Research Article

Sang Jo Yun and Jin-Woo Park*

Study of degenerate derangement polynomials by λ-umbral calculus

https://doi.org/10.1515/dema-2022-0240
received October 1, 2022; accepted May 2, 2023

Abstract: In the 1970s, Rota began to build completely rigid foundations for the theory of umbral calculus based on relatively modern ideas of linear functions and linear operators. Since then, umbral calculus has been used in the study of special functions in various fields. In this article, we derive some new and interesting identities related to degenerate derangement polynomials and some special polynomials by using λ-Sheffer sequences and λ-umbral calculus, which are defined by Kim-Kim (*Degenerate Sheffer sequences and λ-Sheffer sequences*, J. Math. Anal. Appl. **493** (2021), 124521, 21pp).

Keywords: degenerate derangement numbers and polynomials, special polynomials, λ-umbral calculus

MSC 2020: 33E20, 11B83, 05A40

1 Intorduction

For a given set A with n elements, a *derangement* is a permutation of the elements in A such that no element appears in its original position. The number of derangements of A is called the nth derangement number, denoted by d_n. By the definition of the derangement, we see that

$$d_n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}.$$

Derangement numbers are computed recursively or by means of their exponential generating function in general, and one of the important and widely covered fields in various fields such as combinatorics, applied mathematics, and engineering (see [1–3]). In [4], the authors found closed forms for derangement numbers with Hessenberg determinants and derivatives of the exponential generating function of these numbers. In [5], the authors investigated several interesting combinatorial, analytic, and number theoretic properties of r-derangement numbers, which are a permutation on $n + r$ elements, such that in its cycle decomposition, the first r elements appear in distinct cycles. Clarke and Sved found a relation between the derangement numbers and the Bell numbers by using the inclusion-exclusion principle in [6]. In [7], the authors defined p, q-analog of the derangement numbers, which is a generalization of derangement numbers, and derived a recurrence for them.

For a given $\lambda \in \mathbb{R} - \{0\}$, the *degenerate exponential function* is defined to be

$$e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = (1 + \lambda t)^{\frac{1}{\lambda}}, \ \text{(see [8])}. \tag{1}$$

Note that $\lim_{\lambda\to 0} e_\lambda^x = e^{xt}$ and $\lim_{\lambda\to 0} e_\lambda(t) = e^t$.

*** Corresponding author: Jin-Woo Park,** Department of Mathematics Education, Daegu University, Gyeongsan 38453, Republic of Korea, e-mail: a0417001@daegu.ac.kr
Sang Jo Yun: Department of Information Science and Mathematics, Dong-A University, Busan 604-714, Republic of Korea, e-mail: sjyun@dau.ac.kr

Open Access. © 2023 the author(s), published by De Gruyter. This work is licensed under the Creative Commons Attribution 4.0 International License.

The study of the degenerate version of functions was first initiated by Carlitz (see [8]), and since then, degenerations of various special functions have been defined and their properties have been actively studied by many researchers. In [9], the authors defined the partially degenerate Laguerre-Bernoulli polynomials of the first kind and derived some theorems on implicit summation formulae and symmetry identities for these polynomials. In [10], the authors derived some identities and properties on the degenerate Fubini polynomials and the higher-order degenerate Fubini polynomials by using generating functions and certain differential operators. Kwon et al. defined the modified type 2 degenerate poly-Bernoulli polynomials and derived some explicit expressions and their representations by using λ-umbral calculus (see [11]). In [12], the authors defined the degenerate poly-Genocchi polynomials and numbers by using the degenerate polylogarithm function and gave some identities of those polynomials and explicit expressions of degenerate unipoly polynomials related to special polynomials. Kim and Khan introduced a new type of degenerate poly-Frobenius-Euler polynomials and numbers, and derived some combinatorial identities related to these polynomials in [13]. Acikgoz and Duran introduced unified degenerate central Bell polynomials and studied many relations and formulae including the summation formula and derivative properties (see [14]).

For nonzero integers n and k, the *Stirling numbers of the first kind* $S_1(n,k)$ and the *Stirling numbers of the second kind* $S_2(n,k)$, respectively, are given by

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \quad (\text{see [1,3,8,15,16]}), \tag{2}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$ are the falling factorial sequences.

By using the degenerate exponential function, the *degenerate derangement polynomials* are defined by the generating function to be

$$\frac{1}{1-t}e_\lambda^{x-1}(t) = \sum_{n=0}^{\infty} d_{n,\lambda}(x)\frac{t^n}{n!}, \quad (\text{see [2]}).$$

When $x = 0$, $d_{n,\lambda}(0) = d_{n,\lambda}$ are called the *degenerate derangement numbers*.

By the definition of degenerate derangement polynomials,

$$\begin{aligned}\sum_{n=0}^{\infty} d_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{1}{(1-t)e_\lambda(t)}e_\lambda^x(t)\\ &= \left(\sum_{n=0}^{\infty} d_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r}d_{n-r,\lambda}(x)_{r,\lambda}\right)\frac{t^n}{n!},\end{aligned}$$

and thus, we see that

$$d_{n,\lambda}(x) = \sum_{r=0}^{n}\binom{n}{r}d_{n-r,\lambda}(x)_{r,\lambda}. \tag{3}$$

As the degenerate version of the Stirling numbers of the first and the second kind, the *degenerate Stirling numbers of the first kind* $S_{1,\lambda}(n,k)$ and the *degenerate Stirling numbers of the second kind* $S_{2,\lambda}(n,k)$ are, respectively, introduced by Kim et al. (see [9,16]) as follows:

$$\frac{1}{k!}(\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} \quad \text{and} \quad \frac{1}{k!}(e_\lambda(t)-1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}. \tag{4}$$

Let $\mathbb{C}$ be the field of complex numbers,

$$\mathcal{F} = \left\{ f(t) = \sum_{n=0}^{\infty} a_n\frac{t^n}{n!} \;\middle|\; a_k \in \mathbb{C} \right\},$$

and let

$$\mathbb{P} = \mathbb{C}[x] = \left\{ \sum_{k=0}^{\infty} a_k x^k \,\middle|\, a_k \in \mathbb{C} \text{ with } a_k = 0 \text{ for all but finite number of } k \right\}.$$

Let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$.

For given $\lambda \in \mathbb{R} - \{0\}$, the linear functional $\langle f(t)|\cdot\rangle_\lambda$ on $\mathbb{P}$, called *λ-linear functional* given by $f(t)$, is defined by

$$\langle f(t)|(x)_{n,\lambda}\rangle_\lambda = a_n, \ (n \geq 0), \quad (\text{see [17]}), \tag{5}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$. From (5), we have

$$\langle t^k|(x)_{n,\lambda}\rangle_\lambda = n!\delta_{n,k}, \quad (n, k \geq 0), \tag{6}$$

where $\delta_{n,k}$ is the Kronecker's symbol (see [17]).

For each real number λ and each positive integer k, Kim and Kim defined the differential operator on $\mathbb{P}$ in [17] by

$$(t^k)_\lambda(x)_{n,\lambda} = \begin{cases} (n)_k(x)_{n-k,\lambda}, & \text{if } k \leq n, \\ 0, & \text{if } k \geq n, \end{cases}$$

and for any $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$,

$$(f(t))_\lambda(x)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} a_k (x)_{n-k,\lambda}. \tag{7}$$

In addition, they showed that for $f(t), g(t) \in \mathcal{F}$, and $p(x) \in \mathbb{P}$,

$$\langle f(t)g(t)|p(x)\rangle_\lambda = \langle g(t)|(f(t))_\lambda p(x)\rangle_\lambda = \langle f(t)|(g(t))_\lambda p(x)\rangle_\lambda. \tag{8}$$

The *order* $o(f(t))$ of $f(t) \in \mathcal{F} - \{0\}$ is the smallest integer k for which the coefficient of t^k does not vanish. If $o(f(t)) = 0$, then $f(t)$ is said to be *invertible*, and such series has a multiplicative inverse $\frac{1}{f(t)}$ of $f(t)$. If $o(f(t)) = 1$, then $f(t)$ is called *delta series*, and it has a compositional inverse $\bar{f}(t)$ of $f(t)$ with $\bar{f}(f(t)) = f(\bar{f}(t)) = t$ (see [17–19]).

Let $f(t)$ be a delta series, and let $g(t)$ be an invertible series. Then there exists a unique sequence $S_{n,\lambda}(x)$ ($\deg S_{n,\lambda}(x) = n$) of polynomials satisfying the orthogonality conditions:

$$\langle g(t)(f(t))^k|S_{n,\lambda}(x)\rangle_\lambda = n!\delta_{n,k}, \quad (n, k \geq 0). \tag{9}$$

(see [17]). Here $S_{n,\lambda}(x)$ is called the λ-Sheffer sequence for $(g(t), f(t))$, which is denoted by $S_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$. The sequence $S_{n,\lambda}(x)$ is the λ-Sheffer sequence for $(g(t), f(t))$ if and only if

$$\frac{1}{g(\bar{f}(t))} e_\lambda^y(\bar{f}(t)) = \sum_{n=0}^{\infty} S_{n,\lambda}(y)\frac{t^n}{n!}, \tag{10}$$

for all $y \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ such that $f(\bar{f}(t)) = \bar{f}(f(t)) = t$ (see [17]).

Let $S_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ and let $h(x) = \sum_{l=0}^{n} a_l S_{l,\lambda}(x) \in \mathbb{P}$. Then by (9), we have

$$\langle g(t)(f(t))^k|h(x)\rangle_\lambda = \sum_{l=0}^{n} a_l \langle g(t)(f(t))^k|S_{l,\lambda}(x)\rangle_\lambda = k!a_k,$$

and thus, we know that

$$a_k = \frac{1}{k!}\langle g(t)(f(t))^k|h(x)\rangle_\lambda. \tag{11}$$

The following theorem is proved by Kim and Kim [17] and is very useful tools for researching degenerate versions of special polynomials and numbers.

Theorem 1.1. *Let* $s_{n,\lambda} \sim (g(t), f(t))_\lambda$, $r_{n,\lambda} = (h(t), l(t))_\lambda$. *Then we have*

$$s_{n,\lambda} = \sum_{k=0}^{n} c_{n,k} r_{k,\lambda},$$

where

$$c_{n,k} = \frac{1}{k!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))}(l(\bar{f}(t)))^k \,\middle|\, (x)_{n,\lambda} \right\rangle_\lambda .$$

Let $(x)_n = \sum_{k=0}^{n} c_{n,k}(x)_{k,\lambda}$. Since

$$(x)_n \sim (1, e_\lambda(t) - 1)_\lambda \quad \text{and} \quad (x)_{n,\lambda} \sim (1, t)_\lambda,$$

by Theorem 1.1, we obtain

$$c_{n,k} = \frac{1}{k!}\langle(\log_\lambda(1+t))^k|(x)_{n,\lambda}\rangle_\lambda = \sum_{l=k}^{\infty} S_{1,\lambda}(l,k)\frac{1}{l!}\langle t^l|(x)_{n,\lambda}\rangle_\lambda = S_{1,\lambda}(n,k),$$

and thus, we know that

$$(x)_n = \sum_{k=0}^{n} S_{1,\lambda}(n,k)(x)_{k,\lambda}. \tag{12}$$

In the similar way, we also know that

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k)(x)_k. \tag{13}$$

Umbral calculus consisted of primarily symbolic techniques for sequence manipulation with little mathematical rigor. In the 1970s, Rota began to build a completely rigid foundation for theories based on relatively modern ideas of linear functions and linear operators (see [19]). Umbral calculus contributed to the generalization of Lagrange inversion formula and has been applied in many fields such as combinatorial counting with linear recurrences and lattice path counting, graph theory using chromatic polynomials, probability theory, link invariant theory, statistics, topology, physics, etc. (see [18,19]). In addition, it is being actively applied in various fields by researchers (see [17–25]).

In the past few years, various different umbral calculus has been studied (see [17–20]). In particular, Kim and Kim defined the degenerate Sheffer sequences, λ-Sheffer sequence, a family of λ-linear functionals, and λ-differential operators in [17].

In this article, we derive some interesting identities related to degenerate derangement polynomials, degenerate Bernoulli polynomials, degenerate Euler polynomials, degenerate Daehee polynomials, Changhee polynomials, degenerate Bell polynomials, degenerate Lah-Bell polynomials, Mittag-Leffer polynomials, and degenerate Frobenius-Euler polynomials by using λ-umbral calculus.

2 Main results

By the definition of degenerate derangement polynomials, we note that

$$d_{n,\lambda}(x) \sim ((1-t)e_\lambda(t), t)_\lambda. \tag{14}$$

In addition,

$$\frac{1}{1-t}e_\lambda^{x-1}(t) = \left(\sum_{n=0}^{\infty} t^n\right)\left(\sum_{n=0}^{\infty}(x-1)_{n,\lambda}\frac{t^n}{n!}\right) = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{n!}{l!}(x-1)_{l,\lambda}\frac{t^n}{n!}\right). \tag{15}$$

From (2) and (13), we obtain

$$
\begin{aligned}
d_{n,\lambda}(x) &= \sum_{l=0}^{n} \frac{n!}{l!}(x-1)_{l,\lambda} \\
&= n! \sum_{l=0}^{n} \sum_{k=0}^{l} \sum_{m=0}^{k} \frac{S_{2,\lambda}(l,k) S_1(k,m)}{l!}(x-1)^m \\
&= n! \sum_{l=0}^{n} \sum_{k=0}^{l} \sum_{m=0}^{k} \sum_{a=0}^{m} \frac{S_{2,\lambda}(l,k) S_1(k,m)}{l!}(-1)^{m-a} x^a.
\end{aligned} \tag{16}
$$

Let $d_{n,\lambda}(x) = \sum_{l=0}^{n} a_{n,l}(x)_{l,\lambda}$. Since $(x)_{n,\lambda} \sim (1,t)_\lambda$, by Theorem 1.1, we have

$$
a_{n,l} = \frac{1}{l!} \left\langle \frac{1}{(1-t)e_\lambda(t)} t^l \,\middle|\, (x)_{n,\lambda} \right\rangle_\lambda = \binom{n}{l} \left\langle \sum_{m=0}^{\infty} d_{m,\lambda} \frac{t^m}{m!} \,\middle|\, (x)_{n-l,\lambda} \right\rangle_\lambda = \binom{n}{l} d_{n-l,\lambda}, \tag{17}
$$

and by (15),

$$
\frac{1}{(1-t)e_\lambda(t)} = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \frac{(-1)^l \langle 1 \rangle_{l,\lambda}}{l!} \right) \frac{t^n}{n!}. \tag{18}
$$

By (17) and (18), we have

$$
d_{n,\lambda} = \binom{n}{l} \sum_{m=0}^{n-l} \frac{(-1)^m \langle 1 \rangle_{m,\lambda}}{m!}. \tag{19}
$$

Conversely, we may assume that $(x)_{n,\lambda} = \sum_{l=0}^{n} b_{n,l} d_{l,\lambda}(x)$. Note that

$$
(1-t)e_\lambda(t) = (1-t)\sum_{n=0}^{\infty} (1)_{n,\lambda} \frac{t^n}{n!} = \sum_{n=0} (1)_{n,\lambda} \frac{t^n}{n!} - \sum_{n=0}^{\infty} (1)_{n,\lambda} \frac{t^{n+1}}{n!} = 1 + \sum_{n=1}^{\infty} ((1)_{n,\lambda} - n(1)_{n-1,\lambda}) \frac{t^n}{n!}. \tag{20}
$$

By (19) and (20),

$$
\begin{aligned}
b_{n,l} &= \frac{1}{l!} \langle (1-t)e_\lambda(t) t^l | (x)_{n,\lambda} \rangle_\lambda \\
&= \binom{n}{l} \langle (1-t)e_\lambda(t) | (x)_{n-l,\lambda} \rangle_\lambda \\
&= \binom{n}{l} \langle 1 | (x)_{n-l,\lambda} \rangle_\lambda + \binom{n}{l} \left\langle \sum_{m=1}^{\infty} ((1)_{m,\lambda} - m(1)_{m-1,\lambda}) \frac{t^m}{m!} \,\middle|\, (x)_{n-l,\lambda} \right\rangle_\lambda \\
&= 1 + \binom{n}{l} ((1)_{n-l,\lambda} - (n-l)(1)_{n-l-1,\lambda})(1 - \delta_{n,l}).
\end{aligned} \tag{21}
$$

By (17), (19), and (21), we obtain the following theorem.

Theorem 2.1. *For each nonnegative integer n, we have*

$$
d_{n,\lambda}(x) = \sum_{l=0}^{n} \binom{n}{l} d_{n-l,\lambda}(x)_{l,\lambda} = \sum_{l=0}^{n} \left(\sum_{m=0}^{n-l} \binom{n}{l} \frac{(-1)^m \langle 1 \rangle_{m,\lambda}}{m!} \right) (x)_{l,\lambda},
$$

and

$$
(x)_{n,\lambda} = \sum_{l=0}^{n} \left(1 + \binom{n}{l} ((1)_{n-l,\lambda} - (n-l)(1)_{n-l-1,\lambda})(1 - \delta_{n,l}) \right) d_{l,\lambda}(x).
$$

The *degenerate Bernoulli polynomials* are defined by the generating function to be

$$\frac{t}{e_\lambda(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!},\quad \text{(see [8,17,22])}.$$

In the special case $x=0$, $\beta_{n,\lambda}(0)=\beta_{n,\lambda}$ are called the *degenerate Bernoulli numbers*. By the definition of degenerate Bernoulli polynomials, the λ-Sheffer sequences of these polynomials are as follows:

$$\beta_{n,\lambda}(x)\sim\left(\frac{e_\lambda(t)-1}{t},t\right)_\lambda.$$

Note that

$$\frac{e_\lambda(t)-1}{t}=\sum_{n=0}^{\infty}\frac{(1)_{n+1,\lambda}}{n+1}\frac{t^n}{n!}. \tag{22}$$

Let $d_{n,\lambda}(x)=\sum_{l=0}^{n}a_{n,l}\beta_{l,\lambda}(x)$. By Theorem 1.1 and (22), we have

$$\begin{aligned}a_{n,l}&=\frac{1}{l!}\left\langle\frac{\frac{e_\lambda(t)-1}{t}}{(1-t)e_\lambda(t)}t^l\,\middle|\,(x)_{n,\lambda}\right\rangle_\lambda\\&=\binom{n}{l}\left\langle\frac{1}{(1-t)e_\lambda(t)}\,\middle|\,\left(\frac{e_\lambda(t)-1}{t}\right)_\lambda(x)_{n-l,\lambda}\right\rangle_\lambda\\&=\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}\frac{(1)_{k+1,\lambda}}{k+1}\left\langle\frac{1}{(1-t)e_\lambda(t)}\,\middle|\,(x)_{n-l-k,\lambda}\right\rangle_\lambda\\&=\sum_{k=0}^{n-l}\sum_{r=0}^{n-l-k}\binom{n}{l}\binom{n-l}{k}\frac{(1)_{k+1,\lambda}(-1)^r\langle 1\rangle_{r,\lambda}}{(k+1)r!}.\end{aligned} \tag{23}$$

In addition, by (3), we obtain

$$\begin{aligned}a_{n,l}&=\frac{1}{l!}\left\langle\frac{\frac{e_\lambda(t)-1}{t}}{(1-t)e_\lambda(t)}t^l\,\middle|\,(x)_{n,\lambda}\right\rangle_\lambda\\&=\binom{n}{l}\left\langle\frac{e_\lambda(t)-1}{t}\,\middle|\,\left(\frac{1}{(1-t)e_\lambda(t)}\right)_\lambda(x)_{n-l,\lambda}\right\rangle_\lambda\\&=\binom{n}{l}\sum_{m=0}^{n-l}\binom{n-l}{m}d_{m,\lambda}\left\langle\frac{e_\lambda(t)-1}{t}\,\middle|\,(x)_{n-l-m,\lambda}\right\rangle_\lambda\\&=\sum_{m=0}^{n-l}\binom{n}{l}\binom{n-l}{m}\frac{(1)_{n-l-m+1,\lambda}}{n-l-m+1}d_{m,\lambda}.\end{aligned} \tag{24}$$

Conversely, we assume that $\beta_{n,\lambda}(x)=\sum_{l=0}^{n}b_{n,l}d_{l,\lambda}(x)$. Then, by (20), we obtain

$$\begin{aligned}b_{n,l}&=\frac{1}{l!}\left\langle\frac{(1-t)e_\lambda(t)}{\frac{e_\lambda(t)-1}{t}}t^l\,\middle|\,(x)_{n,\lambda}\right\rangle_\lambda\\&=\binom{n}{l}\left\langle\frac{t}{e_\lambda(t)-1}(1-t)e_\lambda(t)\,\middle|\,(x)_{n-l,\lambda}\right\rangle_\lambda\\&=\sum_{m=0}^{n-l}\binom{n-l}{m}\beta_{m,\lambda}\langle(1-t)e_\lambda(t)|(x)_{n-l-m,\lambda}\rangle_\lambda\\&=\beta_{n-l,\lambda}+\sum_{m=0}^{n-l-1}\binom{n-l}{m}\beta_{m,\lambda}((1)_{n-l-m,\lambda}-(n-l-m)(1)_{n-l-m-1,\lambda}).\end{aligned} \tag{25}$$

By (23), (24), and (25), we obtain the following theorem.

Theorem 2.2. *For each nonnegative integer n, we have*

$$d_{n,\lambda}(x) = \sum_{l=0}^{n}\left(\sum_{k=0}^{n-l}\sum_{r=0}^{n-l-k}\binom{n}{l}\binom{n-l}{k}\frac{(1)_{k+1,\lambda}(-1)^r\langle 1\rangle_{r,\lambda}}{(k+1)r!}\right)\beta_{l,\lambda}(x)$$
$$= \sum_{l=0}^{n}\left(\sum_{m=0}^{n-l}\binom{n}{l}\binom{n-l}{m}\frac{(1)_{n-l-m+1,\lambda}}{n-l-m+1}d_{m,\lambda}\right)\beta_{l,\lambda}(x)$$
$$\beta_{n,\lambda}(x) = \sum_{l=0}^{n}\left(\beta_{n-l,\lambda} + \sum_{m=0}^{n-l-1}\binom{n-l}{m}\beta_{m,\lambda}((1)_{n-l-m,\lambda} - (n-l-m)(1)_{n-l-m-1,\lambda})\right)d_{l,\lambda}(x).$$

The *degenerate Euler polynomials* are defined by the generating function to be

$$\frac{2}{e_\lambda(t)+1}e_\lambda^x(t) = \sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [8])}. \tag{26}$$

When $x = 0$, $E_{n,\lambda} = E_{n,\lambda}(0)$ are called the degenerate Euler numbers. By the definition of degenerate Euler polynomials, we see that

$$E_{n,\lambda}(x) \sim \left(\frac{e_\lambda(t)+1}{2}, t\right)_\lambda.$$

Let $d_{n,\lambda}(x) = \sum_{l=0}^{n} a_{n,l}E_{l,\lambda}(x)$. Since

$$\frac{e_\lambda(t)+1}{2} = 1 + \frac{1}{2}\sum_{m=1}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!}, \tag{27}$$

by Theorem 1.1, (22), and (27), we obtain

$$\begin{aligned} a_{n,l} &= \frac{1}{l!}\left\langle \frac{\frac{e_\lambda(t)+1}{2}}{(1-t)e_\lambda(t)}t^l \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\ &= \binom{n}{l}\left\langle \frac{1}{(1-t)e_\lambda(t)}\frac{e_\lambda(t)+1}{2} \,\middle|\, (x)_{n-l,\lambda}\right\rangle_\lambda \\ &= \sum_{a=0}^{n-l}\binom{n}{l}\binom{n-l}{a}d_{a,\lambda}\left\langle \frac{e_\lambda(t)+1}{2} \,\middle|\, (x)_{n-l-a,\lambda}\right\rangle_\lambda \\ &= \binom{n}{l}d_{n-l,\lambda} + \sum_{a=0}^{n-l-1}\binom{n}{l}\binom{n-l}{a}\frac{(1)_{n-l-a,\lambda}}{2}d_{a,\lambda}. \end{aligned} \tag{28}$$

Conversely, assume that $E_{n,\lambda}(x) = \sum_{l=0}^{n} a_{n,l}d_{l,\lambda}(x)$. Then, by (20), we obtain

$$\begin{aligned} a_{n,l} &= \frac{1}{l!}\left\langle \frac{(1-t)e_\lambda(t)}{\frac{e_\lambda(t)+1}{2}}t^l \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\ &= \binom{n}{l}\left\langle \frac{2}{e_\lambda(t)+1}(1-t)e_\lambda(t) \,\middle|\, (x)_{n-l,\lambda}\right\rangle_\lambda \\ &= \sum_{a=0}^{n-l}\binom{n}{l}\binom{n-l}{a}E_{a,\lambda}\langle (1-t)e_\lambda(t) | (x)_{n-l-a,\lambda}\rangle_\lambda \\ &= \binom{n}{l}E_{n-l,\lambda} + \sum_{a=0}^{n-l-a}\binom{n}{l}\binom{n-l}{a}((1)_{n-l-a,\lambda} - (n-l-a)(1)_{n-l-a-1,\lambda}). \end{aligned} \tag{29}$$

By (28) and (29), we obtain the following theorem.

Theorem 2.3. *For each nonnegative integer n, we have*

$$d_{n,\lambda}(x)=\sum_{l=0}^{n}\left(\binom{n}{l}d_{n-l,\lambda}+\sum_{a=0}^{n-l-1}\binom{n}{l}\binom{n-l}{a}\frac{(1)_{n-l-a,\lambda}}{2}d_{a,\lambda}\right)E_{l,\lambda}(x)$$

and

$$E_{n,\lambda}(x)=\sum_{l=0}^{n}\left(\binom{n}{l}E_{n-l,\lambda}+\sum_{a=0}^{n-l-a}\binom{n}{l}\binom{n-l}{a}((1)_{n-l-a,\lambda}-(n-l-a)(1)_{n-l-a-1,\lambda})\right)d_{l,\lambda}(x).$$

The *degenerate Daehee polynomials* are defined by the generating function to be

$$\frac{\log_\lambda(1+t)}{t}e_\lambda^x(\log_\lambda(1+t))=\sum_{n=0}^{\infty}D_{n,\lambda}(x)\frac{t^n}{n!},\quad \text{(see [26,27])}.$$

In the special case $x=0$, $D_{n,\lambda}=D_{n,\lambda}(0)$ are called the *degenerate Daehee numbers*. Note that, by (4),

$$\begin{aligned}\frac{\log_\lambda(1+t)}{t}e_\lambda^x(\log_\lambda(1+t))&=\left(\sum_{n=0}^{\infty}D_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{1}{n!}(\log_\lambda(1+t))^n\right)\\&=\left(\sum_{n=0}^{\infty}D_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x)_{n,\lambda}\sum_{l=n}^{\infty}S_{1,\lambda}(l,n)\frac{t^l}{l!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\sum_{k=0}^{r}\binom{n}{r}S_{1,\lambda}(r,k)D_{n-r,\lambda}(x)_{k,\lambda}\right)\frac{t^n}{n!},\end{aligned}$$

and so we see that

$$D_{n,\lambda}(x)=\sum_{r=0}^{n}\sum_{k=0}^{r}\binom{n}{r}S_{1,\lambda}(r,k)D_{n-r,\lambda}(x)_{k,\lambda}. \tag{30}$$

In addition,

$$D_{n,\lambda}(x)\sim\left(\frac{e_\lambda(t)-1}{t},e_\lambda(t)-1\right)_\lambda.$$

Let $d_{n,\lambda}(x)=\sum_{l=0}^{n}a_{n,l}D_{l,\lambda}(x)$. By Theorem 1.1, (22), and (27), we obtain

$$\begin{aligned}a_{n,l}&=\frac{1}{l!}\left\langle\frac{\frac{e_\lambda(t)-1}{t}}{(1-t)e_\lambda(t)}(e_\lambda(t)-1)^l\,\middle|\,(x)_{n,\lambda}\right\rangle_\lambda\\&=\sum_{m=l}^{n}S_{2,\lambda}(m,l)\binom{n}{m}\left\langle\frac{1}{(1-t)e_\lambda(t)}\frac{e_\lambda(t)-1}{t}\,\middle|\,(x)_{n-m,\lambda}\right\rangle_\lambda\\&=\sum_{m=0}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a}S_{2,\lambda}(m,l)d_{a,\lambda}\left\langle\frac{e_\lambda(t)-1}{t}\,\middle|\,(x)_{n-m-a,\lambda}\right\rangle_\lambda\\&=\sum_{m=0}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a}S_{2,\lambda}(m,l)d_{a,\lambda}\frac{(1)_{n-m-a+1,\lambda}}{n-m-a+1}.\end{aligned} \tag{31}$$

Conversely, assume that $D_{n,\lambda}(x)=\sum_{l=0}^{n}b_{n,l}d_{l,\lambda}(x)$. By (17), (20), and (30), we have

$$\begin{aligned}b_{n,l}&=\frac{1}{l!}\langle(1-t)e_\lambda(t)t^l|D_{n,\lambda}(x)\rangle_\lambda\\&=\sum_{r=0}^{n}\sum_{k=l}^{r}\binom{n}{r}S_{1,\lambda}(r,k)D_{n-r,\lambda}\frac{1}{l!}\langle(1-t)e_\lambda(t)t^l|(x)_{k,\lambda}\rangle_\lambda\\&=\sum_{r=0}^{n}\sum_{k=l}^{r}\binom{n}{r}S_{1,\lambda}(r,k)D_{n-r,\lambda}\binom{k}{l}\langle(1-t)e_\lambda(t)|(x)_{k-l,\lambda}\rangle_\lambda\\&=\sum_{r=0}^{n}\binom{n}{r}S_{1,\lambda}(r,l)D_{n-r,\lambda}+\sum_{r=0}^{n}\sum_{k=l+1}^{r}\binom{n}{r}\binom{k}{l}S_{1,\lambda}(r,k)D_{n-r,\lambda}((1)_{k-l,\lambda}-(k-l)(1)_{k-l-1,\lambda}).\end{aligned} \tag{32}$$

By (31) and (32), we obtain the following theorem.

Theorem 2.4. *For each nonnegative integer n, we have*

$$d_{n,\lambda}(x)=\sum_{l=0}^{n}\left(\sum_{m=0}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a}S_{2,\lambda}(m,l)d_{a,\lambda}\frac{(1)_{n-m-a+1,\lambda}}{n-m-a+1}\right)D_{l,\lambda}(x)$$

and

$$D_{n,\lambda}(x)=\sum_{l=0}^{n}\left(\sum_{r=0}^{n}\binom{n}{r}S_{1,\lambda}(r,l)D_{n-r,\lambda}+\sum_{r=0}^{n}\sum_{r=l+1}^{r}\binom{n}{r}\binom{k}{l}S_{1,\lambda}(r,k)D_{n-r,\lambda}((1)_{k-l,\lambda}-(k-l)(1)_{k-l-1,\lambda})\right)d_{l,\lambda}(x).$$

The *Changhee polynomials* are defined by the generating function to be

$$\frac{2}{2+t}e_{\lambda}^{x}(\log_{\lambda}(1+t))=\sum_{n=0}^{\infty}\mathrm{Ch}_n(x)\frac{t^n}{n!},\quad \text{(see [28,29])}.$$

When $x=0$, $\mathrm{Ch}_n=\mathrm{Ch}_n(0)$ are called the *Changhee numbers*. Note that, by (4),

$$\begin{aligned}\frac{2}{2+t}e_{\lambda}^{x}(\log_{\lambda}(1+t))&=\left(\sum_{n=0}^{\infty}\mathrm{Ch}_n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{1}{n!}(\log_{\lambda}(1+t))^n\right)\\&=\left(\sum_{n=0}^{\infty}\mathrm{Ch}_n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x)_{n,\lambda}\sum_{l=n}^{\infty}S_{1,\lambda}(l,n)\frac{t^l}{l!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\sum_{k=0}^{r}\binom{n}{r}S_{1,\lambda}(r,k)\mathrm{Ch}_{n-r}(x)_{k,\lambda}\right)\frac{t^n}{n!},\end{aligned}$$

and so we see that

$$\mathrm{Ch}_n(x)=\sum_{r=0}^{n}\sum_{k=0}^{r}\binom{n}{r}S_{1,\lambda}(r,k)\mathrm{Ch}_{n-r}(x)_{k,\lambda}. \tag{33}$$

In addition, by the definition of Changhee polynomials, we see the λ-Sheffer sequences of the Changhee polynomials are as follows:

$$\mathrm{Ch}_n(x)\sim\left(\frac{1+e_{\lambda}(t)}{2},e_{\lambda}(t)-1\right)_{\lambda}.$$

Let $d_{n,\lambda}(x)=\sum_{l=0}^{n}a_{n,l}\mathrm{Ch}_l(x)$. By (4), Theorem 1.1, (22), and (27), we obtain

$$\begin{aligned}a_{n,l}&=\frac{1}{l!}\left\langle\frac{\frac{e_{\lambda}(t)+1}{2}}{(1-t)e_{\lambda}(t)}(e_{\lambda}(t)-1)^l\,\middle|\,(x)_{n,\lambda}\right\rangle_{\lambda}\\&=\sum_{m=l}^{n}\binom{n}{m}S_{2,\lambda}(m,l)\left\langle\frac{e_{\lambda}(t)+1}{2}\frac{1}{(1-t)e_{\lambda}(t)}\,\middle|\,(x)_{n-m,\lambda}\right\rangle_{\lambda}\\&=\sum_{m=l}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a}S_{2,\lambda}(m,l)d_{a,\lambda}\left\langle\frac{e_{\lambda}(t)+1}{2}\,\middle|\,(x)_{n-m-a,\lambda}\right\rangle_{\lambda}\\&=\sum_{m=l}^{n}\sum_{a=0}^{n-m-1}\binom{n}{m}\binom{n-m}{a}\frac{S_{2,\lambda}(m,l)d_{a,\lambda}(1)_{n-m-a,\lambda}}{2}+\sum_{m=l}^{n}\binom{n}{m}S_{2,\lambda}(m,l)d_{n-m,\lambda}.\end{aligned} \tag{34}$$

Conversely, assume that $\mathrm{Ch}_n(x)=\sum_{l=0}^{n}b_{n,l}d_{l,\lambda}(x)$. Then, by (20) and (33), we obtain

$$\begin{aligned} b_{n,l} &= \frac{1}{l!}\langle (1-t)e_\lambda(t)t^l | \mathrm{Ch}_n(x)\rangle_\lambda \\ &= \sum_{r=0}^{n}\sum_{k=0}^{r}\binom{n}{r}S_{1,\lambda}(r,k)\mathrm{Ch}_{n-r}\frac{1}{l!}\langle (1-t)e_\lambda(t)t^l | (x)_{k,\lambda}\rangle_\lambda \\ &= \sum_{r=0}^{n}\sum_{k=0}^{r}\binom{n}{r}\binom{k}{l}S_{1,\lambda}(r,k)\mathrm{Ch}_{n-r}\langle (1-t)e_\lambda(t) | (x)_{k-l,\lambda}\rangle_\lambda \\ &= \sum_{r=0}^{n}\binom{n}{r}S_{1,\lambda}(r,l)\mathrm{Ch}_{n-r} + \sum_{r=0}^{n}\sum_{k=l+1}^{r}\binom{n}{r}\binom{k}{l}S_{1,\lambda}\mathrm{Ch}_{n-r}((1)_{k-l,\lambda} - (k-l)(1)_{k-l-1,\lambda}). \end{aligned} \tag{35}$$

By (34) and (35), we obtain the following theorem.

Theorem 2.5. *For each nonnegative integer n, we have*

$$d_{n,\lambda}(x) = \sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{a=0}^{n-m-1}\binom{n}{m}\binom{n-m}{a}\frac{S_{2,\lambda}(m,l)d_{a,\lambda}(1)_{n-m-a,\lambda}}{2} + \sum_{m=l}^{n}\binom{n}{m}S_{2,\lambda}(m,l)d_{n-m,\lambda}\right)\mathrm{Ch}_l(x),$$

and

$$\mathrm{Ch}_n(x) = \sum_{l=0}^{n}\left(\sum_{r=0}^{n}\binom{n}{r}S_{1,\lambda}(r,l)\mathrm{Ch}_{n-r} + \sum_{r=0}^{n}\sum_{k=l+1}^{r}\binom{n}{r}\binom{k}{l}S_{1,\lambda}\mathrm{Ch}_{n-r}((1)_{k-l,\lambda} - (k-l)(1)_{k-l-1,\lambda})\right)d_{l,\lambda}.$$

The *degenerate Bell polynomials* are defined by the generating function to be

$$e_\lambda^x(e_\lambda(t)-1) = \sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [17,30])}.$$

In the special case $x = 1$, $\mathrm{Bel}_{n,\lambda} = \mathrm{Bel}_{n,\lambda}(1)$ are called the *degenerate Bell numbers*. By the definition of the Bell polynomials, we see that

$$\mathrm{Bel}_{n,\lambda}(x) \sim (1, \log_\lambda(1+t))_\lambda. \tag{36}$$

Note that

$$\begin{aligned} e_\lambda^x(e_\lambda(t)-1) &= \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{1}{n!}(e_\lambda(t)-1)^n \\ &= \sum_{n=0}^{\infty}(x)_{n,\lambda}\sum_{l=n}^{\infty}S_{2,\lambda}(l,n)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_{2,\lambda}(n,m)(x)_{m,\lambda}\right)\frac{t^n}{n!}, \end{aligned}$$

and thus, see that

$$\mathrm{Bel}_{n,\lambda} = \sum_{m=0}^{n}S_{2,\lambda}(n,m)(x)_{m,\lambda}. \tag{37}$$

Let $d_{n,\lambda}(x) = \sum_{l=0}^{n}a_{n,l}\mathrm{Bel}_{l,\lambda}(x)$. Then by Theorem 1.1, (2), and (36), we obtain

$$\begin{aligned} a_{n,l} &= \frac{1}{l!}\left\langle \frac{1}{(1-t)e_\lambda(t)}(\log_\lambda(1+t))^l \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\ &= \sum_{m=l}^{n}\binom{n}{m}S_{1,\lambda}(m,l)\left\langle \frac{1}{(1-t)e_\lambda(t)} \,\middle|\, (x)_{n-m,\lambda}\right\rangle_\lambda \\ &= \sum_{m=l}^{n}\binom{n}{m}S_{1,\lambda}(m,l)d_{n-m,\lambda}, \end{aligned} \tag{38}$$

and by (19) and (38), we have

$$a_{n,l} = \sum_{m=l}^{n}\sum_{r=0}^{n-m}\binom{n}{m}S_{1,\lambda}(m,l)\frac{(-1)^r\langle 1\rangle_{r,\lambda}}{r!}. \tag{39}$$

Conversely, assume that $\mathrm{Bel}_{n,\lambda}(x) = \sum_{l=0}^{n} b_{n,l}d_{l,\lambda}(x)$. Since

$$e_\lambda(e_\lambda(t)-1) = \sum_{n=0}^{\infty}(1)_{n,\lambda}\frac{1}{n!}(e_\lambda(t)-1)^n = \sum_{n=0}^{\infty}(1)_{n,\lambda}\sum_{l=m}^{\infty}S_{2,\lambda}(l,m)\frac{t^l}{l!} = \sum_{m=0}^{\infty}\sum_{r=0}^{m}(1)_{r,\lambda}S_{2,\lambda}(m,r)\frac{t^m}{m!}, \tag{40}$$

by (37) and (40), we have

$$\begin{aligned}
b_{n,l} &= \frac{1}{l!}\langle(1-(e_\lambda(t)-1))e_\lambda(e_\lambda(t)-1)(e_\lambda(t)-1)^l|(x)_{n,\lambda}\rangle_\lambda \\
&= \sum_{m=l}^{n}\binom{n}{m}S_{2,\lambda}(m,l)\langle(1-(e_\lambda(t)-1))e_\lambda(e_\lambda(t)-1)|(x)_{n-m,\lambda}\rangle_\lambda \\
&= \sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{r=0}^{a}\binom{n}{m}\binom{n-m}{a}S_{2,\lambda}(m,l)S_{2,\lambda}(a,r)(1)_{r,\lambda}\langle(1-(e_\lambda(t)-1))|(x)_{n-m-a,\lambda}\rangle_\lambda \\
&= 2\sum_{m=l}^{n}\sum_{r=0}^{n-m}\binom{n}{m}S_{2,\lambda}(m,l)S_{2,\lambda}(n-m,r)(1)_{r,\lambda} \\
&\quad - \sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{r=0}^{a}\binom{n}{m}\binom{n-m}{a}S_{2,\lambda}(m,l)S_{2,\lambda}(a,r)(1)_{r,\lambda}(1)_{n-m-a,\lambda},
\end{aligned} \tag{41}$$

and by (20) and (37), we obtain

$$\begin{aligned}
a_{n,l} &= \frac{1}{l!}\langle(1-t)e_\lambda(t)t^l|\mathrm{Bel}_{n,\lambda}(x)\rangle_\lambda \\
&= \sum_{m=0}^{n}\binom{m}{l}S_{2,\lambda}(n,m)\langle(1-t)e_\lambda(t)|(x)_{m-l,\lambda}\rangle_\lambda \\
&= S_{2,\lambda}(n,l) + \sum_{m=l+1}^{n}\binom{m}{l}S_{2,\lambda}(n,m)((1)_{m-l,\lambda}(m-l)(1)_{m-l-1,\lambda}).
\end{aligned} \tag{42}$$

By (38), (39), (41), and (42), we obtain the following theorem.

Theorem 2.6. *For each nonnegative integer n, we have*

$$\begin{aligned}
d_{n,\lambda}(x) &= \sum_{l=0}^{n}\left(\sum_{m=l}^{n}\binom{n}{m}S_{1,\lambda}(m,l)d_{n-m,\lambda}\right)\mathrm{Bel}_{l,\lambda}(x) \\
&= \sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{r=0}^{n-m}\binom{n}{m}S_{1,\lambda}(m,l)\frac{(-1)^r\langle 1\rangle_{r,\lambda}}{r!}\right)\mathrm{Bel}_{l,\lambda}(x),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Bel}_{n,\lambda}(x) &= \sum_{l=0}^{n}\left(2\sum_{m=l}^{n}\sum_{r=0}^{n-m}\binom{n}{m}S_{2,\lambda}(m,l)S_{2,\lambda}(n-m,r)(1)_{r,\lambda}\right. \\
&\quad \left. - \sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{r=0}^{a}\binom{n}{m}\binom{n-m}{a}S_{2,\lambda}(m,l)S_{2,\lambda}(a,r)(1)_{r,\lambda}(1)_{n-m-a,\lambda}\right)d_{l,\lambda}(x) \\
&= \sum_{l=0}^{n}\left(S_{2,\lambda}(n,l) + \sum_{m=l+1}^{n}\binom{m}{l}S_{2,\lambda}(n,m)((1)_{m-l,\lambda}(m-l)(1)_{m-l-1,\lambda})\right)d_{l,\lambda}(x).
\end{aligned}$$

The explicit formula of the *unsigned Lah number* $L(n,k)$ is

$$L(n,k)=\binom{n-1}{k-1}\frac{n!}{k!},\quad \text{(see [22,31])}. \tag{43}$$

By (43), we can derive the generating function of $L(n,k)$ to be

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k=\sum_{n=k}^{\infty}L(n,k)\frac{t^n}{n!},\quad (k\geq 0),\quad \text{(see [22,31])}. \tag{44}$$

Recently, Kim and Kim defined the *degenerate Lah-Bell polynomials* by the generating function to be

$$e_\lambda^x\left(\frac{t}{1-t}\right)=\sum_{n=0}^{\infty}B_{n,\lambda}^L(x)\frac{t^n}{n!},\quad \text{(see [32])}.$$

When $x=1$, $B_{n,\lambda}^L=B_{n,\lambda}^L(1)$ are called the *degenerate Lah-Bell numbers*. By the definition of degenerate Lah-Bell polynomials, we see that

$$B_{n,\lambda}^L(x)\sim\left(1,\frac{t}{1+t}\right)_\lambda \quad \text{and} \quad B_{n,\lambda}^L(x)=\sum_{m=0}^{n}L(n,m)(x)_{m,\lambda}. \tag{45}$$

Let $d_{n,\lambda}(x)=\sum_{l=0}^{n}a_{n,l}B_{l,\lambda}^L(x)$. Since

$$\left(\frac{t}{1+t}\right)^l=\sum_{r=l}^{\infty}(-1)^{r-l}\langle l\rangle_{r-l}\frac{t^r}{(r-l)!}, \tag{46}$$

where $\langle x\rangle_0=1$ and $\langle x\rangle_n=x(x+1)\cdots(x+l-1)$, $n\geq 1$. By (46), we obtain

$$\begin{aligned}
a_{n,l}&=\frac{1}{l!}\left\langle \frac{1}{(1-t)e_\lambda(t)}\left(\frac{t}{1+t}\right)^l \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda\\
&=\sum_{r=l}^{n}(-1)^{r-l}\langle l\rangle_{r-l}\binom{r}{l}\binom{n}{r}\left\langle \frac{1}{(1-t)e_\lambda(t)} \,\middle|\, (x)_{n-r,\lambda}\right\rangle_\lambda\\
&=\sum_{r=l}^{n}(-1)^{r-l}\langle l\rangle_{r-l}\binom{n}{r}\binom{r}{l}d_{n-r,\lambda},
\end{aligned} \tag{47}$$

and by (19) and (47), we have

$$a_{n,l}=\sum_{r=l}^{n}\sum_{a=0}^{n-r}\binom{n}{r}\binom{r}{l}(-1)^{r-l}\langle l\rangle_{r-l}\frac{(-1)^a\langle 1\rangle_{a,\lambda}}{a!}. \tag{48}$$

Conversely, assume that $B_{n,\lambda}^L(x)=\sum_{l=0}^{n}b_{n,l}d_{l,\lambda}$. Then

$$\begin{aligned}
b_{n,l}&=\frac{1}{l!}\left\langle \left(1-\frac{t}{1-t}\right)e_\lambda\left(\frac{t}{1-t}\right)\left(\frac{t}{1-t}\right)^l \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda\\
&=\sum_{m=l}^{n}\binom{n}{m}L(m,l)\left\langle \frac{1-2t}{1-t}e_\lambda\left(\frac{t}{1-t}\right) \,\middle|\, (x)_{n-m,\lambda}\right\rangle_\lambda\\
&=\sum_{m=l}^{n}\sum_{r=0}^{n-m}\binom{n}{m}\frac{n-m}{r}L(m,l)B_{r,\lambda}^L\left\langle -1+\frac{2}{1-t} \,\middle|\, (x)_{n-m-r,\lambda}\right\rangle_\lambda\\
&=-\sum_{m=l}^{n}\binom{n}{m}L(m,l)B_{n-m,\lambda}^L+2\sum_{m=l}^{n}\sum_{r=0}^{n-m}\binom{n}{m}\binom{n-m}{r}(n-m-r)!L(m,l)B_{r,\lambda}^L.
\end{aligned} \tag{49}$$

In addition, by (45), we have

$$\begin{aligned}
a_{n,l}&=\frac{1}{l!}\langle(1-t)e_\lambda(t)t^l|B_{n,\lambda}^L(x)\rangle_\lambda\\
&=\sum_{m=0}^{n}L(n,m)\frac{1}{l!}\langle(1-t)e_\lambda(t)t^l|(x)_{m,\lambda}\rangle_\lambda\\
&=\sum_{m=0}^{n}\binom{m}{l}L(n,m)\langle(1-t)e_\lambda(t)|(x)_{m-l,\lambda}\rangle_\lambda\\
&=L(n,l)+\sum_{m=l+1}^{n}\binom{m}{l}L(n,m)((1)_{m-l,\lambda}-(m-l)(1)_{m-l-1,\lambda}).
\end{aligned} \tag{50}$$

By (47), (48), (49), and (50), we obtain the following theorem.

Theorem 2.7. *For each nonnegative integer n, we have*

$$d_{n,\lambda}(x) = \sum_{l=0}^{n}\left(\sum_{r=l}^{n}(-1)^{r-l}\langle l\rangle_{r-l}\binom{n}{r}\binom{r}{l}d_{r-l,\lambda}\right)B_{l,\lambda}^{L}(x)$$
$$= \sum_{l=0}^{n}\left(\sum_{r=l}^{n}\sum_{a=0}^{n-r}\binom{n}{r}\binom{r}{l}(-1)^{r-l}\langle l\rangle_{r-l}\frac{(-1)^{a}\langle 1\rangle_{a,\lambda}}{a!}\right)B_{l,\lambda}^{L}(x).$$

and

$$B_{n,\lambda}^{L}(x) = \sum_{l=0}^{n}\left(-\sum_{m=l}^{n}\binom{n}{m}L(m,l)B_{n-m,\lambda}^{L} + 2\sum_{m=l}^{n}\sum_{r=0}^{n-m}\binom{n}{m}\binom{n-m}{r}(n-m-r)!L(m,l)B_{r,\lambda}^{L}\right)d_{l,\lambda}(x)$$
$$= \sum_{l=0}^{n}\left(L(n,l) + \sum_{m=l+1}^{n}\binom{m}{l}L(n,m)((1)_{m-l,\lambda} - (m-l)(1)_{m-l-1,\lambda})\right)d_{l,\lambda}(x).$$

The *Mittag-Leffer polynomials* are defined by the generating function to be

$$\left(\frac{1+t}{1-t}\right)^{x} = \sum_{n=0}^{\infty}M_{n}(x)\frac{t^{n}}{n!},\quad \text{(see [18,22])}.$$

In the special case $x = 1$, $M_n = M_n(1)$ are called the *Mittag-Leffer numbers*. Note that

$$\left(\frac{1+t}{1-t}\right)^{x} = \sum_{n=0}^{\infty}\binom{x}{n}\left(\frac{2t}{1-t}\right)^{n}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}2^{n}(x)_{n}\frac{(n+m)!}{n!m!}\langle n\rangle_{m}\frac{t^{n+m}}{(n+m)!} \tag{51}$$
$$= \sum_{r=0}^{\infty}\sum_{n=0}^{r}\binom{r}{n}2^{n}(x)_{n}\langle n\rangle_{r-n}\frac{t^{r}}{r!},$$

and so, by the definition of Mittag-Leffer polynomials and (20),

$$M_{n}(x) = \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}2^{m}\langle m\rangle_{n-m}S_{1,\lambda}(m,k)(x)_{k,\lambda} \tag{52}$$

and

$$(e_{\lambda}(t)+1)^{-l} = 2^{-l}\left(1+\frac{e_{\lambda}(t)-1}{2}\right)^{-l}$$
$$= \sum_{r=0}^{\infty}2^{-r-l}(-1)^{r}\langle l\rangle_{r}\frac{1}{r!}(e_{\lambda}(t)-1)^{r} \tag{53}$$
$$= \sum_{b=0}^{\infty}\sum_{r=0}^{b}2^{-r-l}(-1)^{r}\langle l\rangle_{r}S_{2,\lambda}(b,r)\frac{t^{b}}{b!}.$$

Let $d_{n,\lambda}(x) = \sum_{l=0}^{n}a_{n,l}M_{l}(x)$. Since

$$M_{n}(x) \sim \left(1, \frac{e_{\lambda}(t)-1}{e_{\lambda}(t)+1}\right),$$

by (53), we obtain

$$\begin{aligned} a_{n,l} &= \frac{1}{l!}\left\langle \frac{1}{(1-t)e_\lambda(t)}\left(\frac{e_\lambda(t)-1}{e_\lambda(t)+1}\right)^l \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\ &= \left\langle \frac{1}{(1-t)e_\lambda(t)}(e_\lambda(t)+1)^{-l} \,\middle|\, \left(\frac{1}{l!}(e_\lambda(t)-1)^l\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\ &= \sum_{m=l}^{n}\binom{n}{m}S_{2,\lambda}(m,l)\left\langle \frac{1}{(1-t)e_\lambda(t)}(e_\lambda(t)+1)^{-l} \,\middle|\, (x)_{n-m,\lambda}\right\rangle_\lambda \\ &= \sum_{m=l}^{n}\sum_{b=0}^{n-m}\sum_{r=0}^{b}\binom{n}{m}\binom{n-m}{b}2^{-l-r}(-1)^r\langle l\rangle_r S_{2,\lambda}(m,l)S_{2,\lambda}(b,r)\left\langle \frac{1}{(1-t)e_\lambda(t)} \,\middle|\, (x)_{n-m-b,\lambda}\right\rangle_\lambda \\ &= \sum_{m=l}^{n}\sum_{b=0}^{n-m}\sum_{r=0}^{b}\binom{n}{m}\binom{n-m}{b}\frac{(-1)^r\langle l\rangle_r}{2^{l+r}}S_{2,\lambda}(m,l)S_{2,\lambda}(b,r)d_{n-m-b,\lambda}, \end{aligned} \tag{54}$$

and by (19) and (54), we have

$$a_{n,l} = \sum_{m=l}^{n}\sum_{b=0}^{n-m}\sum_{r=0}^{b}\sum_{a=0}^{n-m-b}\binom{n}{m}\binom{n-m}{b}\frac{(-1)^r\langle l\rangle_r}{2^{l+r}}S_{2,\lambda}(m,l)S_{2,\lambda}(b,r)\frac{(-1)^a\langle 1\rangle_{a,\lambda}}{a!}. \tag{55}$$

Conversely, assume that $M_n(x) = \sum_{l=0}^{n} a_{n,l}d_{l,\lambda}(x)$. Then, by (17), we obtain

$$\begin{aligned} a_{n,l} &= \frac{1}{l!}\langle (1-t)e_\lambda(t)t^l | M_n(x)\rangle_\lambda \\ &= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}2^m\langle m\rangle_{n-m}S_{1,\lambda}(m,k)\frac{1}{l!}\langle (1-t)e_\lambda(t)t^l | (x)_{k,\lambda}\rangle_\lambda \\ &= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\binom{k}{l}2^m\langle m\rangle_{n-m}S_{1,\lambda}(m,k)\frac{1}{l!}\langle (1-t)e_\lambda(t) | (x)_{k-l,\lambda}\rangle_\lambda \\ &= \sum_{m=0}^{n}\binom{n}{m}2^m\langle m\rangle_{n-m}S_{1,\lambda}(m,l) + \sum_{m=0}^{n}\sum_{k=l+1}^{m}\binom{n}{m}\binom{k}{l}2^m\langle m\rangle_{n-m}S_{1,\lambda}(m,k)((1)_{k-l,\lambda} - (k-l)(1)_{k-l-1,\lambda}). \end{aligned} \tag{56}$$

By (54) and (56), we obtain the following theorem.

Theorem 2.8. *For each nonnegative integer n, we have*

$$\begin{aligned} d_{n,\lambda}(x) &= \sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{b=0}^{n-m}\sum_{r=0}^{b}\binom{n}{m}\binom{n-m}{b}\frac{(-1)^r\langle l\rangle_r}{2^{l+r}}S_{2,\lambda}(m,l)S_{2,\lambda}(b,r)d_{n-m-b,\lambda}\right)M_l(x) \\ &= \sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{b=0}^{n-m}\sum_{r=0}^{b}\sum_{a=0}^{n-m-b}\binom{n}{m}\binom{n-m}{b}\frac{(-1)^r\langle l\rangle_r}{2^{l+r}}S_{2,\lambda}(m,l)S_{2,\lambda}(b,r)\frac{(-1)^a\langle 1\rangle_{a,\lambda}}{a!}\right)M_l(x). \end{aligned}$$

$$\begin{aligned} M_n(x) = \sum_{l=0}^{n}\Bigg(&\sum_{m=0}^{n}\binom{n}{m}2^m\langle m\rangle_{n-m}S_{1,\lambda}(m,l) + \sum_{m=0}^{n}\sum_{k=l+1}^{m}\binom{n}{m}\binom{k}{l}2^m\langle m\rangle_{n-m}S_{1,\lambda}(m,k)((1)_{k-l,\lambda} \\ &- (k-l)(1)_{k-l-1,\lambda})\Bigg)d_{l,\lambda}(x). \end{aligned}$$

The *degenerate Frobenius-Euler polynomials of order* α, $\alpha \in \{0\}\cup\mathbb{N}$, are defined by the generating function to be

$$\left(\frac{1-u}{e_\lambda(t)-u}\right)^\alpha e_\lambda^x(t) = \sum_{n=0}^{\infty}h_{n,\lambda}^{(\alpha)}(x|u)\frac{t^n}{n!} \quad \text{(see [33,34])}. \tag{57}$$

When $x = 0$, $h_{n,\lambda}^{(\alpha)}(u) = h_{n,\lambda}^{(\alpha)}(u|0)$ are called the *degenerate Frobenius-Euler numbers*. By (57), we see that

$$h_{n,\lambda}^{(\alpha)}(x|u) \sim \left(\left(\frac{e_\lambda(t)-u}{1-u}\right)^{\alpha}, t\right)_{\lambda}.$$

Note that

$$\begin{aligned}\left(\frac{e_\lambda(t)-u}{1-u}\right)^{\alpha} &= \frac{1}{(1-u)^{\alpha}}((e_\lambda(t)-1)+(1-u))^{\alpha}\\ &= \frac{1}{(1-u)^{\alpha}}\sum_{r=0}^{\alpha}\binom{\alpha}{r}(1-u)^{\alpha-r}(e_\lambda(t)-1)^{r}\\ &= \sum_{r=0}^{\alpha}\frac{(\alpha)_r}{(1-u)^r}\frac{1}{r!}(e_\lambda(t)-1)^r\\ &= \sum_{r=0}^{\alpha}\sum_{a=r}^{\infty}\frac{(\alpha)_r}{(1-u)^r}S_{2,\lambda}(a,r)\frac{t^a}{a!}.\end{aligned} \tag{58}$$

By (58), we obtain

$$\begin{aligned}a_{n,l} &= \frac{1}{l!}\left\langle \frac{\left(\frac{e_\lambda(t)-u}{1-u}\right)^{\alpha}}{(1-t)e_\lambda(t)}t^l \,\middle|\, (x)_{n,\lambda}\right\rangle_{\lambda}\\ &= \binom{n}{l}\left\langle \frac{1}{(1-t)e_\lambda(t)}\left(\frac{e_\lambda(t)-u}{1-u}\right)^{\alpha} \,\middle|\, (x)_{n-l,\lambda}\right\rangle_{\lambda}\\ &= \sum_{m=0}^{n-l}\binom{n}{l}\binom{n-l}{m}d_{m,\lambda}\left\langle \left(\frac{e_\lambda(t)-u}{1-u}\right)^{\alpha} \,\middle|\, (x)_{n-l-m,\lambda}\right\rangle_{\lambda}\\ &= \sum_{m=0}^{n-l}\sum_{r=0}^{\alpha}\binom{n}{l}\binom{n-l}{m}\frac{(\alpha)_r}{(1-u)^r}S_{2,\lambda}(n-l-m,r)d_{m,\lambda},\end{aligned} \tag{59}$$

and by (19) and (59), we have

$$a_{n,l} = \sum_{m=0}^{n-l}\sum_{r=0}^{\alpha}\sum_{b=0}^{m}\binom{n}{l}\binom{n-l}{m}S_{2,\lambda}(n-l-m,r)\frac{(\alpha)_r(-1)^b\langle 1\rangle_{b,\lambda}}{(1-u)^r b!}. \tag{60}$$

Conversely, assume that $h_{n,\lambda}^{(\alpha)}(x|u) = \sum_{l=0}^{n} a_{n,l}d_{l,\lambda}(x)$. Then

$$\begin{aligned}a_{n,l} &= \frac{1}{l!}\left\langle \frac{(1-t)e_\lambda(t)}{\left(\frac{e_\lambda(t)-u}{1-u}\right)^{\alpha}}t^l \,\middle|\, (x)_{n,\lambda}\right\rangle_{\lambda}\\ &= \binom{n}{l}\left\langle \left(\frac{1-u}{e_\lambda(t)-u}\right)^{\alpha}(1-t)e_\lambda(t) \,\middle|\, (x)_{n-l,\lambda}\right\rangle_{\lambda}\\ &= \sum_{m=0}^{n-l}\binom{n}{l}\binom{n-l}{m}h_{m,\lambda}^{(\alpha)}(u)\langle(1-t)e_\lambda(t)|(x)_{n-l-m,\lambda}\rangle_{\lambda}\\ &= \binom{n}{l}h_{n-l,\lambda}^{(\alpha)}(u) + \sum_{m=0}^{n-l-1}\binom{n}{l}\binom{n-l}{m}h_{m,\lambda}^{(\alpha)}(u)((1)_{n-l-m,\lambda}-(n-l-m)(1)_{n-l-m-1,\lambda}).\end{aligned} \tag{61}$$

By (59) and (61), we obtain the following theorem.

Theorem 2.9. *For each nonnegative integer n, we have*

$$\begin{aligned}d_{n,\lambda}(x) &= \sum_{l=0}^{n}\left(\sum_{m=0}^{n-l}\sum_{r=0}^{\alpha}\binom{n}{l}\binom{n-l}{m}\frac{(\alpha)_r}{(1-u)^r}S_{2,\lambda}(n-l-m,r)d_{m,\lambda}\right)h_{l,\lambda}^{(\alpha)}(x)\\ &= \sum_{l=0}^{n}\left(\sum_{m=0}^{n-l}\sum_{r=0}^{\alpha}\sum_{b=0}^{m}\binom{n}{l}\binom{n-l}{m}S_{2,\lambda}(n-l-m,r)\frac{(\alpha)_r(-1)^b\langle 1\rangle_{b,\lambda}}{(1-u)^r b!}\right)h_{l,\lambda}^{(\alpha)}(x).\end{aligned}$$

$$h_{n,\lambda}^{(\alpha)}(x) = \sum_{l=0}^{n}\left(\binom{n}{l}h_{n-l,\lambda}^{(\alpha)}(u) + \sum_{m=0}^{n-l-1}\binom{n}{l}\binom{n-l}{m}h_{m,\lambda}^{(\alpha)}(u)((1)_{n-l-m,\lambda} - (n-l-m)(1)_{n-l-m-1,\lambda})\right)d_{l,\lambda}(x).$$

Remark 2.10. Since $\lim_{\lambda\to 0}e_\lambda^x(t) = e^{xt}$, $\lim_{\lambda\to 0}d_{n,\lambda}(x) = d_n(x)$, $\lim_{\lambda\to 0}\beta_{n,\lambda}(x) = B_n(x)$, $\lim_{\lambda\to 0}E_{n,\lambda} = E_n(x)$, $\lim_{\lambda\to 0}D_{n,\lambda}(x) = D_n(x)$, and $\lim_{\lambda\to 0}\mathrm{Bel}_{n,\lambda}(x) = \mathrm{Bel}_n(x)$, where $B_n(x)$, $E_n(x)$, $D_n(x)$, and $\mathrm{Bel}_n(x)$ are the ordinary Bernoulli polynomials, Euler polynomials, Daehee polynomials, and Bell polynomials, respectively. If we take all the degenerate special functions in this article to $\lambda \to 0$, then we obtain the interesting identities which are mentioned in [35,36].

In addition, Kim found relationships between degenerate Lah-Bell polynomials and another degenerate special functions. In this article, we found the relationships between degenerate derangement polynomials and degenerate Lah-Bell polynomials.

3 Conclusion

Derangement numbers are one of the important and widely covered fiel in various fields such as combinatorics, applied mathematics, and engineering.

One of the important tools to study the properties of special polynomials is the umbral calculus, which are built a completely rigid foundation for theories, based on relatively modern ideas of linear functions and linear operators by Gian-Carlo Rota in 1970s. In particular, Kim-Kim defined λ-umbral calculus with degenerate Sheffer sequences, λ-Sheffer sequence, a family of λ-linear functionals, and λ-differential operators in [17].

In this article, we investigated the relationships between degenerate derangement polynomials and some special polynomials, which are λ-falling polynomials, degenerate Bernoulli polynomials, degenerate Euler polynomials, degenerate Daehee polynomials, Changhee polynomials, degenerate Bell polynomials, degenerate Lah-Bell polynomials, Mittag-Leffer polynomials, and degenerate Frobenius-Euler polynomials of order α by using λ-umbral calculus, and derived some interesting identities of those polynomials.

Acknowledgements: The authors would like to thank the referees for their valuable and detailed comments, which have significantly improved the presentation of this article.

Funding information: This research was supported by the Daegu University Research Grant, 2022.

Author contributions: JWP conceived of the framework, and all authors wrote the manuscript; SJY checked the results of the papers. All authors read and approved the final paper.

Conflict of interest: The authors state no conflict of interest.

Ethical approval: The conducted research is not related to either human or animal use.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during this study.

References

[1] L. Comtet, *Advanced Combinatorics: The Aart of Finite and Infinite Expansions*, D. Reidel Publishing Co., Dordrecht, 1974.
[2] T. Kim, D. S. Kim, H. Lee, and L. C. Jang, *A note on degenerate derangement polynomials and numbers*, AIMS Math. **6** (2021), 6469–6481.

[3] J. Quaintance and H. W. Gould, *Combinatorial Identities for Stirling Numbers*, World Scientific Publishing Co. Pte. Ltd, Singapore, 2016.
[4] F. Qi, J. L. Zhao, and B. N. Guo, *Closed forms for derangement numbers in terms of the Hessenberg determinants*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM. **112** (2018), no. 4, 933–944.
[5] C. Wang, P. Misaka, and I. Mezo, *The r-derangement numbers*, Discrete Math. **340** (2017), no. 7, 1681–1692.
[6] R. J. Clarke and M. Sved, *The Derangements and Bell numbers*, Math. Mag. **66** (1993), no. 5, 299–303.
[7] K. S. Briggs and J. B. Remmel, *A p, q-analogue of the generalized derangement numbers*, Ann. Comb. **13** (2009), no. 1, 1–25.
[8] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Util. Math. **15** (1979), 51–88.
[9] W. A. Khan, J. Younis, and M. Nadeem, *Construction of partially degenerate Laguerre-Bernoulli polynomials of the first kind*, Appl. Math. Sci. Eng. **30** (2022), no. 1, 362–375.
[10] T. Kim, D. S. Kim, H. K. Kim, and H. Lee, *Some properties on degenerate Fubini polynomials*, Appl. Math. Sci. Eng. **30** (2022), no. 1, 235–248.
[11] J. Kwon, P. Wongsason, Y. Kim, and D. Kim, *Representations of modified type 2 degenerate poly-Bernoulli polynomials*, AIMS Math. **7** (2022), no. 6, 11443–11463.
[12] W. A. Khan, R. Ai, K. A. H. Alzobydi, and N. Ahmed, *New family of degenerate poly-Genocchi polynomials with its certain properties*, J. Funct. Spaces. **2021** (2021), Art. ID 6660517, 8 pp.
[13] H. K. Kim and W. A. Khan, *Some identities of a new type of degenerate poly-Frobenius-Euler polynomials and numbers*, Proc. Jangjeon Math. Soc. **24**, (2021), no. 1, 33–45.
[14] M. Acikgoz and U. Duran, *Unified degenerate central Bell polynomials*, J. Math. Anal. **11** (2020), no. 2, 18–33.
[15] T. Kim, *A note on degenerate Stirling polynomials of the second kind*, Proc. Jangjeon Math. Soc. **20** (2017), no. 3, 319–331.
[16] D. S. Kim, T. Kim, and G. W. Jang, *A note on degenerate Stirling numbers of the first kind*, Proc. Jangjeon Math. Soc. **21** (2018), no. 3, 393–404.
[17] D. S. Kim and T. Kim, *Degenerate Sheffer sequences and λ -Sheffer sequences*, J. Math. Anal. Appl. **493** (2021), 124521, 21 pp.
[18] K. S. Nisar, *Umbral Calculus*, LAP LAMBERT Academic Publishing GmbH & Co. KG, Germany, 2012.
[19] S. Roman, *The Umbral Calculus*, Dover Publications, Inc., New York, 2005.
[20] S. Araci, M. Acikgoz, T. Diagana, and H. M. Srivastavav, *A novel approach for obtaining new identities for the λ extension of q-Euler polynomials arising from the q-umbral calculus*, J. Nonlinear Sci. Appl. **10** (2017), no. 4, 1316–1325.
[21] R. Dere and Y. Simsek, *Applications of umbral algebra to some special polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **22** (2012), no. 3, 433–438.
[22] H. Kim, *Degenerate Lah-Bell polynomials arising from degenerate Sheffer sequences*, Adv. Differential Equations. **2020** (2020), Paper no. 687, 16 pp.
[23] T. Kim and D. S. Kim, *Some identities of Catalan-Daehee polynomials arising from umbral calculus*, Appl. Comput. Math. **16** (2017), no. 2, 177–189.
[24] T. Kim, D. S. Kim, D. V. Dolgy, and J. W. Park, *On the type 2 poly-Bernoulli polynomials associated with umbral calculus*, Open Math. **19** (2021), no. 1, 878–887.
[25] Y. Simsek, *Special numbers and polynomials including their generating functions in umbral analysis methods*, Axioms **7** (2018), 22.
[26] D. Lim, *Degenerate, partially degenerate and totally degenerate Daehee numbers and polynomials*, Adv. Difference Equ. **2015** (2015), 287, 9 pp.
[27] S. J. Yun and J. W. Park, *On fully degenerate Daehee numbers and polynomials of the second kind*, J. Math. **2020** (2020), Art. ID 7893498, 9 pp.
[28] H. I. Kwon, T. Kim, and J. J. Seo, *A note on degenerate Changhee numbers and polynomials*, Proc. Jangjeon Math. Soc. **18** (2015), no. 3, 295–305.
[29] S. H. Rim, J. W. Park, S. S. Pyo, and J. Kwon, *The nth twisted Changhee polynomials and numbers*, Proc. Jangjeon Math. Soc. **18** (2015), no. 3, 295–305.
[30] T. Kim and D. S. Kim, *Degenerate polyexponential functions and degenerate Bell polynomials*, J. Math. Anal. Appl. **487** (2020), no. 2, 124017, 15 pp.
[31] S. Tauber, *Lah numbers for Fibonacci and Lucas polynomials*, Fibonacci Quart. **6** (1968), no. 5, 93–99.
[32] D. S. Kim and T. Kim, *Lah-Bell numbers and polynomials*, Proc. Jangjeon Math. Soc. **23** (2020), no. 4, 577–586.
[33] J. Choi, D. S. Kim, T. Kim, and Y. H. Kim, *A note on some identities of Frobeniu-Euler numbers and polynomials*, Int. J. Math. Math. Sci. **2012** (2012), Art. ID 861797, 9 pp.
[34] T. Kim and T. Mansour, *Umbral calculus associated with Frobenius-type Eulerian polynomials*, Russ. J. Math. Phys. **21** (2014), no. 4, 484–493.
[35] T. Kim, D. S. Kim, G. W. Jang, and J. Kwon, *A note on some identities of derangement polynomials*, J. Inequal. Appl. **2018** (2018), Paper no. 40, 17 pp.
[36] Y. Ma, D. S. Kim, T. Kim, H. Kim, and H. Lee, *Someidentities of Lah-Bell polynomials*, Adv. Difference Equ. **2020** (2020), Paper no. 510, 10 pp.