

Research Article

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A certain class of fractional difference equations with damping: Oscillatory properties

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Abstract: In this study, we have investigated the oscillatory properties of the following fractional difference equation:

$$\nabla^{\alpha+1}\chi(\kappa) \cdot \nabla^{\alpha}\chi(\kappa) - p(\kappa)r(\nabla^{\alpha}\chi(\kappa)) + q(\kappa)\mathcal{G}\left(\sum_{\mu=\kappa-\alpha+1}^{\infty}(\mu-\kappa-1)^{(-\alpha)}\chi(\mu)\right) = 0,$$

where $\kappa \in \mathbb{N}_0$, ∇^{α} denotes the Liouville fractional difference operator of order $\alpha \in (0, 1)$, p , and q are non-negative sequences, and r and $\mathcal{G}$ are real valued continuous functions, all of which satisfy certain assumptions. Using the generalized Riccati transformation technique, mathematical inequalities, and comparison results, we have found a number of new oscillation results. A few examples have been built up in this context to illustrate the main findings. The conclusion of this study is regarded as an expansion of continuous time to discrete time in fractional contexts.

Keywords: oscillation, fractional difference equation, damping, Riccati transformation

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1 Introduction

For every real-world situation to be physically understood, difference equations are crucial. Recent years have seen a significant increase in scholarly interest in the study of fractional differences, which is a generalization or extension of classical calculus. Compared to continuous fractional calculus, which has a lengthy history spanning several centuries, discrete fractional calculus is a relatively recent idea. It has also been proven that equations with fractional differences perform better than identical equations with integer order differences. Fractional differences are helpful in numerous applications of applied science, including engineering, physics, computer science, chemistry, biology, signal processing, electrochemistry, viscoelasticity, fluid dynamics, and image processing [1–4]. Furthermore, one of the most fascinating topics for scientists and engineers is the

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oscillation/non-oscillation of physical waves, which can be well described by fractional difference equations [1,5].

In [6], Gray *et al.* provided a novel definition of fractional difference in the form of a finite sum over a specified index set, which was understood as the discrete fractional operator in the sense of Riemann-Liouville. This was done because the sum of the discrete fractional operators already in use was infinite. Since then, there have been several distinct definitions of discrete fractional operators of the Riemann-Liouville type. In [7], Miller and Ross proposed a new description of the Riemann-Liouville left fractional difference and sum using a generalization of the Cauchy function, which is similar to the Riemann-Liouville fractional differential and integral. Following that, several facets of this definition have been studied, including some features of fractional finite difference theory and fractional difference equation theory. See [8,9] and the references therein for more details. Moreover, in [10], Atici and Eloe introduced a fractional nabla difference operator that is comparable to the forward fractional difference suggested in a study by Miller and Ross [7].

In recent years, many authors have thought about various features of fractional differential and fractional difference equations. For further information, see [10–26] and references therein, where various qualitative characteristics of solutions have been taken into account. Several authors have also looked into fractional difference equations' oscillatory solutions [27–29]. Particularly, some authors have investigated how fractional nabla difference equations oscillate [30–35].

Before introducing the main problem, we recall some relevant results. In [11], Chen initiated the study of oscillatory behavior for the fractional difference equation of the form

$$[r(\kappa)(D_-^\alpha \chi)(\kappa)]' - q(\kappa) \Gamma \left(\int_{\kappa}^{\infty} (\mu - \kappa)^{-\alpha} \chi(\mu) d\mu \right) = 0,$$

where $\kappa > 0$, $\alpha \in (0, 1)$ is a constant, $D_-^\alpha \chi$ is the Liouville right-sided fractional derivative of order α of χ defined by $(D_-^\alpha \chi)(\kappa) := -\frac{1}{\Gamma(1-\alpha)} \frac{d}{d\kappa} \int_{\kappa}^{\infty} (\mu - \kappa)^{-\alpha} \chi(\mu) d\mu$ for $\kappa \in (0, \infty)$. They used the generalized Riccati transformation technique and Young's inequality to obtain oscillations results.

Furthermore, in [12], Chen obtained some sufficient conditions for the oscillatory behavior of the fractional differential equation with damping

$$(D_-^{1+\alpha} \chi)(\kappa) - p(\kappa)(D_-^\alpha \chi)(\kappa) + q(\kappa) \Gamma \left(\int_{\kappa}^{\infty} (\mu - \kappa)^{-\alpha} \chi(\mu) d\mu \right) = 0,$$

where $\kappa > 0$, $\alpha \in (0, 1)$ is a constant, $D_-^\alpha \chi$ is the Liouville right-sided fractional derivative of order α of χ defined by $(D_-^\alpha \chi)(\kappa) := -\frac{1}{\Gamma(1-\alpha)} \frac{d}{d\kappa} \int_{\kappa}^{\infty} (\mu - \kappa)^{-\alpha} \chi(\mu) d\mu$ for $\kappa \in (0, \infty)$. The main results were obtained by employing a generalized Riccati transformation technique and certain mathematical inequalities.

In [28], Chatzarakis *et al.* studied the oscillatory behavior of solutions of the fractional difference equation of the form

$$\Delta(r(\kappa)g(\Delta^\alpha \chi(\kappa))) + p(\kappa) \Gamma \left(\sum_{\mu=\kappa_0}^{\kappa-1+\alpha} (\kappa - \mu - 1)^{(-\alpha)} \chi(\mu) \right) = 0,$$

where $\kappa \in \mathbb{N}_{\kappa_0+1-\alpha}$, $\alpha \in (0, 1]$ is a constant, and Δ^α denotes the Riemann-Liouville fractional difference of order α , by using the Riccati transformation and Hardy-type inequalities.

In [14], Liu and Xu discussed a class of fractional differential equations of the form

$$D_-^{1+\alpha} \chi(\kappa) \cdot D_-^\alpha \chi(\kappa) - p(\kappa) \Gamma(D_-^\alpha \chi(\kappa)) + q(\kappa) \mathcal{G} \left(\int_{\kappa}^{\infty} (\mu - \kappa)^{-\alpha} \chi(\mu) ds \right) = 0,$$

where $\kappa > 0$, $\alpha \in (0, 1)$ is a constant, and $D_-^\alpha \chi$ denotes the Liouville right-sided fractional derivative of order α of χ . They used a generalized Riccati transformation technique to derive some additional oscillation criteria for the equation.

In [29], Alzabut et al. established new oscillation results for a nonlinear fractional difference equation with a damping term of the form

$$\Delta(a(\kappa)\Delta^\alpha\chi(\kappa)) + p(\kappa)\Delta^\alpha\chi(\kappa) + q(\kappa)\Gamma\left(\sum_{\mu=\kappa_0}^{\kappa-1+\alpha}(\kappa-\mu-1)^{(-\alpha)}\chi(\mu)\right) = 0,$$

where $\kappa \in \mathbb{N}_{\kappa_0+1-\alpha}$, $\alpha \in (0, 1]$ is a constant, and Δ^α denotes the Riemann-Liouville fractional difference of order α . The Riccati transformation, a few mathematical inequalities, and comparative outcomes are used to produce the results.

The works described above served as our inspiration and motivation for this article, which examines the oscillatory behavior of the following fractional difference equation with damping term:

$$\nabla^{\alpha+1}\chi(\kappa) \cdot \nabla^\alpha\chi(\kappa) - p(\kappa)\Gamma(\nabla^\alpha\chi(\kappa)) + q(\kappa)\mathcal{G}\left(\sum_{\mu=\kappa-\alpha+1}^{\infty}(\mu-\kappa-1)^{(-\alpha)}\chi(\mu)\right) = 0, \quad (1)$$

where $\kappa \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and ∇^α denotes the Riemann-Liouville fractional difference operator of order $\alpha \in (0, 1)$ and the following circumstances are assumed to exist:

- (A₁) p is a nonnegative sequence such that $1 - p(\kappa) > 0$ for all $\kappa \in \mathbb{N}_0$.
- (A₂) q is a nonnegative sequence.
- (A₃) $\mathcal{G}, F : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions with $\kappa\mathcal{G}(\kappa) > 0$, $\kappa F(\kappa) > 0$ for $\kappa \neq 0$, and there exist constants k_1 and k_2 such that $\mathcal{G}(\kappa)/\kappa \geq k_1$, $\kappa/F(\kappa) \geq k_2$ for all $\kappa \neq 0$.
- (A₄) $\bar{\nabla}F(u) \leq u$ and $F^{-1}(u) \in C(\mathbb{R}, \mathbb{R})$ are continuous functions with $F^{-1}(u) > 0$ for $u \neq 0$, and there exist some positive constant α_1 such that $F^{-1}(uv) \geq \alpha_1 F^{-1}(u)F^{-1}(v)$ for $uv \neq 0$.

The article is organized as follows: in Section 2, we present some basic definitions to support our key conclusions; in Section 3, we offer some fresh findings on the oscillation of equation (1); and finally, we construct a few examples in Section 4 to show that it is impossible to ignore the presumptions that underlie our primary findings.

2 Preliminaries

In this section, we review certain fundamental notations, definitions, and discrete fractional calculus lemmas that are necessary for the sections to follow.

For arbitrary α , we define

$$\kappa^{(\alpha)} = \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa - \alpha + 1)}, \quad (2)$$

where we have the convention that division at the pole yields zero, i.e., we assume that if $\kappa - \alpha + 1 \in \{0, -1, \dots, -k\}$, then $\kappa^{(\alpha)} = 0$. The backward difference operator is defined by as follows:

$$\nabla F(\kappa) = F(\kappa) - F(\kappa - 1). \quad (3)$$

Throughout the study, we consider the following $\bar{\nabla}$ notation:

$$\nabla F(g(\kappa)) = \frac{\nabla F(g(\kappa))}{\nabla g(\kappa)} \nabla g(\kappa) \quad (4)$$

$$= \bar{\nabla} F(g(\kappa)) \nabla g(\kappa). \quad (5)$$

We refer to a real-valued sequence χ meeting equation (1) as a solution of equation (1). A solution $\chi(\kappa)$ of equation (1) is said to be *oscillatory* if for every positive integer κ_0 , there exists $\kappa \geq \kappa_0$ such that $\chi(\kappa)\chi(\kappa + 1) \leq 0$; otherwise, it is said to be *non-oscillatory*. An equation is *oscillatory* if all its solutions oscillate.

Definition 2.1. [36] Let $\alpha > 0$ and $\rho(\mu) = \mu - 1$ be the backward jumping operator. Then, the α th right fractional sum of F is defined as follows:

$$\nabla^{-\alpha}F(\kappa) = \frac{1}{\Gamma(\alpha)} \sum_{\mu=\kappa+\alpha}^b (\rho(\mu) - \kappa)^{(\alpha-1)}F(\mu), \quad \kappa \in_b \mathbb{N} = \{b, b-1, b-2, \dots\}. \quad (6)$$

Note that $\nabla^{-\alpha}$ maps functions defined on ${}_b\mathbb{N}$ to the functions defined on ${}_{b-a}\mathbb{N}$.

Definition 2.2. [36] Let $\alpha > 0$ and $[n] = \alpha + 1$. The Riemann-Liouville right fractional difference of F is defined as follows:

$$\begin{aligned} \nabla^{\alpha}F(\kappa) &= (-1)^n \nabla^n \nabla^{-(n-\alpha)}F(\mu), \quad \kappa \in_{b+(n-\alpha)} \mathbb{N}. \\ &= (-1)^n \frac{\nabla^n}{\Gamma(n-\alpha)} \sum_{\mu=\kappa+(n-\alpha)}^b (\rho(\mu) - \kappa)^{(n-\alpha-1)}F(\mu), \quad \kappa \in_{b+(n-\alpha)} \mathbb{N}. \end{aligned} \quad (7)$$

Using Definitions 2.1 and 2.2, we can obtain the definition for Liouville right-sided fractional sum and Liouville right-sided fractional difference on the whole real axis $\mathbb{R}$ of order $\alpha \in (0, 1)$ for a function F as follows:

$$\nabla^{-\alpha}F(\kappa) = \frac{1}{\Gamma(\alpha)} \sum_{\mu=\kappa+\alpha}^{\infty} (\rho(\mu) - \kappa)^{(\alpha-1)}F(\mu), \quad \kappa \in \mathbb{N}_0. \quad (8)$$

$$\nabla^{\alpha}F(\kappa) = (-1) \nabla \nabla^{-(1-\alpha)}F(\mu) = -\frac{\nabla}{\Gamma(1-\alpha)} \sum_{\mu=\kappa+(1-\alpha)}^{\infty} (\rho(\mu) - \kappa)^{(-\alpha)}F(\mu), \quad \kappa \in \mathbb{N}_0. \quad (9)$$

The following relation is also valid:

$$(\nabla^{(1+\alpha)}\chi)(\kappa) = -\nabla(\nabla^{\alpha}\chi)(\kappa), \quad \alpha \in (0, 1), \quad \kappa \in \mathbb{N}_0. \quad (10)$$

Set

$$G(\kappa) := \sum_{\mu=\kappa-\alpha+1}^{\infty} (\mu - \kappa - 1)^{(-\alpha)}\chi(\mu), \quad (11)$$

and then

$$\nabla G(\kappa) = -\Gamma(1-\alpha)(\nabla^{\alpha}\chi)(\kappa). \quad (12)$$

3 Main results

First, we study the oscillation of equation (1) under the following condition:

$$\sum_{\mu=\kappa_0}^{\infty} F^{-1}\left(\frac{1}{v(\mu)}\right) = \infty, \quad \kappa_0 \in \mathbb{N}_0, \quad (13)$$

where $v(\mu) := \Pi_{k=\kappa_0}^{\mu}(1 + p(k))$, $\mu \in \mathbb{N}_{\kappa_0}$.

Theorem 3.1. Suppose that (A_1) – (A_4) and (13) hold. Assume that there exists a positive decreasing sequence r such that

$$\limsup_{\kappa \rightarrow \infty} \sum_{\mu=\kappa_0+1}^{\kappa} \left[k_1 r(\mu) q(\mu) v(\mu-1) - \frac{(\nabla r(\mu))^2 v(\mu)}{4k_2 \Gamma(1-\alpha) r(\mu)} \right] = \infty, \quad (14)$$

where k_1 and k_2 are defined as in (A_3) . Then, every solution of (1) is oscillatory.

Proof. Suppose that χ is a nonoscillatory solution of (1). Without loss of generality, we may assume that χ is an eventually positive solution of (1). Then, there exists $\kappa_1 \in \mathbb{N}_{\kappa_0}$ such that $\chi(\kappa) > 0$ and $G(\kappa) > 0$ for all $\kappa \in \mathbb{N}_{\kappa_1}$, where G is defined in (11). It follows that

$$\nabla(F(\nabla^a \chi(\kappa))v(\kappa)) = \bar{\nabla}F(\nabla^a \chi(\kappa))v(\kappa - 1)\nabla(\nabla^a \chi(\kappa)) + F(\nabla^a \chi(\kappa))\nabla v(\kappa).$$

In view of v above, we have

$$\begin{aligned} \nabla(F(\nabla^a \chi(\kappa))v(\kappa)) &= -\bar{\nabla}F(\nabla^a \chi(\kappa))v(\kappa - 1)(\nabla^{a+1} \chi(\kappa)) + F(\nabla^a \chi(\kappa))[\Pi_{\mu=\kappa_0}^{\kappa}(1 + p(\mu)) - \Pi_{\mu=\kappa_0}^{\kappa-1}(1 + p(\mu))] \\ &= -\bar{\nabla}F(\nabla^a \chi(\kappa))(\nabla^{a+1} \chi(\kappa))v(\kappa - 1) + F(\nabla^a \chi(\kappa))(1 - 1 + p(\kappa))v(\kappa - 1) \\ &= -\bar{\nabla}F(\nabla^a \chi(\kappa))(\nabla^{a+1} \chi(\kappa))v(\kappa - 1) + F(\nabla^a \chi(\kappa))p(\kappa)v(\kappa - 1). \end{aligned}$$

From (A_4) , the above equation becomes

$$\nabla(F(\nabla^a \chi(\kappa))v(\kappa)) \geq -\nabla^a \chi(\kappa)(\nabla^{a+1} \chi(\kappa))v(\kappa - 1) + F(\nabla^a \chi(\kappa))p(\kappa)v(\kappa - 1).$$

From (1), we have

$$\nabla(F(\nabla^a \chi(\kappa))v(\kappa)) \geq -\nabla^a \chi(\kappa)(\nabla^{a+1} \chi(\kappa))v(\kappa - 1) + F(\nabla^a \chi(\kappa))p(\kappa)v(\kappa - 1) = q(\kappa)\mathcal{G}(G(\kappa))v(\kappa - 1) > 0. \quad (15)$$

That is, $\nabla(F(\nabla^a \chi(\kappa))v(\kappa)) > 0$ $\kappa \in \mathbb{N}_{\kappa_1}$. Then, $F(\nabla^a \chi(\kappa))v(\kappa)$ is strictly increasing for all $\kappa \in \mathbb{N}_{\kappa_1}$. Since, $v(\kappa - 1) > 0$ for all κ and from (A_4) , $\nabla^a \chi(\kappa)$ is eventually of one sign. Now, we claim that

$$\nabla^a \chi(\kappa) < 0, \quad \kappa \in \mathbb{N}_{\kappa_1}. \quad (16)$$

If not, then there exists $\kappa_2 \in \mathbb{N}_{\kappa_1}$ such that $\nabla^a \chi(\kappa_2) > 0$. Since $F(\nabla^a \chi(\kappa))v(\kappa)$ is strictly increasing for all $\kappa \in \mathbb{N}_{\kappa_1}$, we have

$$F(\nabla^a \chi(\kappa))v(\kappa) \geq F(\nabla^a \chi(\kappa_2))v(\kappa_2) =: c > 0 \quad \text{for all } \kappa \in \mathbb{N}_{\kappa_2}. \quad (17)$$

Then, we have

$$\nabla^a \chi(\kappa) \geq F^{-1}\left(\frac{c}{v(\kappa)}\right) = F^{-1}\left(\frac{c}{\Pi_{\mu=\kappa_0}^{\kappa}(1 + p(\mu))}\right).$$

From (A_4) , we obtain

$$F^{-1}\left(\frac{c}{\Pi_{\mu=\kappa_0}^{\kappa}(1 + p(\mu))}\right) \geq \alpha_1 F^{-1}(c) F^{-1}\left(\frac{1}{\Pi_{\mu=\kappa_0}^{\kappa}(1 + p(\mu))}\right).$$

Then, we obtain

$$F^{-1}\left(\frac{1}{\Pi_{\mu=\kappa_0}^{\kappa}(1 + p(\mu))}\right) \leq \frac{-\nabla G(\kappa)}{\alpha_1 F^{-1}(c)\Gamma(1 - \alpha)}.$$

Summing the above inequality from $\kappa_2 + 1$ to κ , we have

$$\begin{aligned} \sum_{\mu=\kappa_2+1}^{\kappa} F^{-1}\left(\frac{1}{\Pi_{\mu=\kappa_0}^{\mu}(1 + p(\mu))}\right) &\leq - \sum_{\mu=\kappa_2+1}^{\kappa} \frac{\nabla G(\mu)}{\alpha_1 F^{-1}(c)\Gamma(1 - \alpha)}, \quad \kappa \geq \kappa_2. \\ &= - \left[\frac{G(\kappa) - G(\kappa_2)}{\alpha_1 F^{-1}(c)\Gamma(1 - \alpha)} \right] \\ &< \frac{G(\kappa_2)}{\alpha_1 F^{-1}(c)\Gamma(1 - \alpha)} < \infty, \end{aligned}$$

which is a contradiction with (13). Therefore, (16) holds. Define the function $w(\kappa)$ as generalized Riccati substitution

$$w(\kappa) = r(\kappa) \frac{-v(\kappa)F(\nabla^a \chi(\kappa))}{G(\kappa)}, \quad \kappa \in \mathbb{N}_{\kappa_1}. \quad (18)$$

Then, we have $w(\kappa) > 0$ for $\kappa \in \mathbb{N}_{\kappa_1}$. Now,

$$\begin{aligned}\nabla w(\kappa) &= \nabla \left[\frac{-v(\kappa)F(\nabla^a \chi(\kappa))}{G(\kappa)} \right] r(\kappa - 1) + \nabla r(\kappa) \left[\frac{-v(\kappa)F(\nabla^a \chi(\kappa))}{G(\kappa)} \right] \\ &= \left[\frac{\nabla(-v(\kappa)F(\nabla^a \chi(\kappa)))}{G(\kappa)} + \nabla G(\kappa) \frac{v(\kappa - 1)F(\nabla^a \chi(\kappa - 1))}{G(\kappa)G(\kappa - 1)} \right] r(\kappa - 1) + \frac{\nabla r(\kappa)}{r(\kappa)} w(\kappa) \\ &\leq -\frac{r(\kappa)q(\kappa)\mathcal{G}(G(\kappa))v(\kappa - 1)}{G(\kappa)} - \frac{r(\kappa)F(\nabla^a \chi(\kappa - 1))v(\kappa - 1)\Gamma(1 - \alpha)\nabla^a \chi(\kappa)}{G(\kappa)G(\kappa - 1)} + \frac{\nabla r(\kappa)}{r(\kappa)} w(\kappa).\end{aligned}$$

Since $F(\nabla^a \chi(\kappa))v(\kappa)$ is strictly increasing on $\mathbb{N}_{\kappa_1}$, $F(\nabla^a \chi(\kappa - 1))v(\kappa - 1) < F(\nabla^a \chi(\kappa))v(\kappa)$ for all $\kappa \in \mathbb{N}_{\kappa_1}$. Then, the above inequality becomes

$$\nabla w(\kappa) < -\frac{r(\kappa)q(\kappa)\mathcal{G}(G(\kappa))v(\kappa - 1)}{G(\kappa)} - \frac{r(\kappa)F(\nabla^a \chi(\kappa))v(\kappa)\Gamma(1 - \alpha)\nabla^a \chi(\kappa)}{G(\kappa)G(\kappa - 1)} + \frac{\nabla r(\kappa)}{r(\kappa)} w(\kappa).$$

Since $\nabla G(\kappa) > 0$, $G(\kappa)$ is increasing for all $\kappa \in \mathbb{N}_{\kappa_1}$. Then,

$$\begin{aligned}\nabla w(\kappa) &< -\frac{r(\kappa)q(\kappa)\mathcal{G}(G(\kappa))v(\kappa - 1)}{G(\kappa)} - \frac{r(\kappa)F(\nabla^a \chi(\kappa))v(\kappa)\Gamma(1 - \alpha)\nabla^a \chi(\kappa)}{G^2(\kappa)} + \frac{\nabla r(\kappa)}{r(\kappa)} w(\kappa) \\ &= -\frac{r(\kappa)q(\kappa)\mathcal{G}(G(\kappa))v(\kappa - 1)}{G(\kappa)} - \frac{\Gamma(1 - \alpha)\nabla^a \chi(\kappa)}{r(\kappa)v(\kappa)F(\nabla^a \chi(\kappa))} w^2(\kappa) + \frac{\nabla r(\kappa)}{r(\kappa)} w(\kappa).\end{aligned}$$

Using (A_3) , we have

$$\begin{aligned}\nabla w(\kappa) &< -\frac{r(\kappa)q(\kappa)\mathcal{G}(G(\kappa))v(\kappa - 1)}{G(\kappa)} - \frac{\Gamma(1 - \alpha)\nabla^a \chi(\kappa)}{r(\kappa)v(\kappa)F(\nabla^a \chi(\kappa))} w^2(\kappa) + \frac{\nabla r(\kappa)}{r(\kappa)} w(\kappa) \\ &\leq -k_1 r(\kappa)q(\kappa)v(\kappa - 1) - \frac{k_2 \Gamma(1 - \alpha)}{r(\kappa)v(\kappa)} w^2(\kappa) + \frac{\nabla r(\kappa)}{r(\kappa)} w(\kappa).\end{aligned}$$

Let $x = w(\kappa)$, $a = -\frac{k_2 \Gamma(1 - \alpha)}{r(\kappa)v(\kappa)}$, and $b = \frac{\nabla r(\kappa)}{r(\kappa)}$. By using the inequality $ax^2 + bx \leq -b^2/4a$, we have

$$\nabla w(\kappa) \leq -k_1 r(\kappa)q(\kappa)v(\kappa - 1) + \frac{(\nabla r(\kappa))^2 v(\kappa)}{4k_2 \Gamma(1 - \alpha)r(\kappa)}$$

or

$$-\nabla w(\kappa) \geq k_1 r(\kappa)q(\kappa)v(\kappa - 1) - \frac{(\nabla r(\kappa))^2 v(\kappa)}{4k_2 \Gamma(1 - \alpha)r(\kappa)}.$$

Summing from $\kappa_0 + 1$ to κ on both sides of the above inequality, we obtain

$$w(\kappa_0) \geq w(\kappa_0) - w(\kappa) \geq \sum_{\mu=\kappa_0+1}^{\kappa} \left[k_1 r(\mu)q(\mu)v(\mu - 1) - \frac{(\nabla r(\mu))^2 v(\mu)}{4k_2 \Gamma(1 - \alpha)r(\mu)} \right].$$

Taking $\limsup_{\kappa \rightarrow \infty}$ on both sides of the above inequality, then we obtain

$$\limsup_{\kappa \rightarrow \infty} \sum_{\mu=\kappa_0+1}^{\kappa} \left[k_1 r(\mu)q(\mu)v(\mu - 1) - \frac{(\nabla r(\mu))^2 v(\mu)}{4k_2 \Gamma(1 - \alpha)r(\mu)} \right] < w(\kappa_0),$$

which is a contradiction with (14). If χ is eventually a negative solution of (1), the proof is similar; hence, we omit it. The proof is complete. $\square$

Theorem 3.2. Suppose that (A_1) – (A_4) and (13) hold. Assume that there exists a positive decreasing sequence r such that

$$\limsup_{\kappa \rightarrow \infty} \sum_{\mu=\kappa_0+1}^{\kappa} \left[k_1 r(\mu)q(\mu) - \frac{[\nabla r(\mu) - r(\mu)p(\mu)]^2}{4k_2 \Gamma(1 - \alpha)r(\mu)} \right] = \infty, \quad (19)$$

where $v(\mu)$ is defined in Theorem 3.1. Then, every solution of (1) is oscillatory.

Proof. Suppose that $\chi(\kappa)$ is a nonoscillatory solution of (1). Without loss of generality, we may assume that $\chi(\kappa)$ is an eventually positive solution of (1). Then, there exists $\kappa_1 \in \mathbb{N}_{\kappa_0}$ such that $\chi(\kappa) > 0$ and $G(\kappa) > 0$ for all $\kappa \in \mathbb{N}_{\kappa_1}$. Proceeding as in the proof of Theorem 3.1, we obtain (16). Now, define the function $w(\kappa)$ as follows:

$$w(\kappa) = r(\kappa) \frac{-F(\nabla^a \chi(\kappa))}{G(\kappa)}, \quad \kappa \in \mathbb{N}_{\kappa_1}. \quad (20)$$

Then, we have

$$\begin{aligned} \nabla w(\kappa) &= \nabla \left[\frac{-F(\nabla^a \chi(\kappa))}{G(\kappa)} \right] r(\kappa - 1) + \nabla r(\kappa) \left[\frac{-F(\nabla^a \chi(\kappa))}{G(\kappa)} \right] \\ &= \left[\frac{\bar{\nabla}(-F(\nabla^a \chi(\kappa))) \nabla(\nabla^a \chi(\kappa))}{G(\kappa)} + \nabla G(\kappa) \frac{F(\nabla^a \chi(\kappa - 1))}{G(\kappa)G(\kappa - 1)} \right] r(\kappa - 1) + \frac{\nabla r(\kappa)}{r(\kappa)} w(\kappa) \\ &= \frac{\bar{\nabla}(F(\nabla^a \chi(\kappa))) \nabla^{a+1} \chi(\kappa) r(\kappa - 1)}{G(\kappa)} - \frac{r(\kappa - 1) F(\nabla^a \chi(\kappa - 1)) \Gamma(1 - a) \nabla^a \chi(\kappa)}{G(\kappa)G(\kappa - 1)} + \frac{\nabla r(\kappa)}{r(\kappa)} w(\kappa) \\ &\leq \frac{\bar{\nabla}(F(\nabla^a \chi(\kappa))) \nabla^{a+1} \chi(\kappa) r(\kappa)}{G(\kappa)} - \frac{r(\kappa) F(\nabla^a \chi(\kappa - 1)) \Gamma(1 - a) \nabla^a \chi(\kappa)}{G(\kappa)G(\kappa - 1)} + \frac{\nabla r(\kappa)}{r(\kappa)} w(\kappa). \end{aligned}$$

From (A4) and (1), we obtain

$$\begin{aligned} \nabla w(\kappa) &\leq \frac{[p(\kappa)F(\nabla^a \chi(\kappa)) - q(\kappa)\mathcal{G}(G(\kappa))]r(\kappa)}{G(\kappa)} - \frac{r(\kappa)F(\nabla^a \chi(\kappa - 1))\Gamma(1 - a)\nabla^a \chi(\kappa)}{G(\kappa)G(\kappa - 1)} + \frac{\nabla r(\kappa)}{r(\kappa)} w(\kappa) \\ &= -p(\kappa)w(\kappa) - \frac{q(\kappa)\mathcal{G}(G(\kappa))r(\kappa)}{G(\kappa)} - \frac{r(\kappa)F(\nabla^a \chi(\kappa - 1))\Gamma(1 - a)\nabla^a \chi(\kappa)}{G(\kappa)G(\kappa - 1)} + \frac{\nabla r(\kappa)}{r(\kappa)} w(\kappa). \end{aligned}$$

Since $G(\kappa) > 0$, $\nabla^a \chi(\kappa) < 0$, and $\bar{\nabla}F(\nabla^a \chi(\kappa)) \leq \nabla^a \chi(\kappa)$, we have $F(\nabla^a \chi(\kappa))$ is strictly decreasing on $\mathbb{N}_{\kappa_1}$. Therefore, we have $-F(\nabla^a \chi(\kappa - 1)) < -F(\nabla^a \chi(\kappa))$ for all $\kappa \in \mathbb{N}_{\kappa_1}$. Thus,

$$\nabla w(\kappa) \leq -p(\kappa)w(\kappa) - \frac{q(\kappa)\mathcal{G}(G(\kappa))r(\kappa)}{G(\kappa)} - \frac{r(\kappa)F(\nabla^a \chi(\kappa))\Gamma(1 - a)\nabla^a \chi(\kappa)}{G(\kappa)G(\kappa - 1)} + \frac{\nabla r(\kappa)}{r(\kappa)} w(\kappa).$$

Since $\nabla G(\kappa) > 0$, $G(\kappa)$ is increasing for all $\kappa \in \mathbb{N}_{\kappa_1}$. Then

$$\nabla w(\kappa) \leq -p(\kappa)w(\kappa) - \frac{q(\kappa)\mathcal{G}(G(\kappa))r(\kappa)}{G(\kappa)} - \frac{r(\kappa)F(\nabla^a \chi(\kappa))\Gamma(1 - a)\nabla^a \chi(\kappa)}{G^2(\kappa)} + \frac{\nabla r(\kappa)}{r(\kappa)} w(\kappa).$$

Using (A₃), we have,

$$\nabla w(\kappa) \leq \frac{-p(\kappa)w(\kappa)r(\kappa)}{r(\kappa)} - k_1 q(\kappa)r(\kappa) - \frac{k_2 \Gamma(1 - a)w^2(\kappa)}{r(\kappa)} + \frac{\nabla r(\kappa)}{r(\kappa)} w(\kappa). \quad (21)$$

Taking

$$x = w(\kappa), \quad a = -\frac{k_2 \Gamma(1 - a)}{r(\kappa)} (a \neq 0), \quad b = \frac{\nabla r(\kappa) - p(\kappa)r(\kappa)}{r(\kappa)},$$

and by using the inequality $ax^2 + bx \leq -b^2/4a$, we have

$$\nabla w(\kappa) \leq -k_1 r(\kappa) q(\kappa) + \frac{[\nabla r(\kappa) - r(\kappa)p(\kappa)]^2}{4k_2 \Gamma(1 - a)r(\kappa)}.$$

Summing both sides of the above inequality from $\kappa_0 + 1$ to κ , we obtain

$$\begin{aligned} w(\kappa) - w(\kappa_0) &\leq \sum_{\mu=\kappa_0+1}^{\kappa} \left[-k_1 r(\mu) q(\mu) + \frac{[\nabla r(\mu) - r(\mu)p(\mu)]^2}{4k_2 \Gamma(1 - a)r(\mu)} \right] \\ w(\kappa_0) &\geq \sum_{\mu=\kappa_0+1}^{\kappa} \left[k_1 r(\mu) q(\mu) - \frac{[\nabla r(\mu) - r(\mu)p(\mu)]^2}{4k_2 \Gamma(1 - a)r(\mu)} \right]. \end{aligned}$$

Taking the limit supremum of both sides of the above inequality as $\kappa \rightarrow \infty$, we obtain

$$\sum_{\mu=\kappa_0+1}^{\kappa} \left[k_1 r(\mu) q(\mu) - \frac{[\nabla r(\mu) - r(\mu)p(\mu)]^2}{4k_2\Gamma(1-\alpha)r(\mu)} \right] \leq w(\kappa_0) < \infty,$$

which is a contradiction with (19). If $\chi(\kappa)$ is eventually a negative solution of (1), the proof is similar; hence, we omit it. The proof is complete. $\square$

Theorem 3.3. Assume that (A_1) – (A_4) and (13) hold, and there exists a positive sequence $H(\kappa, \mu)$ such that

$$\begin{aligned} H(\kappa, \kappa) &= 0 \quad \text{for } \kappa \geq \kappa_0, \\ H(\kappa, \mu) &> 0 \quad \text{for } \kappa > \mu \geq \kappa_0, \\ \nabla_2 H(\kappa, \mu) &= H(\kappa, \mu) - H(\kappa, \mu - 1) < 0 \quad \text{for } \kappa \geq \mu \geq \kappa_0. \end{aligned} \quad (22)$$

If there exists a positive decreasing sequence r such that

$$\limsup_{\kappa \rightarrow \infty} \frac{1}{H(\kappa, \kappa_0)} \sum_{\mu=\kappa_0}^{\kappa} \left[q(\mu)r(\mu)H(\kappa, \mu) - \frac{(\nabla r(\mu))^2 H(\kappa, \mu)}{4k_1 k_2 \Gamma(1-\alpha)r(\mu)} \right] = \infty, \quad (23)$$

then (1) is oscillatory.

Proof. Suppose that $\chi(\kappa)$ is a nonoscillatory solution of (1). Without loss of generality, we may assume that $\chi(\kappa)$ is an eventually positive solution of (1). Then, there exists $\kappa_1 \in \mathbb{N}_{\kappa_0}$ such that $\chi(\kappa) > 0$ and $G(\kappa) > 0$ for all $\kappa \in \mathbb{N}_{\kappa_1}$. Proceeding as in the proof of Theorem 3.1, we obtain (16). Also, as in the proof of Theorem 3.2, we obtain (18). We can rewrite (18) as follows:

$$\nabla w(\kappa) \leq -k_1 q(\kappa) r(\kappa) - \frac{k_2 \Gamma(1-\alpha) w^2(\kappa)}{r(\kappa)} + \frac{\nabla r(\kappa)}{r(\kappa)} w(\kappa).$$

Multiplying the above inequality by $H(\kappa, \mu)$ and summing from κ_1 to κ , we obtain

$$\sum_{\mu=\kappa_1}^{\kappa} k_1 q(\mu) r(\mu) H(\kappa, \mu) \leq + \sum_{\mu=\kappa_1}^{\kappa} H(\kappa, \mu) \nabla w(\mu) + \sum_{\mu=\kappa_1}^{\kappa} \frac{\nabla r(\mu)}{r(\mu)} H(\kappa, \mu) w(\mu) - \sum_{\mu=\kappa_1}^{\kappa} \frac{k_2 \Gamma(1-\alpha) w^2(\mu)}{r(\mu)} H(\kappa, \mu). \quad (24)$$

Using summation by parts, we obtain

$$\begin{aligned} - \sum_{\mu=\kappa_1}^{\kappa} H(\kappa, \mu) \nabla w(\mu) &= -[H(\kappa, \mu) w(\mu)]_{\mu=\kappa_1}^{\kappa} + \sum_{\mu=\kappa_1}^{\kappa} \nabla H(\kappa, \mu) w(\mu - 1) \\ &\leq -[H(\kappa, \mu) w(\mu)]_{\mu=\kappa_1}^{\kappa} + \sum_{\mu=\kappa_1}^{\kappa} \nabla H(\kappa, \mu) w(\mu) \\ &= [H(\kappa, \kappa_1) w(\kappa_1)] + \sum_{\mu=\kappa_1}^{\kappa} \nabla H(\kappa, \mu) w(\mu). \end{aligned}$$

Substituting the above inequality into (24), we have

$$\sum_{\mu=\kappa_1}^{\kappa} k_1 q(\mu) r(\mu) H(\kappa, \mu) \leq H(\kappa, \kappa_1) w(\kappa_1) + \sum_{\mu=\kappa_1}^{\kappa} \left[\nabla H(\kappa, \mu) + \frac{\nabla r(\mu)}{r(\mu)} H(\kappa, \mu) \right] w(\mu) - \frac{k_2 \Gamma(1-\alpha) H(\kappa, \mu)}{r(\mu)} w^2(\mu). \quad (25)$$

Taking

$$x = w(\kappa), \quad a = -\frac{k_2 \Gamma(1-\alpha) H(\kappa, \mu)}{r(\mu)}, \quad b = \nabla H(\kappa, \mu) + \frac{\nabla r(\mu)}{r(\mu)} H(\kappa, \mu),$$

and by using the inequality $ax^2 + bx \leq -b^2/4a$, we have

$$\begin{aligned}
& \sum_{\mu=\kappa_1}^{\kappa} \left\{ \left[\nabla H(\kappa, \mu) + \frac{\nabla r(\mu)}{r(\mu)} H(\kappa, \mu) \right] w(\mu) - \frac{k_2 \Gamma(1-\alpha) H(\kappa, \mu)}{r(\mu)} w^2(\mu) \right\} \\
& \leq \sum_{\mu=\kappa_1}^{\kappa} \frac{(r(\mu) \nabla H(\kappa, \mu) + \nabla r(\mu) H(\kappa, \mu))^2}{4k_2 \Gamma(1-\alpha) H(\kappa, \mu) r(\mu)} \\
& \leq \sum_{\mu=\kappa_1}^{\kappa} \frac{(\nabla r(\mu) H(\kappa, \mu))^2}{4k_2 \Gamma(1-\alpha) H(\kappa, \mu) r(\mu)} \\
& = \sum_{\mu=\kappa_1}^{\kappa} \frac{(\nabla r(\mu))^2 H(\kappa, \mu)}{4k_2 \Gamma(1-\alpha) r(\mu)}.
\end{aligned}$$

Substituting the above inequality in (25), we obtain

$$\sum_{\mu=\kappa_1}^{\kappa} k_1 q(\mu) r(\mu) H(\kappa, \mu) \leq H(\kappa, \kappa_1) w(\kappa_1) + \sum_{\mu=\kappa_1}^{\kappa} \frac{(\nabla r(\mu))^2 H(\kappa, \mu)}{4k_2 \Gamma(1-\alpha) r(\mu)}.$$

Since $\nabla H(\kappa, \mu) \leq 0$ for $\kappa > \mu \geq \kappa_0$, we have $0 < H(\kappa, \kappa_1) \leq H(\kappa, \kappa_0)$ for $\kappa > \kappa_1 \geq \kappa_0$. Therefore, from the previous inequality, we obtain for $\kappa \in \mathbb{N}_{\kappa_1}$

$$\sum_{\mu=\kappa_1}^{\kappa} \left[q(\mu) r(\mu) H(\kappa, \mu) - \frac{(\nabla r(\mu))^2 H(\kappa, \mu)}{4k_1 k_2 \Gamma(1-\alpha) r(\mu)} \right] \leq \frac{1}{k_1} H(\kappa, \kappa_1) w(\kappa_1) \leq \frac{1}{k_1} H(\kappa, \kappa_0) w(\kappa_1). \quad (26)$$

Since $0 < H(\kappa, \kappa_1) \leq H(\kappa, \kappa_0)$ for $\kappa > \mu \geq \kappa_0$, we have $0 < H(\kappa, \mu)/H(\kappa, \kappa_0) \leq 1$ for $\kappa > \mu \geq \kappa_0$. Hence, from (26), we have

$$\begin{aligned}
& \frac{1}{H(\kappa, \kappa_0)} \sum_{\mu=\kappa_0}^{\kappa} \left[q(\mu) r(\mu) H(\kappa, \mu) - \frac{(\nabla r(\mu))^2 H(\kappa, \mu)}{4k_1 k_2 \Gamma(1-\alpha) r(\mu)} \right] \\
& = \frac{1}{H(\kappa, \kappa_0)} \sum_{\mu=\kappa_0}^{\kappa_1-1} \left[q(\mu) r(\mu) H(\kappa, \mu) - \frac{(\nabla r(\mu))^2 H(\kappa, \mu)}{4k_1 k_2 \Gamma(1-\alpha) r(\mu)} \right] \\
& \quad + \frac{1}{H(\kappa, \kappa_0)} \sum_{\mu=\kappa_1}^{\kappa} \left[q(\mu) r(\mu) H(\kappa, \mu) - \frac{(\nabla r(\mu))^2 H(\kappa, \mu)}{4k_1 k_2 \Gamma(1-\alpha) r(\mu)} \right] \\
& \leq \frac{1}{H(\kappa, \kappa_0)} \left(\sum_{\mu=\kappa_0}^{\kappa_1-1} r(\mu) q(\mu) H(\kappa, \mu) + \frac{1}{k_1} H(\kappa, \kappa_0) w(\kappa_1) \right) \\
& \leq \sum_{\mu=\kappa_0}^{\kappa_1-1} r(\mu) q(\mu) + \frac{1}{k_1} w(\kappa_1).
\end{aligned}$$

Letting $\kappa \rightarrow \infty$, we obtain

$$\limsup_{\kappa \rightarrow \infty} \frac{1}{H(\kappa, \kappa_0)} \sum_{\mu=\kappa_0}^{\kappa} \left[q(\mu) r(\mu) H(\kappa, \mu) - \frac{(\nabla r(\mu))^2 H(\kappa, \mu)}{4k_1 k_2 \Gamma(1-\alpha) r(\mu)} \right] \leq \sum_{\mu=\kappa_0}^{\kappa_1-1} r(\mu) q(\mu) + \frac{1}{k_1} w(\kappa_1) < \infty,$$

which yields a contradiction to (23). The proof is complete. $\square$

Next, we study the oscillation of (1) under the following condition:

$$\sum_{\mu=\kappa_0+1}^{\infty} F^{-1} \left(\frac{1}{\Pi_{v=\kappa_0}^{\mu} (1 + p(v))} \right) < \infty. \quad (27)$$

Theorem 3.4. Suppose that (A_1) – (A_4) and (27) hold. Assume that there exists a positive decreasing sequence r such that (14) holds and also, for every $T \geq \kappa_0$,

$$\sum_{\kappa=T}^{\infty} F^{-1} \left[\frac{1}{v(\kappa)} \sum_{\mu=T}^{\kappa} q(\mu)v(\mu-1) \right] = \infty, \quad (28)$$

where $v(\mu)$ is defined in Theorem 3.1 holds. Then, every solution of (1) is oscillatory or satisfies $\lim_{\kappa \rightarrow \infty} G(\kappa) = 0$.

Proof. Suppose that $\chi(\kappa)$ is a nonoscillatory solution of (1). Without loss of generality, we may assume that $\chi(\kappa)$ is an eventually positive solution of (1). Then, there exists $\kappa_1 \in \mathbb{N}_{\kappa_0}$ such that $\chi(\kappa) > 0$ and $G(\kappa) > 0$ for all $\kappa \in \mathbb{N}_{\kappa_1}$. Proceeding as in the proof of Theorem 3.1, we know that $\nabla^a \chi(\kappa)$ is eventually one sign; then, there are two cases for the sign of $\nabla^a \chi(\kappa)$. If $\nabla^a \chi(\kappa)$ is eventually negative, similar to Theorem 3.1, we have the oscillation of (1). Next, $\nabla^a \chi(\kappa)$ is eventually positive, then there exists $\kappa_2 \in \mathbb{N}_{\kappa_1}$ such that $\nabla^a \chi(\kappa) > 0$ for $\kappa \geq \kappa_2$. From (12), we obtain $\nabla G(\kappa) < 0$ for $\kappa \in \mathbb{N}_{\kappa_2}$. Thus, we obtain $\lim_{\kappa \rightarrow \infty} G(\kappa) = L$. We now claim that $L = 0$. Otherwise, assume that $L > 0$, then from (12) and (A_3) , we obtain for $\kappa \in \mathbb{N}_{\kappa_2}$

$$\nabla(F(\nabla^a \chi(\kappa))v(\kappa)) \geq q(\kappa)\mathcal{G}(G(\kappa))v(\kappa-1) \geq k_1 q(\kappa)G(\kappa)v(\kappa-1) \geq k_1 L q(\kappa)v(\kappa-1).$$

Summing both sides of the above inequality from $\kappa_2 + 1$ to κ , we have

$$F(\nabla^a \chi(\kappa))v(\kappa) \geq F(\nabla^a \chi(\kappa_2))v(\kappa_2) + k_1 L \sum_{\mu=\kappa_2+1}^{\kappa} q(\mu)v(\mu-1) > k_1 L \sum_{\mu=\kappa_2+1}^{\kappa} q(\mu)v(\mu-1).$$

Hence, from (12), we obtain

$$-\frac{\nabla G(\kappa)}{\Gamma(1-\alpha)} = \nabla^a \chi(\kappa) \geq F^{-1} \left[\frac{k_1 L \sum_{\mu=\kappa_2+1}^{\kappa} q(\mu)v(\mu-1)}{v(\kappa)} \right] \geq \alpha_1 F^{-1}(k_1 L) F^{-1} \left[\frac{\sum_{\mu=\kappa_2+1}^{\kappa} q(\mu)v(\mu-1)}{v(\kappa)} \right].$$

Summing both sides of the last inequality from $\kappa_2 + 1$ to κ , we obtain

$$G(\kappa) \leq G(\kappa_2) - \alpha_1 \Gamma(1-\alpha) F^{-1}(k_1 L) \sum_{u=\kappa_2+1}^{\kappa} F^{-1} \left[\frac{\sum_{\mu=\kappa_2+1}^{\kappa} q(\mu)v(\mu-1)}{v(u)} \right].$$

Letting $\kappa \rightarrow \infty$, from (28), we obtain $\lim_{\kappa \rightarrow \infty} G(\kappa) = -\infty$. This is a contradiction with $G(\kappa) > 0$. Therefore, we have $L = 0$, i.e., $\lim_{\kappa \rightarrow \infty} G(\kappa) = 0$. The proof is complete. $\square$

4 Applications

In this section, we construct some numerical examples regarding our results.

Example 4.1. Consider the following fractional difference equation:

$$(\nabla^{5/3} \chi)(\kappa) \nabla^{2/3} \chi(\kappa) - \frac{(\nabla^{2/3} \chi)(\kappa)}{2(\kappa+2)} + \frac{\kappa^2 + \kappa + 3}{4\Gamma(1/3)(\kappa+2)\kappa^2} \sum_{\mu=\kappa+1/3}^{\infty} (\mu - \kappa - 1)^{(-2/3)} \chi(\mu) = 0, \quad \kappa \in \mathbb{N}_0. \quad (29)$$

This corresponds to equation (1) with $\alpha = 2/3$, $p(\kappa) = \frac{1}{\kappa+2}$, $q(\kappa) = \frac{\kappa^2 + \kappa + 3}{8\Gamma(1/3)(\kappa+2)\kappa^2}$, $F(x) = \frac{x}{2}$, and $\mathcal{G}(x) = 2x$. For $\kappa_0 = 0$,

$$\sum_{\mu=\kappa_0}^{\infty} F^{-1} \left(\frac{1}{\prod_{k=\kappa_0}^{\mu} (1+p(k))} \right) = 2 \sum_{\mu=0}^{\infty} \frac{1}{\prod_{k=0}^{\mu} \left(1 + \frac{1}{k+2} \right)} = 4 \sum_{\mu=0}^{\infty} \frac{1}{\mu+3} = \infty.$$

Therefore, (13) holds. Choose $k_1 = 2$ and $k_2 = 1$. It is clear that the conditions $(A1)$ – $(A4)$ hold. Furthermore, taking $r(\kappa) = \frac{1}{\kappa+1}$, we have

$$\begin{aligned}
& \limsup_{\kappa \rightarrow \infty} \sum_{\mu=\kappa_0+1}^{\kappa} \left[k_1 r(\mu) q(\mu) v(\mu-1) - \frac{(\nabla r(\mu))^2 v(\mu)}{4k_2 \Gamma(1-\alpha) r(\mu)} \right] \\
&= \limsup_{\kappa \rightarrow \infty} \sum_{\mu=1}^{\kappa} \left[\frac{2}{\mu+1} \frac{1}{8\Gamma(1/3)} \frac{\mu^2 + \mu + 3}{(\mu+2)\mu^2} \frac{\mu+2}{2} - \frac{\mu+3}{8\Gamma(1/3)\mu^2(\mu+1)} \right] \\
&= \frac{1}{8\Gamma(1/3)} \limsup_{\kappa \rightarrow \infty} \sum_{\mu=1}^{\kappa} \frac{1}{\mu+1} = \infty.
\end{aligned}$$

Then, (14) holds. Therefore, by Theorem 3.1, every solution of (29) is oscillatory.

Example 4.2. Consider the following fractional difference equation:

$$(\nabla^{3/2} \chi)(\kappa) \cdot (\nabla^{1/2} \chi)(\kappa) - \frac{1}{\kappa} (\nabla^{1/2} \chi)(\kappa) + \kappa^2 \sum_{\mu=\kappa+1/2}^{\infty} (\mu - \kappa - 1)^{(-1/2)} \chi(\mu) = 0, \quad \kappa \in \mathbb{N}_0. \quad (30)$$

This corresponds to equation (1) with $\alpha = 1/2$, $p(\kappa) = 1/\kappa$, $q(\kappa) = \kappa^2$, and $F(x) = \mathcal{G}(x) = x$. For $\kappa_0 = 1$,

$$\sum_{\mu=\kappa_0}^{\infty} F^{-1} \left(\frac{1}{\prod_{k=\kappa_0}^{\mu} (1 + p(k))} \right) = \sum_{\mu=1}^{\infty} \left(\frac{1}{\prod_{k=1}^{\mu} (1 + \frac{1}{k})} \right) = \sum_{\mu=1}^{\infty} \frac{1}{\mu+1} = \infty.$$

Therefore, (13) holds. Choose $k_1 = k_2 = 1$. It is clear that the conditions (A_1) – (A_4) hold. Furthermore, taking $r(\kappa) = 1/\kappa$, we have

$$\begin{aligned}
\limsup_{\kappa \rightarrow \infty} \sum_{\mu=\kappa_0+1}^{\kappa} \left[k_1 r(\mu) q(\mu) - \frac{[\nabla r(\mu) - r(\mu)p(\mu)]^2}{4k_2 \Gamma(1-\alpha) r(\mu)} \right] &= \limsup_{\kappa \rightarrow \infty} \sum_{\mu=2}^{\kappa} \left[(1/\mu)\mu^2 - \frac{(-1/\mu(\mu-1) - 1/\mu^2)^2}{4\Gamma(1/2)\mu^2} \right] \\
&= \limsup_{\kappa \rightarrow \infty} \sum_{\mu=2}^{\kappa} \left[\mu - \frac{1}{4\sqrt{\pi}} \frac{1}{\mu} \left(\frac{1}{\mu-1} + \frac{1}{\mu} \right)^2 \right] \\
&= \infty.
\end{aligned}$$

Then, (19) holds. Therefore, by Theorem 3.2, every solution of (29) is oscillatory.

Example 4.3. Consider the following fractional difference equation:

$$(\nabla^{5/4} \chi)(\kappa) \cdot (\nabla^{1/4} \chi)(\kappa) - \frac{1}{\kappa} (\nabla^{1/4} \chi)(\kappa) + \kappa \sum_{\mu=\kappa+3/4}^{\infty} (\mu - \kappa - 1)^{-1/2} \chi(\mu) = 0, \quad \kappa \in \mathbb{N}_0. \quad (31)$$

This corresponds to equation (1) with $\alpha = 1/4$, $p(\kappa) = 1/\kappa$, $q(\kappa) = \kappa$, and $F(x) = \mathcal{G}(x) = x$. Taking $\kappa_0 = 1$, $k_1 = k_2 = 1$, and $r(\kappa) = 1$, we see that the conditions (13) and (A_1) – (A_4) hold. Furthermore, taking $H(\kappa, \mu) = \kappa - \mu^2$, then

$$\nabla H(\kappa, \mu) = \kappa - \mu^2 - (\kappa - (\mu-1)^2) = 1 - 2\mu < 0,$$

for $\kappa > \mu \geq 1$. Now,

$$\limsup_{\kappa \rightarrow \infty} \frac{1}{H(\kappa, \kappa_0)} \sum_{\mu=\kappa_0}^{\kappa} \left[q(\mu) r(\mu) H(\kappa, \mu) - \frac{(\nabla r(\mu))^2 H(\kappa, \mu)}{4k_2 \Gamma(1-\alpha) r(\mu)} \right] = \limsup_{\kappa \rightarrow \infty} \frac{1}{\kappa - 1^2} \sum_{\mu=1}^{\kappa} \mu(\kappa - \mu^2) = \infty.$$

Thus, by Theorem 3.3, equation (30) is oscillatory.

Example 4.4. Consider the following fractional difference equation:

$$\nabla^{4/3} \chi(\kappa) \cdot \nabla^{1/3} \chi(\kappa) - \nabla^{1/3} \chi(\kappa) + 2\kappa \sum_{\mu=\kappa+2/3}^{\infty} (\mu - \kappa - 1)^{(-1/3)} \chi(\mu) = 0, \quad \kappa \in \mathbb{N}_0. \quad (32)$$

This corresponds to equation (1) with $\alpha = 1/3$, $p(\kappa) = 1/q(\kappa) = 2\kappa$, and $F(x) = \mathcal{G}(x) = x$. For $\kappa_0 = 1$,

$$\sum_{\mu=\kappa_0}^{\infty} F^{-1}\left(\frac{1}{\Pi_{k=\kappa_0}^{\mu}(1+p(k))}\right) = \sum_{\mu=1}^{\infty} \frac{1}{\Pi_{k=1}^{\mu} 2} = \sum_{\mu=1}^{\infty} \frac{1}{2^{\mu}} < \infty.$$

Thus, (27) holds. Choose $k_1 = k_2 = 1$. It is clear that the conditions (A1)–(A4) hold. Furthermore, $r(\kappa) = \frac{1}{2^{\kappa}}$, then we have

$$\limsup_{\kappa \rightarrow \infty} \sum_{\mu=\kappa_0+1}^{\kappa} \left[k_1 r(\mu) q(\mu) v(\mu-1) - \frac{(\nabla r(\mu))^2 v(\mu)}{4k_2 \Gamma(1-\alpha) r(\mu)} \right] = \limsup_{\kappa \rightarrow \infty} \sum_{\mu=1}^{\kappa} \left[2^{\mu} \frac{1}{2} \frac{1}{2^{\mu+2} \Gamma(2/3)} \right] = \infty.$$

Then, (14) holds. In addition, for $T = 1$,

$$\sum_{\kappa=1}^{\infty} F^{-1}\left(\frac{1}{v(\kappa)} \sum_{\mu=T}^{\kappa} q(\mu) v(\mu-1)\right) = \sum_{\kappa=1}^{\infty} \frac{1}{2^{\kappa}} \sum_{\mu=1}^{\kappa} 2^{\mu} 2^{\mu-1} = \frac{4}{3} \sum_{\kappa=1}^{\infty} \frac{(2^{2\kappa} - 1)}{2^{\kappa+1}} = \infty.$$

Thus, (28) holds. Therefore, by Theorem 3.4, equation (30) is oscillatory or $\lim_{\kappa \rightarrow \infty} G(\kappa) = 0$.

5 Conclusion

In this article, we have established some oscillation criteria for fractional difference equations with a damping term. In particular, nothing is yet known about the oscillatory properties of the following fractional difference equation:

$$\nabla^{\alpha+1} \chi(\kappa) \cdot \nabla^{\alpha} \chi(\kappa) - p(\kappa) F(\nabla^{\alpha} \chi(\kappa)) + q(\kappa) \mathcal{G} \left(\sum_{\mu=\kappa-\alpha+1}^{\infty} (\mu - \kappa - 1)^{(-\alpha)} \chi(\mu) \right) = 0.$$

Based on the transformation used in G , we have obtained a relationship between fractional- and integer-order difference. We employed the generalized Riccati transformation technique, some mathematical inequalities, and comparison results to prove four oscillation theorems for the proposed equation. To evaluate the validity of the proposed results in this research, we offered some numerical examples that indicate consistency with the theoretical results. The approach in obtaining the main theorems above can be generalized to research oscillatory solutions of fractional difference equations with more complicated forms, which are expected to be researched further.

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