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Research Article

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On the stability of a strongly stabilizing control for degenerate systems in Hilbert spaces

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Abstract: In this article, we explain how a recent Lyapunov theorem on stability plays a role in the study of the strong stabilizability problem in Hilbert spaces. We explore a degenerate controlled system and investigate the properties of a feedback control to stabilize such system in depth. The spectral theory of an appropriate pencil operator is used to generate robustness constraints for a stabilizing control.

Keywords: stabilization of systems by feedback, operators theory, degenerate systems

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1 Introduction

In the present article, we study the controlled system described by the degenerate differential equation:

$$\begin{aligned} &L\xi'(t) = M\xi(t) + Cu(t), \quad \xi, u \in \mathcal{H}, \quad t \geq 0 \\ &\xi(0) = \xi_0, \end{aligned} \tag{1}$$

where L, M, and C are bounded operators in Hilbert spaces $\mathcal{H}$ on the complex number $\mathbb{C}$. The operator L is not necessarily invertible and u is the control.

$$\xi(t) = e^{\lambda t}\xi_0,$$

we obtain the equation

$$(\lambda L - M)\xi_0 = Cu(t)e^{-\lambda t},$$

where $\lambda L - M$ is the pencil operator related to the linear part of system (1).

Stability and stabilization is one of the important concepts of control theory in mathematics. Our purpose in the present work is to check if the characteristic of feedback control $u = \langle \xi, q\rangle$, $q \in \mathcal{H}$ is stabilizing. The literature on controlled systems is more appropriate to study stabilization since the conditions of form are usually too strong for infinite dimensional spaces. Numerous researchers focused on the stability and stabilizability problems, see [1,2] and the references therein.

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Many articles dealt with the study of strong stabilizability and steady Riccati equation approach. Moreover, the linear and nonlinear system of delay differential equations (DDEs) with constant time retardation, new approach to qualitative analyses of differential systems with time-varying delays via Lyapunov-Krasovskii have proved some results to the distinguishability of descriptor systems with a regular pencil [3–6]. Korobov and Sklyar have used analytic semigroups governed by explicit differential equations, with the strong stabilizability of contractive systems in Hilbert spaces [7]. Recently, the approximate controllability of Sobolev-type Hilfer neutral fractional stochastic differential inclusions in Hilbert spaces was examined [8,9]. The stability theory for quasi-linear implicit differential equations was considered in terms exponential solution, while the present work deals with the strongly stabilizing control for degenerate systems, which led to more efficient consequences.

This article is organized as follows: in Section 2, we use spectral theory of operators and an appropriate conformal mapping to generalize Lyapunov's famous result for the spectrum of the pencil operator $\lambda L - M$. The stability of the related stationary degenerate systems of the form:

$$L\xi'(t) = M\xi(t), \quad t \geq 0, \tag{2}$$

in particular, the linear system with the identity operator $L = I$ and $M = T$, i.e.,

$$\xi'(t) = T\xi(t), \quad t \geq 0, \tag{3}$$

is said to be explicit, where T is a linear bounded operator, with $\mathrm{Ker} L \neq \{0\}$, the degenerate system (1) is said to be implicit [10]. In Section 3, we show that operator $\tilde{\lambda}_n L - \tilde{M}$ preserves the Riesz basis property of its eigenelements if $\|c\| \cdot \|q\| < \frac{\Delta_\sigma}{2}$. (Theorem 4.2). This fact, in particular implies (compare Theorem 3.3) that for the strong stability of the semigroup $\{e^{(M+cq^*)t}\}_{t\geq 0}$. The result of Sections 3 and 4 is a development for the case of semigroup $\{e^{(M+cq^*)t}\}_{t\geq 0}$ by degenerate differential equation [7]. In Section 5, we give a robustness analysis for a stabilizing control $u = \langle \xi, q + p\rangle, q, p \in \mathcal{H}$.

2 Preliminaries

We consider the following assumptions:

(1) The operator $\lambda L - M$ is bounded by the discrete spectrum of simple eigenvalues denoted by $\{\lambda_k\}_{k=1}^{\infty}$.
(2) Let M be the generator of a C_0-semigroup $\{e^{Mt}\}_{t\geq 0}$ on a Hilbert space $\mathcal{H}$, and
  (i) the set $\sigma(L, M) \cap (i\mathbb{R})$ is at most countable,
  (ii) if adjoint operator M^* has no pure imaginary eigenvalues, then the semigroup $\{e^{Mt}\}_{t\geq 0}$ is strongly exponentially stable if and only if it is uniformly bounded.
(3) The constant $\Delta_\sigma = \frac{1}{2}\min_{i\neq j}|\lambda_i - \lambda_j| > 0$ exists.
(4) The space $\mathcal{H}$ is one dimensional, so we associate C with a vector $c \in \mathcal{H}$; besides, if $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal eigenbasisi $(\lambda_n L - M)\varphi_n = 0$, then $c_n = \langle c, \varphi_n\rangle \neq 0$, $n \in \mathbb{N}^*$.

The following notations are used in the sequel:

(1) $\mathcal{L}(\mathcal{H}, \mathcal{H})$: denotes Hilbert space of all bounded linear operators.
(2) $\mathbb{C}_- = \{\lambda \in \mathbb{C}, \mathrm{Re}\lambda < 0\}$: open half-plane in $\mathbb{C}$.
(3) $\sigma(L, M)$: spectrum of pencil operator $\lambda L - M$, $\lambda \in \mathbb{C}$, and $L, M \in \mathcal{L}(\mathcal{H}, \mathcal{H})$.
(4) $\sigma_p(L, M)$: point spectrum of pencil operator $\lambda L - M$, i.e., the set of eigenvalues of $\lambda L - M$.
(5) $\mathfrak{H}$: the subspace of $\mathcal{H}$.
(6) The restriction of the operator L, i.e., $L|_{\mathfrak{H}} = L_1$.

Definition 2.1. [11] A pencil of matrices $\lambda L - M$ is polynomial matrix whose coefficients are polynomials of degree less than or equal to 1, λ is a variable indeterminate, where L and M are any two matrices of the same order $m \times n$.

Definition 2.2. [11] A pencil of matrices $\lambda L - M$ is called regular if
- L and M are square matrices of the same order n, and
- The determinant $|\lambda L - M|$ does not vanish identically.

In all other case ($m \neq n$ or $m = n$ but $|\lambda L - M| = 0$), the pencil is called singular.

Definition 2.3. [12,13] The complex number $\lambda \in \mathbb{C}$ is called a regular value of the pencil $\lambda L - M$, if the resolvent $(\lambda L - M)^{-1}$ exists and is bounded. The set of all regular values is denoted by $\rho(L, M)$, and its complement $\sigma(L, M) = \mathbb{C} \backslash \rho(L, M)$ is called the spectrum of the pencil $\lambda L - M$. The set of all eigenvalues of the pencil $\lambda L - M$ is denoted by

$$\sigma_p(L, M) = \{\lambda \in \mathbb{C} / \exists v \neq 0 : (\lambda L - M)v = 0\},$$

where $\rho(L, M)$ indicates the pencil resolvent set

$$\rho(L, M) = \{\lambda \in \mathbb{C} : \lambda \text{ is finite and } (\lambda L - M)^{-1} \text{ is bounded}\}.$$

3 Stability of stationary degenerate systems

For the stationary degenerate system (2), we can define the following condition for exponential stability.

Definition 3.1. [12]. If there are two constants $K > 0$ and $\alpha > 0$ such that for each solution $\xi(t)$, $(t \geq 0)$, the system (2) is called exponentially stable

$$||\xi(t)|| \leq K e^{-\alpha t} ||\xi_0||, \quad \text{for any } t \geq 0. \tag{4}$$

Definition 3.2. [12,14]. If the system (2) satisfies the following qualities:
(i) for any solution $\xi(\cdot)$ such that $\xi(0) = \xi_0 = 0$ then $\xi(t) = 0$, for all $t \geq 0$,
(ii) $S_M(t)$ is strongly stable semigroup if for all

$$\xi \in \mathcal{H}, \quad S_M(t)\,\xi \longrightarrow 0 \quad \text{as } t \longrightarrow \infty,$$

then it is considered to be well posed.

Theorem 3.3. [1]. *The strongly exponentially stable of* $\{e^{Mt}\}_{t \geq 0}$ *occurs if and only if there is a norm* $||\cdot||_1$ *equivalent to the initial norm* $||\cdot||$ *of* $\mathcal{H}$ *such that M is dissipative in* $||\cdot||_1$*, i.e.,*

$$||e^{Mt}\xi||_1 \leq ||\xi||_1, \quad \xi \in \mathcal{H}, \quad t \geq 0.$$

Remark 3.4. [1]. An operator M is said to be dissipative if

$$\langle(\mathrm{Re}M)\xi, \xi\rangle = \mathrm{Re}\langle M\xi, \xi\rangle \leq 0.$$

We call an operator M uniformly dissipative if $\mathrm{Re}(M) \ll 0$.

Theorem 3.5. [12]. *If the system* (2) *is exponentially stable, then all eigenvalues of the pencil* $\lambda A - B$ *are in the half-plane* $\mathrm{Re}\lambda \leq \alpha$, *where* α *is the constant defined in* (4).

Theorem 3.6. *Assume that the pencil operator of bounded operators L and M have a spectrum* $\sigma(L, M)$ *in the left half-plane. Then* $G \gg 0$ *is used for any nonnegative uniform operator*[1], *and there exists an operator* $W \gg 0$ *such that.*

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1 It indicates G is a self-adjoint operator ($G = G^*$), and the inner product is defined nonnegative ($\langle G\xi, \xi\rangle \geq a||\xi||^2$, $a > 0$).

$$L^*WM + M^*WL = -G. \tag{5}$$

Proof. We suppose that

$$\sigma(L, M) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}.$$

Then i is a regular point, and $T = i(iL + M)(iL - M)^{-1}$ is a bounded operator. By using the conformal mapping method, we obtain

$$\mu = \varphi(\lambda) = \frac{-i\lambda + 1}{\lambda - i},$$

then

$$\mu I - T = \left(\frac{-i\lambda + 1}{\lambda - i}\right)(iL - M)(iL - M)^{-1} - i(iL + M)(iL - M)^{-1} = \frac{2}{\lambda - i}(\lambda L - M)(iL - M)^{-1}.$$

The operator $\mu I - T$ is invertible. As a result, $\sigma(T) = \sigma(I, T) = \varphi\sigma(L, M)$ is the spectrum of T. Hence, $\sigma(T)$ is contained in the unit disk. We have the following as a consequence of the general Lyapunov theorem [1]. For each operator $H \gg 0$, there is an operator $W \gg 0$ such that:

$$\begin{aligned}
\operatorname{Re}(WT) &= \frac{WT + T^*W}{2} \\
&= \frac{1}{2}[iW(iL + M)(iL - M)^{-1} - i(-iL^* - M^*)^{-1}(-iL^* + M^*)W] \\
&= \frac{i}{2}(-iL^* - M^*)^{-1}[(-iL^* - M^*)W(iL + M) - (-iL^* + M^*)W(iL - M)](-iL - M)^{-1} \\
&= (-iL^* - M^*)^{-1}(L^*WM + M^*WL)(iL - M)^{-1} = -H.
\end{aligned}$$

It is the same as

$$L^*WM + M^*WL = -G,$$

where $-G = (iL^* + M^*)H(iL - M) \ll 0$. In fact,

$$G^* = G \quad \text{and} \quad \langle G\xi, \xi\rangle \geq a\|\xi\|^2, a > 0.$$

Therefore, the equality is proved. □

Theorem 3.7. *Assume that i is a regular point for the bounded pencil operator $\lambda L - M$, there is an operator $W \gg 0$ such that*

$$L^*WM + M^*WL \ll 0. \tag{6}$$

Then the pencil operator $\lambda L - M$ spectrum $\sigma(L, M)$ is in the left half-plane.

Proof. If $i \in \rho(L, M)$ the operator

$$T = i(iL + M)(iL - M)^{-1}$$

is bounded, and the equality becomes

$$-(iL^* + M^*)\operatorname{Re}(WT)(iL - M) = L^*WM + M^*WL \ll 0.$$

Therefore,

$$\operatorname{Re}(WT) \ll 0.$$

By using the general Lyapunov theorem [1], the spectrum $\sigma(T)$ will be in the unit disk, then we conclude that

$$\sigma(L, M) = \varphi^{-1}(\sigma(T)) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\},$$

where $\lambda = \varphi^{-1}(\mu) = \frac{i\mu + 1}{\mu + i}$.

Consequently, Theorem 3.7 is proved. □

Theorem 3.8. [12] *If* (5) *is satisfied for the couple of positive uniform operators* (W, G), *then* $\lambda = i$ *is not an eigenvalue for the pencil* $\lambda L - M$.

Remark 3.9. The necessary and sufficient conditions of stability for stationary degenerate systems can be obtained by using Theorems 3.6–3.8. We obtain the following crucial result in finite dimensional spaces using the elementary divisors of the pencil of matrices [11].

Corollary 3.10. [12] *The following assertions are equivalents in finite dimensional spaces*:
(*i*) *the system* (2) *is exponentially stable*;
(*ii*) $\sigma(L, M) = \sigma_p(L, M) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\}$;
(*iii*) *there is nonnegative definite matrix* $W \gg 0$ *that has the property*

$$L^* WM + M^* WL \ll 0.$$

Example 3.11. Considering the system (2) in the finite-dimensional spaces.

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \lambda L - M = \begin{pmatrix} \lambda + 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

So,

$$\det(\lambda L - M) = \lambda + 1 \neq 0,$$

it is a regular pencil.
• Finite elementary dividers (FED)

$$D_2(\lambda) = \lambda + 1, \quad D_1(\lambda) = 1, \quad D_0(\lambda) = 1.$$

• Invariant polynomials are given by

$$i_1(\lambda) = \frac{D_2(\lambda)}{D_1(\lambda)} = \lambda + 1, \quad i_2(\lambda) = \frac{D_1(\lambda)}{D_0(\lambda)} = 1.$$

Then, there is only one FED, so according to Weierstrass's theorem [11], we obtain

$$\lambda L - M \equiv \begin{pmatrix} \lambda + 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since

$$\sigma(L; M) = \sigma_p(L; M) = \{-1\} \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\},$$

then the system (2) is exponentially stable.

4 Basis and spectral properties of pencil operator $\tilde{\lambda}_n L - \tilde{M}$

We need to introduce the set:

$$\mathfrak{H} \equiv \left\{ c \in \mathcal{H} : \sum_{i,k=1}^{\infty} |\langle c, \Phi_k \rangle \langle c, \Phi_i \rangle| < \infty;\ \langle c, \Phi_j \rangle \neq 0,\ j = 1, 2, \ldots \right\}.$$

Remark 4.1. If a vector $c \in \mathcal{H}$ such that $\langle c, \Phi_k \rangle \leq \frac{1}{k^2}$, then $c \in \mathfrak{H}$.

In this section, we are interested to study the system of one dimensional feedback control in Hilbert space $\mathcal{H}$ given by

$$L\xi'(t) = M\xi(t) + cu(t), \quad c \in \mathcal{H}, \ \xi(0) = \xi_0, \ t \geq 0. \tag{7}$$

Theorem 4.2. *Let* $\|c\|\cdot\|q\| < \frac{\Delta_\sigma}{2}$, *where* $\Delta_\sigma = \frac{1}{2}\min_{i\neq j}|\lambda_i - \lambda_j| > 0$. *Then the eigenvectors* Φ_n *of the pencil operator* $\tilde{\lambda}_n L - \tilde{M}$, *where* $\tilde{M} = M + cq^*$ *construct a Riesz basis* $\mathcal{H}$.

Proof. Let us consider the spectral equation for the eigenvectors Φ_n:

$$(\tilde{\lambda}_n L - \tilde{M})\Phi_n = 0,$$

or

$$(M + cq^* - \tilde{\lambda}_n L)\Phi_n = 0,$$

and apply the pencil resolvent $(\lambda_n L - M)^{-1}$ of the operator $\lambda L - M$. We obtain

$$\Phi_n = \theta_n \cdot (\lambda_n L - M)^{-1} c,$$

where $\theta_n = -\langle \Phi_n, q\rangle$ and λ_n is eigenvalue of pencil operator $\lambda L - M$.

We obtain

$$\langle (\lambda_n L - M)^{-1} c, q\rangle = -1,$$

with the property $|\lambda_n - \tilde{\lambda}_j| > \Delta_\sigma$, for all $n \neq j$, the resolvent

$$(\lambda_n L - M)^{-1} c = \sum_{n=1}^{\infty} \frac{c_n \varphi_n}{\lambda_n - \tilde{\lambda}_j},$$

we have

$$\frac{\langle c, \varphi_n\rangle\langle \varphi_n, q\rangle}{\lambda_n - \tilde{\lambda}_j} = -1 - \sum_{n\neq j}^{\infty} \frac{\langle c_n, \varphi_n\rangle\langle \varphi_n, q\rangle}{\lambda_n - \tilde{\lambda}_j}. \qquad \square$$

In the sequel, we need the following theorem.

Theorem 4.3. *Let* $\{\tilde{\lambda}_n\}_{n=1}^{\infty}$ *any set of complex numbers such that*:
(i) $|\lambda_n - \tilde{\lambda}_n| < \Delta_\sigma$, $n \in \mathbb{N}^*$,
(ii) $\sum_{n=1}^{\infty} \frac{|\lambda_n - \tilde{\lambda}_n|^2}{|c_n|^2} < \frac{\Delta_\sigma}{\|c\|^2}$,

where Δ_σ, $c_n \equiv \langle c, \varphi_n\rangle$, *and* λ_n *are as in* Theorem 4.2. *Then there exists a unique control* $u(\xi) = q^*\xi$ *such that the spectrum* $\sigma(L, \tilde{M})$ *of the pencil operator* $\tilde{\lambda}_n L - \tilde{M}$ *is* $\{\tilde{\lambda}_n\}_{n=1}^{\infty}$ *the corresponding eigenvectors* $(\tilde{\lambda}_n L - \tilde{M})\Phi_n = 0$.

5 Strong stability of stationary degenerate systems

We provide a detailed analysis and precise description of the norms that guarantee the dissipativity of the operator $\tilde{M}$.

Definition 5.1. The controlled degenerate system (7) is said to be exponentially stabilizable by means of a direct feedback $u(t) = q^*\xi(t)$, $q^* \in \mathcal{H}$ if the corresponding system

$$L\xi'(t) = (M + cq^*)\xi(t) = \tilde{M}\xi(t), \quad \xi(0) = \xi_0 \in \mathcal{H}, \ t \geq 0, \tag{8}$$

is strongly exponentially stable.

Theorem 5.2. *Let system* (8) *is strongly exponentially stable and* $\|c\|\cdot\|q\| < \frac{\Delta_\sigma}{2}$. *Then there exists a Hilbert norm* $\|\cdot\|_W = \langle W\cdot,\cdot\rangle^{1/2}$ *with positive definite* W *such that the operator* $\tilde{M}$ *is dissipative then*:

(*a*) *the system* (8) *is well posed*,[2]

(*b*) *for any solution* $\xi(t)$ *of the system* (8), *one has*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\|_W^2 = \frac{\mathrm{d}}{\mathrm{d}t}\langle W\xi(t), \xi(t)\rangle = -\langle G_0\xi(t), \xi(t)\rangle,$$

where G_0 *is self-adjoint nonnegative compact operator* $G_0 = \sum_{i=1}^{N}\zeta_i\omega_i\omega_i^*$, $\{\omega_i\}_{i=1}^{\infty}$ *is orthonormal basis of eigenvectors corresponding to eigenvalues* $\zeta_i \geq 0$; $i > 1$, *such that*:

(*i*) $\sum_{i=1}^{N}\zeta_i < \infty$,

(*ii*) *there exist* $\alpha_1, \alpha_2 > 0$ *such that for any normed eigenelement* Φ_n *of pencil operator* $\tilde{\lambda}_n L - \tilde{M}$ *and* $(\tilde{\lambda}_n L - \tilde{M})\Phi_n = 0$; $n \geq 1$, *the following estimate holds*

$$\alpha_1 \leq \sum_{i=1}^{N}\frac{\zeta_i\,|\langle\omega_i, \Phi_i\rangle|^2}{|\mathrm{Re}\tilde{\lambda}_i|} \leq \alpha_2, \quad i = 1, \dots, N,$$

(*c*) *the operator* W *is given by*

$$W = \int_0^\infty \sum_{i=1}^{N}\zeta_i e^{L_1^{-1}\tilde{M}^* t}\omega_i\omega_i^* e^{L_1^{-1}\tilde{M}t}\mathrm{d}t, \quad t \geq 0.$$

Corollary 5.3. *If* $\|c\|\cdot\|q\| < \frac{\Delta_\sigma}{2}$, *then the following assertions are equivalents*

(*a*) *the system* (8) *is strongly exponentially stable*;

(*b*) *all the eigenvalues* $\tilde{\lambda}_n$ *of pencil operator* $\tilde{\lambda}_n L - \tilde{M}$ *have a negative real part* $\mathrm{Re}\tilde{\lambda}_n < 0$, *i.e.*,

$$\sigma(L, \tilde{M}) = \sigma_p(L, \tilde{M}) = \{\tilde{\lambda}_n \in \mathbb{C}, n \geq 1 : \mathrm{Re}\tilde{\lambda}_n < 0\};$$

(*c*) *operator* $L_1^{-1}\tilde{M}$ *is uniformly W-dissipative in some Hilbert norm and has not pure imaginary eigenvalues*;

(*d*) *there exists a nonnegative matrix* $W \gg 0$

$$Q_1 \equiv L^* W\tilde{M} + \tilde{M}^* WL = -G_0 \ll 0.$$

6 Robustness of a stabilizing control for degenerate systems

In this section, we deal with system (7) with a feedback control is as follows:

$$u(\xi) = q^*\xi + p^*\xi = \langle\xi, q + p\rangle,$$

it takes the form

$$L\xi'(t) = (M + cq^* + cp^*)\xi(t) = (\tilde{M} + cp^*)\xi(t), \quad c, q, p \in \mathcal{H}, \ t \geq 0. \tag{9}$$

Considering an arbitrary finite or infinite orthonormal system $\{\omega_i\}_{i=1}^{N} \subset \mathcal{H}$ and $\{\varsigma_i\}_{i=1}^{N} \subset \mathcal{H}; \varsigma_i \geq 0$. We define a self-adjoint compact operator by

$$G_1 = \sum_{i=1}^{N}\varsigma_i\omega_i\omega_i^* \gg 0.$$

We examine the Lyapunov equation

$$L^* W(\tilde{M} + cp^*)\xi + (\tilde{M} + cp^*)^* WL\xi = -G_1\xi, \tag{10}$$

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2 In particular, if the system (8) is well posed, then $L_1 = L_{|\mathfrak{H}}$ is invertible.

and it has unique operator solution $W \gg 0$

$$W\xi = \int_0^\infty e^{L_1^{-1}(\tilde{M}+cp^*)t} G_1 e^{L_1^{-1}(\tilde{M}+cp^*)^* t} \xi \mathrm{d}t, \quad t \geq 0. \tag{11}$$

using (11) and fact that the family $\{\omega_i\}_{i=1}^N$ is a basis in $\mathcal{H}$ for more details see [1]. The operator $W \gg 0$ satisfies the condition of Theorem 3.7:

$$\operatorname{Re}(WL_1^{-1}(\tilde{M} + cp^*)) \ll 0.$$

Theorem 6.1. *Suppose that*

(*i*) *the system* (8) *is strongly exponentially stable and* $\|c\| \cdot \|q\| < \frac{\Delta_\sigma}{2}$;

(*ii*) *for any vector p for which there exist a finite or infinite orthonormal system* $\{\omega_i\}_{i=1}^N \subset \mathfrak{H}$, *and* $\{\zeta_i\}_{i=1}^N \subset \mathfrak{H}$, $\zeta_i \geq 0$ *such that*

$$p, W, c \in \{\omega_i\}_{i=1}^N, \quad L_1 = L\,|_{\mathfrak{H}}$$

and the condition

$$\lambda_+ |\langle \xi_+, \omega_i \rangle| < \zeta_i \|\xi_+\|, \quad i = 1, 2, \ldots, N.$$

Then the system (7) *is exponentially stabilizable.*

Proof. The norm $\|\cdot\|_1$ satisfies $\|\xi(t)\|_1 \leq \|\xi(0)\|_1$, $t \geq 0$ for all solutions of (9) and the compact nonnegative operator G_1, constructed by $\{\omega_i\}_{i=1}^N$, $\{\varsigma_i\}_{i=1}^N$ and the operator G_0 (Theorem 5.2). This yields for the eigenelements Φ_n of pencil operator $\tilde{\lambda}_n L - \tilde{M}$:

$$\begin{aligned}
\langle Q_1 \Phi_n, \Phi_n \rangle &= -\langle G_0 \Phi_n, \Phi_n \rangle \\
&= -\langle [L^* W \tilde{M} + \tilde{M}^* W L] \Phi_n, \Phi_n \rangle \\
&= -\langle [L^* W \tilde{\lambda}_n L] \Phi_n + [\tilde{\lambda}_n L^* W L] \Phi_n, \Phi_n \rangle \\
&= -2 \operatorname{Re} \tilde{\lambda}_n \langle W \Phi_n, \Phi_n \rangle \\
&= -\sum_{i=1}^N \varsigma_i |\langle \omega_i, \Phi_n \rangle|^2, \quad n \geq 1.
\end{aligned}$$

The existence of such constants $\alpha_1, \alpha_2 > 0$ with $\frac{\alpha_1}{2} \leq \langle W\Phi_n, \Phi_n \rangle \leq \frac{\alpha_2}{2}$, and since $\operatorname{Re}\tilde{\lambda}_n = -|\operatorname{Re}\tilde{\lambda}_n|$ (see property ii of Theorem 5.2), by Corollary 5.3, equation (10) is equivalent to

$$(L^* W \tilde{M} + \tilde{M}^* W L + L^* W c p^* + p c^* W L)\xi = -G_1 \xi,$$

or

$$(Q_1 + Q_2)\xi = -G_1 \xi \ll 0.$$

We denote by $\lambda_\pm$ and $\xi_\pm$ the eigenvalues and eigenvectors of the two-dimensional self-adjoint operator

$$Q_2 \equiv (L^* W c p^* + p c^* W L) \ll 0$$

given by

$$\lambda_\pm = \langle L_1 W c, p \rangle \pm \|W L_1 c\|.\ \|p\|, \lambda_+ > 0, \lambda_- < 0, L_1 = L_{|\mathfrak{H}}$$

with $\lambda_+ \leq \varepsilon \tilde{\lambda}_n$, we obtain

$$\tilde{\lambda}_n \equiv \min\{-\lambda_1, -\lambda_2, -\lambda_-\}, \quad \exists \varepsilon \in [0, 1],$$

then, for all $\xi \in \mathcal{H}$, we have

$$\xi_\pm = L_1 W c \|p\| \pm p \|W L_1 c\|, \quad \lim_{t \to \infty} \langle S_{\tilde{M}+cp^*}(t) \xi_+, \xi_- \rangle = 0.$$

The eigenvalues $\lambda_1 < 0$ and $\lambda_2 < 0$ of the self-adjoint nonpositive operator $Q_2 \equiv -L_1^* Wcp^* - pc^* WL_1$ given by

(i) if $\langle W, c\rangle \neq 0$, then

$$\lambda_{1,2} = -\|c\|^2 - \frac{\|W\|^2}{2} \pm \sqrt{\left(\|c\|^2 - \frac{\|W\|^2}{2}\right)^2 + 4\langle W, c\rangle};$$

(ii) if $\langle W, c\rangle = 0$, then

$$\lambda_1 = -2\|c\|^2 \quad \text{and} \quad \lambda_2 = -\|W\|^2.$$

It is easy to notice that the condition $\lambda_+|\langle \xi_+, \omega_i\rangle| < \zeta_i\|\xi_+\|$, $i \geq 1$ is sufficient for the from

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\|_1^2 = \langle (Q_1 + Q_2)\xi, \xi\rangle. \tag{12}$$

It is used to prove that $\tilde{M} + cp^*$ has no imaginary eigenvalues. Assume that exists $\tilde{\lambda}_n \in i\mathbb{R}$ such that

$$(\tilde{M} + cp^*)\hat{\xi} = \tilde{\lambda}_n L\hat{\xi}.$$

It is easy to deduce from (12) that $\langle (Q_1 + Q_2)\hat{\xi}, \hat{\xi}\rangle = 0$. If $p, W, c \in \{\omega_i\}_{i=1}^N$, then $\hat{\xi}$ is orthogonal to $\{\omega_i\}_{i=1}^N$. Hence, $\langle \hat{\xi}, p\rangle = 0$ implies that $\tilde{M}\hat{\xi} = \tilde{\lambda}_n L\hat{\xi}$, which contradicts the exponential stability of $\tilde{M}$ since $\tilde{\lambda}_n$ is a pure imaginary eigenvalue. Consequently, Theorem 6.1 is proved. □

Example 6.2. Considering system (1) in the infinite-dimensional Hilbert spaces.

Let us denote by S_n^- the set

$$S_n^- = \left\{\lambda \in \mathbb{C}, c \in \mathfrak{H}, \mathrm{Re}\lambda < 0 : \frac{5}{3}|\langle c, \varphi_n\rangle\langle q, \varphi_n\rangle| \leq |\lambda - \lambda_n| \leq 5|\langle c, \varphi_n\rangle\langle q, \varphi_n\rangle|\right\}.$$

It can be shown that system (8) is strongly exponentially stable if and only if

$$\langle (\lambda_n L - M)^{-1} c, q\rangle = -1.$$

We study system (8) with the feedback control $u(\xi) = \langle \xi, q + p\rangle$, $q, p \in \mathcal{H}$. Let

$$L(\gamma_n) = (\tilde{\lambda}_n \gamma_n) \Rightarrow L_1(\gamma_n) = \delta_n \quad \text{if } L_1 = L|_{\mathfrak{H}} \quad \text{for } n \in \mathbb{N}^*,$$
$$\tilde{M}(\varpi_n) = (\tilde{\lambda}_n \varpi_n), \quad Cu \equiv (c_n)u = c\langle \xi, q + p\rangle \quad \text{and} \quad u = \langle \xi, q + p\rangle,$$

where $\tilde{\lambda}_n \to 0$; $c = (c_n) \in \mathfrak{H}$, $c_n \neq 0$ for $n \in \mathbb{N}^*$, and $\sum_{n=1}^{\infty}\left|\frac{1}{\tilde{\lambda}_n}c_n\right|^2 < +\infty$, with $c_n \neq 0$ for $n \in \mathbb{N}^*$, that system (7) is approximately controllable for any feedback law, on the other hand, the system

$$L\xi' = M\xi + c\langle \xi, q + p\rangle, \quad \xi(0) \in \mathcal{H},$$

is not exponentially stable, and

$$\delta_n = \tilde{\lambda}_n \varpi_n + c_n p^*, \quad n \in \mathbb{N}^*,$$

or

$$\varpi_n = \frac{1}{\tilde{\lambda}_n}\delta_n - \frac{1}{\tilde{\lambda}_n}c_n p^*, \quad n \in \mathbb{N}^*.$$

Finally, $(\varpi_n) \in \mathcal{H}$ is the solution only for some $(\delta_n) \in \mathcal{H}$, such that

$$\sum_{n=1}^{\infty}\left|\frac{1}{\tilde{\lambda}_n}\delta_n\right|^2 < +\infty.$$

7 Conclusion

This article has discussed the stability of a new class for control systems defined by the degenerate differential equation in Hilbert spaces, and this was the major focus of this work. By using the spectral theory of pencil operator $\lambda L - M$, we have established some requirements for the stability of Lyapunov results, and an example was presented to illustrate the main results. Furthermore, the subjects that will be considered in the future are as follows: non local degenerate fractional integro-differential inclusions with Brownian motion, approximate boundary strong stabilizability, and controllability.

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