Research Article

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The study of solutions for several systems of PDDEs with two complex variables

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Abstract: The purpose of this article is to describe the properties of the pair of solutions of several systems of Fermat-type partial differential difference equations. Our theorems exhibit the forms of finite order transcendental entire solutions for these systems, which are some extensions and improvement of the previous theorems given by Xu, Cao, Liu, etc. Furthermore, we give a series of examples to show that the existence conditions and the forms of transcendental entire solutions with finite order of such systems are precise.

Keywords: partial differential difference equation, entire solution, Nevanlinna theory

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1 Introduction

Let us first recall the classical results that the entire solutions of the functional equation

$$f^2 + g^2 = 1, \tag{1.1}$$

are $f = \cos a(z), g = \sin a(z)$ was proved by Gross [1], where $a(z)$ is an entire function. This simple-looking nonlinear functional equation (1.1) can be called as the Fermat-type functional equation, analogous with the equation $x^2 + y^2 = z^2$ in Fermat's last theorem in number theory. As a matter of fact, we can find that the study of the Fermat-type functional equations can be tracked back to more than 60 years ago or even earlier [2,3].

In the past 30 years, there were lots of research focusing on the solutions of functional equation (1.1), readers can refer to [4–17]. For example, Khavinson [11] in 1995 proved that any entire solutions of the partial differential equations

$$\left(\frac{\partial f}{\partial z_1}\right)^2 + \left(\frac{\partial f}{\partial z_2}\right)^2 = 1, \tag{1.2}$$

in $\mathbb{C}^2$ are necessarily linear. It should be noted that equation (1.2) is called as eiconal equation. Later, Saleeby [18,19] further proved that the entire solution of equation (1.2) is of the form $f(z_1, z_2) = c_1z_1 + c_2z_2 + \eta$. After theirs works, Li and co-authors [20–22] further discussed a series of deformation forms of Fermat-type partial differential equations and gave a number of important and interesting results about the existence and the forms of solutions for these partial differential equations.

Theorem A. [20, Corollary 2.3] *Let $P(z_1, z_2)$ and $Q(z_1, z_2)$ be arbitrary polynomials in C^2. Then, f is an entire solution of the equation*

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$$\left(P\frac{\partial f(z_1,z_2)}{\partial z_1}\right)^2+\left(Q\frac{\partial f(z_1,z_2)}{\partial z_2}\right)^2=1, \tag{1.3}$$

if and only if $f=c_1z_1+c_2z_2+c_3$ *is a linear function, where* c_j*'s are constants, and exactly one of the following holds:*

(*i*) $c_1=0$ *and* Q *is a constant satisfying that* $(c_2Q)^2=1$;
(*ii*) $c_2=0$ *and* P *is a constant satisfying that* $(c_1P)^2=1$;
(*iii*) $c_1c_2\neq 0$ *and* P *and* Q *are both constants satisfying that* $(c_1P)^2+(c_2Q)^2=1$.

In the past 40 years, the Nevanlinna theory and the difference Nevanlinna theory of Meromorphic function with several complex variables have been developed rapidly [23–29]. Especially, Korhonen [30, Theorem 3.1] in 2012 established a logarithmic difference lemma for meromorphic functions in several variables of hyper order <2/3. Later, Cao and Korhonen [23] proved that the logarithmic difference lemma for meromorphic functions holds under the condition "hyper order <1." By making use of the Nevanlinna theory and difference Nevanlinna theory of several complex variables [23,30], Xu and Cao [31,32] discussed the transcendental solutions of several Fermat-type partial differential difference equations. An equation is called partial differential difference equation, if the equation includes partial derivatives, shifts or differences of f, which can be called PDDE for short.

Theorem B. [31, Theorem 1.2] *Let* $c=(c_1,c_2)\in\mathbb{C}^2$. *Then, any transcendental entire solution with finite order of the PDEE*

$$\left(\frac{\partial f(z_1,z_2)}{\partial z_1}\right)^2+f(z_1+c_1,z_2+c_2)^2=1, \tag{1.4}$$

has the form of $f(z_1,z_2)=\sin(Az_1+B)$, *where* A *is a constant on* $\mathbb{C}$ *satisfying* $Ae^{iAc_1}=1$, *and* B *is a constant on* $\mathbb{C}$; *as a special case, whenever* $c_1=0$, *we have* $f(z_1,z_2)=\sin(z_1+B)$.

In 2020, the author of this article and his colleagues [33] studied the finite order transcendental entire solutions when equation (1.4) turns to the system of Fermat-type PDDEs and obtained Theorem C.

Theorem C. [33, Theorem 1.3] *Let* $c=(c_1,c_2)\in\mathbb{C}^2$. *Then, any pair of transcendental entire solutions with finite order for the system of Fermat-type PDEEs*

$$\begin{cases}\left(\dfrac{\partial f_1(z_1,z_2)}{\partial z_1}\right)^2+f_2(z_1+c_1,z_2+c_2)^2=1,\\[2ex] \left(\dfrac{\partial f_2(z_1,z_2)}{\partial z_1}\right)^2+f_1(z_1+c_1,z_2+c_2)^2=1\end{cases} \tag{1.5}$$

have the following forms:

$$(f_1(z),f_2(z))=\left(\frac{e^{L(z)+B_1}+e^{-(L(z)+B_1)}}{2},\frac{A_{21}e^{L(z)+B_1}+A_{22}e^{-(L(z)+B_1)}}{2}\right),$$

where $L(z)=a_1z_1+a_2z_2$, B_1 *is a constant in* $\mathbb{C}$, *and* a_1,c,A_{21},A_{22} *satisfy one of the following cases*:

(*i*) $A_{21}=-i$, $A_{22}=i$, *and* $a_1=i$, $L(c)=\left(2k+\frac{1}{2}\right)\pi i$, or $a_1=-i$, $L(c)=\left(2k-\frac{1}{2}\right)\pi i$;
(*ii*) $A_{21}=i$, $A_{22}=-i$, *and* $a_1=i$, $L(c)=\left(2k-\frac{1}{2}\right)\pi i$, or $a_1=-i$, $L(c)=\left(2k+\frac{1}{2}\right)\pi i$;
(*iii*) $A_{21}=1$, $A_{22}=1$, *and* $a_1=i$, $L(c)=2k\pi i$, *or* $a_1=-i$, $L(c)=(2k+1)\pi i$;
(*iv*) $A_{21}=-1$, $A_{22}=-1$, *and* $a_1=i$, $L(c)=(2k+1)\pi i$, *or* $a_1=-i$, $L(c)=2k\pi i$.

To the best of our knowledge, there are few results about the study of systems of this Fermat-type PDDE with several complex variables. Moreover, it appears that the study of such fields has not been addressed in the literature before. Based on these, we are mainly concerned with the solutions of complex Fermat-type PDDEs, and describe the existence and form of the pair of the finite order transcendental solutions of the systems of PDDEs with constant coefficients

$$\begin{cases} \left[\mu f_1(z) + \lambda \dfrac{\partial f_1}{\partial z_1}\right]^2 + [\alpha f_2(z+c) - \beta f_1(z)]^2 = 1, \\ \left[\mu f_2(z) + \lambda \dfrac{\partial f_2}{\partial z_1}\right]^2 + [\alpha f_1(z+c) - \beta f_2(z)]^2 = 1, \end{cases} \tag{1.6}$$

and

$$\begin{cases} \left[\mu f_1(z) + \lambda_1 \dfrac{\partial f_1}{\partial z_1} + \lambda_2 \dfrac{\partial f_1}{\partial z_2}\right]^2 + [\alpha f_2(z+c) - \beta f_1(z)]^2 = 1, \\ \left[\mu f_2(z) + \lambda_1 \dfrac{\partial f_2}{\partial z_1} + \lambda_2 \dfrac{\partial f_2}{\partial z_2}\right]^2 + [\alpha f_1(z+c) - \beta f_2(z)]^2 = 1, \end{cases} \tag{1.7}$$

where $\alpha, \beta, \mu, \lambda, \lambda_1, \lambda_2, c_1, c_2$ are constants in $\mathbb{C}$. Obviously, equation (1.4) and system (1.5) are the special cases of systems (1.6) and (1.7). The article is organized as follows. We will introduce our main results about the existence and the forms of entire solutions for (1.6) and (1.7) in Section 2, which generalize the previous theorems given by Xu et al. [31–33]. Meantime, we give a series of examples to explain that our results about the forms of solutions of such systems are precise. The proofs of Theorems 1.6 and 1.7 are given in Sections 4 and 5, respectively.

2 Results and examples

The first main theorem is about the existence and the forms of the solutions for system (1.6).

Theorem 2.1. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$, $c_2 \neq 0$, *and* $\alpha, \beta, \mu, \lambda$ *be nonzero constants in* $\mathbb{C}$. *Let* $(f_1(z_1, z_2), f_2(z_1, z_2))$ *be a pair of transcendental entire solution with finite order of system* (1.6). *Then,* $(f_1(z_1, z_2), f_2(z_1, z_2))$ *must satisfy one of the following cases*:
(*i*)

$$f_1(z_1, z_2) = \frac{\eta_1}{\mu} - \frac{1}{\mu} e^{-\frac{\mu}{\lambda} z_1 + \gamma z_2 + D_1}, \quad f_2(z_1, z_2) = \frac{\delta_1}{\mu} - \frac{1}{\mu} e^{-\frac{\mu}{\lambda} z_1 + \gamma z_2 + D_2},$$

where $\eta_1, \delta_1, \gamma, D_1, D_2 \in \mathbb{C}$ *satisfy*

$$e^{D_1 - D_2} = 1, \quad \gamma = \frac{\frac{\mu}{\lambda} c_1 + \log\frac{\beta}{\alpha} + k\pi i}{c_2}, \quad k \in \mathbb{Z}, \tag{2.1}$$

and one of the following cases:

(i_1) $\delta_1 = -\eta_1 = \pm 1$ *and* $\alpha = -\beta$ *or* $\delta_1 = \eta_1 = \pm 1$, $\alpha = \beta$;
(i_2) $\delta_1 = \eta_1$ *and* $\eta_1^2 = \frac{\mu^2}{\mu^2 + (\alpha - \beta)^2}$, *or* $\delta_1 = -\eta_1$ *and* $\eta_1^2 = \frac{\mu^2}{\mu^2 + (\alpha + \beta)^2}$;
(i_3) $\mu^2 = \alpha^2 - \beta^2$, $\delta_1^2 + \delta_2^2 = 1$, $\eta_1^2 + \eta_2^2 = 1$, *and* $\alpha\delta_2 = -\beta\eta_2 \pm \sqrt{(\eta_2^2 - 1)(\beta^2 - \alpha^2)}$;
(*ii*) *if* $a_1 \neq \pm\frac{\mu}{\lambda}$, *then*

$$f_1(z_1, z_2) = \frac{1}{2(\lambda a_1 + \mu)} e^{a_1z_1+a_2z_2+b_1} - \frac{1}{2(\lambda a_1 - \mu)} e^{-a_1z_1-a_2z_2-b_1} - \vartheta(z_2)e^{-\frac{\mu}{\lambda}z_1+\gamma z_2+D_1},$$
$$f_2(z_1, z_2) = \frac{1}{2(\lambda a_1 + \mu)} e^{a_1z_1+a_2z_2+b_2} - \frac{1}{2(\lambda a_1 - \mu)} e^{-a_1z_1-a_2z_2-b_2} - \vartheta(z_2 + c_2)e^{-\frac{\mu}{\lambda}z_1+\gamma z_2+D_2},$$

where $\vartheta(z_2)$ is a finite order period entire function with period $2c_2$, and $a_1, a_2, b_1, b_2, \gamma, D_1, D_2 \in \mathbb{C}$ satisfy (2.1),

$$e^{2(b_1-b_2)} = 1, \tag{2.2}$$

and

$$a_1 = \frac{-\beta i \pm \sqrt{\mu^2 - \alpha^2}}{\lambda}, \quad e^{2(a_1c_1+a_2c_2)} = \frac{(\lambda a_1 + \mu + \beta i)^2}{-\alpha^2};$$

if $a_1 = \frac{\mu}{\lambda}$, then

$$f_1(z_1, z_2) = \frac{1}{4\mu} e^{a_1z_1+a_2z_2+b_1} + \frac{z_1}{2\lambda} e^{-a_1z_1-a_2z_2-b_1} - \vartheta(z_2)e^{-\frac{\mu}{\lambda}z_1+\gamma z_2+D_1},$$
$$f_2(z_1, z_2) = \frac{1}{4\mu} e^{a_1z_1+a_2z_2+b_2} + \frac{z_1}{2\lambda} e^{-a_1z_1-a_2z_2-b_2} - \vartheta(z_2 + c_2)e^{-\frac{\mu}{\lambda}z_1+\gamma z_2+D_2},$$

where $a_1, a_2, b_1, b_2, \gamma \in \mathbb{C}$ satisfy (2.1), (2.2) *and*

$$2\mu\beta i = \beta^2 - \alpha^2, \quad \beta c_1 = \lambda i, \quad e^{2(a_1c_1+a_2c_2)} = 1 - \frac{2\mu}{\beta} i; \tag{2.3}$$

if $a_1 = -\frac{\mu}{\lambda}$, then

$$f_1(z_1, z_2) = \frac{z_1}{2\lambda} e^{a_1z_1+a_2z_2+b_1} + \frac{1}{4\mu} e^{-a_1z_1-a_2z_2-b_1} - \vartheta(z_2)e^{-\frac{\mu}{\lambda}z_1+\gamma z_2+D_1},$$
$$f_2(z_1, z_2) = \frac{z_1}{2\lambda} e^{a_1z_1+a_2z_2+b_2} + \frac{1}{4\mu} e^{-a_1z_1-a_2z_2-b_2} - \vartheta(z_2 + c_2)e^{-\frac{\mu}{\lambda}z_1+\gamma z_2+D_2},$$

where $a_1, a_2, b_1, b_2, \gamma \in \mathbb{C}$ satisfy (2.1), (2.2) *and*

$$-2\mu\beta i = \beta^2 - \alpha^2, \quad \beta c_1 = -\lambda i, \quad e^{-2(a_1c_1+a_2c_2)} = 1 + \frac{2\mu}{\beta} i. \tag{2.4}$$

The following examples show the existence of transcendental entire solutions of system (1.6).

Example 2.1. Let

$$f_1(z_1, z_2) = \frac{1}{2} - \frac{1}{2} e^{-2z_1+3z_2}, \quad f_2(z_1, z_2) = -\frac{1}{2} - \frac{1}{2} e^{-2z_1+3z_2}.$$

Thus, (f_1, f_2) is a pair of finite order transcendental entire solution of system (1.6) with $\lambda = 1$, $\mu = 2$, $\alpha = 1$, $\beta = -1$, and $(c_1, c_2) = (\pi i, \pi i)$.

Example 2.2. Let

$$f_1(z_1, z_2) = 1 - e^{-z_1+z_2}, \quad f_2(z_1, z_2) = 1 - e^{-z_1+z_2}.$$

Thus, (f_1, f_2) is a pair of finite order transcendental entire solution of system (1.6) with $\lambda = \mu = \alpha = \beta = 1$, and $(c_1, c_2) = (\pi i, \pi i)$.

Example 2.3. Let

$$f_1(z_1, z_2) = \frac{1}{\sqrt{3}} - e^{-\frac{1}{2}z_1+z_2}, \quad f_2(z_1, z_2) = -\frac{1}{\sqrt{3}} - e^{-\frac{1}{2}z_1+z_2}.$$

Thus, (f_1, f_2) is a pair of finite order transcendental entire solution of system (1.6) with $\lambda = 2$, $\mu = 1$, $\alpha = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $\beta = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$, and $(c_1, c_2) = (\pi i, \pi i)$.

Example 2.4. Let

$$f_1(z_1, z_2) = \frac{\sqrt{2}}{2} + \frac{i}{\sqrt{2}}e^{-\sqrt{2}iz_1+(\sqrt{2}-0.5)z_2}, \quad f_2(z_1, z_2) = 1 - \frac{\sqrt{2}}{2}i + \frac{i}{\sqrt{2}}e^{-\sqrt{2}iz_1+(\sqrt{2}-0.5)z_2}.$$

Thus, (f_1, f_2) is a pair of finite order transcendental entire solution of system (1.6) with $\lambda = 1$, $\mu = \sqrt{2}i$, $\alpha = i$, $\beta = 1$, and $(c_1, c_2) = (\pi, \pi i)$.

Example 2.5. Let

$$f_1(z_1, z_2) = \frac{1}{2\sqrt{2}(\sqrt{2}+1)}e^{2z_1+a_2z_2} - \frac{1}{2\sqrt{2}(\sqrt{2}-1)}e^{-2z_1-a_2z_2} - \sin(\pi i z_2)e^{-\sqrt{2}z_1+(\sqrt{2}+0.5)\pi i z_2},$$

$$f_2(z_1, z_2) = -\frac{1}{2\sqrt{2}(\sqrt{2}+1)}e^{2z_1+a_2z_2} + \frac{1}{2\sqrt{2}(\sqrt{2}-1)}e^{-2z_1-a_2z_2} + \sin(\pi i z_2)e^{-\sqrt{2}z_1+(\sqrt{2}+0.5)\pi i z_2},$$

where $a_2 = \log(\sqrt{2}+1) - \frac{\pi}{2}i$. Thus, (f_1, f_2) is a pair of finite order transcendental entire solution of system (1.6) with $\lambda = 1$, $\mu = \sqrt{2}$, $\alpha = 1$, $\beta = i$, and $(c_1, c_2) = (\pi i, 1)$.

Example 2.6. Let $\gamma = -\frac{1}{4}\ln 2 + \frac{\pi}{8}i + \frac{1}{2}i$, $a_2 = \frac{1}{4}\ln 2 - \frac{\pi}{8}i - \frac{1}{2}i$ and

$$f_1(z_1, z_2) = \frac{1}{4}e^{\frac{i}{2}z_1+a_2z_2} - \frac{iz_1}{4}e^{-\frac{i}{2}z_1-a_2z_2} - \cos(\pi i z_2)e^{-\frac{i}{2}z_1+\gamma z_2},$$

$$f_2(z_1, z_2) = \frac{1}{4}e^{\frac{i}{2}z_1+a_2z_2} - \frac{iz_1}{4}e^{-\frac{i}{2}z_1-a_2z_2} + \cos(\pi i z_2)e^{-\frac{i}{2}z_1+\gamma z_2}.$$

Thus, (f_1, f_2) is a transcendental entire solution of (1.6) with $\lambda = -2i$, $\mu = 1$, $\alpha = 2^{\frac{5}{4}}e^{-\frac{\pi}{8}i}$, $\beta = 2$, $(c_1, c_2) = (1, 1)$, and $\rho(f) = 1$.

For $\alpha = 1$ and $\beta = 0$ in system (1.6), we have

Corollary 2.1. *Let $c = (c_1, c_2) \in \mathbb{C}^2$, $c_2 \neq 0$, and μ, λ be nonzero constants. If (f_1, f_2) are a pair of finite order transcendental entire solution of the following system*:

$$\begin{cases} \left[\mu f_1(z) + \lambda\frac{\partial f_1}{\partial z_1}\right]^2 + f_2(z+c)^2 = 1, \\ \left[\mu f_2(z) + \lambda\frac{\partial f_2}{\partial z_1}\right]^2 + f_1(z+c)^2 = 1, \end{cases} \tag{2.5}$$

then, (f_1, f_2) must be of the form

$$f_1(z_1, z_2) = \frac{1}{2(\lambda a_1 + \mu)}e^{a_1z_1+a_2z_2+b_1} + \frac{1}{2(\mu - \lambda a_1)}e^{-a_1z_1-a_2z_2-b_1},$$

$$f_2(z_1, z_2) = \frac{1}{2(\lambda a_1 + \mu)}e^{a_1z_1+a_2z_2+b_2} + \frac{1}{2(\mu - \lambda a_1)}e^{-a_1z_1-a_2z_2-b_2},$$

where $a_1, a_2, b_1, b_2 \in \mathbb{C}$ satisfy $a_1^2 = \frac{\mu^2-1}{\lambda^2}$ and

$$e^{2(a_1c_1+a_2c_2)} = (\mu + \lambda a_1)^2 = \frac{1}{(\mu - \lambda a_1)^2}, \quad e^{2(b_1-b_2)} = -1, \tag{2.6}$$

or

$$e^{2(a_1c_1+a_2c_2)} = -(\mu + \lambda a_1)^2 = -\frac{1}{(\mu - \lambda a_1)^2}, \quad e^{2(b_1-b_2)} = 1. \tag{2.7}$$

Theorem 2.2. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$, $\alpha, \beta, \mu, \lambda_1, \lambda_2$ *be nonzero constants in* $\mathbb{C}$, $s_1 := \lambda_2 z_1 - \lambda_1 z_2$, *and* $s_0 := \lambda_1 c_2 - \lambda_2 c_1 \neq 0$. *Let* (f_1, f_2) *be a pair of the finite order transcendental entire solutions of system* (1.7). *Then,* (f_1, f_2) *must satisfy one of the following cases*:

(*i*)

$$f_1(z_1, z_2) = \frac{\eta_1}{\mu} - \frac{1}{\mu} e^{-\frac{\mu}{\lambda_1} z_1 + \gamma(\lambda_1 z_2 - \lambda_2 z_1) + D_1},$$

$$f_2(z_1, z_2) = \frac{\delta_1}{\mu} - \frac{1}{\mu} e^{-\frac{\mu}{\lambda_1} z_1 + \gamma(\lambda_1 z_2 - \lambda_2 z_1) + D_2},$$

where $\eta_1, \delta_1, \gamma, D_1, D_2 \in \mathbb{C}$ *satisfy*

$$e^{D_1 - D_2} = 1, \quad \gamma = \frac{\frac{\mu}{\lambda_1} c_1 + \log\frac{\beta}{\alpha} + 2k\pi i}{\lambda_1 c_2 - \lambda_2 c_1}, \quad k \in \mathbb{Z}, \tag{2.8}$$

and one of the following cases:

(i_1) $\delta_1 = -\eta_1 = \pm 1$ *and* $\alpha = -\beta$ *or* $\delta_1 = \eta_1 = \pm 1$, $\alpha = \beta$;

(i_2) $\delta_1 = \eta_1$ *and* $\eta_1^2 = \frac{\mu^2}{\mu^2 + (\alpha - \beta)^2}$ *or* $\delta_1 = -\eta_1$ *and* $\eta_1^2 = \frac{\mu^2}{\mu^2 + (\alpha + \beta)^2}$;

(i_3) $\mu^2 = \alpha^2 - \beta^2$, $\delta_1^2 + \delta_2^2 = 1$, $\eta_1^2 + \eta_2^2 = 1$, *and* $\alpha\delta_2 = -\beta\eta_2 \pm \sqrt{(\eta_2^2 - 1)(\beta^2 - \alpha^2)}$;

(*ii*) *if* $\mu^2 \neq (\lambda_1 a_1 + \lambda_2 a_2)^2$, *then*

$$f_1(z_1, z_2) = \frac{1}{2(\lambda_1 a_1 + \lambda_2 a_2 + \mu)} e^{a_1 z_1 + a_2 z_2 + b_1} - \frac{1}{2(\lambda_1 a_1 + \lambda_2 a_2 - \mu)} e^{-a_1 z_1 - a_2 z_2 - b_1} - \vartheta(s_1) e^{-\frac{\mu}{\lambda_1} z_1 + \gamma s_1 + D_1},$$

$$f_2(z_1, z_2) = \frac{1}{2(\lambda_1 a_1 + \lambda_2 a_2 + \mu)} e^{a_1 z_1 + a_2 z_2 + b_2} - \frac{1}{2(\lambda_1 a_1 + \lambda_2 a_2 - \mu)} e^{-a_1 z_1 - a_2 z_2 - b_2} - \vartheta(s_1 + s_0) e^{-\frac{\mu}{\lambda_1} z_1 + \gamma s_1 + D_2},$$

where $\vartheta(s_1)$ *is a finite order period entire functions with period* $2s_0$, *and* $a_1, a_2, b_1, b_2, \gamma, D_1, D_2$ *satisfy* (2.2), (2.8), *and*

$$(\lambda_1 a_1 + \lambda_2 a_2 + \beta i)^2 = \mu^2 - \alpha^2, \quad e^{2(a_1c_1+a_2c_2)} = \frac{\lambda_1 a_1 + \lambda_2 a_2 + \beta i + \mu}{\lambda_1 a_1 + \lambda_2 a_2 + \beta i - \mu}; \tag{2.9}$$

if $\mu = \lambda_1 a_1 + \lambda_2 a_2$, *then*

$$f_1(z_1, z_2) = \frac{1}{4\mu} e^{a_1 z_1 + a_2 z_2 + b_1} + \frac{z_1}{2\lambda_1} e^{-a_1 z_1 - a_2 z_2 - b_1} - \vartheta(s_1) e^{-\frac{\mu}{\lambda_1} z_1 + \gamma s_1 + D_1},$$

$$f_2(z_1, z_2) = \frac{1}{4\mu} e^{a_1 z_1 + a_2 z_2 + b_2} + \frac{z_1}{2\lambda_1} e^{-a_1 z_1 - a_2 z_2 - b_2} - \vartheta(s_1 + s_0) e^{-\frac{\mu}{\lambda_1} z_1 + \gamma s_1 + D_2},$$

where $a_1, a_2, b_1, b_2, \gamma \in \mathbb{C}$ *satisfy* (2.2), (2.8), *and*

$$2\mu\beta i = \beta^2 - \alpha^2, \quad \beta c_1 = \lambda_1 i, \quad e^{2(a_1c_1+a_2c_2)} = 1 - \frac{2\mu}{\beta} i; \tag{2.10}$$

if $\mu = -(\lambda_1 a_1 + \lambda_2 a_2)$, *then*

$$f_1(z_1, z_2) = \frac{z_1}{2\lambda_1} e^{a_1 z_1 + a_2 z_2 + b_1} + \frac{1}{4\mu} e^{-a_1 z_1 - a_2 z_2 - b_1} - \vartheta(s_1) e^{-\frac{\mu}{\lambda_1} z_1 + \gamma s_1 + D_1},$$

$$f_2(z_1, z_2) = \frac{z_1}{2\lambda_1} e^{a_1 z_1 + a_2 z_2 + b_2} + \frac{1}{4\mu} e^{-a_1 z_1 - a_2 z_2 - b_2} - \vartheta(s_1 + s_0) e^{-\frac{\mu}{\lambda_1} z_1 + \gamma s_1 + D_2},$$

where $a_1, a_2, b_1, b_2, \gamma \in \mathbb{C}$ *satisfy* (2.2), (2.8), *and*

$$-2\mu\beta i = \beta^2 - \alpha^2, \quad \beta c_1 = -\lambda_1 i, \quad e^{-2(a_1c_1+a_2c_2)} = 1 + \frac{2\mu}{\beta}i. \tag{2.11}$$

The following examples show the existence of transcendental entire solutions of system (1.7).

Example 2.7. Let

$$f_1(z_1, z_2) = -\frac{1}{2} - \frac{1}{2}e^{-z_1+z_2}, \quad f_2(z_1, z_2) = -\frac{1}{2} - \frac{1}{2}e^{-z_1+z_2}.$$

Thus, (f_1, f_2) is a pair of finite order transcendental entire solution of system (1.7) with $\lambda_1 = 1$, $\lambda_2 = -1$, $\mu = 2$, $\alpha = 1$, $\beta = 1$, and $(c_1, c_2) = (\kappa, \kappa)$, $\kappa \in \{\mathbb{C}\} - \{0\}$.

Example 2.8. Let

$$f_1(z_1, z_2) = \frac{\sqrt{2}}{4} - \frac{1}{2}e^{-z_1+z_2}, \quad f_2(z_1, z_2) = -\frac{\sqrt{2}}{4} - \frac{1}{2}e^{-z_1+z_2}.$$

Thus, (f_1, f_2) is a pair of finite order transcendental entire solution of system (1.7) with $\lambda_1 = 1$, $\lambda_2 = -1$, $\mu = 2$, $\alpha = 1$, $\beta = 1$, and $(c_1, c_2) = (\kappa, \kappa)$, $\kappa \in \{\mathbb{C}\} - \{0\}$.

Example 2.9. Let

$$f_1(z_1, z_2) = \frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}}e^{\left(-\frac{\sqrt{2}}{2}+\frac{1}{4}\right)z_1+\left(\frac{\sqrt{2}}{2}+\frac{1}{4}\right)z_2},$$

$$f_2(z_1, z_2) = \frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{2}}i - \frac{1}{\sqrt{2}}e^{\left(-\frac{\sqrt{2}}{2}+\frac{1}{4}\right)z_1+\left(\frac{\sqrt{2}}{2}+\frac{1}{4}\right)z_2}.$$

Thus, (f_1, f_2) is a pair of finite order transcendental entire solution of system (1.7) with $\lambda_1 = 1$, $\lambda_2 = -1$, $\mu = \sqrt{2}$, $\alpha = 1$, $\beta = i$, and $(c_1, c_2) = (\pi i, \pi i)$.

Example 2.10. Let

$$f_1(z_1, z_2) = \frac{1}{2\sqrt{3}}e^{\frac{4}{3}z_1-\frac{5}{6}z_2} + \frac{z_1}{2\sqrt{3}}e^{-\frac{4}{3}z_1+\frac{5}{6}z_2} - \cos\left[\frac{\pi i}{\kappa}(z_1 - z_2)\right]e^{-\frac{1}{2}z_1+A(z_1-z_2)},$$

$$f_2(z_1, z_2) = \frac{1}{2\sqrt{3}}e^{\frac{4}{3}z_1-\frac{5}{6}z_2} + \frac{z_1}{2\sqrt{3}}e^{-\frac{4}{3}z_1+\frac{5}{6}z_2} - \cos\left[\frac{\pi i}{\kappa}(z_1 - z_2)\right]e^{-\frac{1}{2}z_1+A(z_1-z_2)},$$

where

$$A = \frac{\frac{\sqrt{3}}{2} + \frac{1}{2}i + \frac{\pi}{3\sqrt{3}}i}{-9 + (3\sqrt{3} + 4\pi)i}, \quad \kappa = \sqrt{3}\left(\frac{9}{10} - \frac{\sqrt{3} + 4\pi}{10}i\right).$$

Thus, (f_1, f_2) is a pair of finite order transcendental entire solution of system (1.7) with $\lambda_1 = \sqrt{3}$, $\lambda_2 = \sqrt{3}$, $\mu = \frac{\sqrt{3}}{2}$, $\alpha = 1$, $\beta = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, and

$$(c_1, c_2) = \left(\frac{3}{2} + \frac{\sqrt{3}}{2}i, \frac{3}{5} + \left(\frac{4\sqrt{3} + 2\pi}{5}\right)i\right).$$

For $\alpha = 1$ and $\beta = 0$ in system (1.7), we have

Corollary 2.2. *Let $c = (c_1, c_2) \in \mathbb{C}^2$, μ, λ_1, λ_2 be nonzero constants, and $\lambda_1 c_2 - \lambda_2 c_1 \neq 0$. If (f_1, f_2) are a pair of finite order transcendental entire solution of the following system*:

$$\begin{cases} \left[\mu f_1(z) + \lambda_1 \frac{\partial f_1}{\partial z_1} + \lambda_2 \frac{\partial f_1}{\partial z_2}\right]^2 + f_2(z+c)^2 = 1, \\ \left[\mu f_2(z) + \lambda_1 \frac{\partial f_2}{\partial z_1} + \lambda_2 \frac{\partial f_2}{\partial z_2}\right]^2 + f_1(z+c)^2 = 1, \end{cases} \tag{2.12}$$

then, (f_1, f_2) must be of the form

$$f_1(z_1, z_2) = \frac{1}{2(\lambda_1 a_1 + \lambda_2 a_2 + \mu)} e^{a_1 z_1 + a_2 z_2 + b_1} + \frac{1}{2(\mu - \lambda_1 a_1 - \lambda_2 a_2)} e^{-a_1 z_1 - a_2 z_2 - b_1},$$

$$f_2(z_1, z_2) = \frac{1}{2(\lambda_1 a_1 + \lambda_2 a_2 + \mu)} e^{a_1 z_1 + a_2 z_2 + b_2} + \frac{1}{2(\mu - \lambda_1 a_1 - \lambda_2 a_2)} e^{-a_1 z_1 - a_2 z_2 - b_2},$$

where $a_1, a_2, b_1, b_2 \in \mathbb{C}$ satisfy $(\lambda_1 a_1 + \lambda_2 a_2)^2 = \mu^2 - 1$ and

$$e^{2(a_1 c_1 + a_2 c_2)} = (\mu + \lambda_1 a_1 + \lambda_2 a_2)^2 = \frac{1}{(\mu - \lambda_1 a_1 - \lambda_2 a_2)^2}, \quad e^{2(b_1 - b_2)} = -1, \tag{2.13}$$

or

$$e^{2(a_1 c_1 + a_2 c_2)} = -(\mu + \lambda_1 a_1 + \lambda_2 a_2)^2 = -\frac{1}{(\mu - \lambda_1 a_1 - \lambda_2 a_2)^2}, \quad e^{2(b_1 - b_2)} = 1. \tag{2.14}$$

3 Some lemmas

The following lemmas play the key role in proving our results.

Lemma 3.1. [27,28] *For an entire function F on $\mathbb{C}^n$, $F(0) \neq 0$ and put $\rho(n_F) = \rho < \infty$. Then, there exist a canonical function f_F and a function $g_F \in \mathbb{C}^n$ such that $F(z) = f_F(z)e^{g_F(z)}$. For the special case $n = 1$, f_F is the canonical product of Weierstrass.*

Remark 3.1. Here denote $\rho(n_F)$ to be the order of the counting function of zeros of F.

Lemma 3.2. [3] *If g and h are entire functions on the complex plane $\mathbb{C}$ and $g(h)$ is an entire function of finite order, then there are only two possible cases: either*
(*a*) *the internal function h is a polynomial and the external function g is of finite order*; *or else*
(*b*) *the internal function h is not a polynomial but a function of finite order, and the external function g is of zero order.*

Lemma 3.3. [34, Lemma 3.1] *Let $f_j(\not\equiv 0)$, $j = 1, 2, 3$, be meromorphic functions on $\mathbb{C}^m$ such that f_1 is not constant, and $f_1 + f_2 + f_3 = 1$, and such that*

$$\sum_{j=1}^{3} \left\{ N_2\left(r, \frac{1}{f_j}\right) + 2\overline{N}(r, f_j) \right\} < \lambda T(r, f_1) + O(\log^+ T(r, f_1)),$$

for all r outside possibly a set with finite logarithmic measure, where $\lambda < 1$ is a positive number. Then, either $f_2 = 1$ or $f_3 = 1$.

Remark 3.2. Here $N_2(r, \frac{1}{f})$ is the counting function of the zeros of f in $|z| \leq r$, where the simple zero is counted once, and the multiple zero is counted twice.

Lemma 3.4. *Let $c = (c_1, c_2) \in \mathbb{C}^2$ and α, β be two nonzero constants. Let*

$$g_1(z) = \theta_1(z_2)e^{\xi z_1+\psi_1(z_2)}, \quad g_2(z) = \theta_2(z_2)e^{\xi z_1+\psi_2(z_2)},$$

where ξ is a constant, $\theta_j(z_2), j = 1, 2$ are finite order entire functions, and $\psi_j(z_2), j = 1, 2$ are polynomials in z_2. If (g_1, g_2) is a pair of solutions of system

$$\begin{cases} \alpha g_2(z + c) - \beta g_1(z) = 0, \\ \alpha g_1(z + c) - \beta g_2(z) = 0, \end{cases} \tag{3.1}$$

then, (g_1, g_2) can be expressed as the form of

$$g_1(z) = \theta(z_2)e^{\xi z_1+\gamma z_2+D_1}, \quad g_2(z) = \theta(z_2 + c_2)e^{\xi z_1+\gamma z_2+D_2},$$

where $\theta(z_2)$ are finite order entire period function with the period $2c_2$ and γ is a constant such that

$$e^{D_1-D_2} = 1, \quad e^{\gamma c_2} = \frac{\beta}{\alpha}e^{-c_1\xi}.$$

Proof. Substituting g_1, g_2 into system (3.1), we have

$$\begin{cases} e^{\psi_2(z_2+c_2)-\psi_1(z_2)} = \dfrac{\theta_1(z_2)}{\theta_2(z_2 + c_2)}\dfrac{\beta}{\alpha}e^{-c_1\xi}, \\ e^{\psi_1(z_2+c_2)-\psi_2(z_2)} = \dfrac{\theta_2(z_2)}{\theta_1(z_2 + c_2)}\dfrac{\beta}{\alpha}e^{-c_1\xi}, \end{cases} \tag{3.2}$$

which implies

$$e^{\psi_j(z_2+2c_2)-\psi_j(z_2)} = \frac{\theta_j(z_2)}{\theta_j(z_2 + 2c_2)}\frac{\beta^2}{\alpha^2}e^{-2c_1\xi}, \quad j = 1, 2. \tag{3.3}$$

Noting that $\psi_j(z_2)$ is a polynomial, we will consider two cases as follows.

Case 1. Suppose that $e^{\psi_j(z_2+2c_2)-\psi_j(z_2)}$ is a constant. In view of (3.3), it follows that $\frac{\theta_j(z_2)}{\theta_j(z_2+2c_2)}$ is a nonzero constant for $j = 1, 2$. Set

$$\frac{\theta_j(z_2)}{\theta_j(z_2 + 2c_2)} = d, \quad j = 1, 2. \tag{3.4}$$

In view of (3.3), we can deduce that

$$\psi_j(z_2) = hz_2 + D_j, \quad j = 1, 2, \tag{3.5}$$

where $h, D_j, (j = 1, 2)$ are constants and

$$h = \gamma + \frac{\log d}{2c_2}, \quad \gamma = \frac{-c_1\xi + \log\frac{\beta}{\alpha}}{c_2}. \tag{3.6}$$

If $d = 1$, then $\theta_j(z_2)(j = 1, 2)$ are finite order entire period functions with period $2c_2$ and $h = \gamma = \frac{-c_1\xi+\log\frac{\beta}{\alpha}}{c_2}$. Substituting (3.5) and (3.6) in (3.2), we have

$$\theta_2(z_2)e^{D_2} = \theta_1(z_2 + c_2)e^{D_1}.$$

Thus, in view of (3.5) and (3.6), it follows that

$$g_1(z) = \theta_1(z_2)e^{\xi z_1+\gamma z_2+D_1},$$

and

$$g_2(z) = \theta_2(z_2)e^{\xi z_1+hz_2+D_2} = \theta_1(z_2 + c_2)e^{\xi z_1+\gamma z_2+D_1}.$$

If $d \neq 1$, it follows from (3.4) that

$$\theta_j(z_2) = e^{-\frac{\log d}{2c_2}z_2+\varepsilon_j}, \quad j = 1, 2, \tag{3.7}$$

where $\varepsilon_j, j = 1, 2$ are constants. Substituting (3.5)–(3.7) in (3.2), we have $e^{D_2-D_1} = e^{\varepsilon_1-\varepsilon_2}$, which implies

$$e^{D_1+\varepsilon_1-(D_2+\varepsilon_2)} = 1. \tag{3.8}$$

For convenience, let $B_1 = D_1 + \varepsilon_1$ and $B_2 = D_2 + \varepsilon_2$. In view of (3.5)–(3.8), it follows that

$$g_1(z) = \theta_1(z_2)e^{\xi z_1+hz_2+D_1} = e^{\xi z_1+\gamma z_2+B_1}$$

and

$$g_2(z) = \theta_2(z_2)e^{\xi z_1+hz_2+D_2} = e^{\xi z_1+\gamma z_2+B_2},$$

where B_1, B_2 are constants satisfying (3.8).

Case 2. Suppose that $e^{\psi_j(z_2+2c_2)-\psi_j(z_2)}(j = 1, 2)$ are not constants. Noting that $\psi_j(z_2), j = 1, 2$, are nonconstant polynomials, we can deduce from (3.3) that $\frac{\theta_j(z_2)}{\theta_j(z_2+2c_2)}, j = 1, 2$ are finite order transcendental entire functions. Thus, there exist two functions $\varrho_1(z_2), \varrho_2(z_2)$ such that

$$\theta_j(z_2) = e^{\varrho_j(z_2)}, \quad j = 1, 2. \tag{3.9}$$

This leads to

$$g_1(z) = e^{\xi z_1+\mu_1(z_2)}, \quad g_2(z) = e^{\xi z_1+\mu_2(z_2)}, \tag{3.10}$$

where $\mu_j(z_2) = \psi_j(z_2) + \varrho_j(z_2)$. Substituting g_1, g_2 in (3.1), we have

$$\begin{cases} e^{\mu_2(z_2+c_2)-\mu_1(z_2)} = \dfrac{\beta}{\alpha}e^{-c_1\xi}, \\ e^{\mu_1(z_2+c_2)-\mu_2(z_2)} = \dfrac{\beta}{\alpha}e^{-c_1\xi}, \end{cases} \tag{3.11}$$

which implies that

$$\mu_j(z_2) = \gamma z_2 + D_j, \quad j = 1, 2, \tag{3.12}$$

where $\gamma, D_j, (j = 1, 2)$ are constants and

$$\gamma = \frac{-c_1\xi + \log\frac{\beta}{\alpha}}{c_2}, \quad e^{D_1-D_2} = 1. \tag{3.13}$$

In view of (3.10) and (3.12), we have

$$g_1(z) = e^{\xi z_1+\gamma z_2+D_1}, \quad g_2(z) = e^{\xi z_1+\gamma z_2+D_2}, \tag{3.14}$$

where γ, D_1, D_2 satisfy (3.13).

Therefore, this completes the proof of Lemma 3.4. □

4 The proof of Theorem 2.1

Proof. Let (f_1, f_2) be a pair of transcendental entire solutions of finite order for system (1.6). Thus, we will consider the following two cases.

(i) If $\mu f_1(z) + \lambda\frac{\partial f_1}{\partial z_1}$ is a constant. Denote

$$\mu f_1(z) + \lambda\frac{\partial f_1}{\partial z_1} = \eta_1. \tag{4.1}$$

In view of (1.6), it follows that

$$\alpha f_2(z+c) - \beta f_1(z) = \eta_2, \tag{4.2}$$

where η_2 is a constant satisfying

$$\eta_1^2 + \eta_2^2 = 1. \tag{4.3}$$

From (4.1) and (4.2), we have

$$\mu f_2 + \lambda\frac{\partial f_2}{\partial z_1} = \frac{\beta}{\alpha}\left[\mu f_1(z-c) + \lambda\frac{\partial f_1(z-c)}{\partial z_1}\right] + \mu\frac{\eta_2}{\alpha} = \eta_1\frac{\beta}{\alpha} + \mu\frac{\eta_2}{\alpha}. \tag{4.4}$$

This shows that $\mu f_2 + \lambda\frac{\partial f_2}{\partial z_1}$ is a constant. Let

$$\mu f_2 + \lambda\frac{\partial f_2}{\partial z_1} = \delta_1, \tag{4.5}$$

then $\alpha f_1(z+c) - \beta f_2(z)$ is a constant. Denote

$$\alpha f_1(z+c) - \beta f_2(z) = \delta_2, \tag{4.6}$$

then it follows

$$\delta_1^2 + \delta_2^2 = 1, \quad \delta_1 = \frac{\beta}{\alpha}\eta_1 + \frac{\mu}{\alpha}\eta_2. \tag{4.7}$$

Solving equations (4.1) and (4.5), we have

$$f_1(z_1, z_2) = \frac{\eta_1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu}{\lambda}z_1+\phi_1(z_2)}, \quad f_2(z_1, z_2) = \frac{\delta_1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu}{\lambda}z_1+\phi_2(z_2)}, \tag{4.8}$$

where $\phi_1(z_2)$, $\phi_2(z_2)$ are entire functions in z_2. Substituting (4.8) in (4.2) and (4.6), we have

$$\begin{cases} \alpha\left(\frac{\delta_1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu}{\lambda}(z_1+c_1)+\phi_2(z_2+c_2)}\right) - \beta\left(\frac{\eta_1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu}{\lambda}z_1+\phi_1(z_2)}\right) = \eta_2, \\ \alpha\left(\frac{\eta_1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu}{\lambda}(z_1+c_1)+\phi_1(z_2+c_2)}\right) - \beta\left(\frac{\delta_1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu}{\lambda}z_1+\phi_2(z_2)}\right) = \delta_2, \end{cases}$$

which implies

$$\begin{cases} \alpha\delta_1 - \beta\eta_1 = \mu\eta_2, \\ e^{\phi_2(z_2+c_2)-\phi_1(z_2)} = \frac{\beta}{\alpha}e^{\frac{\mu}{\lambda}c_1}, \\ \alpha\eta_1 - \beta\delta_1 = \mu\delta_2, \\ e^{\phi_1(z_2+c_2)-\phi_2(z_2)} = \frac{\beta}{\alpha}e^{\frac{\mu}{\lambda}c_1}. \end{cases} \tag{4.9}$$

Thus, it yields that

$$\phi_1(z_2) = \gamma z_2 + D_1, \quad \phi_2(z_2) = \gamma z_2 + D_2, \tag{4.10}$$

where γ, D_1, D_2 are constants satisfying

$$e^{2(D_1-D_2)} = 1, \quad \gamma = \frac{\frac{\mu}{\lambda}c_1 - \log\frac{\alpha}{\beta} + k\pi i}{c_2}, k \in \mathbb{Z}. \tag{4.11}$$

Moreover, it follows from (4.7) and (4.9) that

$$\eta_1 = \frac{\beta}{\alpha}\delta_1 + \frac{\mu}{\alpha}\delta_2, \quad (\delta_2^2 - \eta_2^2)[\mu^2 - (\alpha^2 - \beta^2)]\frac{\alpha^2 - \beta^2}{\mu^2} = 0. \tag{4.12}$$

(a) If $\delta_2 = \eta_2$, it follows from (4.7) and (4.9) that

$$\delta_1 = \pm\eta_1, \quad (\eta_1 - \delta_1)\left(1 + \frac{\beta}{\alpha}\right) = 0. \tag{4.13}$$

If $\delta_1 = \eta_1$ and $\alpha = \beta$, then $\delta_2 = \eta_2 = 0$ and $\delta_1 = \eta_1 = \pm 1$.

If $\delta_1 = \eta_1$ and $\alpha \neq \beta$, then it follows from (4.12) that $\eta_2 = \frac{\alpha-\beta}{\mu}\eta_1$. Substituting this into (4.3), we have

$$\delta_1^2 = \eta_1^2 = \frac{\mu^2}{\mu^2 + (\alpha - \beta)^2}.$$

If $\delta_1 = -\eta_1$, then it follows from (4.13) that $\alpha = -\beta$. Thus, we can deduce from (4.7) or (4.12) that $\delta_2 = \eta_2 = 0$, which implies that $\delta_1 = -\eta_1 = 1$ or $\delta_1 = -\eta_1 = -1$.

(b) If $\delta_2 = -\eta_2$, it follows from (4.2), (4.7), and (4.9) that

$$\delta_1 = \pm\eta_1, \quad (\eta_1 + \delta_1)\left(1 - \frac{\beta}{\alpha}\right) = 0. \tag{4.14}$$

If $\delta_1 = -\eta_1$ and $\alpha = -\beta$, then $\delta_2 = \eta_2 = 0$. Then, it yields that $\delta_1 = -\eta_1 = 1$ or $\delta_1 = -\eta_1 = -1$.

If $\delta_1 = -\eta_1$ and $\alpha \neq -\beta$, then it follows from (4.12) that $\eta_2 = -\frac{\alpha+\beta}{\mu}\eta_1$. Substituting this into (4.3), we have

$$\delta_1^2 = \eta_1^2 = \frac{\mu^2}{\mu^2 + (\alpha + \beta)^2}.$$

If $\delta_1 = \eta_1$, then it follows from (4.14) that $\alpha = \beta$. Thus, we can deduce from (4.7) or (4.12) that $\delta_2 = \eta_2 = 0$, which implies that $\delta_1 = \eta_1 = \pm 1$.

(c) If $\mu^2 - (\alpha^2 - \beta^2) = 0$, it follows from $\mu \neq 0$ that $\alpha \neq \pm\beta$. Then we have

$$\alpha\delta_2 = -\beta\eta_2 \pm \sqrt{(\eta_2^2 - 1)(\beta^2 - \alpha^2)}. \tag{4.15}$$

Therefore, from (4.8), (4.10), (4.11) and (a), (b), (c), we obtain the conclusion (i) of Theorem 2.1.

(ii) If $\mu f_1(z) + \lambda\frac{\partial f_1}{\partial z_1}$ is a nonconstant, then it yields that $\alpha f_2(z+c) - \beta f_1(z)$, $\mu f_2 + \lambda\frac{\partial f_2}{\partial z_1}$, and $\alpha f_1(z+c) - \beta f_2(z)$ are all nonconstant. Otherwise, if one of these terms is a constant, we can deduce that $\mu f_1(z) + \lambda\frac{\partial f_1}{\partial z_1}$ is a constant. This is a contradiction. Thus, we can rewrite (1.6) as the form

$$\begin{cases} \left[\mu f_1(z) + \lambda\dfrac{\partial f_1}{\partial z_1} + i(\alpha f_2(z+c) - \beta f_1(z))\right]\left[\mu f_1(z) + \lambda\dfrac{\partial f_1}{\partial z_1} - i(\alpha f_2(z+c) - \beta f_1(z))\right] = 1, \\ \left[\mu f_2(z) + \lambda\dfrac{\partial f_2}{\partial z_1} + i(\alpha f_1(z+c) - \beta f_2(z))\right]\left[\mu f_2(z) + \lambda\dfrac{\partial f_2}{\partial z_1} - i(\alpha f_1(z+c) - \beta f_2(z))\right] = 1. \end{cases}$$

Since f_1, f_2 are entire functions, it follows that $\mu f_1(z) + \lambda\frac{\partial f_1}{\partial z_1} + i(\alpha f_2(z+c) - \beta f_1(z))$, $\mu f_1(z) + \lambda\frac{\partial f_1}{\partial z_1} - i(\alpha f_2(z+c) - \beta f_1(z))$, $\mu f_2(z) + \lambda\frac{\partial f_2}{\partial z_1} - i(\alpha f_1(z+c) - \beta f_2(z))$ and $\mu f_2(z) + \lambda\frac{\partial f_2}{\partial z_1} + i(\alpha f_1(z+c) - \beta f_2(z))$ do not exist zeros and poles. By Lemmas 3.1 and 3.2, there exist two nonconstant polynomials $p(z), q(z)$ in $\mathbb{C}^2$ such that

$$\begin{cases} \mu f_1(z) + \lambda\dfrac{\partial f_1}{\partial z_1} + i(\alpha f_2(z+c) - \beta f_1(z)) = e^{p(z)}, \\ \mu f_1(z) + \lambda\dfrac{\partial f_1}{\partial z_1} - i(\alpha f_2(z+c) - \beta f_1(z)) = e^{-p(z)}, \\ \mu f_2(z) + \lambda\dfrac{\partial f_2}{\partial z_1} + i(\alpha f_1(z+c) - \beta f_2(z)) = e^{q(z)}, \\ \mu f_2(z) + \lambda\dfrac{\partial f_2}{\partial z_1} - i(\alpha f_1(z+c) - \beta f_2(z)) = e^{-q(z)}. \end{cases}$$

The above equations lead to

$$\mu f_1(z) + \lambda\frac{\partial f_1}{\partial z_1} = \frac{1}{2}(e^{p} + e^{-p}), \tag{4.16}$$

$$\alpha f_2(z+c) - \beta f_1(z) = \frac{1}{2i}(e^{p} - e^{-p}), \tag{4.17}$$

$$\mu f_2(z) + \lambda\frac{\partial f_2}{\partial z_1} = \frac{1}{2}(e^{q} + e^{-q}), \tag{4.18}$$

$$\alpha f_1(z+c) - \beta f_2(z) = \frac{1}{2i}(e^{q} - e^{-q}). \tag{4.19}$$

In view of (4.17) and (4.18), we can deduce that

$$\alpha\mu f_2(z+c) + \beta\lambda\frac{\partial f_1}{\partial z_1} = \frac{\alpha}{2}(e^{q(z+c)} + e^{-q(z+c)}) - \frac{\lambda}{2i}\frac{\partial p}{\partial z_1}(e^{p(z)} + e^{-p(z)}). \tag{4.20}$$

In view of (4.16) and (4.17), we have

$$\alpha\mu f_2(z+c) + \beta\lambda\frac{\partial f_1}{\partial z_1} = \frac{\beta i + \mu}{2i}e^{p(z)} + \frac{\beta i - \mu}{2i}e^{-p(z)}. \tag{4.21}$$

By combining with (4.20) and (4.21), we have

$$\frac{\lambda\frac{\partial p}{\partial z_1} + \beta i + \mu}{\alpha i}e^{q(z+c)+p(z)} + \frac{\lambda\frac{\partial p}{\partial z_1} + \beta i - \mu}{\alpha i}e^{q(z+c)-p(z)} - e^{2q(z+c)} \equiv 1. \tag{4.22}$$

Similar to the above argument, we can deduce from (4.16), (4.18), and (4.19) that

$$\frac{\lambda\frac{\partial q}{\partial z_1} + \beta i + \mu}{\alpha i}e^{p(z+c)+q(z)} + \frac{\lambda\frac{\partial q}{\partial z_1} + \beta i - \mu}{\alpha i}e^{p(z+c)-q(z)} - e^{2p(z+c)} \equiv 1. \tag{4.23}$$

By Lemma 3.3, we can deduce from (4.22) and (4.23) that

$$\frac{\lambda\frac{\partial p}{\partial z_1} + \beta i - \mu}{\alpha i}e^{q(z+c)-p(z)} \equiv 1 \quad \text{or} \quad \frac{\lambda\frac{\partial p}{\partial z_1} + \beta i + \mu}{\alpha i}e^{q(z+c)+p(z)} \equiv 1,$$

and

$$\frac{\lambda\frac{\partial q}{\partial z_1} + \beta i - \mu}{\alpha i}e^{p(z+c)-q(z)} \equiv 1 \quad \text{or} \quad \frac{\lambda\frac{\partial q}{\partial z_1} + \beta i + \mu}{\alpha i}e^{p(z+c)+q(z)} \equiv 1.$$

Now we will consider four cases as follows.

Case 1.

$$\begin{cases} \dfrac{\lambda\frac{\partial p}{\partial z_1} + \beta i - \mu}{\alpha i}e^{q(z+c)-p(z)} \equiv 1, \\ \dfrac{\lambda\frac{\partial q}{\partial z_1} + \beta i - \mu}{\alpha i}e^{p(z+c)-q(z)} \equiv 1. \end{cases} \tag{4.24}$$

In view of (4.24), it follows that $q(z+c) - p(z) = d_1$ and $p(z+c) - q(z) = d_2$, where d_1, d_2 are constants in $\mathbb{C}$. Thus, it yields that $q(z+2c) - q(z) = d_1 + d_2$ and $p(z+2c) - p(z) = d_1 + d_2$. Since p, q are polynomials in $\mathbb{C}^2$, it follows that $p(z) = L(z) + H(c_2z_1 - c_1z_2) + b_1$ and $q(z) = L(z) + H(c_2z_1 - c_1z_2) + b_2$, where $L(z)$ is a linear form of $L(z) = a_1z_1 + a_2z_2$, $H(s)$ is a polynomial in $s := c_2z_1 - c_1z_2$, a_1, a_2, b_1, b_2 are constants. Substituting $p(z), q(z)$ into (4.24), we have

$$\frac{\lambda a_1 + \lambda c_2 H' + \beta i - \mu}{\alpha i} e^{L(c)+b_2-b_1} \equiv 1, \quad \frac{\lambda a_1 + \lambda c_2 H' + \beta i - \mu}{\alpha i} e^{L(c)+b_1-b_2} \equiv 1. \tag{4.25}$$

By combining with $\lambda \neq 0$ and $c_2 \neq 0$, it follows from (4.25) that $\deg_s H \leq 1$. Thus, we still write $p(z)$, $q(z)$ as the forms of $p(z) = L(z) + b_1$ and $q(z) = L(z) + b_2$. In view of (4.22)–(4.24), we have

$$\begin{cases} \dfrac{\lambda a_1 + \beta i + \mu}{\alpha i} e^{-L(c)+b_1-b_2} \equiv 1, \\ \dfrac{\lambda a_1 + \beta i + \mu}{\alpha i} e^{-L(c)+b_2-b_1} \equiv 1, \\ \dfrac{\lambda a_1 + \beta i - \mu}{\alpha i} e^{L(c)+b_2-b_1} \equiv 1, \\ \dfrac{\lambda a_1 + \beta i - \mu}{\alpha i} e^{L(c)+b_1-b_2} \equiv 1. \end{cases} \tag{4.26}$$

Thus, we can deduce from (4.26) that

$$(\lambda a_1 + \beta i)^2 = \mu^2 - \alpha^2, \quad e^{2(b_1-b_2)} = 1, \quad e^{2L(c)} = \frac{-\alpha^2}{(\lambda a_1 + \beta i - \mu)^2} = \frac{(\lambda a_1 + \beta i + \mu)^2}{-\alpha^2}. \tag{4.27}$$

If $a_1 \neq \pm\frac{\mu}{\lambda}$, solving equations (4.16) and (4.18), we have

$$f_1(z_1, z_2) = \frac{e^{a_1z_1+a_2z_2+b_1}}{2(\lambda a_1 + \mu)} - \frac{e^{-a_1z_1-a_2z_2-b_1}}{2(\lambda a_1 - \mu)} - \vartheta_1(z_2)e^{-\frac{\mu}{\lambda}z_1+\phi_1(z_2)}, \tag{4.28}$$

$$f_2(z_1, z_2) = \frac{e^{a_1z_1+a_2z_2+b_2}}{2(\lambda a_1 + \mu)} - \frac{e^{-a_1z_1-a_2z_2-b_2}}{2(\lambda a_1 - \mu)} - \vartheta_2(z_2)e^{-\frac{\mu}{\lambda}z_1+\phi_2(z_2)}, \tag{4.29}$$

where $\vartheta_1(z_2)$, $\vartheta_2(z_2)$ are finite order entire functions and $\phi_1(z_2)$, $\phi_2(z_2)$ are polynomials in z_2. Substituting (4.28) and (4.29) in (4.17) and (4.19), and combining with (4.26) and (4.27), by Lemma 3.4, we have

$$f_1(z_1, z_2) = \frac{e^{a_1z_1+a_2z_2+b_1}}{2(\lambda a_1 + \mu)} - \frac{e^{-a_1z_1-a_2z_2-b_1}}{2(\lambda a_1 - \mu)} - \vartheta(z_2)e^{-\frac{\mu}{\lambda}z_1+\gamma z_2+D_1}, \tag{4.30}$$

$$f_2(z_1, z_2) = \frac{e^{a_1z_1+a_2z_2+b_2}}{2(\lambda a_1 + \mu)} - \frac{e^{-a_1z_1-a_2z_2-b_2}}{2(\lambda a_1 - \mu)} - \vartheta(z_2 + c_2)e^{-\frac{\mu}{\lambda}z_1+\gamma z_2+D_2}, \tag{4.31}$$

where $\vartheta(z_2)$ is a finite order period entire function with period $2c_2$, and γ, D_1, D_2 satisfy (4.11) and (4.27).

If $a_1 = \frac{\mu}{\lambda}$, solving equations (4.16) and (4.18), similar to the argument as in case $a_1 \neq \frac{\mu}{\lambda}$, we have

$$f_1(z_1, z_2) = \frac{1}{4\mu}e^{a_1z_1+a_2z_2+b_1} + \frac{z_1}{2\lambda}e^{-a_1z_1-a_2z_2-b_1} - \vartheta(z_2)e^{-\frac{\mu}{\lambda}z_1+\gamma z_2+D_1}, \tag{4.32}$$

$$f_2(z_1, z_2) = \frac{1}{4\mu}e^{a_1z_1+a_2z_2+b_2} + \frac{z_1}{2\lambda}e^{-a_1z_1-a_2z_2-b_2} - \vartheta(z_2 + c_2)e^{-\frac{\mu}{\lambda}z_1+\gamma z_2+D_2}, \tag{4.33}$$

where $\vartheta(z_2)$ is a finite order period entire function with period $2c_2$, and γ, D_1, D_2 satisfy (4.11) and (4.27). Substituting (4.32) and (4.33) in (4.17) and (4.19), and combining with (4.26) and (4.27), we have $\beta c_1 = \lambda i$.

If $a_1 = -\frac{\mu}{\lambda}$, solving equations (4.16) and (4.18), we have

$$f_1(z_1, z_2) = \frac{1}{4\mu}e^{-a_1z_1-a_2z_2-b_1} + \frac{z_1}{2\lambda}e^{a_1z_1+a_2z_2+b_1} - \vartheta(z_2)e^{-\frac{\mu}{\lambda}z_1+\gamma z_2+D_1}, \tag{4.34}$$

$$f_2(z_1, z_2) = \frac{1}{4\mu}e^{-a_1z_1-a_2z_2-b_2} + \frac{z_1}{2\lambda}e^{a_1z_1+a_2z_2+b_2} - \vartheta(z_2 + c_2)e^{-\frac{\mu}{\lambda}z_1+\gamma z_2+D_2}, \tag{4.35}$$

where $\vartheta(z_2)$ is a finite order period entire function with period $2c_2$, and γ, D_1, D_2 satisfy (4.11) and (4.27). Similar to the above argument, we have $\beta c_1 = -\lambda i$.

Case 2.

$$\begin{cases} \dfrac{\lambda \frac{\partial p}{\partial z_1} + \beta i - \mu}{\alpha i} e^{q(z+c)-p(z)} \equiv 1, \\ \dfrac{\lambda \frac{\partial q}{\partial z_1} + \beta i + \mu}{\alpha i} e^{p(z+c)+q(z)} \equiv 1. \end{cases}$$

Thus, it follows that $q(z + c) - p(z) = d_1$ and $p(z + c) + q(z) = d_2$ where d_1, d_2 are constants. Hence, we have $q(z + 2c) + q(z) = d_1 + d_2$, which is a contradiction with the assumption of $q(z)$ being nonconstant polynomial in $\mathbb{C}^2$.

Case 3.

$$\begin{cases} \dfrac{\lambda \frac{\partial p}{\partial z_1} + \beta i + \mu}{\alpha i} e^{q(z+c)+p(z)} \equiv 1, \\ \dfrac{\lambda \frac{\partial q}{\partial z_1} + \beta i - \mu}{\alpha i} e^{p(z+c)-q(z)} \equiv 1. \end{cases}$$

Thus, it follows that $q(z + c) + p(z) = d_1$ and $p(z + c) - q(z) = d_2$ where d_1, d_2 are constants. Hence, we have $p(z + 2c) + p(z) = d_1 + d_2$, which is a contradiction with the assumption of $p(z)$ being nonconstant polynomial in $\mathbb{C}^2$.

Case 4.

$$\begin{cases} \dfrac{\lambda \frac{\partial p}{\partial z_1} + \beta i + \mu}{\alpha i} e^{q(z+c)+p(z)} \equiv 1, \\ \dfrac{\lambda \frac{\partial q}{\partial z_1} + \beta i + \mu}{\alpha i} e^{p(z+c)+q(z)} \equiv 1. \end{cases} \tag{4.36}$$

In view of (4.36), it follows that $q(z + c) + p(z) = d_1$ and $p(z + c) + q(z) = d_2$, where d_1, d_2 are constants in $\mathbb{C}$. Thus, it yields that $q(z + 2c) - q(z) = d_1 - d_2$ and $p(z + 2c) - p(z) = d_1 - d_2$. Since p, q are polynomials in $\mathbb{C}^2$, it follows that $p(z) = L(z) + H(c_2z_1 - c_1z_2) + b_1$ and $q(z) = -L(z) - H(c_2z_1 - c_1z_2) + b_2$, where $L(z)$ is a linear form of $L(z) = a_1z_1 + a_2z_2$, $H(s)$ is a polynomial in $s := c_2z_1 - c_1z_2$, a_1, a_2, b_1, b_2 are constants. Similar to the argument as in Case 1, we can obtain that $p(z) = L(z) + b_1 = a_1z_1 + a_2z_2 + b_1$ and $q(z) = -L(z) + b_2 = -a_1z_1 - a_2z_2 + b_2$. In view of (4.22), (4.23), and (4.36), it follows

$$\begin{cases} \dfrac{\lambda a_1 + \beta i + \mu}{\alpha i} e^{-L(c)+b_2+b_1} \equiv 1, \\ \dfrac{-\lambda a_1 + \beta i + \mu}{\alpha i} e^{L(c)+b_1+b_2} \equiv 1, \\ \dfrac{\lambda a_1 + \beta i - \mu}{\alpha i} e^{L(c)-b_2-b_1} \equiv 1, \\ \dfrac{-\lambda a_1 + \beta i - \mu}{\alpha i} e^{-L(c)-b_1-b_2} \equiv 1. \end{cases} \tag{4.37}$$

Thus, it leads to

$$\frac{-\lambda a_1 + \beta i + \mu}{\alpha i} \frac{-\lambda a_1 + \beta i - \mu}{\alpha i} = \frac{\lambda a_1 + \beta i + \mu}{\alpha i} \frac{\lambda a_1 + \beta i - \mu}{\alpha i}.$$

By combining with $\alpha \neq 0$, $\beta \neq 0$, and $\lambda \neq 0$, we have $a_1 = 0$. Then, $p(z) = a_2z_2 + b_1$ and $q(z) = -a_2z_2 + b_2$. In view of (4.37), it follows

$$\mu^2 = \alpha^2 - \beta^2, \quad e^{2a_2c_2} \equiv 1, \quad e^{2(b_1+b_2)} = \frac{-\alpha^2}{(\beta i + \mu)^2} = \frac{(\beta i - \mu)^2}{-\alpha^2}. \tag{4.38}$$

Solving equations (4.16) and (4.18), we have

$$f_1(z_1, z_2) = \frac{1}{2\mu}(e^{a_2z_2+b_1} + e^{-a_2z_2-b_1}) - \vartheta_1(z_2)e^{-\frac{\mu}{\lambda}z_1+\phi_1(z_2)}, \tag{4.39}$$

$$f_2(z_1, z_2) = \frac{1}{2\mu}(e^{-a_2z_2+b_2} + e^{a_2z_2-b_2}) - \vartheta_2(z_2)e^{-\frac{\mu}{\lambda}z_1+\phi_2(z_2)}, \tag{4.40}$$

where $\vartheta_1(z_2), \vartheta_2(z_2)$ are finite order entire functions and $\phi_1(z_2), \phi_2(z_2)$ are polynomials in z_2. Similar to the above argument in Case 1, it follows from (4.39) and (4.40) that

$$f_1(z_1, z_2) = \frac{1}{2\mu}(e^{a_2z_2+b_1} + e^{-a_2z_2-b_1}) - \vartheta(z_2)e^{-\frac{\mu}{\lambda}z_1+\gamma z_2+D_1},$$
$$f_2(z_1, z_2) = \frac{1}{2\mu}(e^{-a_2z_2+b_2} + e^{a_2z_2-b_2}) - \vartheta(z_2 + c_2)e^{-\frac{\mu}{\lambda}z_1+\gamma z_2+D_2},$$

where $\vartheta(z_2)$ is a finite order period entire function with period $2c_2$, and γ, D_1, D_2 satisfy (4.11) and (4.27). In fact, we can see that the forms of solutions are included in case that $a_1 \neq \pm\frac{\mu}{\lambda}$ in Case 1.

Therefore, this completes the proof of Theorem 2.1. □

5 The proof of Theorem 2.2

Proof. Let (f_1, f_2) be a pair of transcendental entire solutions of finite order for system (1.7). Thus, we will consider the following two cases.

(i) If $\mu f_1(z) + \lambda_1\frac{\partial f_1}{\partial z_1} + \lambda_2\frac{\partial f_1}{\partial z_2}$ is a constant. Denote

$$\mu f_1(z) + \lambda_1\frac{\partial f_1}{\partial z_1} + \lambda_2\frac{\partial f_1}{\partial z_2} = \eta_1. \tag{5.1}$$

In view of (1.7), it follows that

$$\alpha f_2(z + c) - \beta f_1(z) = \eta_2, \tag{5.2}$$

where η_2 is a constant satisfying (4.3).

From (5.1) and (5.2), we have

$$\begin{aligned}\mu f_2 + \lambda_1\frac{\partial f_2}{\partial z_1} + \lambda_2\frac{\partial f_2}{\partial z_2} &= \frac{\beta}{\alpha}\left[\mu f_1(z - c) + \lambda_1\frac{\partial f_1(z - c)}{\partial z_1} + \lambda_2\frac{\partial f_1(z - c)}{\partial z_2}\right] + \mu\frac{\eta_2}{\alpha}\\ &= \eta_1\frac{\beta}{\alpha} + \mu\frac{\eta_2}{\alpha}.\end{aligned} \tag{5.3}$$

This shows that $\mu f_2 + \lambda_1\frac{\partial f_2}{\partial z_1} + \lambda_2\frac{\partial f_2}{\partial z_2}$ is a constant. Let

$$\mu f_2 + \lambda_1\frac{\partial f_2}{\partial z_1} + \lambda_2\frac{\partial f_2}{\partial z_2} = \delta_1, \tag{5.4}$$

then $\alpha f_1(z + c) - \beta f_2(z)$ is a constant. Denote

$$\alpha f_1(z + c) - \beta f_2(z) = \delta_2, \tag{5.5}$$

then, we have (4.7). The characteristic equations of (5.1) are

$$\frac{dz_1}{dt} = \lambda_1, \quad \frac{dz_2}{dt} = \lambda_2, \quad \frac{df_1}{dt} = \eta_1 - \mu f_1.$$

Using the initial conditions: $z_1 = 0, z_2 = s_1$, and $f_1 = f_1(0, s_1) := \psi(s_1)$ with a parameter s, we obtain the following parametric representation for the solutions of the characteristic equations: $z_1 = \lambda_1 t$, $z_2 = \lambda_2 t + s_1$, and

$$f_1(z_1, z_2) = \frac{\eta_1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu}{\lambda_1}z_1+\varphi_1(s_1)}, \tag{5.6}$$

where $\varphi_1(s_1)$ is an entire function in $s_1 := \lambda_2 z_1 - \lambda_1 z_2$. Similarly, solving equation (5.4), we have

$$f_2(z_1, z_2) = \frac{\delta_1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu}{\lambda_1}z_1+\varphi_2(s_1)}, \tag{5.7}$$

where $\varphi_2(s_1)$ is an entire function in $s_1 := \lambda_2 z_1 - \lambda_1 z_2$. Substituting (5.6) and (5.7) in (5.2) and (5.5), we have

$$\begin{cases} \alpha\left(\frac{\delta_1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu}{\lambda}(z_1+c_1)+\varphi_2(s_1+s_0)}\right) - \beta\left(\frac{\eta_1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu}{\lambda}z_1+\varphi_1(s_1)}\right) = \eta_2, \\ \alpha\left(\frac{\eta_1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu}{\lambda}(z_1+c_1)+\varphi_1(s_1+s_0)}\right) - \beta\left(\frac{\delta_1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu}{\lambda}z_1+\varphi_2(s_1)}\right) = \delta_2, \end{cases}$$

which implies

$$\begin{cases} \alpha\delta_1 - \beta\eta_1 = \mu\eta_2, \\ e^{\varphi_2(s_1+s_0)-\varphi_1(s_1)} = \frac{\beta}{\alpha}e^{\frac{\mu}{\lambda}c_1}, \\ \alpha\eta_1 - \beta\delta_1 = \mu\delta_2, \\ e^{\varphi_1(s_1+s_0)-\varphi_2(s_1)} = \frac{\beta}{\alpha}e^{\frac{\mu}{\lambda}c_1}. \end{cases} \tag{5.8}$$

Thus, it yields that

$$\varphi_1(s_1) = \gamma s_1 + D_1, \quad \varphi_2(s_1) = \gamma s_1 + D_2, \tag{5.9}$$

where D_1, D_2 are constants and satisfying

$$e^{2(D_1-D_2)} = 1, \quad \gamma = \frac{\frac{\mu}{\lambda}c_1 - \log\frac{\alpha}{\beta} + k\pi i}{\lambda_2 c_1 - \lambda_1 c_2}, k \in \mathbb{Z}. \tag{5.10}$$

Moreover, from (4.7) and (5.8), we have (4.12). Thus, from (5.6), (5.7), and (5.9), by using the same argument as in the proof of Theorem 2.1 (i), we can obtain the conclusions of Theorem 2.2 (i).

(ii) If $\mu f_1(z) + \lambda_1\frac{\partial f_1}{\partial z_1} + \lambda_2\frac{\partial f_1}{\partial z_2}$ is a nonconstant, then it yields that $\alpha f_2(z+c) - \beta f_1(z)$, $\mu f_2 + \lambda_1\frac{\partial f_2}{\partial z_1} + \lambda_2\frac{\partial f_2}{\partial z_2}$ and $\alpha f_1(z+c) - \beta f_2(z)$ are all nonconstant. Otherwise, if one of these terms is a constant, we can deduce that $\mu f_1(z) + \lambda_1\frac{\partial f_1}{\partial z_1} + \lambda_2\frac{\partial f_1}{\partial z_2}$ is a constant. This is a contradiction. Thus, similar to the argument as in the proof of Theorem 2.1 (ii), there exists two nonconstant polynomials $p(z), q(z)$ in $\mathbb{C}^2$ such that

$$\mu f_1(z) + \lambda_1\frac{\partial f_1}{\partial z_1} + \lambda_2\frac{\partial f_1}{\partial z_2} = \frac{1}{2}(e^p + e^{-p}), \tag{5.11}$$

$$\alpha f_2(z+c) - \beta f_1(z) = \frac{1}{2i}(e^p - e^{-p}), \tag{5.12}$$

$$\mu f_2(z) + \lambda_1\frac{\partial f_2}{\partial z_1} + \lambda_2\frac{\partial f_2}{\partial z_2} = \frac{1}{2}(e^q + e^{-q}), \tag{5.13}$$

$$\alpha f_1(z+c) - \beta f_2(z) = \frac{1}{2i}(e^q - e^{-q}). \tag{5.14}$$

In view of (5.11) and (5.14), we have

$$\frac{\lambda_1\frac{\partial p}{\partial z_1} + \lambda_2\frac{\partial p}{\partial z_2} + \beta i + \mu}{\alpha i}e^{q(z+c)+p(z)} + \frac{\lambda_1\frac{\partial p}{\partial z_1} + \lambda_2\frac{\partial p}{\partial z_2} + \beta i - \mu}{\alpha i}e^{q(z+c)-p(z)} - e^{2q(z+c)} \equiv 1. \tag{5.15}$$

$$\frac{\lambda_1\frac{\partial q}{\partial z_1}+\lambda_2\frac{\partial q}{\partial z_2}+\beta i+\mu}{\alpha i}e^{p(z+c)+q(z)}+\frac{\lambda_1\frac{\partial q}{\partial z_1}+\lambda_2\frac{\partial q}{\partial z_2}+\beta i-\mu}{\alpha i}e^{p(z+c)-q(z)}-e^{2p(z+c)}\equiv 1. \tag{5.16}$$

By Lemma 3.3, we can deduce from (5.15) and (5.16) that

$$\frac{\lambda_1\frac{\partial p}{\partial z_1}+\lambda_2\frac{\partial p}{\partial z_2}+\beta i-\mu}{\alpha i}e^{q(z+c)-p(z)}\equiv 1 \quad \text{or} \quad \frac{\lambda_1\frac{\partial p}{\partial z_1}+\lambda_2\frac{\partial p}{\partial z_2}+\beta i+\mu}{\alpha i}e^{q(z+c)+p(z)}\equiv 1,$$

and

$$\frac{\lambda_1\frac{\partial q}{\partial z_1}+\lambda_2\frac{\partial q}{\partial z_2}+\beta i-\mu}{\alpha i}e^{p(z+c)-q(z)}\equiv 1 \quad \textit{or} \quad \frac{\lambda_1\frac{\partial q}{\partial z_1}+\lambda_2\frac{\partial q}{\partial z_2}+\beta i+\mu}{\alpha i}e^{p(z+c)+q(z)}\equiv 1.$$

Now we will consider four cases as follows.

Case 1.

$$\begin{cases}\frac{\lambda_1\frac{\partial p}{\partial z_1}+\lambda_2\frac{\partial p}{\partial z_2}+\beta i-\mu}{\alpha i}e^{q(z+c)-p(z)}\equiv 1,\\ \frac{\lambda_1\frac{\partial q}{\partial z_1}+\lambda_2\frac{\partial q}{\partial z_2}+\beta i-\mu}{\alpha i}e^{p(z+c)-q(z)}\equiv 1.\end{cases} \tag{5.17}$$

In view of (5.17), it follows that $q(z+c)-p(z)=d_1$ and $p(z+c)-q(z)=d_2$, where d_1,d_2 are constants in $\mathbb{C}$. Thus, it yields that $q(z+2c)-q(z)=d_1+d_2$ and $p(z+2c)-p(z)=d_1+d_2$. Since p,q are polynomials in $\mathbb{C}^2$, it follows that $p(z)=L(z)+H(c_2z_1-c_1z_2)+b_1$ and $q(z)=L(z)+H(c_2z_1-c_1z_2)+b_2$, where $L(z)$ is a linear form of $L(z)=a_1z_1+a_2z_2$, $H(s)$ is a polynomial in s and a_1,a_2,b_1,b_2 are constants. Substituting $p(z),q(z)$ in (5.17), we have

$$\frac{\lambda_1a_1+\lambda_2a_2+(\lambda_1c_2-\lambda_2c_1)H'+\beta i-\mu}{\alpha i}e^{L(c)+b_2-b_1}\equiv 1, \tag{5.18}$$

$$\frac{\lambda_1a_1+\lambda_2a_2+(\lambda_1c_2-\lambda_2c_1)H'+\beta i-\mu}{\alpha i}e^{L(c)+b_1-b_2}\equiv 1. \tag{5.19}$$

By combining with $\lambda_1c_2-\lambda_2c_1\neq 0$, it follows from (5.18) and (5.19) that $\deg_s H\leq 1$. Thus, we still write $p(z),q(z)$ as the forms of $p(z)=L(z)+b_1$ and $q(z)=L(z)+b_2$. In view of (5.15)–(5.17), we have

$$\begin{cases}\frac{\lambda_1a_1+\lambda_2a_2+\beta i+\mu}{\alpha i}e^{-L(c)+b_1-b_2}\equiv 1,\\ \frac{\lambda_1a_1+\lambda_2a_2+\beta i+\mu}{\alpha i}e^{-L(c)+b_2-b_1}\equiv 1,\\ \frac{\lambda_1a_1+\lambda_2a_2+\beta i-\mu}{\alpha i}e^{L(c)+b_2-b_1}\equiv 1,\\ \frac{\lambda_1a_1+\lambda_2a_2+\beta i-\mu}{\alpha i}e^{L(c)+b_1-b_2}\equiv 1.\end{cases} \tag{5.20}$$

Thus, we can deduce from (5.20) that

$$(\lambda_1a_1+\lambda_2a_2+\beta i)^2=\mu^2-\alpha^2,\quad e^{2(b_1-b_2)}=1, \tag{5.21}$$

and

$$e^{2L(c)}=\frac{-\alpha^2}{(\lambda_1a_1+\lambda_2a_2+\beta i-\mu)^2}=\frac{(\lambda_1a_1+\lambda_2a_2+\beta i+\mu)^2}{-\alpha^2}. \tag{5.22}$$

If $\lambda_1a_1+\lambda_2a_2\neq\pm\mu$, solving equations (5.11) and (5.13), we have

$$f_1(z_1,z_2)=\frac{e^{a_1z_1+a_2z_2+b_1}}{2(\lambda_1a_1+\lambda_2a_2+\mu)}-\frac{e^{-a_1z_1-a_2z_2-b_1}}{2(\lambda_1a_1+\lambda_2a_2-\mu)}-\vartheta_1(s_1)e^{-\frac{\mu}{\lambda}z_1+\phi_1(s_1)}, \tag{5.23}$$

$$f_2(z_1, z_2) = \frac{e^{a_1z_1+a_2z_2+b_2}}{2(\lambda_1a_1 + \lambda_2a_2 + \mu)} - \frac{e^{-a_1z_1-a_2z_2-b_2}}{2(\lambda_1a_1 + \lambda_2a_2 - \mu)} - \vartheta_2(s_1)e^{-\frac{\mu}{\lambda}z_1+\phi_2(s_1)}, \tag{5.24}$$

where $\vartheta_1(s_1)$, $\vartheta_2(s_1)$ are finite order entire functions and $\phi_1(s_1)$, $\phi_2(s_1)$ are polynomials in s_1. Substituting (5.23) and (5.24) in (5.12) and (5.14), and combining with (5.20)–(5.22), by Lemma 3.4 we have

$$f_1(z_1, z_2) = \frac{e^{a_1z_1+a_2z_2+b_1}}{2(\lambda_1a_1 + \lambda_2a_2 + \mu)} - \frac{e^{-a_1z_1-a_2z_2-b_1}}{2(\lambda_1a_1 + \lambda_2a_2 - \mu)} - \vartheta(s_1)e^{-\frac{\mu}{\lambda_1}z_1+\gamma s_1+D_1}, \tag{5.25}$$

$$f_2(z_1, z_2) = \frac{e^{a_1z_1+a_2z_2+b_2}}{2(\lambda_1a_1 + \lambda_2a_2 + \mu)} - \frac{e^{-a_1z_1-a_2z_2-b_2}}{2(\lambda_1a_1 + \lambda_2a_2 - \mu)} - \vartheta(s_1 + s_0)e^{-\frac{\mu}{\lambda_1}z_1+\gamma s_1+D_2}, \tag{5.26}$$

where $\vartheta(s_1)$ is a finite order period entire function with period $2s_0$, and $a_1, a_2, b_1, b_2, \gamma, D_1, D_2$ satisfy (5.10), (5.21), and (5.22).

If $\lambda_1a_1 + \lambda_2a_2 = \mu$, solving equations (5.11) and (5.13), similar to the argument as in case $\lambda_1a_1 + \lambda_2a_2 \neq \mu$, we have

$$f_1(z_1, z_2) = \frac{1}{4\mu}e^{a_1z_1+a_2z_2+b_1} + \frac{z_1}{2\lambda_1}e^{-a_1z_1-a_2z_2-b_1} - \vartheta(s_1)e^{-\frac{\mu}{\lambda_1}z_1+\gamma s_1+D_1}, \tag{5.27}$$

$$f_2(z_1, z_2) = \frac{1}{4\mu}e^{a_1z_1+a_2z_2+b_2} - \frac{z_1}{2\lambda_1}e^{-a_1z_1-a_2z_2-b_2} - \vartheta(s_1 + s_0)e^{-\frac{\mu}{\lambda_1}z_1+\gamma s_1+D_2}, \tag{5.28}$$

where $\vartheta(s_1)$ is a finite order period entire function with period $2s_0$, and γ, D_1, D_2 satisfy (5.10), (5.21), and (5.22). Substituting (5.27) and (5.28) in (5.12) and (5.14), and combining with (5.20)–(5.22), we have $\beta c_1 = \lambda_1 i$.

If $\lambda_1a_1 + \lambda_2a_2 = -\mu$, solving equations (5.11) and (5.13), we have

$$f_1(z_1, z_2) = \frac{1}{4\mu}e^{-a_1z_1-a_2z_2-b_1} + \frac{z_1}{2\lambda_1}e^{a_1z_1+a_2z_2+b_1} - \vartheta(s_1)e^{-\frac{\mu}{\lambda_1}z_1+\gamma s_1+D_1}, \tag{5.29}$$

$$f_2(z_1, z_2) = \frac{1}{4\mu}e^{-a_1z_1-a_2z_2-b_2} + \frac{z_1}{2\lambda_1}e^{a_1z_1+a_2z_2+b_2} - \vartheta(s_1 + s_0)e^{-\frac{\mu}{\lambda_1}z_1+\gamma s_1+D_2}, \tag{5.30}$$

where $\vartheta(s_1)$ is a finite order period entire function with period $2s_0$, and γ, D_1, D_2 satisfy (5.10), (5.21), and (5.22). Similar to the above argument, we have $\beta c_1 = -\lambda_1 i$.

Case 2.

$$\begin{cases} \dfrac{\lambda_1\frac{\partial p}{\partial z_1} + \lambda_2\frac{\partial p}{\partial z_2} + \beta i - \mu}{ai}e^{q(z+c)-p(z)} \equiv 1, \\ \dfrac{\lambda_1\frac{\partial q}{\partial z_1} + \lambda_2\frac{\partial q}{\partial z_2} + \beta i + \mu}{ai}e^{p(z+c)+q(z)} \equiv 1. \end{cases}$$

Thus, it follows that $q(z + c) - p(z) = d_1$ and $p(z + c) + q(z) = d_2$ where d_1, d_2 are constants. Hence, we have $q(z + 2c) + q(z) = d_1 + d_2$, which is a contradiction with the assumption of $q(z)$ being nonconstant polynomial in $\mathbb{C}^2$.

Case 3.

$$\begin{cases} \dfrac{\lambda_1\frac{\partial p}{\partial z_1} + \lambda_2\frac{\partial p}{\partial z_2} + \beta i + \mu}{ai}e^{q(z+c)+p(z)} \equiv 1, \\ \dfrac{\lambda_1\frac{\partial q}{\partial z_1} + \lambda_2\frac{\partial q}{\partial z_2} + \beta i - \mu}{ai}e^{p(z+c)-q(z)} \equiv 1. \end{cases}$$

Thus, it follows that $q(z + c) + p(z) = d_1$ and $p(z + c) - q(z) = d_2$ where d_1, d_2 are constants. Hence, we have $p(z + 2c) + p(z) = d_1 + d_2$, which is a contradiction with the assumption of $p(z)$ being nonconstant polynomial in $\mathbb{C}^2$.

Case 4.

$$\begin{cases} \dfrac{\lambda_1 \frac{\partial p}{\partial z_1} + \lambda_2 \frac{\partial p}{\partial z_2} + \beta i + \mu}{\alpha i} e^{q(z+c)+p(z)} \equiv 1, \\ \dfrac{\lambda_1 \frac{\partial q}{\partial z_1} + \lambda_2 \frac{\partial q}{\partial z_2} + \beta i + \mu}{\alpha i} e^{p(z+c)+q(z)} \equiv 1. \end{cases} \tag{5.31}$$

In view of (5.31), it follows that $q(z + c) + p(z) = d_1$ and $p(z + c) + q(z) = d_2$, where d_1, d_2 are constants in $\mathbb{C}$. Thus, it yields that $q(z + 2c) - q(z) = d_1 - d_2$ and $p(z + 2c) - p(z) = d_1 - d_2$. Since p, q are polynomials in $\mathbb{C}^2$, it follows that $p(z) = L(z) + H(c_2z_1 - c_1z_2) + b_1$ and $q(z) = -L(z) - H(c_2z_1 - c_1z_2) + b_2$, where $L(z)$ is a linear form of $L(z) = a_1z_1 + a_2z_2$, $H(s)$ is a polynomial in s and a_1, a_2, b_1, b_2 are constants. Similar to the argument as in Case 1, we can obtain that $p(z) = L(z) + b_1 = a_1z_1 + a_2z_2 + b_1$ and $q(z) = -L(z) + b_2 = -a_1z_1 - a_2z_2 + b_1$.

By using the same argument as in Case 4 of Theorem 2.1, we can obtain that the forms of solutions are included in case that $\lambda_1 a_1 + \lambda_2 a_2 \neq \pm\mu$ in Case 1 of Theorem 2.2.

Therefore, this completes the proof of Theorem 2.2. □

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References

[1] F. Gross, *On the equation $f^n + g^n = 1$*, Bull. Amer. Math. Soc. **72** (1966), 86–88.
[2] P. Montel, *Lecons sur les familles normales de fonctions analytiques et leurs applications*, Gauthier-Villars, Paris, 1927, 135–136.
[3] G. Pólya, *On an integral function of an integral function*, J. Lond. Math. Soc. **1** (1926), 12–15.
[4] T. B. Cao and L. Xu, *Logarithmic difference lemma in several complex variables and partial difference equations*, Ann. Mat. Pura Appl. **199** (2020), 767–794.
[5] S. G. Georgiev, K. Bouhali, and K. Zennir, *A new topological approach to target the existence of solutions for nonlinear fractional impulsive wave equations*, Axioms **11** (2022), no. 721, 1–21.
[6] B. Q. Li, *On entire solutions of Fermat-type partial differential equations*, Int. J. Math. **15** (2004), 473–485.
[7] B. Q. Li, *Entire solutions of certain partial differential equations and factorization of partial derivatives*, Tran. Amer. Math. Soc. **357** (2004), 3169–3177.
[8] K. Liu and T. B. Cao, *Entire solutions of Fermat-type difference differential equations*, Electron. J. Diff. Equ. **2013** (2013), no. 59, 1–10.
[9] Z. H. Liu, D. Motreanu and S. D. Zeng, *Generalized Penalty and Regularization Method for Differential Variational-Hemivariational Inequalities*, SIAM J. Optim. **31** (2021), 1158–1183.

[10] F. Lü and Z. Li, *Meromorphic solutions of Fermat-type partial differential equations*, J. Math. Anal. Appl. **478** (2019), 864–873.
[11] D. Khavinson, *A note on entire solutions of the Eiconal equation*, Amer. Math. Month. **102** (1995), 159–161.
[12] H. Y. Xu and Y. Y. Jiang, *Results on entire and meromorphic solutions for several systems of quadratic trinomial functional equations with two complex variables*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM. **116** (2022), no. 8, 1–19.
[13] H. Y. Xu, H. Li and X. Ding, *Entire and meromorphic solutions for systems of the differential difference equations*, Demonstratio Mathematica **55** (2022), 676–694.
[14] H. Y. Xu and L. Xu, *Transcendental entire solutions for several quadratic binomial and trinomial PDEs with constant coefficients*, Anal. Math. Phys. **12** (2022), no. 64, 1–21.
[15] H. Y. Xu, X. L. Liu, and Y. H. Xu, *On solutions for several systems of complex nonlinear partial differential equations with two variables*, Anal. Math. Phys. **13** (2023), no. 47, 1–24.
[16] S. D. Zeng, S. Migórski and Z. H. Liu, *Well-Posedness, optimal control, and sensitivity analysis for a class of differential variational hemivariational inequalities*, SIAM J. Optim. **31** (2021), 2829–2862.
[17] S. D. Zeng, S. Migórski, and A. A. Khan, *Nonlinear quasi-hemivariational inequalities: existence and optimal control*, SIAM J. Control. Optim. **59**,(2021), 1246–1274.
[18] E. G. Saleeby, *Entire and meromorphic solutions of Fermat-type partial differential equations*, Analysis **19** (1999), 369–376.
[19] E. G. Saleeby, *On entire and meromorphic solutions of* $\lambda u^k + \sum_{i=1}^{n} u_{z_i}^m = 1$, *Complex*, Variables Theory Appl. **49** (2004), 101–107.
[20] D. D. Chang and B. Q. Li, *Description of entire solutions of Eiconal type equations*, Canad. Math. Bull. **55** (2012), 249–259.
[21] B. Q. Li, *Entire solutions of* $(u_{z_1})^m + (u_{z_2})^n = e^g$, Nagoya Math. J. **178** (2005), 151–162.
[22] B. Q. Li, *Entire solutions of Eiconal type equations*, Arch. Math. **89** (2007), 350–357.
[23] T. B. Cao and R. J. Korhonen, *A new version of the second main theorem for meromorphic mappings intersecting hyperplanes in several complex variables*, J. Math. Anal. Appl. **444** (2016), 1114–1132.
[24] Y. M. Chiang and S. J. Feng, *On the Nevanlinna characteristic of* $f(z+\eta)$ *and difference equations in the complex plane*, Ramanujan J. **16** (2008), 105–129.
[25] R. G. Halburd and R. Korhonen, *Finite-order meromorphic solutions and the discrete Painlevé equations*, Proc. London Math. Soc. **94** (2007), 443–474.
[26] R. G. Halburd and R. J. Korhonen, *Nevanlinna theory for the difference operator*, Ann. Acad. Sci. Fenn. Math. **31** (2006), 463–478.
[27] L. I. Ronkin, *Introduction to the Theory of Entire Functions of Several Variables*, Nauka, Moscow, 1971, (Russian). American Mathematical Society, Providence (1974).
[28] W. Stoll, *Holomorphic Functions of Finite Order in Several Complex Variables*, American Mathematical Society, Providence, 1974.
[29] A. Vitter, *The lemma of the logarithmic derivative in several complex variables*, Duke Math. J. **44** (1977), 89–104.
[30] R. J. Korhonen, *A difference Picard theorem for meromorphic functions of several variables*, Comput. Methods Funct. Theory **12** (2012), 343–361.
[31] L. Xu and T. B. Cao, *Solutions of complex Fermat-type partial difference and differential-difference equations*, Mediterr. J. Math. **15** (2018), 1–14.
[32] L. Xu and T. B. Cao, *Correction to: Solutions of complex Fermat-type partial difference and differential-difference equations*, Mediterr. J. Math. **17** (2020), 1–4.
[33] H. Y. Xu, S. Y. Liu, and Q. P. Li, *Entire solutions for several systems of nonlinear difference and partial differential-difference equations of Fermat-type*, J. Math. Anal. Appl. **483** (2020), no. 123641, 1–22.
[34] P. C. Hu, P. Li, and C. C. Yang, *Unicity of Meromorphic Mappings, Advances in Complex Analysis and its Applications*, vol. 1, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.