DE GRUYTER Demonstratio Mathematica 2023; 56: 20220235

Research Article

Evrim Toklu*

On some geometric results for generalized *k*-Bessel functions

https://doi.org/10.1515/dema-2022-0235
received October 18, 2021; accepted April 28, 2023

Abstract: The main aim of this article is to present some novel geometric properties for three distinct normalizations of the generalized k-Bessel functions, such as the radii of uniform convexity and of α-convexity. In addition, we show that the radii of α-convexity remain in between the radii of starlikeness and convexity, in the case when $\alpha \in [0, 1]$, and they are decreasing with respect to the parameter α. The key tools in the proof of our main results are infinite product representations for normalized k-Bessel functions and some properties of real zeros of these functions.

Keywords: k-Bessel function, univalent, starlike and convex functions, α-convex functions, radius of uniform convexity, radius of α-convexity

MSC 2020: 30C45, 30D15, 33C10

1 Introduction

For $r > 0$, we denote by $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ the open disk with radius r centered at the origin. Let $f : \mathbb{D}_r \to \mathbb{C}$ be the function defined as follows:

$$f(z) = z + \sum_{n\geq 2} a_n z^n, \tag{1}$$

where r is equal or less than the radius of convergence of the above power series. Let $\mathcal{A}$ be the class of analytic functions of the form (1), i.e., normalized by the condition $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions. In addition, we say that the function f is starlike of order $\alpha(\alpha \in [0, 1))$ in $\mathbb{D}_r$ if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad \text{for all } z \in \mathbb{D}_r.$$

The radius of starlikeness of order α of the function f is defined as the real number

$$r_\alpha^\star(f) = \sup\left\{r > 0 \,\middle|\, \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \text{ for all } z \in \mathbb{D}_r\right\}.$$

Note that $r^\star(f) = r_0^\star(f)$ is in fact the largest radius such that the image region $f(\mathbb{D}_{r^\star(f)})$ is a starlike domain with respect to the origin. For $\alpha \in [0, 1)$, we say that the function f is convex of order α in $\mathbb{D}_r$ if and only if

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad \text{for all } z \in \mathbb{D}_r.$$

* **Corresponding author: Evrim Toklu,** Department of Mathematics, Faculty of Education, Ağrı İbrahim Çeçen University, Ağrı, Turkey, e-mail: evrimtoklu@gmail.com

We shall denote the radius of convexity of order α of the function f by the real number

$$r_\alpha^c(f) = \sup\left\{r > 0 \,\middle|\, \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \text{ for all } z \in \mathbb{D}_r\right\}.$$

Note that $r^c(f) = r_0^c(f)$ is the largest radius such that the image region $f(\mathbb{D}_{r^c(f)})$ is a convex domain. For more details about convex and starlike functions, one can refer to [1–3] and the references therein. A function $f \in \mathcal{A}$ is said to be uniformly convex in $\mathbb{D}$, if $f(z)$ is in class of convex function in $\mathbb{D}$ and has the property that for every circular arc γ contained in $\mathbb{D}$, with center ℓ also in $\mathbb{D}$, the arc $f(\gamma)$ is a convex arc. In 1993, Rønning [4] determined the necessary and sufficient conditions for analytic functions to be uniformly convex in the open unit disk, while in 2002, Ravichandran [5] also came up with a simpler criterion for uniform convexity. Analytically, the function f is uniformly convex in $\mathbb{D}_r$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(x)}\right|, \quad z \in \mathbb{D}_r.$$

The radius of uniform convexity is given by the real number

$$r^{\mathrm{uc}}(f) = \sup\left\{r > 0 \,\middle|\, \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right| \text{ for all } z \in \mathbb{D}_r\right\}.$$

Moreover, a function f is said to be in the class of β-uniformly convex function of order α, denoted by $\mathcal{UCV}(\beta, \alpha)$, in [6] if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \left|\frac{zf''(z)}{f'(z)}\right| + \alpha, \quad z \in \mathbb{D}.$$

It is obvious that these classes give a unified presentation of various subclasses. The class $\mathcal{UCV}(\beta, 0)$ is the class of β-uniformly convex functions [7] (see also [8,9]) and $\mathcal{UCV}(1, 0)$ is the class of uniformly convex functions defined by Goodman [1] and Rønning [4]. The real number

$$r_{\beta,\alpha}^{ucv}(f) = \sup\left\{r > 0 \,\middle|\, \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \left|\frac{zf''(z)}{f'(z)}\right| + \alpha \text{ for all } z \in \mathbb{D}_r\right\}$$

is called the radius of β-uniform convexity of order α of the function f.

Finally, by $\mathcal{M}(\alpha, \beta)$, we mean the subclasses of $\mathcal{S}$ consisting of functions that are α-convex of order β in $\mathbb{D}_r$, where $\alpha \in \mathbb{R}$ and $0 \le \beta < 1$. For $\alpha \in \mathbb{R}$ and $\beta \in [0, 1)$, we say that f is α-convex of order β in $\mathbb{D}_r$ if and only if

$$\operatorname{Re}\left((1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf'(z)}{f'(z)}\right)\right) > \beta, \quad z \in \mathbb{D}_r.$$

The radius of α-convexity of order β of the function f is given by the real number

$$r_{\alpha,\beta}(f) = \sup\left\{r > 0 \,\middle|\, \operatorname{Re}\left((1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf'(z)}{f'(z)}\right)\right) > \beta, \quad \alpha \in [0, 1), z \in \mathbb{D}_r\right\}.$$

It is clear that radius of α-convexity of order β of the function f give a unified presentation of the radius of starlikeness of order β and of the radius of convexity of order β. That is, we have the relations $r_{0,\beta}(f) = r_\beta^\star(f)$ and $r_{1,\beta}(f) = r_\beta^c(f)$. For more details on starlike, convex, and α-convex functions, one can refer to [2,3,10,11] and the references therein.

As is well known, certain recent extensions of the gamma functions, such as the q-gamma functions and its generalizations ((p, q)-gamma functions) [12,13] and also k-gamma functions, have been of interest to a wide audience in recent years. The main reason is that these topics stand for a meeting point of today's fast-developing areas in mathematics and physics, like the theory of quantum orthogonal polynomials and special functions, quantum groups, conformal field theories, and statistics. Diaz and Pariguan [14] introduced and investigated k-gamma functions when they were evaluating Feynman integrals. These integrals play a significant role in high-energy physics because they offer a general integral representation of the involved functions [15]. Since then, k-gamma functions have played an important role in the theory of special functions, are closely related to factorial, fractional differential equations, and mathematical physics, and have cropped up in many unexpected places in analysis. For more comprehensive and detailed studies of related works, we can refer the interested reader to [14–19] and the associated references therein. From the above series of articles, many generalizations about special functions arise. Hence, the authors from geometric function theory field continued the study of this family of generalized functions and suggested that many geometric properties of classical special functions have a counterpart in this more general setting. For the studies concerning special functions seen in geometric function theory, one may refer to the works [19–27] and the references therein.

By taking inspiration from the above series of articles, in our current investigation, our main aim is to determine the radii of uniform convexity and α-convexity for three different kinds of normalized k-Bessel functions.

1.1 The generalized k-Bessel function

We shall focus on a generalization of the k-Bessel function of order ν defined by the series

$$ {}_kW_{\nu,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\Gamma_k(nk+\nu+k)}\left(\frac{z}{2}\right)^{2n+\frac{\nu}{k}}, $$

where $k > 0$, $\nu > -1$, $c \in \mathbb{R}$, and Γ_k stands for the k-gamma functions studied in [14] and defined as

$$ \Gamma_k(z) = \int_0^{\infty} t^{z-1}e^{-\frac{t^k}{k}}\mathrm{d}t, $$

for $\mathrm{Re}(z) > 0$. For several intriguing properties of k-gamma and k-Bessel functions, we refer the readers to [18,28,29].

Observe that as $k \to 1$, the k-Bessel function ${}_1W_{\nu,1}$ is reduced to the classical Bessel function J_ν, whereas ${}_1W_{\nu,-1}$ coincides with the modified Bessel function I_ν.

It is easy to check that the function $z \mapsto {}_kW_{\nu,c}$ does not belong to class $\mathcal{A}$. Thus, first we shall perform some natural normalization. We define three functions originating from ${}_kW_{\nu,c}(\cdot)$:

$$ \begin{aligned} {}_kf_{\nu,c}(z) &= \left(2^{\frac{\nu}{k}}\Gamma_k(\nu+k)\,{}_kW_{\nu,c}(z)\right)^{\frac{k}{\nu}}, \\ {}_kg_{\nu,c}(z) &= 2^{\frac{\nu}{k}}\Gamma_k(\nu+k)z^{1-\frac{\nu}{k}}\,{}_kW_{\nu,c}(z), \\ {}_kh_{\nu,c}(z) &= 2^{\frac{\nu}{k}}\Gamma_k(\nu+k)z^{1-\frac{\nu}{2k}}\,{}_kW_{\nu,c}(\sqrt{z}). \end{aligned} $$

It is obvious that each of these functions is of the class $\mathcal{A}$. It is convenient to mention here that, in fact,

$$ {}_kf_{\nu,c}(z) = \exp\left[\frac{k}{\nu}\mathrm{Log}(2^{\frac{\nu}{k}}\Gamma_k(\nu+k)\,{}_kW_{\nu,c}(z))\right], $$

where Log represents the principle branch of the logarithm function, and every many-valued function considered in this article is taken with the principal branch.

2 Preliminary results

In order to prove our main results in the next section, each of the following lemmas will be needed. Some of these lemmas are well known; however, to the best of our knowledge, the first three lemmas are quite new and may be of independent interest.

Lemma 2.1. *If $k > 0$, $c > 0$, $\nu + k > 0$, and $a, b \in \mathbb{R}$, then $z \mapsto a_k W_{\nu,c}(z) + bz_k W'_{\nu,c}(z)$ has all its zeros real, except in the case when $\frac{a}{b} + \frac{\nu}{k} < 0$. Moreover, the zeros of $z \mapsto a_k W_{\nu,c}(z) + bz_k W'_{\nu,c}(z)$ are interlacing with the zeros of $z \mapsto {}_kW_{\nu,c}(z)$.*

Proof. From [19, Lemma 1.1], we know that for $k > 0$, $c > 0$, and $\nu + k > 0$, the function $z \mapsto {}_kW_{\nu,c}(z)$ has infinitely many zeros, which are all real. Moreover, denoting by ${}_k\omega_{\nu,c,n}$, the nth positive zero of ${}_kW_{\nu,c}(z)$, under the same conditions, the Weierstrassian decomposition

$$ {}_kW_{\nu,c}(z) = \frac{\left(\frac{z}{2}\right)^{\frac{\nu}{k}}}{\Gamma_k(\nu + k)} \prod_{n\geq 1}\left(1 - \frac{z^2}{{}_k\omega^2_{\nu,c,n}}\right) \tag{2} $$

is valid. Let us consider the function $z \mapsto {}_k\phi_{\nu,c}(z) = z^{\frac{a}{b}}{}_kW_{\nu,c}(z)$. It is obvious that, in light of Eq. (2), the following equality immediately takes places:

$$ {}_k\phi_{\nu,c}(z) = \frac{\left(\frac{z}{2}\right)^{\frac{\nu}{k}+\frac{a}{b}}}{\Gamma_k(\nu + k)} \prod_{n\geq 1}\left(1 - \frac{z^2}{{}_k\omega^2_{\nu,c,n}}\right). $$

Taking this into consideration, we obtain

$$ {}_k\phi'_{\nu,c}(z) = \frac{\left(\frac{z}{2}\right)^{\frac{\nu}{k}+\frac{a}{b}} \prod_{n\geq 1}\left(1 - \frac{z^2}{{}_k\omega^2_{\nu,c,n}}\right)}{\Gamma_k(\nu + k)} \left(\frac{\frac{\nu}{k} + \frac{a}{b}}{z} - \sum_{n\geq 1} \frac{2z}{{}_k\omega^2_{\nu,c,n} - z^2}\right), $$

which gives

$$ \frac{{}_k\phi'_{\nu,c}(z)}{{}_k\phi_{\nu,c}(z)} = \frac{\frac{\nu}{k} + \frac{a}{b}}{z} - \sum_{n\geq 1} \frac{2z}{{}_k\omega^2_{\nu,c,n} - z^2}, $$

where $k > 0$, $c > 0$, and $\nu + k > 0$ such that $z \neq 0$, $z \neq {}_k\omega_{\nu,c,n}$, and $n \in \mathbb{N}_0$. Now, we are going to show that for $\frac{\nu}{k} + \frac{a}{b} > 0$, all the zeros of ${}_k\phi'_{\nu,c}(z)$ are real, provided that for $k > 0$, $c > 0$, and $\nu + k > 0$, all the zeros of ${}_kW_{\nu,c}(z)$ are real. To do this, we show that for $\frac{\nu}{k} + \frac{a}{b} > 0$, the zeros of ${}_kW'_{\nu,c}(z)$ cannot be purely imaginary. Letting ${}_k\phi'_{\nu,c}(iy) = 0$, where $y \in \mathbb{R}$ and $y \neq 0$, and from (2), we conclude that

$$ 0 = -i\left(\frac{\nu}{k} + \frac{a}{b} + \sum_{n\geq 1} \frac{2y^2}{{}_k\omega^2_{\nu,c,n} + y^2}\right), $$

which is a contradiction since $\frac{\nu}{k} + \frac{a}{b} > 0$ and the zeros ${}_k\omega_{\nu,c,n}$ are real. Finally, we show that for $\frac{\nu}{k} + \frac{a}{b} > 0$, the zeros of ${}_k\phi'_{\nu,c}(z)$ cannot be complex. Letting $z = x + iy$, where $xy \neq 0$ and $\gamma = {}_k\omega^2_{\nu,c,n} - x^2 + y^2$

$$ z\frac{{}_k\phi'_{\nu,c}(z)}{{}_k\phi_{\nu,c}(z)} = \frac{\nu}{k} + \frac{a}{b} - 2\sum_{n\geq 1} \frac{\gamma(x^2 - y^2) - 4x^2y^2}{\gamma^2 + 4x^2y^2} - 4xyi\sum_{n\geq 1} \frac{\gamma + x^2 - y^2}{\gamma^2 + 4x^2y^2} = 0, $$

which implies that

$$ \sum_{n\geq 1} \frac{\gamma + x^2 - y^2}{a^2 + 4x^2y^2} = \sum_{n\geq 1} \frac{{}_k\omega^2_{\nu,c,n}}{\gamma^2 + 4x^2y^2} = 0, $$

which is a contradiction. Thus, indeed for $\frac{\nu}{k} + \frac{a}{b} > 0$, the zeros of ${}_k\phi'_{\nu,c}(z)$ are all real. On the other hand, in view of [19], we have that ${}_kW_{\nu,c}(z)$ real entire function of growth order $\frac{1}{2}$ and of genus 0. It is well known from Laguerre's theorem on separation of zeros that, if $z \mapsto f(z)$ is an entire function, not a constant, which is real for real z and has only real zeros, and is of genus 0 or 1, then the zeros of f' are also real and are separated by the zeros of f. Thus, by using the fact that the growth order of the real entire function $z \mapsto z^{\frac{a}{b}}{}_kW_{\nu,c}(z)$ is $\frac{1}{2}$, and by using Laguerre's separation theorem, we deduce that the zeros of $z \mapsto a_k W_{\nu,c}(z) + bz_k W'_{\nu,c}(z)$ are real when $\frac{a}{b} + \frac{\nu}{k} > 0$ and are interlacing with the zeros of $z \mapsto {}_kW_{\nu,c}(z)$. □

Lemma 2.2. *Let $k > 0$, $c > 0$, and $\nu + k > 0$. Suppose that ${}_k\theta_{\nu,c,n}$ s are the nth positive roots of ${}_kW_{\nu,c}(z) - cz_kW_{\nu+k,c}(z) = 0$. Then, the following development holds for all $z \neq {}_k\theta_{\nu,c,n}$, $n \in \mathbb{N}$,*

$$\frac{{}_kg''_{\nu,c}(z)}{{}_kg'_{\nu,c}(z)} = \frac{c^2z_kW_{\nu+2k,c}(z) - 3c_kW_{\nu+k,c}(z)}{{}_kW_{\nu,c}(z) - cz_kW_{\nu+k,c}(z)} = -\sum_{n\geq 1}\frac{2z}{{}_k\theta^2_{\nu,c,n} - z^2}.$$

Proof. Let us consider the integral

$$I_r = \frac{1}{2\pi i}\int_{Q_r}\frac{F(\zeta)}{\zeta(\zeta - z)}\mathrm{d}\zeta, \tag{3}$$

where

$$F(\zeta) = \frac{c^2\zeta_kW_{\nu+2k,c}(\zeta) - 3c_kW_{\nu+k,c}(\zeta)}{{}_kW_{\nu,c}(\zeta) - cz_kW_{\nu+k,c}(\zeta)}.$$

$\{Q_r\}$ is the sequence of circles about origin such that Q_r includes all nonzero zeros $({}_k\theta_{\nu,c,n})(\forall n = 1, 2, \ldots r)$. Moreover, we assume that inside Q_r there are m positive and m negative roots of ${}_kW_{\nu,c}(z) - cz_kW_{\nu+k,c}(z)$. Of course, it is possible to make this assumption since $(1 - \frac{\nu}{k})_kW_{\nu,c}(z) + z_kW'_{\nu,c}(z) = {}_kW_{\nu,c}(z) - cz_kW_{\nu+k,c}(z)$ and according to Lemma 2.1, the zeros of $2k_kW_{\nu,c}(z) - kz_kW_{\nu+k,c}(z)$ are simple and real. Furthermore, Q_r is not passing through any zeros. Then, F is uniformly bounded on these circles. So, let $|F| \leq M$. Therefore, equation (3) provides us $|I_r| \leq \frac{M}{R_r}$, where R_r denotes the radius of Q_r. Hence, $\lim_{r\to\infty}I_r = 0$ as $R_r \to \infty$ for $r \to \infty$. A residue integration shows that (see [30, p. 111], [31, p. 254])

$$\begin{aligned} I_r &= -\frac{F(0)}{z} + \frac{F(z)}{z} + \sum_{n=1}^{r}\frac{1}{{}_k\theta_{\nu,c,n}({}_k\theta_{\nu,c,n} - z)} + \sum_{n=1}^{r}\frac{1}{{}_k\theta_{\nu,c,n}({}_k\theta_{\nu,c,n} + z)} \\ &= -\frac{F(0)}{z} + \frac{F(z)}{z} + 2\sum_{n=1}^{r}\frac{1}{{}_k\theta^2_{\nu,c,n} - z^2}. \end{aligned}$$

The above series will converge uniformly, and we can take the limit. It follows that since $F(0) = 0$,

$$F(z) = -2\sum_{n\geq 1}\frac{z}{{}_k\theta^2_{\nu,c,n} - z^2}.$$

□

Lemma 2.3. *Let $k > 0$, $c > 0$, and $\nu + k > 0$. Suppose that ${}_k\tau_{\nu,c,n}$s are the nth positive roots of $(2k - \nu)_kW_{\nu,c}(z) + kz_kW'_{\nu,c}(z) = 0$. Then, the following equality holds for all $z \neq {}_k\tau_{\nu,c,n}$, $n \in \mathbb{N}$:*

$$z\frac{{}_kh''_{\nu,c}(z)}{{}_kh'_{\nu,c}(z)} = \frac{\nu(\nu - 2k)_kW_{\nu,c}(z^{1/2}) + k(3k - 2\nu)z^{1/2}{}_kW'_{\nu,c}(z^{1/2}) + k^2z_kW''_{\nu,c}(z^{1/2})}{2k(2k - \nu)_kW_{\nu,c}(z^{1/2}) + 2k^2z^{1/2}{}_kW'_{\nu,c}(z^{1/2})} = -\sum_{n\geq 1}\frac{z}{{}_k\tau^2_{\nu,c,n} - z}.$$

Proof. For the sake of convenience, we shall first prove the following formula:

$$\frac{\nu(\nu - 2k)_kW_{\nu,c}(z) + k(3k - 2\nu)z_kW'_{\nu,c}(z) + k^2z^2{}_kW''_{\nu,c}(z)}{2k(2k - \nu)_kW_{\nu,c}(z) + 2k^2z_kW'_{\nu,c}(z)} = -\sum_{n\geq 1}\frac{z^2}{{}_k\tau^2_{\nu,c,n} - z^2}.$$

By making use of the recurrence formula (see [28]) $z_kW'_{\nu,c}(z) = \frac{\nu}{k}{}_kW_{\nu,c}(z) - zc_kW_{\nu+k,c}(z)$, we have

$$(2k - \nu)_kW_{\nu,c}(z) + kz_kW'_{\nu,c}(z) = 2k_kW_{\nu,c}(z) - zkc_kW_{\nu+k,c}(z)$$

and

$$\frac{\nu(\nu - 2k)_kW_{\nu,c}(z) + k(3k - 2\nu)z_kW'_{\nu,c}(z) + k^2z^2{}_kW''_{\nu,c}(z)}{2k(2k - \nu)_kW_{\nu,c}(z) + 2k^2z_kW'_{\nu,c}(z)} = \frac{z}{2}\frac{c^2z_kW_{\nu+2k,c}(z) - 4c_kW_{\nu+k,c}(z)}{2_kW_{\nu,c}(z) - cz_kW_{\nu+k,c}(z)}.$$

We consider the integral

$$I_s = \frac{1}{2\pi i}\int_{Q_s}\frac{F(\zeta)}{\zeta(\zeta - z)}\mathrm{d}\zeta, \tag{4}$$

where

$$F(\zeta) = \frac{c^2\zeta_k W_{\nu+2k,c}(\zeta) - 4c_k W_{\nu+k,c}(\zeta)}{2_k W_{\nu,c}(\zeta) - cz_k W_{\nu+k,c}(\zeta)}.$$

$\{Q_s\}$ is the sequence of circles about origin such that Q_s includes all nonzero zeros $({}_k\tau_{\nu,c,n})$ ($\forall n = 1, 2, \ldots s$). Moreover, we assume that inside Q_s there are m positive and m negative roots of $(2k-\nu)_k W_{\nu,c}(z) + kz_k W'_{\nu,c}(z)$. Of course, it is possible to make this assumption since $(2k-\nu)_k W_{\nu,c}(z) + kz_k W'_{\nu,c}(z) = 2k_k W_{\nu,c}(z) - zkc_k W_{\nu+k,c}(z)$, and according to Lemma 2.1, the zeros of $2k_k W_{\nu,c}(z) - zkc_k W_{\nu+k,c}(z)$ are simple and real. Furthermore, Q_s is not passing through any zeros. Then, F is uniformly bounded on these circles. So, let $|F| \le M$. Therefore, equation (4) provides us $|I_s| \le \frac{M}{R_s}$, where R_s denotes the radius of Q_s. Hence, $\lim_{s\to\infty} I_s = 0$ as $R_s \to \infty$ for $s \to \infty$. A residue integration shows that

$$\begin{aligned} I_s &= -\frac{F(0)}{z} + \frac{F(z)}{z} + \sum_{n=1}^{s}\frac{1}{{}_k\tau_{\nu,c,n}({}_k\tau_{\nu,c,n} - z)} + \sum_{n=1}^{s}\frac{1}{{}_k\tau_{\nu,c,n}({}_k\tau_{\nu,c,n} + z)} \\ &= -\frac{F(0)}{z} + \frac{F(z)}{z} + 2\sum_{n=1}^{s}\frac{1}{{}_k\tau^2_{\nu,c,n} - z^2}. \end{aligned}$$

The above series will converge uniformly, and we can take the limit. It follows that since $F(0) = 0$,

$$F(z) = -2\sum_{n\ge1}\frac{z}{{}_k\tau^2_{\nu,c,n} - z^2},$$

which implies that

$$\frac{z}{2}\frac{c^2 z_k W_{\nu+2k,c}(z) - 4c_k W_{\nu+k,c}(z)}{2_k W_{\nu,c}(z) - cz_k W_{\nu+k,c}(z)} = -\sum_{n\ge1}\frac{z^2}{{}_k\tau^2_{\nu,c,n} - z^2},$$

and finally, we obtain

$$\frac{\nu(\nu-2k)_k W_{\nu,c}(z^{1/2}) + k(3k-2\nu)z^{1/2}{}_k W'_{\nu,c}(z^{1/2}) + k^2 z_k W''_{\nu,c}(z^{1/2})}{2k(2k-\nu)_k W_{\nu,c}(z^{1/2}) + 2k^2 z^{1/2}{}_k W'_{\nu,c}(z^{1/2})} = -\sum_{n\ge1}\frac{z}{{}_k\tau^2_{\nu,c,n} - z}. \square$$

Lemma 2.4. (see [24,32]) *If $a > b > r \ge |z|$ and $\lambda \le 1$, then*

$$\left|\frac{z}{b-z} - \lambda\frac{z}{a-z}\right| \le \frac{r}{b-r} - \lambda\frac{r}{a-r}. \tag{5}$$

The following results can be obtained as a natural consequence of this inequality:

$$\mathrm{Re}\left(\frac{z}{b-z} - \lambda\frac{z}{a-z}\right) \le \frac{r}{b-r} - \lambda\frac{r}{a-r} \tag{6}$$

and

$$\mathrm{Re}\left(\frac{z}{b-z}\right) \le \left|\frac{z}{b-z}\right| \le \frac{r}{b-r}. \tag{7}$$

It is important to note here that in [32, Lemma 2.1], it was assumed that $\lambda \in [0,1]$, but following the proof of [32, Lemma 2.1], it is obvious that we do not need the assumption $\lambda \ge 0$. This means that the inequality given by (6) remains valid for $\lambda \le 1$.

3 Main results

We are now in a position to build up our main results.

3.1 Radii of β-uniformly convexity of order α of functions ${}_kf_{\nu,c}$, ${}_kg_{\nu,c}$, and ${}_kh_{\nu,c}$

The aim of this section is to determine the radii of β-uniformly convex of order α of the functions ${}_kf_{\nu,c}$, ${}_kg_{\nu,c}$ and ${}_kh_{\nu,c}$.

Theorem 3.1. *Let $k > 0$, $c > 0$, and $\alpha \in [0, 1)$.*

(*a*) *If $\frac{\nu}{k} > 0$, then the radius of β-uniform convexity of order α of the function ${}_kf_{\nu,c}$ is the smallest positive root of the equation*

$$1 - \alpha + (1 + \beta)\left(r\frac{{}_kW''_{\nu,c}(r)}{{}_kW'_{\nu,c}(r)} + \left(\frac{k}{\nu} - 1\right)r\frac{{}_kW'_{\nu,c}(r)}{{}_kW_{\nu,c}(r)}\right) = 0.$$

(*b*) *If $\nu + k > 0$, then the radius of β-uniform convexity of order α of the function ${}_kg_{\nu,c}$ is the smallest positive root of the equation*

$$1 - \alpha + (1 + \beta)r\frac{{}_kg''_{\nu,c}}{{}_kg'_{\nu,c}} = 0.$$

(*c*) *If $\nu + k > 0$, then the radius of β-uniform convexity of order α of the function ${}_kh_{\nu,c}$ is the smallest positive root of the equation*

$$1 - \alpha + (1 + \beta)r\frac{{}_kh''_{\nu,c}}{{}_kh'_{\nu,c}} = 0.$$

Proof. (**a**) Let ${}_k\omega_{\nu,c,n}$ and ${}_k\omega'_{\nu,c,n}$ be the nth positive roots of ${}_kW_{\nu,c}(z)$ and ${}_kW'_{\nu,c}(z)$, respectively. In [19, Theorem 1.3(a)], the following equality was proved:

$$1 + z\frac{{}_kf''_{\nu,c}(z)}{{}_kf'_{\nu,c}(z)} = 1 - \left(\frac{k}{\nu} - 1\right)\sum_{n\geq 1}\frac{2z^2}{{}_k\omega^2_{\nu,c,n} - z^2} - \sum_{n\geq 1}\frac{2z^2}{{}_k\omega'^2_{\nu,c,n} - z^2}.$$

We will prove the theorem in two steps. First, suppose $\frac{\nu}{k} > 1$. In this case, we will use the property of the zeros ${}_k\omega_{\nu,c,n}$, ${}_k\omega'_{\nu,c,n}$ that interlace according to the inequalities

$${}_k\omega'_{\nu,c,1} < {}_k\omega_{\nu,c,1} < {}_k\omega'_{\nu,c,2} < {}_k\omega_{\nu,c,2} < \cdots$$

Setting $\lambda = 1 - \frac{k}{\nu}$, in light of the inequality (6), for $|z| \leq r < {}_k\omega'_{\nu,c,1} < {}_k\omega_{\nu,c,1}$, we obtain

$$\operatorname{Re}\left(\frac{2z^2}{{}_k\omega'^2_{\nu,c,n} - z^2} - \left(1 - \frac{k}{\nu}\right)\frac{2z^2}{{}_k\omega^2_{\nu,c,n} - z^2}\right) \leq \frac{2r^2}{{}_k\omega'^2_{\nu,c,n} - r^2} - \left(1 - \frac{k}{\nu}\right)\frac{2r^2}{{}_k\omega^2_{\nu,c,n} - r^2}.$$

Furthermore, we have

$$\begin{aligned}\operatorname{Re}\left(1 + z\frac{{}_kf''_{\nu,c}(z)}{{}_kf'_{\nu,c}(z)}\right) &= 1 - \sum_{n\geq 1}\operatorname{Re}\left(\frac{2z^2}{{}_k\omega'^2_{\nu,c,n} - z^2} - \left(1 - \frac{k}{\nu}\right)\frac{2z^2}{{}_k\omega^2_{\nu,c,n} - z^2}\right)\\ &\geq 1 - \sum_{n\geq 1}\left(\frac{2r^2}{{}_k\omega'^2_{\nu,c,n} - r^2} - \left(1 - \frac{k}{\nu}\right)\frac{2r^2}{{}_k\omega^2_{\nu,c,n} - r^2}\right) \qquad (8)\\ &= 1 + r\frac{{}_kf''_{\nu,c}(r)}{{}_kf'_{\nu,c}(r)}.\end{aligned}$$

On the other hand, if in inequality (5) we replace z by z^2 and put $\lambda = 1 - \frac{k}{\nu}$, then it follows that

$$\left|\frac{2z^2}{{}_k\omega'^2_{\nu,c,n} - z^2} - \left(1 - \frac{k}{\nu}\right)\frac{2z^2}{{}_k\omega^2_{\nu,c,n} - z^2}\right| \leq \frac{2r^2}{{}_k\omega'^2_{\nu,c,n} - r^2} - \left(1 - \frac{k}{\nu}\right)\frac{2r^2}{{}_k\omega^2_{\nu,c,n} - r^2},$$

where $|z| \leq r < {}_k\omega'_{\nu,c,1} < {}_k\omega_{\nu,c,1}$. Consequently, for $\beta \geq 0$ and $|z| \leq r < {}_k\omega'_{\nu,c,1} < {}_k\omega_{\nu,c,1}$, we obtain

$$\begin{aligned}\beta\left|z\frac{{}_kf''_{\nu,c}(z)}{{}_kf'_{\nu,c}(z)}\right| &= \beta\left|\sum_{n\geq1}\frac{2z^2}{{}_k\omega'^2_{\nu,c,n}-z^2}-\left(1-\frac{k}{\nu}\right)\frac{2z^2}{{}_k\omega^2_{\nu,c,n}-z^2}\right|\\ &\leq \beta\sum_{n\geq1}\left|\frac{2z^2}{{}_k\omega'^2_{\nu,c,n}-z^2}-\left(1-\frac{k}{\nu}\right)\frac{2z^2}{{}_k\omega^2_{\nu,c,n}-z^2}\right|\\ &\leq \beta\sum_{n\geq1}\left(\frac{2r^2}{{}_k\omega'^2_{\nu,c,n}-r^2}-\left(1-\frac{k}{\nu}\right)\frac{2r^2}{{}_k\omega^2_{\nu,c,n}-r^2}\right)\\ &= -\beta\frac{r_kf''_{\nu,c}(r)}{{}_kf'_{\nu,c}(r)}.\end{aligned} \tag{9}$$

In the second step, we will show that inequalities (8) and (9) hold true in the case $\frac{\nu}{k} \in (0,1)$. Inequality (7) implies

$$\mathrm{Re}\frac{2z^2}{{}_k\omega'^2_{\nu,c,n}-z^2} \leq \left|\frac{2z^2}{{}_k\omega'^2_{\nu,c,n}-z^2}\right| \leq \frac{2r^2}{{}_k\omega'^2_{\nu,c,n}-r^2}, \quad |z| \leq r < {}_k\omega'_{\nu,c,1} < {}_k\omega_{\nu,c,1}$$

and

$$\mathrm{Re}\frac{2z^2}{{}_k\omega^2_{\nu,c,n}-z^2} \leq \left|\frac{2z^2}{{}_k\omega^2_{\nu,c,n}-z^2}\right| \leq \frac{2r^2}{{}_k\omega^2_{\nu,c,n}-r^2}, \quad |z| \leq r < {}_k\omega'_{\nu,c,1} < {}_k\omega_{\nu,c,1}.$$

Since $\frac{k}{\nu}-1>0$, by means of the previous inequalities, we deduce that

$$\begin{aligned}\mathrm{Re}\left(1+z\frac{{}_kf''_{\nu,c}(z)}{{}_kf'_{\nu,c}(z)}\right) &= 1-\sum_{n\geq1}\mathrm{Re}\left(\frac{2z^2}{{}_k\omega^2_{\nu,c,n}-z^2}\right)-\left(\frac{k}{\nu}-1\right)\sum_{n\geq1}\mathrm{Re}\left(\frac{2z^2}{{}_k\omega'^2_{\nu,c,n}-z^2}\right)\\ &\geq 1-\sum_{n\geq1}\frac{2r^2}{{}_k\omega'^2_{\nu,c,n}-r^2}-\left(\frac{k}{\nu}-1\right)\sum_{n\geq1}\frac{2r^2}{{}_k\omega^2_{\nu,c,n}-r^2}\\ &= 1+r\frac{{}_kf''_{\nu,c}(r)}{{}_kf'_{\nu,c}(r)}.\end{aligned} \tag{10}$$

Moreover, for $\beta \geq 0$ and $|z| \leq r < {}_k\omega'_{\nu,c,1} < {}_k\omega_{\nu,c,1}$, we have

$$\begin{aligned}\beta\left|z\frac{{}_kf''_{\nu,c}(z)}{{}_kf'_{\nu,c}(z)}\right| &= \left|\sum_{n\geq1}\left(\frac{2z^2}{{}_k\omega^2_{\nu,c,n}-z^2}\right)+\left(\frac{k}{\nu}-1\right)\sum_{n\geq1}\left(\frac{2z^2}{{}_k\omega'^2_{\nu,c,n}-z^2}\right)\right|\\ &\leq \beta\sum_{n\geq1}\left|\frac{2z^2}{{}_k\omega^2_{\nu,c,n}-z^2}\right|+\left(\frac{k}{\nu}-1\right)\sum_{n\geq1}\left|\frac{2z^2}{{}_k\omega'^2_{\nu,c,n}-z^2}\right|\\ &\leq \beta\sum_{n\geq1}\left(\frac{2r^2}{{}_k\omega^2_{\nu,c,n}-r^2}+\left(\frac{k}{\nu}-1\right)\frac{2r^2}{{}_k\omega'^2_{\nu,c,n}-r^2}\right)\\ &= -\beta r\frac{{}_kf''_{\nu,c}(r)}{{}_kf'_{\nu,c}(r)}.\end{aligned} \tag{11}$$

From equations (8)–(11), we obtain

$$\mathrm{Re}\left(1+z\frac{{}_kf''_{\nu,c}(z)}{{}_kf'_{\nu,c}(z)}\right)-\beta\left|z\frac{{}_kf''_{\nu,c}(z)}{{}_kf'_{\nu,c}(z)}\right|-\alpha \geq 1-\alpha+(1+\beta)\frac{r_kf''_{\nu,c}(r)}{{}_kf'_{\nu,c}(r)},$$

where $|z| \leq r < {}_k\omega'_{\nu,c,1}$. By means of the minimum principle for harmonic functions, equality holds if and only if $z = r$. The above obtained inequality implies that for $r \in (0, {}_k\omega'_{\nu,c,1})$,

$$\inf_{|z|<r}\left\{\mathrm{Re}\left(1 + z\frac{{}_kf''_{\nu,c}(z)}{{}_kf'_{\nu,c}(z)}\right) - \beta\left|z\frac{{}_kf''_{\nu,c}(z)}{{}_kf'_{\nu,c}(z)}\right| - \alpha\right\} = 1 - \alpha + (1+\beta)\frac{r\,{}_kf''_{\nu,c}(r)}{{}_kf'_{\nu,c}(r)}.$$

The function ${}_ku_{\nu,c} : (0, {}_k\omega'_{\nu,c,1}) \to \mathbb{R}$, defined by

$${}_ku_{\nu,c}(r) = 1 - \alpha + (1+\beta)\frac{r\,{}_kf''_{\nu,c}(r)}{{}_kf'_{\nu,c}(r)} = 1 - \alpha - (1+\beta)\sum_{n\geq 1}\left(\frac{2r^2}{{}_k\omega'^2_{\nu,c,n} - r^2} - \left(1 - \frac{k}{\nu}\right)\frac{2r^2}{{}_k\omega^2_{\nu,c,n} - r^2}\right),$$

is strictly decreasing when $k > 0$, $\frac{\nu}{k} > 0$, and $\beta \geq 0$, $\alpha \in [0, 1)$. We also observe that

$$\lim_{r\nearrow 0} {}_ku_{\nu,c}(r) = 1 - \alpha > 0 \quad \text{and} \quad \lim_{r\searrow {}_k\omega'_{\nu,c,1}} {}_ku_{\nu,c}(r) = -\infty.$$

Thus, it follows that the equation

$$1 + (1+\beta)r\frac{{}_kf''_{\nu,c}(r)}{{}_kf'_{\nu,c}(r)} = \alpha, \quad \beta \geq 0, \ \alpha \in [0, 1)$$

has a unique root situated in $r_1 \in (0, {}_k\omega'_{\nu,c,1})$.

(**b**) Suppose that ${}_k\theta_{\nu,c,n}$s are the real zeros of the function ${}_kg'_{\nu,c}$. In view of Lemma 2.2 (see also [19, Theorem 1.3(b)]), we have the following equality:

$$1 + z\frac{{}_kg''_{\nu,c}(z)}{{}_kg'_{\nu,c}(z)} = 1 - 2\sum_{n\geq 1}\frac{z^2}{{}_k\theta^2_{\nu,c,n} - z^2},$$

and it was shown in [19] that

$$\mathrm{Re}\left(1 + z\frac{{}_kg''_{\nu,c}(z)}{{}_kg'_{\nu,c}(z)}\right) \geq 1 - 2\sum_{n\geq 1}\frac{r^2}{{}_k\theta^2_{\nu,c,n} - r^2}, \tag{12}$$

where $|z| \leq r < {}_k\theta_{\nu,c,1}$. From equation (3.1) and $\beta \geq 0$, we obtain

$$\begin{aligned}\beta\left|z\frac{{}_kg''_{\nu,c}(z)}{{}_kg'_{\nu,c}(z)}\right| &= \beta\left|2\sum_{n\geq 1}\frac{z^2}{{}_k\theta^2_{\nu,c,n} - z^2}\right| \\ &\leq 2\beta\sum_{n\geq 1}\left|\frac{z^2}{{}_k\theta^2_{\nu,c,n} - z^2}\right| \\ &\leq 2\beta\sum_{n\geq 1}\frac{r^2}{{}_k\theta^2_{\nu,c,n} - r^2} \\ &= -\beta r\frac{{}_kg''_{\nu,c}(r)}{{}_kg'_{\nu,c}(r)}, \quad |z| \leq r < {}_k\theta_{\nu,c,1}.\end{aligned} \tag{13}$$

With the aid of equations (12) and (13), we obtain

$$\mathrm{Re}\left(1 + z\frac{{}_kg''_{\nu,c}(z)}{{}_kg'_{\nu,c}(z)}\right) - \beta\left|z\frac{{}_kg''_{\nu,c}(z)}{{}_kg'_{\nu,c}(z)}\right| - \alpha \geq 1 - \alpha + (1+\beta)r\frac{{}_kg''_{\nu,c}(r)}{{}_kg'_{\nu,c}(r)},$$

where $|z| \leq r < {}_k\theta_{\nu,c,1}$, $\beta \geq 0$, and $\alpha \in [0, 1)$. In light of the minimum principle for harmonic functions, equality holds if and only if $z = r$. Thus, for $r \in (0, {}_k\theta_{\nu,c,1})$, $\beta \geq 0$, and $\alpha \in [0, 1)$, we have

$$\inf_{|z|<r}\left\{\mathrm{Re}\left(1 + z\frac{{}_kg''_{\nu,c}(z)}{{}_kg'_{\nu,c}(z)}\right) - \beta\left|z\frac{{}_kg''_{\nu,c}(z)}{{}_kg'_{\nu,c}(z)}\right| - \alpha\right\} = 1 - \alpha + (1+\beta)r\frac{{}_kg''_{\nu,c}(r)}{{}_kg'_{\nu,c}(r)}.$$

The function ${}_k\phi_{\nu,c} : (0, {}_k\theta_{\nu,c,1}) \to \mathbb{R}$, defined by

$$ {}_k\phi_{\nu,c}(r) = 1 - \alpha + (1+\beta) r \frac{{}_k g''_{\nu,c}(r)}{{}_k g'_{\nu,c}(r)}, $$

is strictly decreasing and $\lim_{r \nearrow 0} {}_k\phi_{\nu,c}(r) = 1 - \alpha > 0$ and $\lim_{r \searrow {}_k\theta_{\nu,c,1}} {}_k\phi_{\nu,c}(r) = -\infty$. Consequently, the equation

$$ 1 - \alpha + (1+\beta) r \frac{{}_k g''_{\nu,c}(r)}{{}_k g'_{\nu,c}(r)} = 0 $$

has a unique root r_2 in $(0, {}_k\theta_{\nu,c,1})$.

(**c**) Let ${}_k\tau_{\nu,c,n}$ be the nth positive zero of the function ${}_k h'_{\nu,c}(z)$. By means of Lemma 2.3, we have

$$ z\frac{{}_k h''_{\nu,c}(z)}{{}_k h'_{\nu,c}(z)} = -\sum_{n\geq 1} \frac{z}{{}_k\tau^2_{\nu,c,n} - z}. \tag{14} $$

In light of equation (7) given in Lemma 2.4, we obtain

$$ \operatorname{Re}\left(1 + z\frac{{}_k h''_{\nu,c}(z)}{{}_k h'_{\nu,c}(z)}\right) = 1 - \operatorname{Re}\left(\sum_{n\geq 1} \frac{z}{{}_k\tau^2_{\nu,c,n} - z}\right) \geq 1 + r\frac{{}_k h''_{\nu,c}(r)}{{}_k h'_{\nu,c}(r)}, \ |z| \leq r < {}_k\tau_{\nu,c,1}. \tag{15} $$

With the help of equation (14), we obtain

$$ \begin{aligned} \beta\left|z\frac{{}_k h''_{\nu,c}(z)}{{}_k h'_{\nu,c}(z)}\right| &= \beta\left|\sum_{n\geq 1} \frac{z}{{}_k\tau^2_{\nu,c,n} - z}\right| \\ &\leq \beta \sum_{n\geq 1} \left|\frac{z}{{}_k\tau^2_{\nu,c,n} - z}\right| \\ &\leq \beta \sum_{n\geq 1} \frac{r}{{}_k\tau^2_{\nu,c,n} - r} \\ &= -\beta r \frac{{}_k h''_{\nu,c}(r)}{{}_k h'_{\nu,c}(r)}, \quad |z| \leq r < {}_k\tau_{\nu,c,1}. \end{aligned} \tag{16} $$

From equations (15) and (16), we arrive at

$$ \operatorname{Re}\left(1 + z\frac{{}_k h''_{\nu,c}(z)}{{}_k h'_{\nu,c}(z)}\right) - \beta\left|z\frac{{}_k h''_{\nu,c}(z)}{{}_k h'_{\nu,c}(z)}\right| - \alpha \geq 1 - \alpha + (1+\beta) r \frac{{}_k h''_{\nu,c}(r)}{{}_k h'_{\nu,c}(r)}, $$

where $|z| \leq r < {}_k\tau_{\nu,c,1}$, $\beta \geq 0$, and $\alpha \in [0,1)$. Because of the minimum principle for harmonic functions, equality holds if and only if $|z| = r$. Thus, we have

$$ \inf_{|z|<r}\left\{\operatorname{Re}\left(1 + z\frac{{}_k h''_{\nu,c}(z)}{{}_k h'_{\nu,c}(z)}\right) - \beta\left|z\frac{{}_k h''_{\nu,c}(z)}{{}_k h'_{\nu,c}(z)}\right| - \alpha\right\} = 1 - \alpha + (1+\beta) r \frac{{}_k h''_{\nu,c}(r)}{{}_k h'_{\nu,c}(r)}. $$

The function ${}_k\psi_{\nu,c} : (0, {}_k\tau_{\nu,c,1}) \to \mathbb{R}$, defined by

$$ {}_k\psi_{\nu,c}(r) = 1 - \alpha + (1+\beta) r \frac{{}_k h''_{\nu,c}(r)}{{}_k h'_{\nu,c}(r)}, $$

is strictly decreasing and $\lim_{r \nearrow 0} {}_k\psi_{\nu,c}(r) = 1 - \alpha > 0$ and $\lim_{r \searrow {}_k\tau_{\nu,c,1}} {}_k\psi_{\nu,c}(r) = -\infty$. Consequently, the equation

$$1-\alpha+(1+\beta)r\frac{{}_k h''_{\nu,c}(r)}{{}_k h'_{\nu,c}(r)}=0$$

has a unique root r_3 in $(0, {}_k\tau_{\nu,c,1})$. □

3.2 Radii of α-convexity of order β of functions ${}_kf_{\nu,c}$, ${}_kg_{\nu,c}$, and ${}_kh_{\nu,c}$

This section is devoted to determining the radii of α-convexity of order β of the functions ${}_kf_{\nu,c}$, ${}_kg_{\nu,c}$, and ${}_kh_{\nu,c}$. The method used in the process of determining the radii of α-convexity of order β is based on the ideas from [22] and [11]. In order to prove our main results, we will take advantage of the following notation:

$$\mathbb{J}(\alpha, u(z))=(1-\alpha)\frac{zu'(z)}{u(z)}+\alpha\left(1+\frac{zu''(z)}{u'(z)}\right).$$

Theorem 3.2. *Let $\nu>0$, $k>0$, $c>0$, $\alpha\geq 0$, and $\beta\in[0,1)$. Then, the radius of α-convexity of order β of the function ${}_kf_{\nu,c}$ is the smallest positive root of the equation*

$$\alpha\left(1+r\frac{{}_kW''_{\nu,c}(r)}{{}_kW'_{\nu,c}(r)}\right)+\left(\frac{k}{\nu}-1\right)r\frac{{}_kW'_{\nu,c}(r)}{{}_kW_{\nu,c}(r)}=\beta.$$

Let ${}_k\omega_{\nu,c,1}$ and ${}_k\omega'_{\nu,c,1}$ be the first positive zeros of ${}_kW_{\nu,c}$ and ${}_kW'_{\nu,c}$, respectively. Then, the radius of α-convexity satisfies the inequalities $r_{\alpha,\beta}({}_kf_{\nu,c})<{}_k\omega'_{\nu,c,1}<{}_k\omega_{\nu,c,1}$. Moreover, the function $\alpha\mapsto r_{\alpha,\beta}({}_kf_{\nu,c})$ is strictly decreasing on $[0,\infty)$, and consequently, we obtain $r^c_\beta<r_{\alpha,\beta}({}_kf_{\nu,c})<r^\star_\beta({}_kf_{\nu,c})$ for all $\alpha\in(0,1)$ and $\beta\in[0,1)$.

Proof. Our interest here is the case when $\alpha>0$ because of the fact that the theorem is proved in the case when $\alpha=0$ in [19]. By virtue of the definition of the function ${}_kf_{\nu,c}(z)$, it is easy to verify that

$$z\frac{{}_kf'_{\nu,c}(z)}{{}_kf_{\nu,c}(z)}=\frac{k}{\nu}z\frac{{}_kW'_{\nu,c}(z)}{{}_kW_{\nu,c}(z)}\quad\text{and}\quad 1+z\frac{{}_kf''_{\nu,c}(z)}{{}_kf'_{\nu,c}(z)}=1+z\frac{{}_kW''_{\nu,c}(z)}{{}_kW'_{\nu,c}(z)}+\left(\frac{k}{\nu}-1\right)z\frac{{}_kW'_{\nu,c}(z)}{{}_kW_{\nu,c}(z)}.$$

From [19], we have the following infinite product representations:

$${}_kW_{\nu,c}(z)=\frac{\left(\frac{z}{2}\right)^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}\prod_{n\geq1}\left(1-\frac{z^2}{{}_k\omega^2_{\nu,c,n}}\right)\quad\text{and}\quad {}_kW'_{\nu,c}(z)=\frac{\nu\left(\frac{z}{2}\right)^{\frac{\nu}{k}-1}}{2k\Gamma_k(\nu+k)}\prod_{n\geq1}\left(1-\frac{z^2}{{}_k\omega'^2_{\nu,c,n}}\right),$$

where ${}_k\omega_{\nu,c,n}$ and ${}_k\omega'_{\nu,c,n}$ denote the positive zeros of ${}_kW_{\nu,c}$ and ${}_kW'_{\nu,c}$, respectively. By using logarithmic differentiation, we arrive at

$$\begin{aligned}\mathbb{J}(\alpha,{}_kf_{\nu,c}(z))&=(1-\alpha)z\frac{{}_kf'_{\nu,c}(z)}{{}_kf_{\nu,c}(z)}+\alpha\left(1+z\frac{{}_kf''_{\nu,c}(z)}{{}_kf'_{\nu,c}(z)}\right)\\&=1+\left(\alpha-\frac{k}{\nu}\right)\sum_{n\geq1}\frac{2z^2}{{}_k\omega^2_{\nu,c,n}-z^2}-\alpha\sum_{n\geq1}\frac{2z^2}{{}_k\omega'^2_{\nu,c,n}-z^2}.\end{aligned}$$

With the help of Lemma 2.4, for all $z\in\mathbb{D}_{{}_k\omega'_{\nu,c,1}}$, we obtain the inequality

$$\frac{1}{\alpha}\mathrm{Re}(\mathbb{J}(\alpha,{}_kf_{\nu,c}(z)))\geq\frac{1}{\alpha}+\left(1-\frac{k}{\alpha\nu}\right)\sum_{n\geq1}\frac{2r^2}{{}_k\omega^2_{\nu,c,n}-r^2}-\sum_{n\geq1}\frac{2r^2}{{}_k\omega'^2_{\nu,c,n}-r^2}=\frac{1}{\alpha}\mathbb{J}(\alpha,{}_kf_{\nu,c}(r)),$$

where $|z|=r$. It is important to note that here we used the zeros ${}_k\omega_{\nu,c,n}$ and ${}_k\omega'_{\nu,c,n}$ to satisfy the interlacing property [19, Lemma 1.1]. That is, we have the relation ${}_k\omega'_{\nu,c,1}<{}_k\omega_{\nu,c,1}<{}_k\omega'_{\nu,c,2}<\dots$ Thus, for

$r \in (0, {}_k\omega'_{\nu,c,1})$, we have $\inf_{z\in\mathbb{D}_r} \mathbb{J}(\alpha, {}_kf_{\nu,c}(z)) = \mathbb{J}(\alpha, {}_kf_{\nu,c}(r))$. Moreover, the function $r \mapsto \mathbb{J}(\alpha, {}_kf_{\nu,c}(r))$ is strictly decreasing on $(0, {}_k\omega'_{\nu,c,1})$ since

$$\frac{\partial}{\partial r}\mathbb{J}(\alpha, {}_kf_{\nu,c}(r)) = -\left(\frac{k}{\nu} - \alpha\right)\sum_{n\geq1}\frac{4r\,{}_k\omega^2_{\nu,c,n}}{({}_k\omega^2_{\nu,c,n} - r^2)^2} - \alpha\sum_{n\geq1}\frac{4r\,{}_k\omega'^2_{\nu,c,n}}{({}_k\omega'^2_{\nu,c,n} - r^2)^2}$$
$$< \alpha\sum_{n\geq1}\frac{4r\,{}_k\omega^2_{\nu,c,n}}{({}_k\omega^2_{\nu,c,n} - r^2)^2} - \alpha\sum_{n\geq1}\frac{4r\,{}_k\omega'^2_{\nu,c,n}}{({}_k\omega'^2_{\nu,c,n} - r^2)^2} < 0$$

for $\nu > 0$, $k > 0$, $c > 0$, and $r \in (0, {}_k\omega'_{\nu,c,1})$. Here, we used tacitly that the zeros ${}_k\omega_{\nu,c,n}$ and ${}_k\omega'_{\nu,c,n}$ interlace, and for $\nu > 0$, $k > 0$, $c > 0$ and $r < \sqrt{{}_k\omega_{\nu,c,1}\,{}_k\omega'_{\nu,c,1}}$ we have

$${}_k\omega^2_{\nu,c,n}({}_k\omega'^2_{\nu,c,n} - r^2)^2 < {}_k\omega'^2_{\nu,c,n}({}_k\omega^2_{\nu,c,n} - r^2)^2.$$

We also observe that

$$\lim_{r\nearrow0}\mathbb{J}(\alpha, {}_kf_{\nu,c}(r)) = 1 > \beta \quad\text{and}\quad \lim_{r\searrow{}_k\omega'_{\nu,c,1}}\mathbb{J}(\alpha, {}_kf_{\nu,c}(r)) = -\infty.$$

This means that for $z \in \mathbb{D}_{r_1}$, we have $\operatorname{Re}\mathbb{J}(\alpha, {}_kf_{\nu,c}(z)) > \beta$ if and only if r_1 is the unique root of $\mathbb{J}(\alpha, {}_kf_{\nu,c}(r)) = \beta$, in $(0, {}_k\omega'_{\nu,c,1})$. Finally, by making use of the relation ${}_k\omega'_{\nu,c,1} < {}_k\omega_{\nu,c,1} < {}_k\omega'_{\nu,c,2} < \dots$, we obtain

$$\frac{\partial}{\partial\alpha}\mathbb{J}(\alpha, {}_kf_{\nu,c}(r)) = \sum_{n\geq1}\frac{2r^2}{{}_k\omega^2_{\nu,c,n} - r^2} - \sum_{n\geq1}\frac{2r^2}{{}_k\omega'^2_{\nu,c,n} - r^2} < 0, \quad r \in (0, {}_k\omega'_{\nu,c,1}).$$

This means that the function $\alpha \mapsto \mathbb{J}(\alpha, {}_kf_{\nu,c}(r))$ is strictly decreasing on $[0, \infty)$ for all $\alpha \geq 0$, $\nu > 0$, $k > 0$, $c > 0$, and $r \in (0, {}_k\omega'_{\nu,c,1})$. Consequently, as a function of α, the unique root of the equation $\mathbb{J}(\alpha, {}_kf_{\nu,c}(r)) = \beta$ is strictly decreasing, where $\beta \in (0, 1)$, $\nu > 0$, $k > 0$, $c > 0$, and $r \in (0, {}_k\omega'_{\nu,c,1})$ are fixed. As a result, for $\alpha \in (0, 1)$, the radius of α-convexity of the function ${}_kf_{\nu,c}$ remains in between the radius of convexity and the radius of starlikeness of the function ${}_kf_{\nu,c}$. This is the desired result. □

Theorem 3.3. *Let $k > 0$, $c > 0$, $\nu + k > 0$, $\alpha \geq 0$, and $\beta \in [0, 1)$. Then, the radius of α-convexity of order β of the function ${}_kg_{\nu,c}$ is the smallest positive root of the equation*

$$1 + (\alpha - 1)r\frac{{}_kW_{\nu+k,c}(r)}{{}_kW_{\nu,c}(r)} + \alpha r\frac{r\,{}_kW_{\nu+2k,c}(r) - 3\,{}_kW_{\nu+k,c}(r)}{{}_kW_{\nu,c}(r) - r\,{}_kW_{\nu+k,c}(r)} = \beta.$$

Let ${}_k\theta_{\nu,c,1}$ denote the first positive zeros of $z \mapsto (1 - \frac{\nu}{k})\,{}_kW_{\nu,c} + z\,{}_kW'_{\nu,c}$. Then, the radius of α-convexity satisfies the inequalities $r_{\alpha,\beta}({}_kg_{\nu,c}) < {}_k\theta_{\nu,c,1} < {}_k\omega_{\nu,c,1}$. Moreover, the function $\alpha \mapsto r_{\alpha,\beta}({}_kg_{\nu,c})$ is strictly decreasing on $[0, \infty)$, and consequently, we obtain $r^c_\beta < r_{\alpha,\beta}({}_kg_{\nu,c}) < r^\star_\beta({}_kf_{\nu,c})$ for all $\alpha \in (0, 1)$ and $\beta \in [0, 1)$.

Proof. Without loss of generality, we assume that $\alpha > 0$. The case $\alpha = 0$ was demonstrated in [19]. From [19] and Lemma 2.2, we have the following equalities:

$$\frac{z\,{}_kg'_{\nu,c}(z)}{{}_kg_{\nu,c}(z)} = 1 - \sum_{n\geq1}\frac{2z^2}{{}_k\omega^2_{\nu,c,n} - z^2} \quad\text{and}\quad 1 + z\frac{{}_kg''_{\nu,c}(z)}{{}_kg'_{\nu,c}(z)} = 1 - \sum_{n\geq1}\frac{2z^2}{{}_k\theta^2_{\nu,c,n} - z^2},$$

where ${}_k\omega_{\nu,c,n}$ and ${}_k\theta_{\nu,c,n}$ stand for the nth positive zeros of the functions ${}_kg_{\nu,c}(z)$ and ${}_kg'_{\nu,c}(z)$, respectively. Thus, we obtain

$$\mathbb{J}(\alpha, {}_kg_{\nu,c}(z)) = (1 - \alpha)z\frac{{}_kg'_{\nu,c}(z)}{{}_kg_{\nu,c}(z)} + \alpha\left(1 + z\frac{{}_kg''_{\nu,c}(z)}{{}_kg'_{\nu,c}(z)}\right)$$
$$= 1 - (1 - \alpha)\sum_{n\geq1}\frac{2z^2}{{}_k\omega^2_{\nu,c,n} - z^2} - \alpha\sum_{n\geq1}\frac{2z^2}{{}_k\theta^2_{\nu,c,n} - z^2}.$$

By making use of the inequality (6) given in Lemma 2.4, we obtain

$$\frac{1}{\alpha}\mathrm{Re}(\mathbb{J}(\alpha, {}_k g_{\nu,c}(z))) \geq \frac{1}{\alpha} + \left(1 - \frac{1}{\alpha}\right)\sum_{n\geq 1}\frac{2r^2}{{}_k\omega^2_{\nu,c,n} - r^2} - \sum_{n\geq 1}\frac{2r^2}{{}_k\theta^2_{\nu,c,n} - r^2} = \frac{1}{\alpha}(\mathbb{J}(\alpha, {}_k g_{\nu,c}(r))),$$

where $|z| = r$. Here, we used tacitly that for all $n \in \{1, 2, \dots\}$, we have ${}_k\theta_{\nu,c,n} \in ({}_k\omega_{\nu,c,n-1}, {}_k\omega_{\nu,c,n})$, where ${}_k\omega_{\nu,c,n}$ is the nth positive zero of ${}_kW_{\nu,c}$. This follows immediately from Lemma 2.1. We also note that the zeros ${}_k\theta_{\nu,c,n}$ are all real when $k > 0$, $c > 0$, and $\nu + k > 0$ (see [19]), and thus the application of the inequality (6) given in Lemma 2.4 is allowed. Thus, for $r \in (0, {}_k\theta_{\nu,c,1})$, we obtain $\inf_{z\in\mathbb{D}_r}\mathrm{Re}(\mathbb{J}(\alpha, {}_k g_{\nu,c}(z))) = \mathbb{J}(\alpha, {}_k g_{\nu,c}(r))$, since according to the minimum principle of harmonic functions, the infimum is taken on the boundary. On the other hand, the function $r \mapsto \mathbb{J}(\alpha, {}_k g_{\nu,c}(r))$ is strictly decreasing on $(0, {}_k\theta_{\nu,c,1})$ since

$$\begin{aligned}\frac{\partial}{\partial r}\mathbb{J}(\alpha, {}_k g_{\nu,c}(r)) &= (\alpha - 1)\sum_{n\geq 1}\frac{4r\,{}_k\omega^2_{\nu,c,n}}{({}_k\omega^2_{\nu,c,n} - r^2)^2} - \alpha\sum_{n\geq 1}\frac{4r\,{}_k\theta^2_{\nu,c,n}}{({}_k\theta^2_{\nu,c,n} - r^2)^2}\\ &\leq \alpha\sum_{n\geq 1}\frac{4r\,{}_k\omega^2_{\nu,c,n}}{({}_k\omega^2_{\nu,c,n} - r^2)^2} - \alpha\sum_{n\geq 1}\frac{4r\,{}_k\theta^2_{\nu,c,n}}{({}_k\theta^2_{\nu,c,n} - r^2)^2} < 0\end{aligned}$$

for $k > 0, c > 0, \nu + k > 0$, and $r \in (0, {}_k\theta_{\nu,c,1})$. Here we used again that the zeros ${}_k\omega_{\nu,c,n}$ and ${}_k\theta_{\nu,c,n}$ interlace, and for all $n \in \mathbb{N}$, $k > 0$, $c > 0$, $\nu + k > 0$, and $r < \sqrt{{}_k\omega_{\nu,c,n}{}_k\theta_{\nu,c,n}}$, we have that

$${}_k\omega^2_{\nu,c,n}({}_k\theta^2_{\nu,c,n} - r^2)^2 < {}_k\theta^2_{\nu,c,n}({}_k\omega^2_{\nu,c,n} - r^2)^2.$$

We also have that $\lim_{r\nearrow 0}\mathbb{J}(\alpha, {}_k g_{\nu,c}(r)) = 1 > \beta$ and $\lim_{r\searrow {}_k\theta_{\nu,c,1}}\mathbb{J}(\alpha, {}_k g_{\nu,c}(r)) = -\infty$, which means that for $z \in \mathbb{D}_{r_2}$, we have $\mathrm{Re}\,\mathbb{J}(\alpha, {}_k g_{\nu,c}(z)) > \beta$ if and only if r_2 is the unique root of $\mathbb{J}(\alpha, {}_k g_{\nu,c}(r)) = \beta$, situated in $(0, {}_k\theta_{\nu,c,1})$. Finally, by making use of the interlacing inequalities ${}_k\omega_{\nu,c,n-1} < {}_k\theta_{\nu,c,n} < {}_k\omega_{\nu,c,n}$ again, we obtain the inequality

$$\frac{\partial}{\partial\alpha}\mathbb{J}(\alpha, {}_k g_{\nu,c}(r)) = \sum_{n\geq 1}\frac{2r^2}{{}_k\omega^2_{\nu,c,n} - r^2} - \sum_{n\geq 1}\frac{2r^2}{{}_k\theta^2_{\nu,c,n} - r^2} < 0,$$

where $k > 0$, $c > 0$, $\nu + k > 0$, $\alpha > 0$, and $r \in (0, {}_k\theta_{\nu,c,1})$. This implies that the function $\alpha \mapsto \mathbb{J}(\alpha, {}_k g_{\nu,c}(r))$ is strictly decreasing on $[0, \infty)$ for all $k > 0$, $c > 0$, $\nu + k > 0$, $\alpha > 0$, and $r \in (0, {}_k\theta_{\nu,c,1})$ fixed. Consequently, as a function of α, the unique root of the equation $\mathbb{J}(\alpha, {}_k g_{\nu,c}(r)) = \beta$ is strictly decreasing, where $\beta \in [0, 1)$, $k > 0$, $c > 0$, $\nu + k > 0$, $\alpha > 0$, and $r \in (0, {}_k\theta_{\nu,c,1})$ are fixed. Consequently, in the case when $\alpha \in (0, 1)$, the radius of α-convexity of the function ${}_k g_{\nu,c}$ remains in between the radius of convexity and the radius of starlikeness of the function ${}_k g_{\nu,c}$. □

Theorem 3.4. *Let $k > 0$, $c > 0$, $\nu + k > 0$, $\alpha \geq 0$, and $\beta \in [0, 1)$. Then, the radius of α-convexity of order β of the function ${}_k h_{\nu,c}$ is the smallest positive root of the equation*

$$(1 - \alpha)\left(1 - \frac{r^{\frac{1}{2}}}{2}\frac{{}_kW_{\nu+k,c}\left(r^{\frac{1}{2}}\right)}{{}_kW_{\nu,c}\left(r^{\frac{1}{2}}\right)}\right) + \alpha\left(1 + \frac{r^{\frac{1}{2}}}{2}\frac{kr^{\frac{1}{2}}{}_kW_{\nu+2k,c}\left(r^{\frac{1}{2}}\right) - 4k\,{}_kW_{\nu+k,c}\left(r^{\frac{1}{2}}\right)}{2\,{}_kW_{\nu,c}\left(r^{\frac{1}{2}}\right) - kr^{\frac{1}{2}}{}_kW_{\nu+k,c}\left(r^{\frac{1}{2}}\right)}\right) = \beta.$$

Let ${}_k\tau_{\nu,c,1}$ be the first positive zeros of $z \mapsto (2k - \nu){}_kW_{\nu,c} + kz\,{}_kW'_{\nu,c}$. Then, the radius of α-convexity satisfies the inequalities $r_{\alpha,\beta}({}_k h_{\nu,c}) < {}_k\tau^2_{\nu,c,1} < {}_k\omega^2_{\nu,c,1}$. Moreover, the function $\alpha \mapsto r_{\alpha,\beta}({}_k h_{\nu,c})$ is strictly decreasing on $[0, \infty)$, and consequently, we obtain $r^c_\beta < r_{\alpha,\beta}({}_k h_{\nu,c}) < r^\star_\beta({}_k f_{\nu,c})$ for all $\alpha \in (0, 1)$ and $\beta \in [0, 1)$.

Proof. Similarly, as in the proof of Theorems 3.2 and 3.3, we suppose that $\alpha > 0$. The case $\alpha = 0$ was showed already in [19]. From the same article and Lemma 2.3, we know that the following equalities are valid:

$$z\frac{{}_k h'_{\nu,c}(z)}{{}_k h_{\nu,c}(z)} = 1 - \sum_{n\geq 1}\frac{z}{{}_k\omega^2_{\nu,c,n} - z} \quad\text{and}\quad z\frac{{}_k h''_{\nu,c}(z)}{{}_k h'_{\nu,c}(z)} = -\sum_{n\geq 1}\frac{z}{{}_k\tau^2_{\nu,c,n} - z},$$

where ${}_k\tau_{\nu,c,n}$ denotes the nth positive zero of $z \mapsto (2k - \nu){}_kW_{\nu,c}(z) + kz\,{}_kW'_{\nu,c}(z)$; it follows that

$$\mathbb{J}(\alpha, {}_kh_{\nu,c}(z)) = (1-\alpha)z\frac{{}_kh'_{\nu,c}(z)}{{}_kh_{\nu,c}(z)} + \alpha\left(1 + z\frac{{}_kh''_{\nu,c}(z)}{{}_kh'_{\nu,c}(z)}\right)$$
$$= 1 - (1-\alpha)\sum_{n\geq 1}\frac{z}{{}_k\omega^2_{\nu,c,n} - z} - \alpha\sum_{n\geq 1}\frac{z}{{}_k\tau^2_{\nu,c,n} - z}.$$

By making use of the inequality (6) in Lemma 2.4, we obtain that

$$\frac{1}{\alpha}\operatorname{Re}\mathbb{J}(\alpha, {}_kh_{\nu,c}(z)) \geq \frac{1}{\alpha} + \left(1 - \frac{1}{\alpha}\right)\sum_{n\geq 1}\frac{r}{{}_k\omega^2_{\nu,c,n} - r} - \sum_{n\geq 1}\frac{r}{{}_k\tau^2_{\nu,c,n} - r} = \frac{1}{\alpha}\mathbb{J}(\alpha, {}_kh_{\nu,c}(r)),$$

where $|z| = r$. It is worth mentioning here that we used tacitly that for all $n \in \{1, 2, \ldots\}$, we have ${}_k\tau_{\nu,c,n} \in ({}_k\omega_{\nu,c,n-1}, {}_k\omega_{\nu,c,n})$, which follows from Lemma 2.1. Note also that the zeros ${}_k\tau_{\nu,c,n}$ are all real when $k > 0$, $c > 0$, and $\nu + k > 0$, and thus the application of the inequality (6) situated in Lemma 2.4 is allowed. Thus, for $r \in (0, {}_k\tau^2_{\nu,c,1})$, we have $\inf_{z\in\mathbb{D}_r}\operatorname{Re}\mathbb{J}(\alpha, {}_kh_{\nu,c}(z)) = \mathbb{J}(\alpha, {}_kh_{\nu,c}(r))$. Moreover, since

$$\frac{\partial}{\partial r}\mathbb{J}(\alpha, {}_kh_{\nu,c}(r)) = (\alpha - 1)\sum_{n\geq 1}\frac{r\,{}_k\omega^2_{\nu,c,n}}{({}_k\omega^2_{\nu,c,n} - r)^2} - \alpha\sum_{n\geq 1}\frac{r\,{}_k\tau^2_{\nu,c,n}}{({}_k\tau^2_{\nu,c,n} - r)^2}$$
$$< \alpha\sum_{n\geq 1}\frac{r\,{}_k\omega^2_{\nu,c,n}}{({}_k\omega^2_{\nu,c,n} - r)^2} - \alpha\sum_{n\geq 1}\frac{r\,{}_k\tau^2_{\nu,c,n}}{({}_k\tau^2_{\nu,c,n} - r)^2} < 0$$

for $k > 0$, $c > 0$, $\nu + k > 0$, and $r \in (0, {}_k\tau^2_{\nu,c,n})$, the function $r \mapsto \mathbb{J}(\alpha, {}_kh_{\nu,c}(r))$ is strictly decreasing on $(0, {}_k\tau^2_{\nu,c,n})$. It is worth mentioning that here we used again that the zeros ${}_k\omega_{\nu,c,n}$ and ${}_k\tau_{\nu,c,n}$ interlace for $k > 0$, $c > 0$, $\nu + k > 0$ and $r < {}_k\omega_{\nu,c,1} < {}_k\tau_{\nu,c,1}$, we have that

$${}_k\omega^2_{\nu,c,n}({}_k\tau^2_{\nu,c,n} - r^2) < {}_k\tau^2_{\nu,c,n}({}_k\omega^2_{\nu,c,n} - r^2).$$

We also observe that

$$\lim_{r\searrow 0}\mathbb{J}(\alpha, {}_kh_{\nu,c}(r)) = 1 > \beta \quad \text{and} \quad \lim_{r\nearrow {}_k\tau_{\nu,c,1}}\mathbb{J}(\alpha, {}_kh_{\nu,c}(r)) = -\infty.$$

This means that for $z \in \mathbb{D}_{r_3}$, we have that $\operatorname{Re}\mathbb{J}(\alpha, {}_kh_{\nu,c}(r)) > \beta$ if and only if r_3 is the unique root of $\mathbb{J}(\alpha, {}_kh_{\nu,c}(r)) = \beta$, in $(0, {}_k\tau^2_{\nu,c,n})$. Finally, with the aid of the interlacing inequalities ${}_k\omega_{\nu,c,n-1} < {}_k\tau_{\nu,c,n} < {}_k\omega_{\nu,c,n}$, it can be seen that the following inequality takes place:

$$\frac{\partial}{\partial\alpha}\mathbb{J}(\alpha, {}_kh_{\nu,c}(r)) = \sum_{n\geq 1}\frac{r}{{}_k\omega^2_{\nu,c,n} - r} - \sum_{n\geq 1}\frac{r}{{}_k\tau^2_{\nu,c,n} - r} < 0$$

for $k > 0$, $c > 0$, $\nu + k > 0$, and $r \in (0, {}_k\tau^2_{\nu,c,n})$. This means that the function $\alpha \mapsto \mathbb{J}(\alpha, {}_kh_{\nu,c}(r))$ is strictly decreasing on $[0, \infty)$ for all $k > 0$, $c > 0$, $\nu + k > 0$, and $r \in (0, {}_k\tau^2_{\nu,c,n})$. Consequently, as a function of α, the unique root of the equation $\mathbb{J}(\alpha, {}_kh_{\nu,c}(r)) = \beta$ is strictly decreasing, where $\beta \in [0, 1)$ $k > 0$, $c > 0$, $\nu + k > 0$, and $r \in (0, {}_k\tau^2_{\nu,c,n})$ are fixed. Thus, in the case when $\alpha \in (0, 1)$, the radius of α-convexity of the function ${}_kh_{\nu,c}$ remains in between the radius of convexity and the radius of starlikeness of the function ${}_kh_{\nu,c}$. □

4 Some particular cases of the main result

This section is devoted to some interesting results obtained by making some comparisons with earlier results. It is possible to say that the generalized k-Bessel function is actually a generalization of the suitable transformation of the Bessel function of the first kind. That is, we have the relation

$${}_1W_{\nu,1}(z) = \mathcal{J}_\nu(z),$$

where $\mathcal{J}_\nu$ stands for the Bessel function of the first kind and order ν. Taking this into account, we see that the main results obtained in this article coincide with the studies listed below.

Remark 4.1. If we choose $\alpha = 0$ and $\beta = 1$ in the parts **a**, **b**, and **c** of Theorem 3.1 and make use of the relations (see [28])

$$z^2{}_kW''_{\nu,c}(z) + z_kW'_{\nu,c}(z) = -\frac{cz^2}{k}{}_kW_{\nu,c}(z) + \frac{\nu^2}{k^2}{}_kW_{\nu,c}(z)$$

and

$$zk_kW'_{\nu,c}(z) = \frac{z}{k}k_kW_{\nu-k,c}(z) - \frac{\nu}{k}k_kW_{\nu,c}(z),$$

the radii of uniform convexity of the functions ${}_1f_{\nu,1}$, ${}_1g_{\nu,1}$, and ${}_1h_{\nu,1}$ coincide with the result in [24, Theorem 3.1], [24, Theorem 3.2], and [24, Theorem 3.3], respectively.

Remark 4.2. it is obvious that our results given in Lemmas 2.2 and 2.3, in particular when $k = 1$ and $c = 1$, are natural generalizations of [32, Lemmas 2.4 and 2.5]. In addition, if we take $\beta = 0$ in Theorems 3.2–3.4, the radii of convexity of order α of the functions ${}_1f_{\nu,1}$, ${}_1g_{\nu,1}$ and ${}_1h_{\nu,1}$ coincide with the results in [32, Theorem 1.1], [32, Theorem 1.2] and [32, Theorem 1.3], respectively.

Remark 4.3. If we choose $k = 1$ and $c = 1$ in Theorems 3.2–3.4, then the radii of α-convexity of order β of the functions ${}_1f_{\nu,1}$, ${}_1g_{\nu,1}$, and ${}_1h_{\nu,1}$ coincide with the result in [22, Theorem 1], [22, Theorem 2], and [22, Theorem 3], respectively.

5 Conclusion

In this article, we investigate the radii of uniform convexity and α-convexity of the k-Bessel function derived from the k-gamma function, which is the extension of the gamma function. In making this investigation, we deal with the normalized k-Bessel functions for three different kinds of normalization in such a way that each of them is analytic in the unit disk of the complex plane. The key tools in the proof of our main results are infinite product representations for normalized k-Bessel functions and some properties of the real zeros of these functions. We also show that for some values of the parameters, our results coincide with those obtained earlier.

Acknowledgements: The author is very grateful to the anonymous referees for their useful comments and suggestions, which helped improve the quality of this article.

Funding information: This research received no external funding.

Conflict of interest: The author declares that he has no conflict of interest.

Data availability statement: Not applicable.

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