that $g(0)=1$, $\Re g(\zeta)>0$ on $\mathbb{D}$. A normalized locally biholomorphic mapping $f:D_{p_1,p_2,\ldots,p_n}\to\mathbb{C}^n$ is called a g-starlike mappings of complex order γ on $D_{p_1,p_2,\ldots,p_n}$ if

$$1+\frac{1}{\gamma}\left(\frac{\rho(z)}{2\frac{\partial\rho(z)}{\partial(z)}[Df(z)]^{-1}f(z)}-1\right)\in g(\mathbb{D}), \quad \forall z\in D_{p_1,p_2,\ldots,p_n}\backslash\{0\}, \tag{2}$$

where ρ is the Minkowski function of $D_{p_1,p_2,\ldots,p_n}$. We denote by $\mathcal{S}^*_{g,\gamma}(D_{p_1,p_2,\ldots,p_n})$ the set of all g-starlike mappings of complex order γ on $D_{p_1,p_2,\ldots,p_n}$ in $\mathbb{C}^n$.

Remark 1.1. (i) If $g(\zeta)=\frac{1+A\zeta}{1+B\zeta}$ $(-1\le A<B\le 1,\ \zeta\in\mathbb{D})$ in Definition 1.1, then we write $\mathcal{S}^*_{\frac{1+A\zeta}{1+B\zeta},\gamma}(D_{p_1,p_2,\ldots,p_n})$ by $\mathcal{S}^*_{\gamma,A,B}(D_{p_1,p_2,\ldots,p_n})$.

(ii) If $n = 1$ and $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$ in Definition 1.1, then it is obvious that Condition (2) is equivalent to

$$1 + \frac{1}{\gamma}\left(\frac{\zeta f'(\zeta)}{f(\zeta)} - 1\right) \prec \frac{1+A\zeta}{1+B\zeta}, \quad -1 \le A < B \le 1,\ \gamma \in \mathbb{C}^*, \zeta \in \mathbb{D}.$$

We denote by $\mathcal{S}^*_{\gamma,A,B}(\mathbb{D})$ the set of all g-starlike mappings of complex order γ on $\mathbb{D}$ in $\mathbb{C}$, where $g(\zeta) = (1 + A\zeta)/(1 + B\zeta)$, $\zeta \in \mathbb{D}$. In particular, $\mathcal{S}^*_{1,A,B}(\mathbb{D})$ is identical with the well-known class of Janowski starlike functions (see, Janowski [35]) and $\mathcal{S}^*_{\gamma,-1,1}(\mathbb{D})$ is the set of all starlike functions of complex order γ in $\mathbb{D}$.

(iii) If $\gamma = 1$ and $A = 2\alpha - 1(\alpha \in [0, 1))$, $B = 1$ in Definition 1.1 (the case g is defined as (ii)), then we obtain the starlike mappings of order α on $D_{p_1,p_2,\dots,p_n}$ (see, e.g., Liu et al. [32]).

(iv) By choosing the suitable functions g and parameters γ in Definition 1.1, we can obtain kinds of subclasses of starlike mappings defined on the Reinhardt domain $D_{p_1,p_2,\dots,p_n}$ in $\mathbb{C}^n$.

In this article, we first extend the definition of g-starlike mappings of complex order γ from the case of one-dimensional space to the case of higher-dimensional space (see Definition 1.1). Next, we obtain the bound of all coefficients of homogeneous expansions for the functions $f \in \mathcal{S}^*_{\gamma,A,B}(\mathbb{D})$ when $\zeta = 0$ is a zero of order $k + 1$ of $f(\zeta) - \zeta$ (see Lemma 2.2). Finally, by applying the results in Section 2, we consider the bound of main coefficients of the homogeneous expansions for the functions $f \in \mathcal{S}^*_{\gamma,A,B}(D_{p_1,p_2,\dots,p_n})$ in several complex variables (Theorems 3.1 and 3.2). Also, our results extend some theorems given in the previous literature (see Remarks 1.1–3.1).

2 Preliminaries

The following lemmas are needed in order to prove our estimates. Actually, we may use the similar way to those in the proof of Liu and Liu [36] (Lemmas 2.1 and 2.3). Here, we give the proof for the sake of completeness.

Lemma 2.1. *Let $k \in \mathbb{N}^+$, $C \ge 0$, $\gamma \in \mathbb{C}^*$. Then, for $q = 2, 3, \dots,$ we have*

$$C^2|\gamma|^2 + C|\gamma|\sum_{m=1}^{q-1}(2mk + C|\gamma|)\left(\frac{1}{m!}\prod_{\mu=0}^{m-1}\left(\mu + \frac{C|\gamma|}{k}\right)\right)^2 = \left(\frac{k}{(q-1)!}\prod_{\mu=0}^{q-1}\left(\mu + \frac{C|\gamma|}{k}\right)\right)^2. \tag{3}$$

Proof. We try to prove this lemma by mathematical induction. First, if $q = 2$, it is easy to see that (3) is true. Next, for all $q = 2, 3, \dots l$, assume that (3) holds true. Then, we need to show that (3) holds true when $q = l + 1$. By a simple computation, we have

$$\begin{aligned} &C^2|\gamma|^2 + C|\gamma|\sum_{m=1}^{l}(2mk + C|\gamma|)\left(\frac{1}{m!}\prod_{\mu=0}^{m-1}\left(\mu + \frac{C|\gamma|}{k}\right)\right)^2 \\ &= \left(\frac{k}{(l-1)!}\prod_{\mu=0}^{l-1}\left(\mu + \frac{C|\gamma|}{k}\right)\right)^2 + C|\gamma|(2lk + C|\gamma|)\left(\frac{1}{l!}\prod_{\mu=0}^{l-1}\left(\mu + \frac{C|\gamma|}{k}\right)\right)^2 \\ &= \left(\frac{k}{l!}\prod_{\mu=0}^{l}\left(\mu + \frac{C|\gamma|}{k}\right)\right)^2. \end{aligned}$$

This completes the proof. □

Lemma 2.2. *Let $f(z) = z + \sum_{m=k+1}^{\infty} a_m z^m \in \mathcal{S}^*_{\gamma,A,B}(\mathbb{D})$ with $k \in \mathbb{N}^+$, $\gamma \in \mathbb{C}^*$, $-1 \le A < B \le 1$. Then, for $s = 1, 2, \dots,$ we have*

$$|a_m| \le \frac{\prod_{\mu=1}^{s}((\mu-1)k + |A-B||\gamma|)}{s!k^s}, \quad sk + 1 \le m \le (s+1)k.$$

In particular, if $k = 1$, then $|a_m| \le \frac{1}{(m-1)!}\prod_{\mu=0}^{m-2}(\mu + |A-B||\gamma|)$, $\quad m = 2, 3, \dots.$

Proof. Since $f \in S^*_{\gamma,A,B}(\mathbb{D})$, so we can write that

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathbb{D}.$$

Thus, there is a function $\varphi \in \mathcal{H}(\mathbb{D}, \mathbb{D})$ with $|\varphi(z)| < 1$, such that

$$\frac{1 + A(\varphi(z))}{1 + B(\varphi(z))} = 1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right), \quad z \in \mathbb{D}. \tag{4}$$

Using (4), a simple computation shows that

$$\varphi(z) = \frac{zf'(z) - f(z)}{((A - B)\gamma + B)f(z) - Bzf'(z)} = b_k z^k + b_{k+1}z^{k+1} + \cdots, \quad z \in \mathbb{D}. \tag{5}$$

(5) is equal to

$$zf'(z) - f(z) = \varphi(z)\{((A - B)\gamma + B)f(z) - Bzf'(z)\} \tag{6}$$

and

$$(m - 1)a_m = (A - B)\gamma b_{m-1}, \quad m = k + 1, k + 2, \ldots, 2k. \tag{7}$$

In view of the relations $|\varphi(z)| < 1$ and $\sum_{m=k}^{\infty}|b_m|^2 \leq 1$, then we have

$$\sum_{m=k}^{2k-1} |b_m|^2 \leq 1. \tag{8}$$

Using (7) and (8), it can be easily shown that

$$\sum_{m=k+1}^{2k} (m - 1)^2|a_m|^2 \leq |A - B|^2|\gamma|^2. \tag{9}$$

By applying (6), then it can also be written as:

$$\begin{aligned}\sum_{m=k+1}^{\infty} (m - 1)a_m z^m &= \varphi(z)\left\{(A - B)\gamma z + \sum_{m=k+1}^{\infty} ((A - B)\gamma + B - mB)a_m z^m\right\} \\ &= \varphi(z)\left\{(A - B)\gamma z + \sum_{m=k+1}^{p-k} ((A - B)\gamma + B - mB)a_m z^m\right\} + \sum_{m=p+1}^{\infty} c_m z^m.\end{aligned} \tag{10}$$

Using the equality in (10), we have

$$\sum_{m=k+1}^{p} (m - 1)a_m z^m + \sum_{m=p+1}^{\infty} d_m z^m = \varphi(z)\left\{(A - B)\gamma z + \sum_{m=k+1}^{p-k} ((A - B)\gamma + B - mB)a_m z^m\right\}. \tag{11}$$

Taking $r \to 1$ in (11) and applying the similar argument being used in Theorem 1 by Boyd [3], we obtain

$$\sum_{m=k+1}^{p} (m - 1)^2|a_m|^2 \leq |A - B|^2|\gamma|^2 + \sum_{m=k+1}^{p-k} |(A - B)\gamma + B - mB|^2|a_m|^2. \tag{12}$$

In fact, with (12), a simple computation shows that

$$\sum_{m=p-k+1}^{p} (m - 1)^2|a_m|^2 \leq |A - B|^2|\gamma|^2 + \sum_{m=k+1}^{p-k} (|(A - B)\gamma + B - mB|^2 - (m - 1)^2)|a_m|^2. \tag{13}$$

Furthermore, in view of (13), we have

$$\sum_{m=p-k+1}^{p} (m - 1)^2|a_m|^2 \leq 2|A - B||\gamma|\left(\frac{|A - B||\gamma|}{2} + \sum_{m=k+1}^{p-k}\left(m - 1 + \frac{|A - B||\gamma|}{2}\right)|a_m|^2\right). \tag{14}$$

Here, we will prove that the following inequalities

$$\sum_{m=sk+1}^{(s+1)k}(m-1)^2|a_m|^2 \le \left(\frac{k}{(s-1)!}\prod_{\mu=0}^{s-1}\left(\mu+\frac{|A-B||\gamma|}{k}\right)\right)^2 \tag{15}$$

and

$$\sum_{m=sk+1}^{(s+1)k}\left(m-1+\frac{|A-B||\gamma|}{2}\right)|a_m|^2 \le \left(sk+\frac{|A-B||\gamma|}{2}\right)\left(\frac{1}{s!}\prod_{\mu=0}^{s-1}\left(\mu+\frac{|A-B||\gamma|}{k}\right)\right)^2 \tag{16}$$

hold true for $s = 1, 2, \ldots$.

If $s = 1$, then (15) holds from (9). Moreover, by using Lemma 2.2 in Liu and Liu [36] and (9), we obtain

$$\begin{aligned}
&\sum_{m=k+1}^{2k}\left(m-1+\frac{|A-B||\gamma|}{2}\right)|a_m|^2\\
&= \frac{k+\frac{|A-B||\gamma|}{2}}{k^2}\sum_{m=k+1}^{2k}\frac{k^2}{k+\frac{|A-B||\gamma|}{2}}\left(m-1+\frac{|A-B||\gamma|}{2}\right)|a_m|^2\\
&\le \frac{k+\frac{|A-B||\gamma|}{2}}{k^2}\sum_{m=k+1}^{2k}(m-1)^2|a_m|^2 \le \frac{k+\frac{|A-B||\gamma|}{2}}{k^2}|A-B|^2|\gamma|^2\\
&= \left(k+\frac{|A-B||\gamma|}{2}\right)\left(\frac{|A-B||\gamma|}{k}\right)^2.
\end{aligned} \tag{17}$$

Thus, (17) implies that (16) is true for $s = 1$.

At last, assume that (15) and (16) are valid for $s = 1, 2, \ldots, q-1$. Taking $p = (q+1)k$ in (14) and using Lemma 2.1, it gives that

$$\begin{aligned}
\sum_{m=qk+1}^{(q+1)k}(m-1)^2|a_m|^2 &\le 2|A-B||\gamma|\left(\frac{|A-B||\gamma|}{2}+\sum_{m=k+1}^{qk}\left(m-1+\frac{|A-B||\gamma|}{2}\right)|a_m|^2\right)\\
&= 2|A-B||\gamma|\left(\frac{|A-B||\gamma|}{2}+\sum_{s=1}^{q-1}\sum_{m=sk+1}^{(s+1)k}\left(m-1+\frac{|A-B||\gamma|}{2}\right)|a_m|^2\right)\\
&\le 2|A-B||\gamma|\left(\frac{|A-B||\gamma|}{2}+\sum_{s=1}^{q-1}\left(sk+\frac{|A-B||\gamma|}{2}\right)\left(\frac{1}{s!}\prod_{\mu=0}^{s-1}\left(\mu+\frac{|A-B||\gamma|}{k}\right)\right)^2\right)\\
&= \left(\frac{k}{(q-1)!}\prod_{\mu=0}^{q-1}\left(\mu+\frac{|A-B||\gamma|}{k}\right)\right)^2.
\end{aligned}$$

The aforementioned inequality shows that (15) holds for $s = q$. On the other hand, when $s = q$, we find that

$$\begin{aligned}
&\sum_{m=qk+1}^{(q+1)k}\left(m-1+\frac{|A-B||\gamma|}{2}\right)|a_m|^2\\
&= \frac{qk+\frac{|A-B||\gamma|}{2}}{(qk)^2}\sum_{m=qk+1}^{(q+1)k}\frac{(qk)^2}{qk+\frac{|A-B||\gamma|}{2}}\left(m-1+\frac{|A-B||\gamma|}{2}\right)|a_m|^2\\
&\le \frac{qk+\frac{|A-B||\gamma|}{2}}{(qk)^2}\sum_{n=qk+1}^{(q+1)k}(m-1)^2|a_m|^2\\
&\le \frac{qk+\frac{|A-B||\gamma|}{2}}{(qk)^2}\left(\frac{k}{(q-1)!}\prod_{\mu=0}^{q-1}\left(\mu+\frac{|A-B||\gamma|}{k}\right)\right)^2\\
&= \left(qk+\frac{|A-B||\gamma|}{2}\right)\left(\frac{1}{q!}\prod_{\mu=0}^{q-1}\left(\mu+\frac{|A-B||\gamma|}{k}\right)\right)^2.
\end{aligned}$$

The aforementioned inequality shows that (16) holds for $s = q$.

In view of (15), for $sk + 1 \le m \le (s+1)k$, we have

$$\begin{aligned}|a_m| &\le \frac{k}{(m-1)(s-1)!}\prod_{\mu=0}^{s-1}\left(\mu + \frac{|A-B||\gamma|}{k}\right)\\ &\le \frac{1}{s!}\prod_{\mu=0}^{s-1}\left(\mu + \frac{|A-B||\gamma|}{k}\right)\\ &= \frac{1}{s!}\prod_{\mu=1}^{s}\left(\mu - 1 + \frac{|A-B||\gamma|}{k}\right)\\ &= \frac{\prod_{\mu=1}^{s}((\mu-1)k + |A-B||\gamma|)}{s!k^s}.\end{aligned}$$

This completes the proof. □

Remark 2.1. If $A = 2\alpha - 1$ ($\alpha \in [0,1)$), $B = 1$, and $\gamma = 1$ in Lemma 2.2, then it reduces to Theorem **A**.

3 Main results

The following theorems give the estimates of main coefficients of homogeneous expansions for the class of g-starlike mappings of complex order γ defined on the bounded complete Reinhardt domain $D_{p_1,p_2,\ldots,p_n}$ in $\mathbb{C}^n$, where $g(\zeta) = (1 + A\zeta)/(1 + B\zeta)$, $-1 \le A < B \le 1$, $\zeta \in \mathbb{D}$. Theorems 3.1 and 3.2 will give generalizations of the results in Boyd [3] and Liu et al. [32].

Theorem 3.1. *Suppose that* $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))' \in S^*_{\gamma,A,B}(D_{p_1,p_2,\ldots,p_n})$ *and* $z = 0$ *is a zero of order* $k+1$ of $f(z) - z$. *Define*

$$\frac{D^m f_q(0)(z^m)}{m!} = \sum_{l_1,l_2,\ldots,l_m=1}^{n} a_{ql_1,l_2,\ldots,l_m} z_{l_1} z_{l_2} \ldots z_{l_m},$$

where $q = 1,2,\ldots,n$, $sk + 1 \le m \le (s+1)k$, $s = 1,2,\ldots$, *and* $k \in \mathbb{N}^+$. *Let* a^m_{qt} *be the* $a_{q\underbrace{t,t\cdots,t}_{m}}$ $(t = \{1,2,\ldots,n\})$. *If* $a^m_{qj} = 0$ $(q \ne j)$, *then we have*

$$|a^m_{jj}| \le \frac{\prod_{\mu=1}^{s}((\mu-1)k + |A-B||\gamma|)}{s!k^s}, \quad sk+1 \le m \le (s+1)k,\ s = 1,2,\ldots, \quad j \in \{1,2,\ldots,n\}.$$

The aforementioned estimates are sharp for $0 < \gamma \le 1$, $A = -1$, $B = 1$, $m = sk+1$, $s = 1,2,\ldots$, *and* $0 < \gamma \le 1$, $A = 2\alpha - 1(\alpha \in [0,1))$, $B = 1$, $m = sk+1$, $s = 1,2,\ldots$. *In particular, if* $k = 1$, *then*

$$|a^m_{jj}| \le \frac{\prod_{\mu=0}^{m-2}(\mu + |A-B||\gamma|)}{(m-1)!}, \quad j \in \{1,2,\ldots,n\}; \quad m = 2,3,\ldots.$$

Proof. Since $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))'$, then

$$Df(z) = \begin{pmatrix} \frac{\partial f_1(z)}{\partial z_1} & \cdots & \frac{\partial f_1(z)}{\partial z_j} & \cdots & \frac{\partial f_1(z)}{\partial z_n}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \frac{\partial f_j(z)}{\partial z_1} & \cdots & \frac{\partial f_j(z)}{\partial z_j} & \cdots & \frac{\partial f_j(z)}{\partial z_n}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \frac{\partial f_n(z)}{\partial z_1} & \cdots & \frac{\partial f_n(z)}{\partial z_j} & \cdots & \frac{\partial f_n(z)}{\partial z_n}\end{pmatrix}.$$

For $(0, \ldots, z_j, \ldots, 0)' \in D_{p_1,p_2,\ldots,p_n}$, it is easy to see that $Df((0, \ldots, z_j, \ldots, 0)')$

$$= \begin{pmatrix} \frac{\partial f_1((0,\ldots,z_j,\ldots,0)')}{\partial z_1} & \cdots & \frac{\partial f_1((0,\ldots,z_j,\ldots,0)')}{\partial z_j} & \cdots & \frac{\partial f_1((0,\ldots,z_j,\ldots,0)')}{\partial z_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_1} & \cdots & \frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_j} & \cdots & \frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial f_n((0,\ldots,z_j,\ldots,0)')}{\partial z_1} & \cdots & \frac{\partial f_n((0,\ldots,z_j,\ldots,0)')}{\partial z_j} & \cdots & \frac{\partial f_n((0,\ldots,z_j,\ldots,0)')}{\partial z_n} \end{pmatrix}. \tag{18}$$

Since $a_{qj}^m = 0$ $(q \neq j)$, it follows that

$$\frac{\partial f_q((0,\ldots,z_j,\ldots,0)')}{\partial z_j} = 0, \quad q = 1, 2, \ldots, n, \, q \neq j. \tag{19}$$

From (18) and (19), then $Df((0, \ldots, z_j, \ldots, 0)')$

$$= \begin{pmatrix} \frac{\partial f_1((0,\ldots,z_j,\ldots,0)')}{\partial z_1} & \cdots & 0 & \cdots & \frac{\partial f_1((0,\ldots,z_j,\ldots,0)')}{\partial z_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_1} & \cdots & \frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_j} & \cdots & \frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial f_n((0,\ldots,z_j,\ldots,0)')}{\partial z_1} & \cdots & 0 & \cdots & \frac{\partial f_n((0,\ldots,z_j,\ldots,0)')}{\partial z_n} \end{pmatrix}. \tag{20}$$

In view that $Df((0, \ldots, z_j, \ldots, 0)')$ is invertible, thus, it implies that

$$\frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_j} \neq 0, \quad j \in \{1, 2, \ldots, n\}. \tag{21}$$

Using (20) and (21), then we obtain

$$(Df((0,\ldots,z_j,\ldots,0)'))^{-1} = \begin{pmatrix} * & \cdots & 0 & \cdots & * \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ * & \cdots & \frac{1}{\frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_j}} & \cdots & * \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ * & \cdots & 0 & \cdots & * \end{pmatrix}, \tag{22}$$

where the symbol * means the unknown term. Furthermore, a simple computation in (22) shows that

$$\begin{aligned} &(Df((0,\ldots,z_j,\ldots,0)'))^{-1} f((0,\ldots,z_j,\ldots,0)') \\ &= \begin{pmatrix} * & \cdots & 0 & \cdots & * \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ * & \cdots & \frac{1}{\frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_j}} & \cdots & * \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ * & \cdots & 0 & \cdots & * \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ f_j((0,\ldots,z_j,\ldots,0)') \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \frac{f_j((0,\ldots,z_j,\ldots,0)')}{\frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_j}} \\ \vdots \\ 0 \end{pmatrix}. \end{aligned}$$

Let $(Df(z)^{-1}f(z)) = W(z) = (W_1(z), \ldots, W_j(z), \ldots, W_n(z))'$ and $h_j(z_j) = f_j((0, \ldots, z_j, \ldots, 0)') \in H(\mathbb{D})$, $z_j \in \mathbb{D}$. Since $f(z) \in S^*_{\gamma,A,B}(D_{p_1,p_2,\ldots,p_n})$ and $z = 0$ is a zero of order $k+1$ of $f(z) - z$, it follows that

$$1+\frac{1}{\gamma}\left(\frac{z_jh_j'(z_j)}{h_j(z_j)}-1\right)=1+\frac{1}{\gamma}\left(\frac{z_j\frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_j}}{f_j((0,\ldots,z_j,\ldots,0)')}-1\right)$$
$$=1+\frac{1}{\gamma}\left(\frac{\rho((0,\ldots,z_j,\ldots,0)')}{2\frac{\partial\rho((0,\ldots,z_j,\ldots,0)')}{\partial z_j}W((0,\ldots,z_j,\ldots,0)')}-1\right)\in g(\mathbb{D}),\quad z_j\neq 0. \tag{23}$$

From (23), we find that $h_j \in S^*_{\gamma,A,B}(\mathbb{D})$, and $z_j = 0$ is a zero of order $\widetilde{k}+1$ of $h_j(z_j) - z_j$, where $\widetilde{k} \geq k$. We note that

$$a^m_{jj} = \frac{h_j(0)^{(m)}}{m!},\quad m = 2, 3, \ldots. \tag{24}$$

Thus, in view of Lemma 2.2 and (24), we obtain the desired results.

Furthermore, let $0 < \gamma \leq 1$ and

$$f(z)=\left(\frac{z_1}{(1-z_1^k)^{\frac{2(1-\alpha)\gamma}{k}}},\frac{z_2}{(1-z_2^k)^{\frac{2(1-\alpha)\gamma}{k}}},\ldots,\frac{z_n}{(1-z_n^k)^{\frac{2(1-\alpha)\gamma}{k}}}\right)',\quad z=(z_1,z_2,\ldots z_n)\in D_{p_1,p_2,\ldots,p_n}\backslash\{0\}. \tag{25}$$

It is easy to check that $f \in S^*_{\gamma,2\alpha-1,1}(D_{p_1,p_2,\ldots,p_n})$. Thus, we have

$$f_j((0,0,\ldots,z_j,\ldots,0)') = z_j + \sum_{s=1}^{\infty} a^{sk+1}_{jj} z_j^{sk+1}, j \in \{1,2,\ldots,n\}$$

and

$$|a^{sk+1}_{jj}| = \frac{\prod_{\mu=1}^{s}((\mu-1)k + 2\gamma(1-\alpha))}{s!k^s},\quad s = 1,2,\ldots,\quad j\in\{1,2,\ldots,n\}.$$

This completes the proof. □

Theorem 3.2. *Suppose that* $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))' \in S^*_{\gamma,A,B}(D_{p_1,p_2,\ldots,p_n})$ *and* $z = 0$ *is a zero of order* $k+1$ *of* $f(z) - z$. *Define*

$$\frac{D^m f_q(0)(z^m)}{m!} = \sum_{l_1,l_2,\ldots,l_m=1}^{n} a_{ql_1,l_2,\ldots,l_m} z_{l_1} z_{l_2}\cdots z_{l_m},$$

where $q = 1,2,\ldots,n$, $sk+1 \leq m \leq (s+1)k$, $s = 1,2,\ldots$, *and* $k \in \mathbb{N}^+$. *Let* a^m_{tq} *be the* $a_{tq\underbrace{t,t\cdots,t}_{m-1}}$ $(t = \{1,2,\ldots,n\})$. *If* $a^m_{jq} = 0$ $(q \neq j)$, *then we have*

$$|a^m_{jj}| \leq \frac{\prod_{\mu=1}^{s}((\mu-1)k + |A-B||\gamma|)}{s!k^s},\quad sk+1\leq m\leq (s+1)k, s=1,2,\ldots,\quad j\in\{1,2,\ldots,n\}.$$

The aforementioned estimates are sharp for $0 < \gamma \leq 1$, $A = -1$, $B = 1$, $m = sk+1$, $s = 1,2,\ldots$ *and* $0 < \gamma \leq 1$, $A = 2\alpha - 1(\alpha \in [0,1))$, $B = 1$, $m = sk+1$, $s = 1,2,\ldots$. *In particular, if* $k = 1$, *then*

$$|a^m_{jj}| \leq \frac{\prod_{\mu=0}^{m-2}(\mu + |A-B||\gamma|)}{(m-1)!},\quad j\in\{1,2,\ldots,n\};\quad m = 2,3,\ldots.$$

Proof. Since $a^m_{jq} = 0$ $(q \neq j)$, it follows that

$$\frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_q} = 0. \tag{26}$$

Using (18) and (26), we have $Df((0,\ldots,z_j,\ldots,0)')$

$$= \begin{pmatrix} \frac{\partial f_1((0,\ldots,z_j,\ldots,0)')}{\partial z_1} & \cdots & \frac{\partial f_1((0,\ldots,z_j,\ldots,0)')}{\partial z_j} & \cdots & \frac{\partial f_1((0,\ldots,z_j,\ldots,0)')}{\partial z_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_j} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial f_n((0,\ldots,z_j,\ldots,0)')}{\partial z_1} & \cdots & \frac{\partial f_n((0,\ldots,z_j,\ldots,0)')}{\partial z_j} & \cdots & \frac{\partial f_n((0,\ldots,z_j,\ldots,0)')}{\partial z_n} \end{pmatrix}. \tag{27}$$

In view that $Df((0,\ldots,z_j,\ldots,0)')$ is invertible, thus, it implies that

$$\frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_j} \neq 0. \tag{28}$$

Thus, (27) and (28) show that

$$(Df((0,\ldots,z_j,\ldots,0)'))^{-1} = \begin{pmatrix} * & \cdots & * & \cdots & * \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_j}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ * & \cdots & * & \cdots & * \end{pmatrix}, \tag{29}$$

where the symbol * means the unknown term. Furthermore, a simple computation in (29) shows that

$$\begin{aligned} &(Df((0,\ldots,z_j,\ldots,0)'))^{-1} f((0,\ldots,z_j,\ldots,0)') \\ &= \begin{pmatrix} * & \cdots & * & \cdots & * \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_j}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ * & \cdots & * & \cdots & * \end{pmatrix} \begin{pmatrix} f_1((0,\ldots,z_j,\ldots,0)') \\ \vdots \\ f_j((0,\ldots,z_j,\ldots,0)') \\ \vdots \\ f_n((0,\ldots,z_j,\ldots,0)') \end{pmatrix} = \begin{pmatrix} * \\ \vdots \\ \frac{f_j((0,\ldots,z_j,\ldots,0)')}{\frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_j}} \\ \vdots \\ * \end{pmatrix}. \end{aligned}$$

Define $(Df(z)^{-1}f(z)) = W(z) = (W_1(z),\ldots,W_j(z),\ldots,W_n(z))'$ and $h_j(z_j) = f_j((0,\ldots,z_j,\ldots,0)') \in H(\mathbb{D})$. Since $f(z) \in S^*_{\gamma,A,B}(D_{p_1,p_2,\ldots,p_n})$ and $z = 0$ is a zero of order $k+1$ of $f(z) - z$, it follows that

$$\begin{aligned} 1 + \frac{1}{\gamma}\left(\frac{z_j h_j'(z_j)}{h_j(z_j)} - 1\right) &= 1 + \frac{1}{\gamma}\left(\frac{z_j \frac{\partial f_j((0,\ldots,z_j,\ldots,0)')}{\partial z_j}}{f_j((0,\ldots,z_j,\ldots,0)')} - 1\right) \\ &= 1 + \frac{1}{\gamma}\left(\frac{\rho((0,\ldots,z_j,\ldots,0)')}{2\frac{\partial \rho((0,\ldots,z_j,\ldots,0)')}{\partial z_j} W((0,\ldots,z_j,\ldots,0)')} - 1\right) \in g(\mathbb{D}), \quad z_j \neq 0. \end{aligned} \tag{30}$$

From (30), we find that $h_j \in S^*_{\gamma,A,B}(\mathbb{D})$, and $z_j = 0$ is a zero of order $\tilde{k}+1$ of $h_j(z_j) - z_j$, where $\tilde{k} \geq k$. We note that

$$a^m_{jj} = \frac{h_j^{(m)}(0)}{m!}, \quad m = 2, 3, \ldots. \tag{31}$$

Thus, in view of Lemma 2.2 and (31), we obtain the desired results. We note that the sharpness of the estimates of Theorem 3.2 is similar to that of Theorem 3.1. This completes the proof. □

Remark 3.1.

(i) If $n = 1$, then Theorem 3.1 (resp. Theorem 3.2) reduces to Lemma 2.2.
(ii) If $\gamma = 1$, $A = 2\alpha - 1$ ($\alpha \in [0, 1)$), and $B = 1$ in Theorems 3.1 and 3.2, then we obtain Theorems 3.1 and 3.2 in Liu et al. [32], respectively.
(iii) If we take different functions g in Theorem 3.1 (resp. Theorem 3.2), then the bounds of homogeneous expansions for kinds of subclasses of g-starlike mappings of complex order γ defined on $D_{p_1,p_2,\ldots,p_n}$ can be obtained immediately.

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