Research Article

Chonjaroen Chairatsiripong, Damrongsak Yambangwai, and Tanakit Thianwan*

Convergence analysis of M-iteration for $\mathcal{G}$-nonexpansive mappings with directed graphs applicable in image deblurring and signal recovering problems

Abstract: In this article, weak and strong convergence theorems of the M-iteration method for $\mathcal{G}$-nonexpansive mapping in a uniformly convex Banach space with a directed graph were established. Moreover, weak convergence theorem without making use of Opial's condition is proved. The rate of convergence between the M-iteration and some other iteration processes in the literature was also compared. Specifically, our main result shows that the M-iteration converges faster than the Noor and SP iterations. Finally, the numerical examples to compare convergence behavior of the M-iteration with the three-step Noor iteration and the SP-iteration were given. As application, some numerical experiments in real-world problems were provided, focused on image deblurring and signal recovering problems.

1 Introduction

The renowned fixed-point result of a contractive mapping in complete metric spaces, which is known as Banach's contraction principle, was introduced in 1922 by Banach [1], an important instrument for solving the existence problems of nonlinear mappings. Since then, various generalizations have been studied in many directions of this principle.

Jachymski [2] proposed a novel notion of $\mathcal{G}$-contraction in 2008, establishing that it was a realistic extension of the Banach contraction principle in a metric space involving a directed graph. Using this notion, he simply proved the Kelisky-Rivlin theorem [3]. By combining graph theory and fixed-point theory, Aleomrainejad et al. [4] provided some iterative scheme findings for $\mathcal{G}$-contractive and $\mathcal{G}$-nonexpansive mappings on graphs. In [5], Alfuraidan and Khamsi introduced the concept of $\mathcal{G}$-monotone nonexpansive multivalued mappings on a metric space endowed with a graph. The existence of fixed-points of monotone nonexpansive mappings on a Banach space endowed with a directed graph was investigated by Alfuraidan [6].

* **Corresponding author: Tanakit Thianwan,** Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand, e-mail: tanakit.th@up.ac.th
Chonjaroen Chairatsiripong: Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand, e-mail: chonjaroen.ch@up.ac.th
Damrongsak Yambangwai: Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand, e-mail: damrong.ya@up.ac.th

In 2015, Tiammee et al. [7] presented Browder's convergence theorem and Halpern iteration process for $\mathcal{G}$-nonexpansive mappings in Hilbert space involving a directed graph. After that, Tripak [8] introduced the Ishikawa iterative scheme to approximate common fixed-points of $\mathcal{G}$-nonexpansive mappings defined on nonempty closed convex subsets of a uniformly convex Banach space endowed with a graph. Recently, various fixed-point iteration processes for $\mathcal{G}$-nonexpansive mappings have been studied extensively by many authors (see, e.g., [9,10] and references cited therein).

In 2000, Noor [11] studied the convergence criteria of the following three-step iteration method for solving general variational inequalities and related problems. The three-step Noor iteration is defined by:

$$
\begin{aligned}
l_n &= (1 - \eta_n)v_n + \eta_n \mathcal{T}v_n, \\
t_n &= (1 - \varrho_n)v_n + \varrho_n \mathcal{T}l_n, \\
v_{n+1} &= (1 - \xi_n)v_n + \xi_n \mathcal{T}t_n, \ \ n \geq 0,
\end{aligned}
\tag{1}
$$

where $\{\eta_n\}$, $\{\varrho_n\}$, and $\{\xi_n\}$ are in $(0, 1)$.

Glowinski and Tallec [12] used the three-step iterative approaches to find solutions for the problem of elastoviscoplasticity, eigenvalue computation, and the theory of liquid crystals. In [12], it was shown that the three-step iterative process yields better numerical results than the estimated iterations in two and one steps. In 1998, Haubruge et al. [13] studied the convergence analysis of three-step methods of Glowinski and Tallec [12] and applied these methods to obtain new splitting-type algorithms for solving variation inequalities, separable convex programming, and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. As a result, we conclude that the three-step approach plays an important and substantial role in the solution of numerous problems in pure and applied sciences.

In 2011, Phuengrattana and Suantai [14] introduced the following new three-step iteration process known as the SP-iteration:

$$
\begin{aligned}
h_n &= (1 - \eta_n)u_n + \eta_n \mathcal{T}u_n, \\
q_n &= (1 - \varrho_n)h_n + \varrho_n \mathcal{T}h_n, \\
u_{n+1} &= (1 - \xi_n)q_n + \xi_n \mathcal{T}q_n, \ \ n \geq 0,
\end{aligned}
\tag{2}
$$

where $\{\eta_n\}$, $\{\varrho_n\}$, and $\{\xi_n\}$ are in $(0, 1)$.

In addition, they showed that the SP-iteration (2) converges faster than the Noor iteration (1) for the class of continuous nondecreasing functions.

Recently in 2018, Ullah and Arshad [15] introduced an iteration, called M-iteration, defined by:

$$
r_n = (1 - \xi_n)w_n + \xi_n \mathcal{T}w_n, \quad s_n = \mathcal{T}r_n, \quad w_{n+1} = \mathcal{T}s_n, \ \ n \geq 0,
\tag{3}
$$

where $\{\xi_n\}$ is in $(0, 1)$.

Ullah and Arshad [15] showed that the iteration process (3) is faster than the Picard-S iteration [16] and the S-iteration [17] for Suzuki generalized nonexpansive mappings. In this direction, some of the notable studies were enhanced and conducted by many works, as seen in [18–20].

The main purpose of this article is to prove some weak and strong convergence theorems of the M-iteration method (3) for $\mathcal{G}$-nonexpansive mapping in a uniformly convex Banach space endowed with a graph. We also show the numerical experiment for supporting our main results and comparing the rate of convergence of the M-iteration (3) with the three-step Noor iteration (1) and the SP-iteration (2). Furthermore, we apply the M-iteration method to solve image deblurring and signal recovering problems.

2 Preliminaries

In this section, we recall a few basic notions concerning the connectivity of graphs. All of them can be found, e.g., in [21].

Let C be a nonempty subset of a real Banach space X. We identify the graph G with the pair $(V(G), E(G))$, where the set $V(G)$ of its vertices coincides with set C and the set of edges $E(G)$ contains $\{(w, w) : w \in C\}$. Also, G is such that no two edges are parallel. A mapping $\mathcal{T} : C \to C$ is said to be G-contraction if $\mathcal{T}$ preserves the edges of G (or $\mathcal{T}$ is edge-preserving), i.e.,

$$(w, s) \in E(G) \Rightarrow (\mathcal{T}w, \mathcal{T}s) \in E(G),$$

and $\mathcal{T}$ decreases the weights of edges of G in the following way: there exists $\psi \in (0, 1)$ such that

$$(w, s) \in E(G) \Rightarrow \|\mathcal{T}w - \mathcal{T}s\| \le \psi\|w - s\|.$$

A mapping $\mathcal{T} : C \to C$ is said to be G-nonexpansive (see [5], Definition 2.3 (iii)) if $\mathcal{T}$ preserves edges of G, i.e.,

$$(w, s) \in E(G) \Rightarrow (\mathcal{T}w, \mathcal{T}s) \in E(G),$$

and T non-increases weights of edges of G in the following way:

$$(w, s) \in E(G) \Rightarrow \|\mathcal{T}w - \mathcal{T}s\| \le \|w - s\|.$$

If w and s are the vertices in a graph G, then a path in G from w to s of length $N (N \in \mathbb{N} \cup \{0\})$ is a sequence $\{w_i\}_{i=0}^{N}$ of $N + 1$ vertices such that $w_0 = w$, $w_N = s$, and $(w_i, w_{i+1}) \in E(G)$ for $i = 0, 1, \dots N - 1$. A graph G is connected if there is a path between any two vertices. A directed graph $G = (V(G), E(G))$ is said to be transitive if, for any $w, s, r \in V(G)$ such that (w, s) and (s, r) are in $E(G)$, we have $(w, r) \in E(G)$. We denote G^{-1} the conversion of a graph G and

$$E(G^{-1}) = \{(w, s) \in X \times X : (s, w) \in E(G)\}.$$

Let $w_0 \in V(G)$ and $\mathcal{A}$ a subset of $V(G)$. We say that $\mathcal{A}$ is dominated by w_0 if $(w_0, w) \in E(G)$ for all $w \in \mathcal{A}$. $\mathcal{A}$ dominates w_0 if for each $w \in \mathcal{A}$, $(w, w_0) \in E(G)$.

In this article, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

A mapping $\mathcal{T} : C \to C$ is said to be G-demiclosed at 0 if, for any sequence $\{w_n\}$ in C such that $(w_n, w_{n+1}) \in E(G)$, $w_n \rightharpoonup w$ and $\mathcal{T}w_n \to 0$ imply $\mathcal{T}w = 0$.

A Banach space X is said to satisfy Opial's condition [22] if $w_n \rightharpoonup w$ and $w \ne s$, implying that

$$\limsup_{n \to \infty}\|w_n - w\| < \limsup_{n \to \infty}\|w_n - s\|.$$

Let C be a nonempty closed convex subset of a real uniformly convex Banach space X. Recall that the mapping $\mathcal{T} : C \to X$ with $\mathcal{F}(\mathcal{T}) \ne \varnothing$ is said to satisfy Condition (A) [23] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$, with $f(0) = 0$ and $f(\hat{t}) > 0$ for all $\hat{t} \in (0, \infty)$ such that

$$\|w - \mathcal{T}w\| \ge f(d(w, \mathcal{F}(\mathcal{T})))$$

for all $w \in C$. Let C be a subset of a metric space (X, d). A mapping $\mathcal{T} : C \to C$ is semi-compact [24] if for a sequence $\{w_n\}$ in C, with $\lim_{n \to \infty} d(w_n, \mathcal{T}w_n) = 0$, there exists a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that $w_{n_j} \to p \in C$.

Let C be a nonempty subset of a normed space X and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Then, C is said to have *Property G* (see [25]) if for each sequence in C converging weakly to $w \in C$ and $(w_n, w_{n+1}) \in E(G)$, there is a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that $(w_{n_j}, w) \in E(G)$ for all $j \in \mathbb{N}$.

Remark 2.1. If G is transitive, then *Property G* is equivalent to the property: if $\{w_n\}$ is a sequence in C with $(w_n, w_{n+1}) \in E(G)$ such that for any subsequence $\{w_{n_j}\}$ of the sequence $\{w_n\}$ converging weakly to w in X, then $(w_n, w) \in E(G)$ for all $n \in \mathbb{N}$.

In the sequel, the following lemmas are needed to prove our main results.

Lemma 2.2. [25] *Suppose that X is a Banach space having Opial's condition, C has Property G, and let $\mathcal{T} : C \to C$ be a G-nonexpansive mapping. Then, $I - \mathcal{T}$ is G-demiclosed at 0, i.e., if $w_n \rightharpoonup w$ and $w_n - \mathcal{T}w_n \to 0$, then $w \in \mathcal{F}(T)$, where $\mathcal{F}(T)$ is the set of fixed-points of $\mathcal{T}$.*

Lemma 2.3. [26] *Let X be a uniformly convex Banach space, and $\{\xi_n\}$ a sequence in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. Suppose that sequences $\{w_n\}$ and $\{s_n\}$ in X are such that $\limsup_{n\to\infty}\|w_n\| \leq c$, $\limsup_{n\to\infty}\|s_n\| \leq c$ and $\limsup_{n\to\infty}\|\xi_n w_n + (1 - \xi_n)s_n\| = c$ for some $c \geq 0$. Then, $\lim_{n\to\infty}\|w_n - s_n\| = 0$.*

Lemma 2.4. [27] *Let X be a Banach space that satisfies Opial's condition and let $\{w_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty}\|w_n - u\|$ and $\lim_{n\to\infty}\|w_n - v\|$ exist. If $\{w_{n_j}\}$ and $\{w_{n_k}\}$ are the subsequences of $\{w_n\}$ that converge weakly to u and v, respectively, then $u = v$.*

Lemma 2.5. [28] *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and suppose that C has Property $\mathcal{G}$. Let $\mathcal{T}$ be a $\mathcal{G}$-nonexpansive mapping on C. Then, $I - \mathcal{T}$ is $\mathcal{G}$-demiclosed at 0.*

Lemma 2.6. [29] *Let $\{w_n\}$ be a bounded sequence in a reflexive Banach space X. If for any weakly convergent subsequence $\{w_{n_j}\}$ of $\{w_n\}$, both $\{w_{n_j}\}$ and $\{w_{n_j+1}\}$ converge weakly to the same point in X, then the sequence $\{w_n\}$ is weakly convergent.*

3 Main results

Throughout the section, let C be a nonempty closed convex subset of a Banach space X endowed with a directed graph $\mathcal{G}$ such that $V(\mathcal{G}) = C$ and $E(\mathcal{G})$ is convex. We also suppose that the graph $\mathcal{G}$ is transitive. The mapping $\mathcal{T}$ is $\mathcal{G}$-nonexpansive from C to C with $\mathcal{F}(\mathcal{T}) \neq \varnothing$. For an arbitrary $w_0 \in C$, define the sequence $\{w_n\}$ by (3).

We start with proving the following useful results.

Proposition 3.1. *Let $\tilde{q} \in \mathcal{F}$ be such that $(w_0, \tilde{q})$ and $(\tilde{q}, w_0)$ are in $E(\mathcal{G})$. Then, $(w_n, \tilde{q})$, $(r_n, \tilde{q})$, $(s_n, \tilde{q})$, $(\tilde{q}, w_n)$, $(\tilde{q}, r_n)$, $(\tilde{q}, s_n)$, (s_n, r_n), (w_n, r_n), and (w_n, w_{n+1}) are in $E(\mathcal{G})$.*

Proof. We proceed by induction. Since $\mathcal{T}$ is edge-preserving and $(w_0, \tilde{q}) \in E(\mathcal{G})$, we have $(\mathcal{T}w_0, \tilde{q}) \in E(\mathcal{G})$ and so $(r_0, \tilde{q}) \in E(\mathcal{G})$, by $E(\mathcal{G})$ is convex. Again, by edge-preserving of $\mathcal{T}$ and $(r_0, \tilde{q}) \in E(\mathcal{G})$, we have $(\mathcal{T}r_0, \tilde{q}) \in E(\mathcal{G})$ and so $(s_0, \tilde{q}) \in E(\mathcal{G})$. Then, since $\mathcal{T}$ is edge-preserving and $(s_0, \tilde{q}) \in E(\mathcal{G})$, we obtain $(\mathcal{T}s_0, \tilde{q}) \in E(\mathcal{G})$, and hence, $(w_1, \tilde{q}) \in E(\mathcal{G})$. Thus, by edge-preserving of $\mathcal{T}$, $(\mathcal{T}w_1, \tilde{q}) \in E(\mathcal{G})$. Again, by the convexity of $E(\mathcal{G})$ and $(\mathcal{T}w_1, \tilde{q}), (w_1, \tilde{q}) \in E(\mathcal{G})$, we have $(r_1, \tilde{q}) \in E(\mathcal{G})$. Then, since $\mathcal{T}$ is edge-preserving, $(\mathcal{T}r_1, \tilde{q}) \in E(\mathcal{G})$, we obtain $(s_1, \tilde{q}) \in E(\mathcal{G})$, and hence, $(\mathcal{T}s_1, \tilde{q}) \in E(\mathcal{G})$.

Next, we assume that $(w_k, \tilde{q}) \in E(\mathcal{G})$. Since $\mathcal{T}$ is edge-preserving, we obtain $(\mathcal{T}w_k, \tilde{q}) \in E(\mathcal{G})$. So $(r_k, \tilde{q}) \in E(\mathcal{G})$, since $E(\mathcal{G})$ is convex. Hence, by edge-preserving of $\mathcal{T}$ and $(r_k, \tilde{q}) \in E(\mathcal{G})$, we have $(\mathcal{T}r_k, \tilde{q}) \in E(\mathcal{G})$, then $(s_k, \tilde{q}) \in E(\mathcal{G})$. Since $\mathcal{T}$ is edge-preserving, we have $(\mathcal{T}s_k, \tilde{q}) \in E(\mathcal{G})$. Thus, $(w_{k+1}, \tilde{q}) \in E(\mathcal{G})$. Hence, by edge-preserving of $\mathcal{T}$, we obtain $(\mathcal{T}w_{k+1}, \tilde{q}) \in E(\mathcal{G})$, and so $(r_{k+1}, \tilde{q}) \in E(\mathcal{G})$, since $E(\mathcal{G})$ is convex. Again, by edge-preserving of $\mathcal{T}$, we obtain $(\mathcal{T}r_{k+1}, \tilde{q}) \in E(\mathcal{G})$, and so $(s_{k+1}, \tilde{q}) \in E(\mathcal{G})$. Therefore, $(w_n, \tilde{q}), (r_n, \tilde{q}), (s_n, \tilde{q}) \in E(\mathcal{G})$ for all $n \geq 1$.

Since $\mathcal{T}$ is edge-preserving and $(\tilde{q}, w_0) \in E(\mathcal{G})$, we have $(\tilde{q}, \mathcal{T}w_0) \in E(\mathcal{G})$, and so $(\tilde{q}, r_0) \in E(\mathcal{G})$, since $E(\mathcal{G})$ is convex. Again, since $\mathcal{T}$ is edge-preserving and $(\tilde{q}, r_0) \in E(\mathcal{G})$, we have $(\tilde{q}, \mathcal{T}r_0) \in E(\mathcal{G})$, and so $(\tilde{q}, s_0) \in E(\mathcal{G})$. Using a similar argument, we can show that $(\tilde{q}, w_n), (\tilde{q}, r_n), (\tilde{q}, s_n) \in E(\mathcal{G})$ for all $n \geq 1$ under the assumption that $(\tilde{q}, w_0) \in E(\mathcal{G})$, $(\tilde{q}, r_0) \in E(\mathcal{G})$, and $(\tilde{q}, s_0) \in E(\mathcal{G})$. By transitivity of $\mathcal{G}$, we obtain $(s_n, r_n), (w_n, r_n)$, $(w_n, w_{n+1}) \in E(\mathcal{G})$. This completes the proof. □

Lemma 3.2. *Let X be a uniformly convex Banach space. Suppose that $\{\xi_n\}$ is a real sequence in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$ and $(w_0, \tilde{q}), (\tilde{q}, w_0) \in E(\mathcal{G})$ for arbitrary $w_0 \in C$ and $\tilde{q} \in \mathcal{F}(\mathcal{T})$. Then,*
(i) $\lim_{n\to\infty}\|w_n - \tilde{q}\|$ exists,
(ii) $\lim_{n\to\infty}\|w_n - \mathcal{T}w_n\| = 0$.

Proof. (i) Let $\tilde{q} \in \mathcal{F}(\mathcal{T})$. By Proposition 3.1, we have $(w_n, \tilde{q}), (s_n, \tilde{q}), (r_n, \tilde{q}), (s_n, r_n), (w_n, r_n) \in E(\mathcal{G})$. Then, by $\mathcal{G}$-nonexpansiveness of $\mathcal{T}$ and using (3), we have

$$
\begin{aligned}
\|r_n - \tilde{q}\| &= \|(1 - \xi_n)w_n + \xi_n \mathcal{T}w_n - \tilde{q}\| \\
&= \|(1 - \xi_n)(w_n - \tilde{q}) + \xi_n(\mathcal{T}w_n - \tilde{q})\| \\
&\leq (1 - \xi_n)\|w_n - \tilde{q}\| + \xi_n\|\mathcal{T}w_n - \tilde{q}\| \\
&\leq (1 - \xi_n)\|w_n - \tilde{q}\| + \xi_n\|w_n - \tilde{q}\| \\
&= \|w_n - \tilde{q}\|,
\end{aligned}
\tag{4}
$$

$$
\|s_n - \tilde{q}\| = \|\mathcal{T}r_n - \tilde{q}\| \leq \|r_n - \tilde{q}\| \leq \|w_n - \tilde{q}\|,
\tag{5}
$$

and so

$$
\|w_{n+1} - \tilde{q}\| = \|\mathcal{T}s_n - \tilde{q}\| \leq \|s_n - \tilde{q}\| \leq \|w_n - \tilde{q}\|.
\tag{6}
$$

Therefore,

$$
\|w_{n+1} - \tilde{q}\| \leq \|w_n - \tilde{q}\| \leq \cdots \leq \|w_1 - \tilde{q}\|, \quad \forall n \in \mathbb{N}.
$$

Since $\{\|w_n - \tilde{q}\|\}$ is monotonically decreasing, we have that the sequence $\{\|w_n - \tilde{q}\|\}$ is convergent. In particular, the sequence $\{w_n\}$ is bounded.

(ii) Assume that $\lim_{n\to\infty}\|w_n - \tilde{q}\| = c$. If $c = 0$, then by $\mathcal{G}$-nonexpansiveness of $\mathcal{T}$, we obtain

$$
\|w_n - \mathcal{T}w_n\| \leq \|w_n - \tilde{q}\| + \|\tilde{q} - \mathcal{T}w_n\| \leq \|w_n - \tilde{q}\| + \|\tilde{q} - w_n\|.
$$

Therefore, the result follows. Suppose that $c > 0$. Taking the lim sup on both sides in Inequality (5), we obtain

$$
\limsup_{n\to\infty}\|s_n - \tilde{q}\| \leq \limsup_{n\to\infty}\|w_n - \tilde{q}\| = c.
\tag{7}
$$

Since $\lim_{n\to\infty}\|w_{n+1} - q_0\| = c$. Letting $n \to \infty$ in Inequality (6), we have

$$
\lim_{n\to\infty}\|\mathcal{T}s_n - \tilde{q}\| = c.
\tag{8}
$$

Taking the lim sup on both sides in the Inequality (4), we obtain

$$
\limsup_{n\to\infty}\|r_n - \tilde{q}\| \leq \limsup_{n\to\infty}\|w_n - \tilde{q}\| = c.
\tag{9}
$$

In addition, by $\mathcal{G}$-nonexpansiveness of $\mathcal{T}$, we have $\|\mathcal{T}r_n - \tilde{q}\| \leq \|r_n - \tilde{q}\|$, taking the lim sup on both sides in this inequality and using (9), we obtain

$$
\limsup_{n\to\infty}\|\mathcal{T}r_n - \tilde{q}\| \leq c.
\tag{10}
$$

Note that $\|w_{n+1} - \tilde{q}\| \leq \|s_n - \tilde{q}\| \leq \|r_n - \tilde{q}\|$ gives that

$$
\liminf_{n\to\infty}\|r_n - \tilde{q}\| \geq c.
\tag{11}
$$

From (9) and (11), we have

$$
\lim_{n\to\infty}\|r_n - \tilde{q}\| = c.
\tag{12}
$$

By (4) and (12), we have

$$
\lim_{n\to\infty}\|(1 - \xi_n)(w_n - \tilde{q}) + \xi_n(\mathcal{T}w_n - \tilde{q})\| = c.
\tag{13}
$$

In addition, $\limsup_{n\to\infty}\|\mathcal{T}w_n - \tilde{q}\| \leq \limsup_{n\to\infty}\|w_n - \tilde{q}\| = c$, using (13) and Lemma 2.3, we have

$$
\lim_{n\to\infty}\|\mathcal{T}w_n - w_n\| = 0.
$$

This completes the proof. □

We now prove the weak convergence of the sequence generated by the M-iteration method (3) for a $\mathcal{G}$-nonexpansive mapping in a uniformly convex Banach space satisfying Opial's condition.

Theorem 3.3. *Let $\mathcal{X}$ be a uniformly convex Banach space that satisfies Opial's condition and C has Property $\mathcal{G}$. Suppose that $\{\xi_n\}$ is a real sequence in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. If $(w_0, \tilde{q}), (\tilde{q}, w_0) \in E(\mathcal{G})$ for arbitrary $w_0 \in C$ and $\tilde{q} \in \mathcal{F}(\mathcal{T})$, then $\{w_n\}$ converges weakly to a fixed-point of $\mathcal{T}$.*

Proof. Let $\tilde{q} \in \mathcal{F}$ be such that $(w_0, \tilde{q}), (\tilde{q}, w_0) \in E(\mathcal{G})$. From Lemma 3.2 (i), $\lim_{n\to\infty}\|w_n - \tilde{q}\|$ exists, so $\{w_n\}$ is bounded. It follows from Lemma 3.2 (ii) that $\lim_{n\to\infty}\|w_n - \mathcal{T}w_n\| = 0$. Since $\mathcal{X}$ is uniformly convex and $\{w_n\}$ is bounded, we may assume that $w_n \rightharpoonup u$ as $n \to \infty$, without loss of generality. By Lemma 2.2, we have $u \in \mathcal{F}$. Suppose that subsequences $\{w_{n_k}\}$ and $\{w_{n_j}\}$ of $\{w_n\}$ converge weakly to u and v, respectively. By Lemma 3.2 (ii), we obtain that $\|w_{n_k} - \mathcal{T}w_{n_k}\| \to 0$ and $\|w_{n_j} - \mathcal{T}w_{n_j}\| \to 0$ as $k, j \to \infty$. Using Lemma 2.2, we have $u, v \in F(T)$. By Lemma 3.2 (i), $\lim_{n\to\infty}\|w_n - u\|$ and $\lim_{n\to\infty}\|w_n - v\|$ exist. It follows from Lemma 2.4 that $u = v$. Therefore, $\{w_n\}$ converges weakly to a fixed-point of $\mathcal{T}$. □

It is worth noting that Opial's condition has remained crucial in proving weak convergence theorems. However, each l^p ($1 \le p < \infty$) satisfies Opial's condition, while all L^p do not have the property unless $p = 2$.

Next, we deal with the weak convergence of the sequence $\{w_n\}$ generated by (3) for $\mathcal{G}$-nonexpansive mapping without assuming Opial's condition in a uniformly convex Banach space with a directed graph.

Theorem 3.4. *Let $\mathcal{X}$ be a uniformly convex Banach space. Suppose that C has Property $\mathcal{G}$, $\{\xi_n\}$ is a real sequence in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$, $\mathcal{F}(\mathcal{T})$ is dominated by w_0, and $\mathcal{F}(\mathcal{T})$ dominates w_0. If $(w_0, \tilde{q}), (\tilde{q}, w_0) \in E(\mathcal{G})$ for arbitrary $w_0 \in C$ and $\tilde{q} \in \mathcal{F}(\mathcal{T})$, then $\{w_n\}$ converges weakly to a fixed-point of $\mathcal{T}$.*

Proof. Let $\tilde{q} \in \mathcal{F}$ be such that (w_0, q_0) and (q_0, w_0) are in $E(\mathcal{G})$. From Lemma 3.2 (i), $\lim_{n\to\infty}\|w_n - \tilde{q}\|$ exists, so $\{w_n\}$ is bounded in C. Since C is nonempty closed convex subset of a uniformly convex Banach space $\mathcal{X}$, it is weakly compact, and hence, there exists a subsequence $\{w_{n_j}\}$ of the sequence $\{w_n\}$ such that $\{w_{n_j}\}$ converges weakly to some point $p \in C$. By Lemma 3.2 (ii), we obtain that

$$\lim_{j\to\infty}\|w_{n_j} - \mathcal{T}w_{n_j}\| = 0. \tag{14}$$

In addition, $\|r_n - w_n\| = \|(1 - \xi_n)w_n + \xi_n\mathcal{T}w_n - w_n\| \le \xi_n\|\mathcal{T}w_n - w_n\|$, using Lemma 3.2 (ii), we have

$$\lim_{n\to\infty}\|r_n - w_n\| = 0. \tag{15}$$

Using Lemma 3.2 (ii) and (15), we have

$$\begin{aligned}
\|\mathcal{T}r_n - r_n\| &= \|\mathcal{T}r_n - \mathcal{T}w_n + \mathcal{T}w_n - w_n + w_n - r_n\| \\
&\le \|\mathcal{T}r_n - \mathcal{T}w_n\| + \|\mathcal{T}w_n - w_n\| + \|w_n - r_n\| \\
&\le \|r_n - w_n\| + \|\mathcal{T}w_n - w_n\| + \|w_n - r_n\| \\
&\to 0 \quad (\text{as } n \to \infty).
\end{aligned} \tag{16}$$

Using (3) and (16), we have

$$\lim_{n\to\infty}\|s_n - r_n\| = \lim_{n\to\infty}\|\mathcal{T}r_n - r_n\| = 0. \tag{17}$$

In addition,

$$\|\mathcal{T}s_n - s_n\| \le \|\mathcal{T}s_n - \mathcal{T}r_n\| + \|\mathcal{T}r_n - r_n\| + \|r_n - s_n\| \le \|s_n - r_n\| + \|\mathcal{T}r_n - r_n\| + \|r_n - s_n\|.$$

Using (16) and (17), we have

$$\lim_{n\to\infty}\|\mathcal{T}s_n - s_n\| = 0. \tag{18}$$

Using Lemma 2.5, we have $I - \mathcal{T}$ is $\mathcal{G}$-demiclosed at 0 so that $p \in \mathcal{F}(\mathcal{T})$. To complete the proof, it suffices to show that $\{w_n\}$ converges weakly to p. To this end, we need to show that $\{w_n\}$ satisfies the hypothesis of Lemma 2.6. Let $\{w_{n_j}\}$ be a subsequence of $\{w_n\}$ that converges weakly to some $q \in C$. By similar aforementioned arguments, q is in $\mathcal{F}(\mathcal{T})$. Now, for each $j \geq 1$, using (3), we have

$$w_{n_j+1} = \mathcal{T}s_{n_j}. \tag{19}$$

It follows from (14) that

$$\mathcal{T}w_{n_j} = (\mathcal{T}w_{n_j} - w_{n_j}) + w_{n_j} \rightharpoonup q. \tag{20}$$

Now from (3) and (20),

$$r_{n_j} = (1 - \xi_{n_j})w_{n_j} + \xi_{n_j}\mathcal{T}w_{n_j} \rightharpoonup q. \tag{21}$$

Using (16) and (21), we have

$$\mathcal{T}r_{n_j} = (\mathcal{T}r_{n_j} - r_{n_j}) + r_{n_j} \rightharpoonup q. \tag{22}$$

Now from (3) and (22),

$$s_{n_j} = \mathcal{T}r_{n_j} \rightharpoonup q. \tag{23}$$

Also, from (18) and (23), we have

$$\mathcal{T}s_{n_j} = (\mathcal{T}s_{n_j} - s_{n_j}) + s_{n_j} \rightharpoonup q. \tag{24}$$

It follows from (19) and (24) that

$$w_{n_j+1} \rightharpoonup q.$$

Therefore, the sequence $\{w_n\}$ satisfies the hypothesis of Lemma 2.6, which in turn implies that $\{w_n\}$ weakly converges to q so that $p = q$. This completes the proof. □

The strong convergence of the sequence generated by the M-iteration method (3) for $\mathcal{G}$-nonexpansive mapping in a uniformly convex Banach space with a directed graph is discussed in the rest of this section.

Theorem 3.5. *Let $\mathcal{X}$ be a uniformly convex Banach space. Suppose that $\{\xi_n\}$ is a real sequence in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$, $\mathcal{T}$ satisfies Condition (A), $\mathcal{F}(\mathcal{T})$ is dominated by w_0 and $\mathcal{F}(\mathcal{T})$ dominates w_0. Then, $\{w_n\}$ converges strongly to a fixed-point of $\mathcal{T}$.*

Proof. By Lemma 3.2 (i), $\lim_{n\to\infty}\|w_n - q\|$ exists and so $\lim_{n\to\infty}d(w_n, \mathcal{F}(\mathcal{T}))$ exists for any $q \in \mathcal{F}(\mathcal{T})$. Also, by Lemma 3.2 (ii), $\lim_{n\to\infty}\|w_n - \mathcal{T}w_n\| = 0$. It follows from Condition (A) that $\lim_{n\to\infty}f(d(w_n, \mathcal{F}(\mathcal{T}))) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$, we obtain that $\lim_{n\to\infty}d(w_n, \mathcal{F}(\mathcal{T})) = 0$. Next, we show that $\{w_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty}d(w_n, \mathcal{F}(\mathcal{T})) = 0$, given any $\varepsilon > 0$, there exists a natural number n_0 such that $d(w_n, \mathcal{F}(\mathcal{T})) < \frac{\varepsilon}{2}$ for all $n \geq n_0$. So we can find $q^* \in \mathcal{F}(\mathcal{T})$ such that $\|w_{n_0} - q^*\| < \frac{\varepsilon}{4}$. For $n \geq n_0$ and $m \geq 1$, we have

$$\|w_{n+m} - w_n\| = \|w_{n+m} - q^*\| + \|w_n - q^*\| \leq \|w_{n_0} - q^*\| + \|w_{n_0} - q^*\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $\{w_n\}$ is a Cauchy sequence and so is convergent since $\mathcal{X}$ is complete. Let $\lim_{n\to\infty}w_n = \tilde{q}$. Then, $d(\tilde{q}, \mathcal{F}(\mathcal{T})) = 0$. It follows that $\tilde{q} \in \mathcal{F}(\mathcal{T})$. This completes the proof. □

Theorem 3.6. *Let $\mathcal{X}$ be a uniformly convex Banach space. Suppose that C has Property $\mathcal{G}$ and $\{\xi_n\}$ is a real sequence in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$, $\mathcal{F}(\mathcal{T})$ is dominated by w_0 and $\mathcal{F}(\mathcal{T})$ dominates w_0. If $\mathcal{T}$ is semi-compact, then $\{w_n\}$ converges strongly to a fixed-point of $\mathcal{T}$.*

Proof. It follows from Lemma 3.2 that $\{w_n\}$ is bounded and $\lim_{n\to\infty}\|w_n - \mathcal{T}w_n\| = 0$. Since $\mathcal{T}$ is semi-compact, then there exists a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that $w_{n_j} \to q \in C$ as $j \to \infty$. Since C has *Property G* and transitivity of graph $\mathcal{G}$, we obtain $(w_{n_j}, q) \in E(\mathcal{G})$. Note that $\lim_{j\to\infty}\|w_{n_j} - \mathcal{T}w_{n_j}\| = 0$. Then,

$$\|q - \mathcal{T}q\| \le \|q - w_{n_j}\| + \|w_{n_j} - \mathcal{T}w_{n_j}\| + \|\mathcal{T}w_{n_j} - \mathcal{T}q\|$$
$$\le \|q - w_{n_j}\| + \|w_{n_j} - \mathcal{T}w_{n_j}\| + \|w_{n_j} - q\|$$
$$\to 0 \quad (\text{as } j \to \infty).$$

Hence, $q \in \mathcal{F}(\mathcal{T})$. Thus, $\lim_{n\to\infty}d(w_n, \mathcal{F}(\mathcal{T}))$ exists by Theorem 3.5. We note that $d(w_{n_j}, \mathcal{F}) \le d(w_{n_j}, q) \to 0$ as $j \to \infty$; hence, $\lim_{n\to\infty}d(w_n, \mathcal{F}(\mathcal{T})) = 0$. It follows, as in the proof of Theorem 3.5, that $\{w_n\}$ converges strongly to a fixed-point of $\mathcal{T}$. This completes the proof. □

4 Rate of convergence and numerical examples

In this section, we show that the M-iteration process converges faster than the iterative scheme due to Phuengrattana et al. and Noor for the class $\mathcal{G}$-contraction mappings. Furthermore, we provide a concrete example, including numerical results, and compare the proposed algorithm (3) with Noor (1) and SP (2) algorithms to declare that our algorithm is more effective. All codes were written in Matlab 2019b.

The following definitions about the rate of convergence are due to Berinde [30].

Definition 4.1. Let $\{\rho_n\}$ and $\{\sigma_n\}$ be two sequences of real numbers converging to ρ and σ, respectively. If $\lim_{n\to\infty}\frac{|\rho_n - \rho|}{|\sigma_n - \sigma|} = 0$, then $\{\rho_n\}$ converges faster than $\{\sigma_n\}$.

Definition 4.2. Suppose that for two fixed-point iteration processes $\{w_n\}$ and $\{u_n\}$, both converging to the same fixed-point $\tilde{q}$, the error estimates

$$\|w_n - \tilde{q}\| \le \rho_n \quad \text{for all } n \ge 1,$$
$$\|u_n - \tilde{q}\| \le \sigma_n \quad \text{for all } n \ge 1,$$

are available, where $\{\rho_n\}$ and $\{\sigma_n\}$ are two sequences of positive numbers converging to zero. If $\{\rho_n\}$ converges faster than $\{\sigma_n\}$, then $\{w_n\}$ *converges faster than* $\{u_n\}$ to $\tilde{q}$.

Definition 4.3. [31] Suppose $\{\zeta_n\}$ is a sequence that converges to ζ, with $\zeta_n \ne \zeta$ for all n. If positive constants ν and ψ exist with

$$\lim_{n\to\infty}\frac{|\zeta_{n+1} - \zeta|}{|\zeta_n - \zeta|^\psi} = \nu,$$

then $\{\zeta_n\}$ converges to ζ of order ψ, with asymptotic error constant ν. If $\psi = 1$ (and $\nu < 1$), the sequence is linearly convergent.

In 2011, Phuengrattana and Suantai [14] showed that the Ishikawa iteration converges faster than the Mann iteration for a class of continuous functions on the closed interval in a real line. In order to study the order of convergence of a real sequence $\{\zeta_n\}$ converging to ζ, we usually use the well-known terminology in numerical analysis (see [31], for example).

Let $\mathcal{T}$ be $\mathcal{G}$-nonexpansive from C to C with $\mathcal{F}(\mathcal{T})$ nonempty, where C is a nonempty closed convex subset of a Banach space X endowed with a directed graph. Let $V(\mathcal{G}) = C$, $E(\mathcal{G})$ is convex and the graph $\mathcal{G}$ is transitive.

The following propositions will be useful in this context.

Proposition 4.4. *For an arbitrary $v_0 \in C$, define the sequence $\{v_n\}$ by the Noor iteration* (1). *Let $\tilde{q} \in \mathcal{F}$ be such that $(v_0, \tilde{q})$ and $(\tilde{q}, v_0)$ are in $E(\mathcal{G})$. Then, $(v_n, \tilde{q})$, $(t_n, \tilde{q})$, $(l_n, \tilde{q})$, $(\tilde{q}, v_n)$, $(\tilde{q}, t_n)$, and $(\tilde{q}, l_n)$ are in $E(\mathcal{G})$.*

Proof. We proceed by induction. Since $\mathcal{T}$ is edge-preserving and $(v_0, \tilde{q}) \in E(\mathcal{G})$, we have $(\mathcal{T}v_0, \tilde{q}) \in E(\mathcal{G})$ and so $(l_0, \tilde{q}) \in E(\mathcal{G})$, by $E(\mathcal{G})$ is convex. Again, by edge-preserving of $\mathcal{T}$ and $(l_0, \tilde{q}) \in E(\mathcal{G})$, we have $(\mathcal{T}l_0, \tilde{q}) \in E(\mathcal{G})$. Since $(v_0, \tilde{q}) \in E(\mathcal{G})$, by the convexity of $E(\mathcal{G})$, we have $(t_0, \tilde{q}) \in E(\mathcal{G})$. Then, since $\mathcal{T}$ is edge-preserving and $(t_0, \tilde{q}) \in E(\mathcal{G})$, we obtain $(\mathcal{T}t_0, \tilde{q}) \in E(\mathcal{G})$, and hence $(v_1, \tilde{q}) \in E(\mathcal{G})$, since $E(\mathcal{G})$ is convex and $(v_0, \tilde{q}) \in E(\mathcal{G})$. Thus, by edge-preserving of $\mathcal{T}$, $(\mathcal{T}v_1, \tilde{q}) \in E(\mathcal{G})$. Again, by the convexity of $E(\mathcal{G})$ and $(\mathcal{T}v_1, \tilde{q})$, $(v_1, \tilde{q})$, we have $(l_1, \tilde{q}) \in E(\mathcal{G})$. Then, since $\mathcal{T}$ is edge-preserving, $(\mathcal{T}l_1, \tilde{q}) \in E(\mathcal{G})$, by $E(\mathcal{G})$ is convex and $(v_0, \tilde{q}) \in E(\mathcal{G})$, we obtain $(t_1, \tilde{q}) \in E(\mathcal{G})$, and hence, $(\mathcal{T}t_1, \tilde{q}) \in E(\mathcal{G})$.

Next, we assume that $(v_k, \tilde{q}) \in E(\mathcal{G})$. Since $\mathcal{T}$ is edge-preserving, we obtain $(\mathcal{T}v_k, \tilde{q}) \in E(\mathcal{G})$. So $(l_k, \tilde{q}) \in E(\mathcal{G})$, since $E(\mathcal{G})$ is convex. Hence, by edge-preserving of $\mathcal{T}$ and $(l_k, \tilde{q}) \in E(\mathcal{G})$, we have $(\mathcal{T}l_k, \tilde{q}) \in E(\mathcal{G})$, then $(t_k, \tilde{q}) \in E(\mathcal{G})$, since $E(\mathcal{G})$ is convex and $(v_k, \tilde{q}) \in E(\mathcal{G})$. Since $\mathcal{T}$ is edge-preserving, we have $(\mathcal{T}t_k, \tilde{q}) \in E(\mathcal{G})$. Thus, $(v_{k+1}, \tilde{q}) \in E(\mathcal{G})$, since $(v_k, \tilde{q})$, $(\mathcal{T}t_k, \tilde{q}) \in E(\mathcal{G})$ and $E(\mathcal{G})$ is convex. Hence, by edge-preserving of $\mathcal{T}$, we obtain $(\mathcal{T}v_{k+1}, \tilde{q}) \in E(\mathcal{G})$, and so $(l_{k+1}, \tilde{q}) \in E(\mathcal{G})$, since $E(\mathcal{G})$ is convex and $(v_{k+1}, \tilde{q}) \in E(\mathcal{G})$. Again, by edge-preserving of $\mathcal{T}$, we obtain $(\mathcal{T}l_{k+1}, \tilde{q}) \in E(\mathcal{G})$, and so $(t_{k+1}, \tilde{q}) \in E(\mathcal{G})$, since $E(\mathcal{G})$ is convex and $(v_{k+1}, \tilde{q}) \in E(\mathcal{G})$. Therefore, $(v_n, \tilde{q})$, $(t_n, \tilde{q})$, $(ln, \tilde{q}) \in E(\mathcal{G})$ for all $n \geq 1$.

Since $\mathcal{T}$ is edge-preserving and $(\tilde{q}, v_0) \in E(\mathcal{G})$, we have $(\tilde{q}, \mathcal{T}v_0) \in E(\mathcal{G})$, and so $(\tilde{q}, l_0) \in E(\mathcal{G})$, since $E(\mathcal{G})$ is convex. Again, since $\mathcal{T}$ is edge-preserving and $(\tilde{q}, l_0) \in E(\mathcal{G})$, we have $(\tilde{q}, \mathcal{T}l_0) \in E(\mathcal{G})$, and so $(\tilde{q}, t_0) \in E(\mathcal{G})$, by $E(\mathcal{G})$ is convex and $(\tilde{q}, v_0) \in E(\mathcal{G})$. Using a similar argument, we can show that $(\tilde{q}, v_n)$, $(\tilde{q}, t_n)$, $(\tilde{q}, l_n) \in E(\mathcal{G})$ for all $n \geq 1$ under the assumption that $(\tilde{q}, v_0) \in E(\mathcal{G})$, $(\tilde{q}, t_0) \in E(\mathcal{G})$, and $(\tilde{q}, l_0) \in E(\mathcal{G})$. This completes the proof. □

Proposition 4.5. *For an arbitrary $u_0 \in C$, define the sequence $\{u_n\}$ by SP-iteration* (2). *Let $\tilde{q} \in \mathcal{F}$ be such that $(u_0, \tilde{q})$ and $(\tilde{q}, u_0)$ are in $E(\mathcal{G})$. Then, $(u_n, \tilde{q})$, $(q_n, \tilde{q})$, $(h_n, \tilde{q})$, $(\tilde{q}, u_n)$, $(\tilde{q}, q_n)$, and $(\tilde{q}, h_n)$ are in $E(\mathcal{G})$.*

Proof. We proceed by induction. Since $\mathcal{T}$ is edge-preserving and $(u_0, \tilde{q}) \in E(\mathcal{G})$, we have $(\mathcal{T}u_0, \tilde{q}) \in E(\mathcal{G})$ and so $(h_0, \tilde{q}) \in E(\mathcal{G})$, by $E(\mathcal{G})$ is convex. Again, by edge-preserving of $\mathcal{T}$ and $(h_0, \tilde{q}) \in E(\mathcal{G})$, we have $(\mathcal{T}h_0, \tilde{q}) \in E(\mathcal{G})$ and so $(q_0, \tilde{q}) \in E(\mathcal{G})$, since $E(\mathcal{G})$ is convex. Then, since $\mathcal{T}$ is edge-preserving and $(q_0, \tilde{q}) \in E(\mathcal{G})$, we obtain $(\mathcal{T}q_0, \tilde{q}) \in E(\mathcal{G})$, and hence, $(u_1, \tilde{q}) \in E(\mathcal{G})$, by $E(\mathcal{G})$ is convex. Thus, by edge-preserving of $\mathcal{T}$, $(\mathcal{T}u_1, \tilde{q}) \in E(\mathcal{G})$. Again, by the convexity of $E(\mathcal{G})$ and $(\mathcal{T}u_1, \tilde{q})$, $(u_1, \tilde{q})$, we have $(h_1, \tilde{q}) \in E(\mathcal{G})$. Then, since $\mathcal{T}$ is edge-preserving, $(\mathcal{T}h_1, \tilde{q}) \in E(\mathcal{G})$, by $E(\mathcal{G})$ is convex, we obtain $(q_1, \tilde{q}) \in E(\mathcal{G})$, and hence, $(\mathcal{T}q_1, \tilde{q}) \in E(\mathcal{G})$.

Next, we assume that $(u_k, \tilde{q}) \in E(\mathcal{G})$. Since $\mathcal{T}$ is edge-preserving, we obtain $(\mathcal{T}u_k, \tilde{q}) \in E(\mathcal{G})$. So $(h_k, \tilde{q}) \in E(\mathcal{G})$, since $E(\mathcal{G})$ is convex. Hence, by edge-preserving of $\mathcal{T}$ and $(h_k, \tilde{q}) \in E(\mathcal{G})$, we have $(\mathcal{T}h_k, \tilde{q}) \in E(\mathcal{G})$, then $(q_k, \tilde{q}) \in E(\mathcal{G})$, since $E(\mathcal{G})$ is convex. Since $\mathcal{T}$ is edge-preserving, we have $(\mathcal{T}q_k, \tilde{q}) \in E(\mathcal{G})$. Again, by the convexity of $E(\mathcal{G})$ and $(q_k, \tilde{q}) \in E(\mathcal{G})$, we have $(u_{k+1}, \tilde{q}) \in E(\mathcal{G})$. Hence, by edge-preserving of $\mathcal{T}$, we obtain $(\mathcal{T}u_{k+1}, \tilde{q}) \in E(\mathcal{G})$, and so $(h_{k+1}, \tilde{q}) \in E(\mathcal{G})$, since $E(\mathcal{G})$ is convex. Again, by edge-preserving of $\mathcal{T}$, we obtain $(\mathcal{T}h_{k+1}, \tilde{q}) \in E(\mathcal{G})$, and so $(q_{k+1}, \tilde{q}) \in E(\mathcal{G})$, since $E(\mathcal{G})$ is convex. Therefore, $(u_n, \tilde{q})$, $(h_n, \tilde{q})$, $(q_n, \tilde{q}) \in E(\mathcal{G})$ for all $n \geq 1$.

Since $\mathcal{T}$ is edge-preserving and $(\tilde{q}, u_0) \in E(\mathcal{G})$, we have $(\tilde{q}, \mathcal{T}u_0) \in E(\mathcal{G})$, and so $(\tilde{q}, h_0) \in E(\mathcal{G})$, since $E(\mathcal{G})$ is convex. Again, since $\mathcal{T}$ is edge-preserving and $(\tilde{q}, h_0) \in E(\mathcal{G})$, we have $(\tilde{q}, \mathcal{T}h_0) \in E(\mathcal{G})$, and so $(\tilde{q}, q_0) \in E(\mathcal{G})$, by $E(\mathcal{G})$ is convex. Using a similar argument, we can show that $(\tilde{q}, u_n)$, $(\tilde{q}, q_n)$, $(\tilde{q}, h_n) \in E(\mathcal{G})$ for all $n \geq 1$ under the assumption that $(\tilde{q}, u_0) \in E(\mathcal{G})$, $(\tilde{q}, q_0) \in E(\mathcal{G})$, and $(\tilde{q}, h_0) \in E(\mathcal{G})$. This completes the proof. □

From the following result, we consider the rate of convergence of the M-iterative method (3) and the well-known iterative methods.

Theorem 4.6. *Let X be a uniformly convex Banach space. Let $\mathcal{T}$ be a $\mathcal{G}$-contraction with a contraction factor $\psi \in (0, 1)$. Suppose that $\{\xi_n\}$, $\{\varrho_n\}$, and $\{\eta_n\}$ are the real sequences in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$, and $(w_0, \tilde{q})$,*

$(\tilde{q}, w_0), (u_0, \tilde{q}), (\tilde{q}, u_0), (v_0, \tilde{q}), (\tilde{q}, v_0) \in E(\mathcal{G})$ *for arbitrary* $w_0, u_0, v_0 \in C$, *and* $\tilde{q} \in \mathcal{F}(\mathcal{T})$. *Then, the sequence* $\{w_n\}$ *generated by M-iteration* (3) *converges faster than Noor iteration* (1) *and SP-iteration* (2).

Proof. First, by the Noor iteration (1),

$$
\begin{aligned}
\|v_{n+1} - \tilde{q}\| &\leq (1 - \xi_n)\|v_n - \tilde{q}\| + \xi_n\|\mathcal{T}t_n - \tilde{q}\| \\
&\leq (1 - \xi_n)\|v_n - \tilde{q}\| + \xi_n\psi\|t_n - \tilde{q}\| \\
&\leq (1 - \xi_n)\|v_n - \tilde{q}\| + \xi_n\psi((1 - \varrho_n)\|v_n - \tilde{q}\| + \varrho_n\psi\|l_n - \tilde{q}\|) \\
&\leq (1 - \xi_n(1 - \psi(1 - \varrho_n) - \varrho_n\psi^2(1 - \eta_n(1 - \psi))))\|v_n - \tilde{q}\| \\
&= (1 - \xi_n(1 - \psi + \varrho_n\psi(1 - \psi(1 - \eta_n(1 - \psi)))))\|v_n - \tilde{q}\| \\
&\leq (1 - \xi_n(1 - \psi))\|v_n - \tilde{q}\| \\
&\vdots \\
&\leq \|v_0 - \tilde{q}\|\prod_{k=0}^{n}(1 - \xi_k(1 - \psi)).
\end{aligned}
$$

Using (3), we have

$$
\begin{aligned}
\|r_n - \tilde{q}\| &= \|(1 - \xi_n)w_n + \xi_n\mathcal{T}w_n - \tilde{q}\| \\
&= \|(1 - \xi_n)(w_n - \tilde{q}) + \xi_n(\mathcal{T}w_n - \tilde{q})\| \\
&\leq (1 - \xi_n)\|w_n - \tilde{q}\| + \xi_n\|\mathcal{T}w_n - \tilde{q}\| \\
&\leq (1 - \xi_n)\|w_n - \tilde{q}\| + \xi_n\psi\|w_n - \tilde{q}\| \\
&= (1 - \xi_n + \psi\xi_n)\|w_n - \tilde{q}\| \\
&= (1 - (1 - \psi)\xi_n)\|w_n - \tilde{q}\|,
\end{aligned}
$$

and so

$$
\|s_n - \tilde{q}\| = \|\mathcal{T}r_n - \tilde{q}\| \leq \psi\|r_n - \tilde{q}\| \leq \psi(1 - (1 - \psi)\xi_n)\|w_n - \tilde{q}\|.
$$

This implies that

$$
\|w_{n+1} - \tilde{q}\| = \|\mathcal{T}s_n - \tilde{q}\| \leq \psi\|s_n - \tilde{q}\| \leq \psi^2(1 - (1 - \psi)\xi_n)\|w_n - \tilde{q}\|.
$$

Repetition of the aforementioned processes gives the following inequality:

$$
\|w_{n+1} - \tilde{q}\| \leq \|w_0 - \tilde{q}\|\psi^{2(n+1)}\prod_{k=0}^{n}(1 - (1 - \psi)\xi_k).
$$

Define

$$
\rho_n := \|w_0 - \tilde{q}\|\psi^{2(n+1)}\prod_{k=0}^{n}(1 - (1 - \psi)\xi_k),
$$

$$
\sigma_n := \|v_0 - \tilde{q}\|\prod_{k=0}^{n}(1 - \xi_k(1 - \psi)),
$$

$$
\theta_n := \frac{\rho_n}{\sigma_n} = \frac{\|w_0 - \tilde{q}\|\psi^{2(n+1)}\prod_{k=0}^{n}(1 - (1 - \psi)\xi_k)}{\|v_0 - \tilde{q}\|\prod_{k=0}^{n}(1 - \xi_k(1 - \psi))}.
$$

Therefore,

$$
\lim_{n \to \infty}\frac{\theta_{n+1}}{\theta_n} = \lim_{n \to \infty}\frac{\psi^2(1 - (1 - \psi)\xi_{n+1})}{(1 - \xi_{n+1}(1 - \psi))} = \psi^2 < 1^2 = 1.
$$

It thus follows from the well-known ratio test that $\sum_{n=0}^{\infty}\theta_n < \infty$. Hence, we have $\lim_{n \to \infty}\theta_n = 0$, which implies that $\{w_n\}$ is faster than $\{v_n\}$. Consequently, the M-iteration (3) converges faster than the Noor iteration (1).

Finally, by the SP-iteration (2),

$$
\begin{aligned}
\|u_{n+1} - \tilde{q}\| &\le (1 - \xi_n)\|q_n - \tilde{q}\| + \xi_n\|\mathcal{T}q_n - \tilde{q}\| \\
&\le (1 - \xi_n)\|q_n - \tilde{q}\| + \xi_n\psi\|q_n - \tilde{q}\| \\
&= (1 - \xi_n(1 - \psi))\|q_n - \tilde{q}\| \\
&\le (1 - \xi_n(1 - \psi))((1 - \varrho_n)\|h_n - \tilde{q}\| + \varrho_n\psi\|h_n - \tilde{q}\|) \\
&= (1 - \xi_n(1 - \psi))(1 - \varrho_n(1 - \psi))\|h_n - \tilde{q}\| \\
&\le (1 - \xi_n(1 - \psi))\|h_n - \tilde{q}\| \\
&\le (1 - \xi_n(1 - \psi))((1 - \eta_n)\|u_n - \tilde{q}\| + \eta_n\psi\|u_n - \tilde{q}\|) \\
&= (1 - \xi_n(1 - \psi))(1 - \eta_n(1 - \psi))\|u_n - \tilde{q}\| \\
&\le (1 - \xi_n(1 - \psi))\|u_n - \tilde{q}\| \\
&\;\;\vdots \\
&\le \|u_0 - \tilde{q}\| \prod_{k=0}^{n} (1 - \xi_k(1 - \psi)).
\end{aligned}
$$

Define

$$
\rho_n := \|w_0 - \tilde{q}\|\psi^{2(n+1)} \prod_{k=0}^{n} (1 - (1 - \psi)\xi_k),
$$

$$
\sigma_n := \|u_0 - \tilde{q}\| \prod_{k=0}^{n} (1 - \xi_k(1 - \psi)),
$$

$$
\theta_n := \frac{\rho_n}{\sigma_n} = \frac{\|w_0 - \tilde{q}\|\psi^{2(n+1)} \prod_{k=0}^{n}(1 - (1 - \psi)\xi_k)}{\|u_0 - \tilde{q}\| \prod_{k=0}^{n}(1 - \xi_k(1 - \psi))}.
$$

By the same argument as earlier, we can show that the sequence generated by the M-iteration (3) converges faster than the SP-iteration (2). This completed the proof. □

Now, we will discuss a numerical experiment that supports our main results.

Example 4.7. Let $X = \mathbb{R}$, $C = [0, 2]$ and $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a directed graph defined by $V(\mathcal{G}) = C$ and $(w, s) \in E(\mathcal{G})$ if and only if $0.50 \le w \ne s \le 1.70$ or $w = s \in C$. In this example, we will present the numerical results of three possible mappings. Define mappings $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : C \to C$, where

$$
\mathcal{T}_1 w = \frac{2}{3}\arcsin(w - 1) + 1, \quad \mathcal{T}_2 w = \frac{1}{3}\tan(w - 1) + 1, \quad \mathcal{T}_3 w = \sqrt{w},
$$

for any $w \in C$. It is easy to show that $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$ are $\mathcal{G}$-nonexpansive but $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$ are not non-expansive because $|\mathcal{T}_1 w - \mathcal{T}_1 s| > 0.50 = |w - s|$, $|\mathcal{T}_2 u - \mathcal{T}_2 v| > 0.07 = |u - v|$, and $|\mathcal{T}_3 p - \mathcal{T}_3 q| > 0.45 = |p - q|$ when $w = 1.95$, $s = 1.45$, $u = 0.08$, $v = 0.01$, $p = 0.5$, and $q = 0.05$. Let
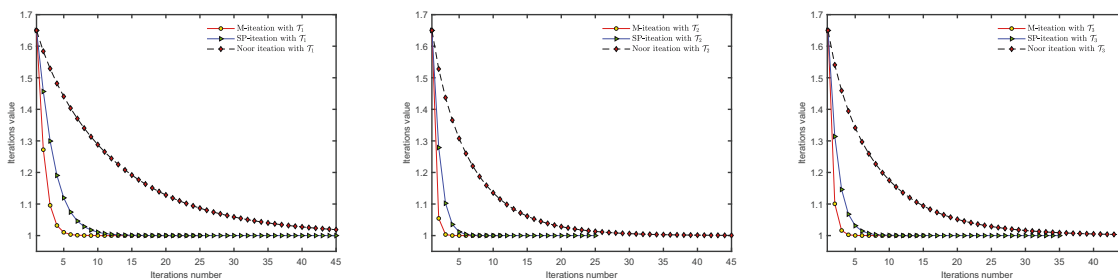

Figure 1: Numerical solution of sequence generated by three comparative methods with operators $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$, respectively.
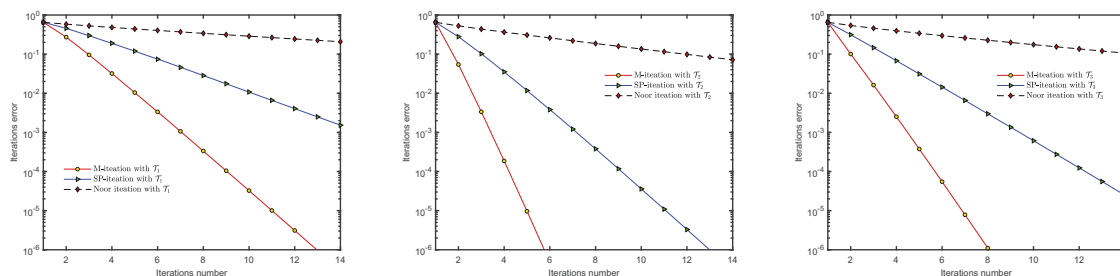
Figure 2: Relative error of sequence generated by three comparative methods with operators $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$, respectively.

$$\xi_n = \frac{n+1}{5n+3}, \varrho_n = \frac{n}{\sqrt{4n^3+2}}, \quad \text{and} \quad \eta_n = \frac{n}{n+1}. \tag{25}$$

Let $\{w_n\}$ be a sequence generated by (3) and $\{s_n\}, \{r_n\}$ be the sequences generated by the three-step Noor iteration (1) and SP-iteration (2), respectively. Example 4.7 shows the convergence behavior of these three comparative methods with the operators $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$. We choose $r_1 = s_1 = w_1 = 1.65$ and set the relative error $|\zeta_n - w|/|w| < 1.00 \times 10^8$ as stopping criterion, with $\{\zeta_n\}$ being all of the comparative sequences.

All numerical experiments for a fixed-point solution of $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$ by using the three-step Noor iteration, SP-iteration and M-iteration are shown in Figure 1–3.

Figures 1 and 2 show the numerical solution and relative error behavior of three comparative methods with operators $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$. It can be seen that all sequences generated by these three methods converge to $w = 1$. The errors of these three comparative methods are also decreased to zero when the number of iterations increased. Figure 3 shows the tendency of the asymptotic error constant σ for sequence $\{\zeta_n\}$ results from the formula $|\zeta_{n+1} - 1|/|\zeta_n - 1|$ of the three-step Noor, SP, and M-iterations. Figure 3 shows that all methods are linearly convergent. This message is being made more confident by using Definition 4.3.

The asymptotic error constants of three comparative methods with the operators $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$ on Figure 3 show that the M-iteration has the smallest asymptotic error constant in all cases. And, the smaller of asymptotic error constant gives us the faster convergence of the considering sequence.

Figure 4 shows that the M-iteration consumes the least amount of time while producing results that are consistent with those obtained earlier. As we evaluate our M-iteration, we observe that by changing only one parameter, we may improve the method's convergence rate. The relative error and asymptotic error constants of the M-iteration impacted by the controlled parameter η_n and operators $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$ are shown in Figures 5 and 6.

We noted from the aforementioned two figures that bringing the parameter η_n closer to 1 enhances the efficiency of our proposed technique for each operator under consideration.
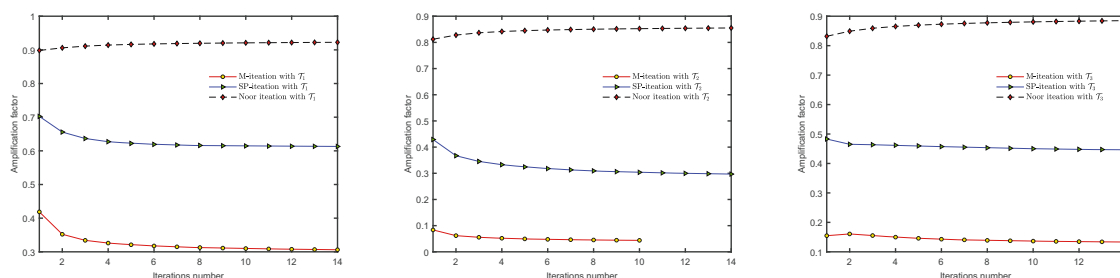

Figure 3: Convergence comparison of sequence generated by three comparative methods with operators with operators $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$, respectively.
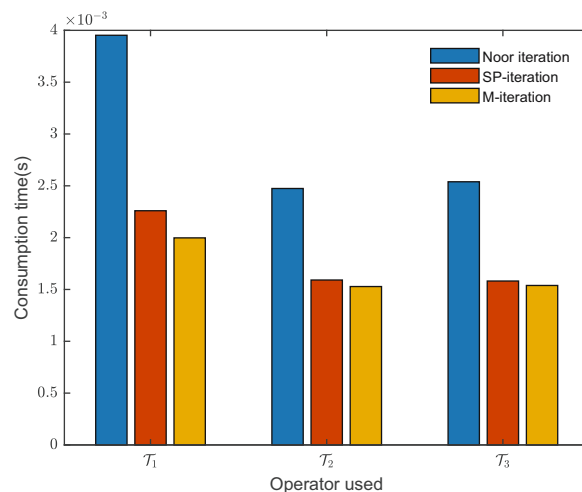
Figure 4: CPU times of sequence generated by three comparative methods with operators $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$, respectively.

5 Simulated results for image deblurring and signal recovering problems

Now, we apply our proposed algorithm to solve image deblurring and signal recovering problems where all codes were written in Matlab and run on laptop Intel core i5, 16.00 GB RAM, windows 10 (64-bit).

The minimization problem of the sum of two functions is to find a solution of

$$\min_{w \in \mathbb{R}^n} \{F(w) := f(w) + h(w)\}, \tag{26}$$

where $h : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is the proper convex and lower semicontinuous function, and $f : \mathbb{R}^n \to \mathbb{R}$ is the convex differentiable function, with gradient ∇f being L-Lipschitz constant for some $L > 0$. The solution of (26) can be characterized by using Fermat's rule, Theorem 16.3 of Bauschke and Combettes [32] as follows:
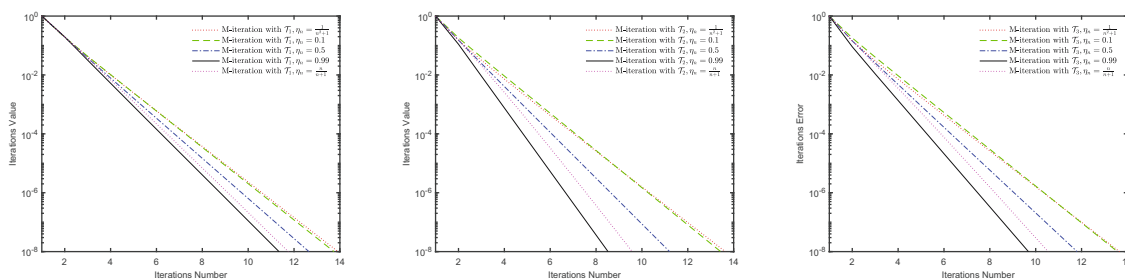

Figure 5: Relative error of sequence generated by the M-iteration impacted with operators $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ and controlled η_n.
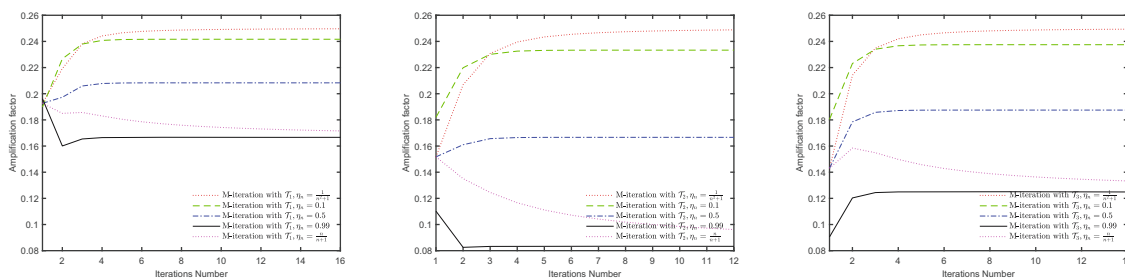

Figure 6: Convergence comparison of sequence generated by the M-iteration impacted with operators $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ and controlled η_n.

$$w^* \text{ is a minimizer of } (f + h) \text{ if and only if } 0 \in \partial h(w^*) + \nabla f(w^*),$$

where ∂h is the subdifferential of h and ∇f is the gradient of f. The subdifferential of h at w^*, denoted by $\partial h(w^*)$, is defined by:

$$\partial h(w^*) \coloneqq \{u : h(w) - h(w^*) \geq \langle u, w - w^* \rangle \text{ for all } w\}.$$

It is also well known that the solution of (26) is characterized by the following fixed-point problem:

$$w^* \text{ is a minimizer of } (f + h) \text{ if and only if } w^* = \text{prox}_{ch}(I - c\nabla f)(w^*),$$

where $c > 0$, prox_h is the proximity operator of h defined by $\text{prox}_h \coloneqq \text{argmin}\{h(s) + \frac{1}{2}\|w - s\|_2^2\}$, (see [33] for more details). It is also known that $\text{prox}_{ch}(I - c\nabla f)$ is a nonexpansive mapping when $c \in (0, \frac{2}{L})$.

5.1 Application to image deblurring problems

Let B be the degraded image of the true image W in the matrix form of $\tilde{m}$ rows and $\tilde{n}$ columns ($B, W \in \mathbb{R}^{\tilde{m} \times \tilde{n}}$). The key to obtaining the image restoration model is to rearrange the elements of the images B and W into the column vectors by stacking the columns of these images into two long vectors $\mathbf{b}$ and $\mathbf{w}$, where $\mathbf{b} = \text{vec}(B)$ and $\mathbf{w} = \text{vec}(W)$, both of length $n = \tilde{m}\tilde{n}$. The image restoration problem can be modeled in one-dimensional vector by the following linear equation system:

$$\mathbf{b} = M\mathbf{w}, \tag{27}$$

where $\mathbf{w} \in \mathbb{R}^n$ is an original image, $\mathbf{b} \in \mathbb{R}^n$ is the observed image, $M \in \mathbb{R}^{n \times n}$ is the blurring operation, and $n = \tilde{m}\tilde{n}$. In order to solve Problem (27), we aim to approximate the original image, vector $\mathbf{b}$, which is known as the following least-squares problem:

$$\min_{\mathbf{w}} \frac{1}{2}\|\mathbf{b} - M\mathbf{w}\|_2^2, \tag{28}$$

where $\|. \|_2$ is defined by $\|\mathbf{w}\|_2 = \sqrt{\sum_{i=1}^{n}|w_i|^2}$. By setting $q(\mathbf{w})$ as equation (28), we will apply our main results for solving the image deblurring problem (27) by setting as follows:

Let $M \in \mathbb{R}^{n \times n}$ be a degraded matrix and $\mathbf{b} \in \mathbb{R}^n$. By applying the M-iteration (3), we obtain the following proposed method to find the common solution of the image deblurring problem:

$$\begin{aligned}
\mathbf{r}_n &= (1 - \eta_n)\mathbf{w}_n + \eta_n(\mathbf{w}_n - \mu M^T(M\mathbf{w}_n - \mathbf{b})), \\
\mathbf{s}_n &= \mathbf{r}_n - \mu M^T(M\mathbf{r}_n - \mathbf{b}), \\
\mathbf{w}_{n+1} &= \mathbf{s}_n - \mu M^T(M\mathbf{s}_n - \mathbf{b}),
\end{aligned} \tag{29}$$

True Image

Figure 7: The original RGB image with matrix size 340 × 354.

where $\mu \in \left(0, \frac{2}{\|M\|_2^2}\right)$ and $\{\eta_n\}$ are the sequences in $[\delta, 1 - \delta]$ for all $n \in \mathbb{N}$ and for some δ in $(0, 1)$. The proposed algorithm (29) is used in solving the image deblurring problem (27) with the default parameter (25) and $\mu = 1/\|M^T M\|_2$. Then, $\{\mathbf{w_n}\}$ converges to its solution.

The goal on image deblurring problem is to find the original image from the observed image without knowing which one is the blurring matrix. However, the blurring matrix M must be known in applying algorithm (29). The original RGB format for color image shown in Figure 7 is used to demonstrate the practicability of the proposed algorithm. The Cauchy and relative errors with the stopping criterion of the proposed algorithms are defined as $\|\mathbf{w_n} - \mathbf{w_{n-1}}\|_\infty < 10^{-7}$ and $\|\mathbf{w_n} - \mathbf{w}\|_\infty/\|\mathbf{w}\|_\infty < 10^{-2}$, respectively. The performance of the comparing algorithms at $\mathbf{w_n}$ on image deblurring process is measured quantitatively by means of the peak signal-to-noise ratio (PSNR) defined by:

$$\text{PSNR}(\mathbf{w_n}) = 20 \log_{10}\left(\frac{255^2}{\text{MSE}}\right),$$

where $\text{MSE} = \|\mathbf{w_n} - \mathbf{w}\|_2^2$ (Figure 8).

Next, we present the restoration of images that have been corrupted by the following blur types:

Type I Gaussian blur of filter size 9×9 with standard deviation $\sigma = 4$ (the original image has been degraded by the blurring matrix M_G).

Type II Out of focus blur (disk) with radius $\hat{r} = 6$ (the original image has been degraded by the blurring matrix M_O).

Type III Motion blur specifying with motion length of 21 pixels (len = 21) and motion orientation $11°$ ($\theta = 11$) (the original image has been degraded by the blurring matrix M_M).

The red-green-blue component represents images W and three different kinds of blurring image B (See Figure 8). Then, we denote W_r, W_g, and W_b, and B_r, B_g, B_b as the gray-scale images that constitute the red-green-blue (RGB) channels of using the image W and the blurring image B, respectively. Thus, we define the column vectors $\mathbf{w} = [\text{vec}(W_r); \text{vec}(W_g); \text{vec}(W_b)]$ and $\mathbf{b} = [\text{vec}(B_r); \text{vec}(B_g); \text{vec}(B_b)]$ from color images W and B and both of length $n = 3\tilde{m}\tilde{n}$. After that, we apply the proposed algorithms in obtaining the solution of deblurring problem with these three blurring matrices.

Figures 9–11 show the reconstructed RGB image using the proposed algorithms in obtaining the solution of the deblurring problem with three blurring matrices M_G, M_O, and M_M for 50th, 1,000th, 20,000th iterations. It can be seen from these figures that the quality of restored image using the proposed algorithms in solving the deblurring problem obtain the quality improvements for the three different types of the degraded images.

Moreover, the behavior of Cauchy error, relative error, and the PSNR of the degraded RGB image by using the proposed algorithms with 100,000th iterations is demonstrated. It is remarkable that the Cauchy and relative error plot of the proposed method is decreased as the number of iterations increased. Thus, the Cauchy and relative error plot shows the validity and confirms the convergence of the proposed methods. Based on the PSNR plots shown in Figure 12, their graphs are also increased as the number of iteration is

Figure 8: The original RGB image degraded by blurred matrices M_G, M_O, and M_M, respectively.

Figure 9: The reconstructed images being 50th, 1,000th, and 20,000th used iterations of the RGB image degraded by blurred matrices M_G.

Figure 10: The reconstructed images being 50th, 1,000th, and 20,000th used iterations of the RGB image degraded by blurred matrices M_O.

Figure 11: The reconstructed images being 50th, 1,000th, and 20,000th used iterations of the RGB image degraded by blurred matrices M_M.

increased. Through these results, it can be concluded that the proposed algorithm produces the quality improvements of the three different types of the original RGB image degraded by the blurring matrices M_G, M_O, and M_M.

5.2 Application to signal recovering problems

In signal processing, compressed sensing can be modeled as the following under the determinated linear equation system:
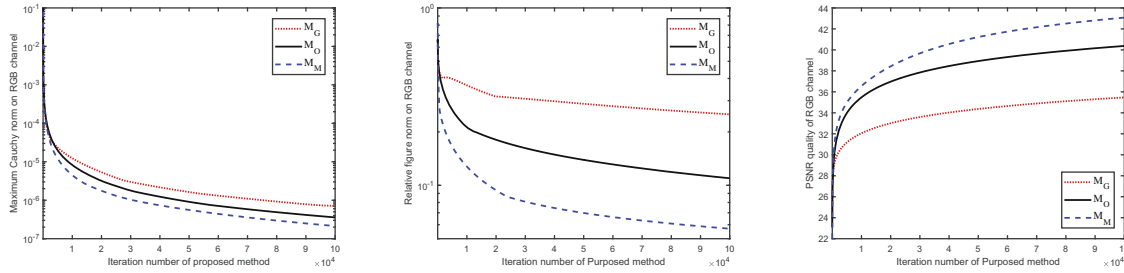
Figure 12: The Cauchy norm, relative figure norm, and PSNR quality plots of the proposed algorithms in all cases of degraded RGB images.

$$\mathbf{y} = A\mathbf{w} + \mathbf{v},$$

where $\mathbf{w} \in \mathbb{R}^n$ is an original signal with n components to be recovered, $\mathbf{v}, \mathbf{y} \in \mathbb{R}^m$ are noise and the observed signal with noisy for m components, respectively, and $A \in \mathbb{R}^{m \times n}$ is a degraded matrix. Finding the solutions of the previously determinated linear equation system can be seen as solving the Lasso problem:

$$\min_{\mathbf{w} \in \mathbb{R}^N} \frac{1}{2}\|\mathbf{y} - A\mathbf{w}\|_2^2 + \lambda\|\mathbf{w}\|_1, \tag{30}$$

where $\lambda > 0$. As a result, various techniques and iterative schemes have been developed to solve the Lasso problem. We can apply the minimization problem of the sum of two functions with our method (3) for solving the Lasso problem (30) by setting $\mathcal{T}\mathbf{w} = \text{prox}_{\lambda g}(\mathbf{w} - \lambda\nabla f(\mathbf{w}))$, where $f(\mathbf{w}) = \frac{1}{2}\|\mathbf{y} - A\mathbf{w}\|_2^2$, $g(\mathbf{w}) = \lambda\|\mathbf{w}\|_1$, and $\nabla f(\mathbf{w}) = A^T(A\mathbf{w} - \mathbf{y})$. And applying the M-iteration (1.3), we obtain the following proposed method to find the solution of the signal recovering problem (30):

$$\begin{aligned}
\mathbf{r}_n &= (1 - \eta_n)\mathbf{w}_n + \eta_n \text{prox}_{\lambda g}(\mathbf{w}_n - \mu A^T(A\mathbf{w}_n - \mathbf{y})), \\
\mathbf{s}_n &= \text{prox}_{\lambda g}(\mathbf{r}_n - \mu A^T(A\mathbf{r}_n - \mathbf{y})), \\
\mathbf{w}_{n+1} &= \text{prox}_{\lambda g}(\mathbf{s}_n - \mu A^T(A\mathbf{s}_n - \mathbf{y})),
\end{aligned} \tag{31}$$

where $\mu \in \left(0, \dfrac{2}{\|A\|_2^2}\right)$ and $\{\eta_n\}$ are the sequences in $[\delta, 1 - \delta]$ for all $n \in \mathbb{N}$ and for some δ in $(0, 1)$.

Next, some experiments are provided to illustrate the convergence and the effectiveness of the proposed algorithm (31). The original signal $\mathbf{w}$ with $n = 1,024$ in Figure 10 generated by the uniform distribution in the interval $[-2, 2]$ with 70 nonzero elements is used to create the observation signal $\mathbf{y}_i = A_i\mathbf{w} + v_i$, $i = 1, 2, 3$ with $m = 512$ (Figure 13).

The observation signal $\mathbf{y}_i$ is shown in Figure 14.

The degraded matrixes A_i that generated by the normal distribution with mean zero and variance one and the white Gaussian noise v_i for all $i = 1, 2, 3$ (Figure 15).

The process is started when the signal initial data $\mathbf{w}_0$, with $n = 1,024$ being picked randomly (Figure 16).

The parameters ξ_n, ϱ_n, and η_n on an implemented algorithm (31) in solving the signal recovering problem are set as equation (25) and $\mu = 1/\|A^TA\|_2$. The Cauchy error and the relative signal error are measured by using max-norm $\|\mathbf{w}_n - \mathbf{w}_{n-1}\|_\infty$ and $\|\mathbf{w}_n - \mathbf{w}\|_\infty/\|\mathbf{w}\|_\infty$, respectively. The performance of the proposed method at nth iteration is measured quantitatively by the means of the signal-to-noise ratio (SNR), which is defined by:

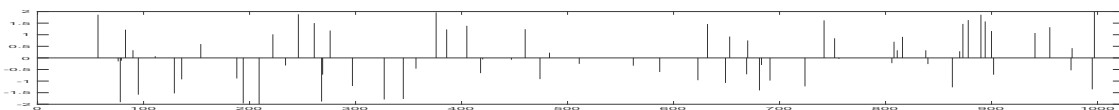$$\text{SNR}(\mathbf{w}_n) = 20\log_{10}\left(\frac{\|\mathbf{w}_n\|_2}{\|\mathbf{w}_n - \mathbf{w}\|_2}\right),$$

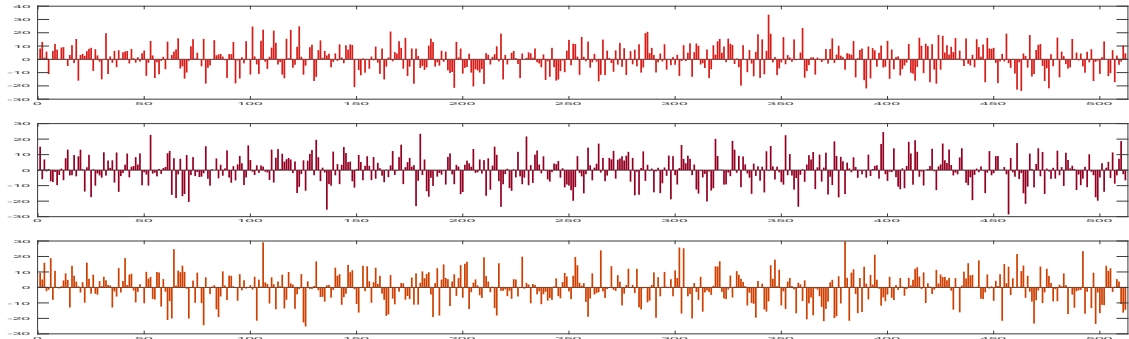**Figure 13:** Original signal (**w**) with 70 nonzero elements.

Figure 14: Degraded signals $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$, respectively.
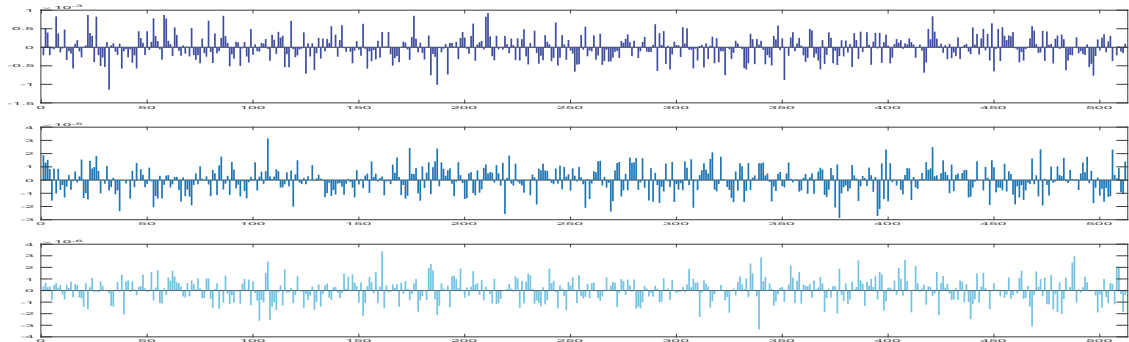

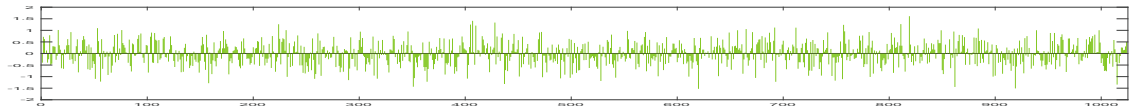**Figure 15:** Noise signals n_1, n_2, and n_3, respectively.
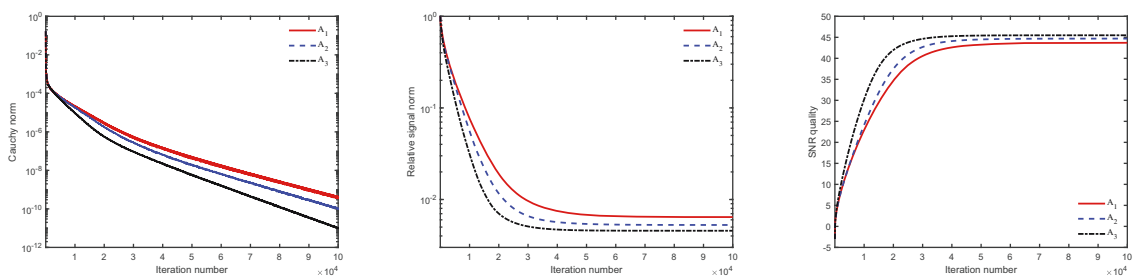

Figure 16: Initial signals w_0.

Figure 17: Cauchy error, signal relative error and SNR quality of the proposed methods in recovering the observed signal with 100,000th.

where $\mathbf{w}_n$ is the recovered signal at nth iteration by using the proposed method. The Cauchy error, signal relative error, and SNR quality of the proposed methods for recovering the degraded signal are shown in Figure 17.

It is remarkable that the Cauchy's error plot of the proposed method decreases as the number of iterations increases. And the signal relative error plot will decrease until it converges to a constant value. Thus, the Cauchy and relative error plots show the validity and confirm the convergence of the proposed methods. For the SNR quality plot, it can be seen that the SNR value increases until it converges to a constant value. Through these results, it can be concluded that the solution of the signal recovering problem solved by the proposed
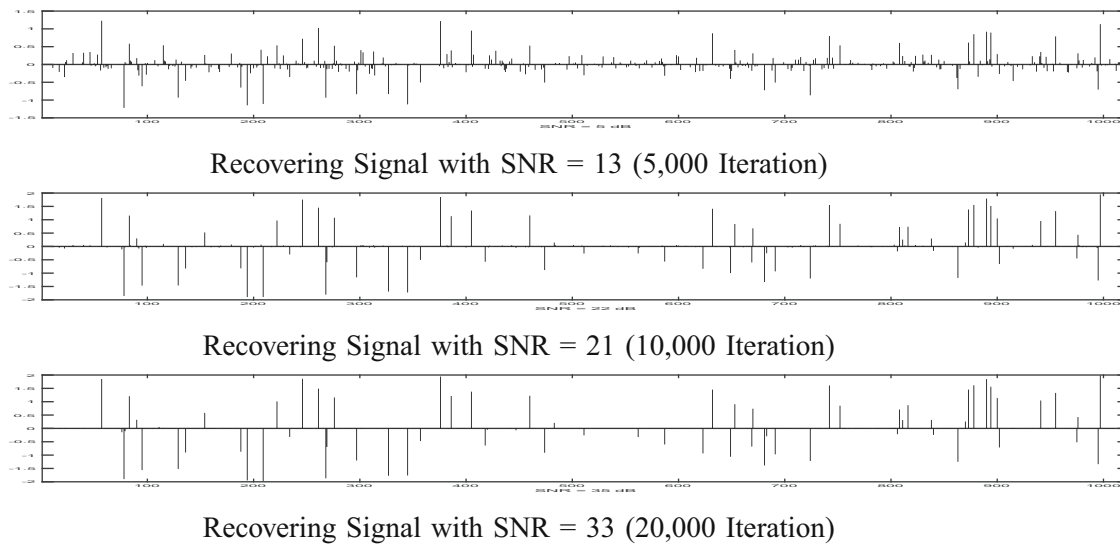
Recovering Signal with SNR = 13 (5,000 Iteration)

Recovering Signal with SNR = 21 (10,000 Iteration)

Recovering Signal with SNR = 33 (20,000 Iteration)

Figure 18: Recovering signals based on SNR quality for the degraded signal with operator A_1 and noise v_1.

Recovering Signal with SNR = 13 (5,000 Iteration)

Recovering Signal with SNR = 23 (10,000 Iteration)
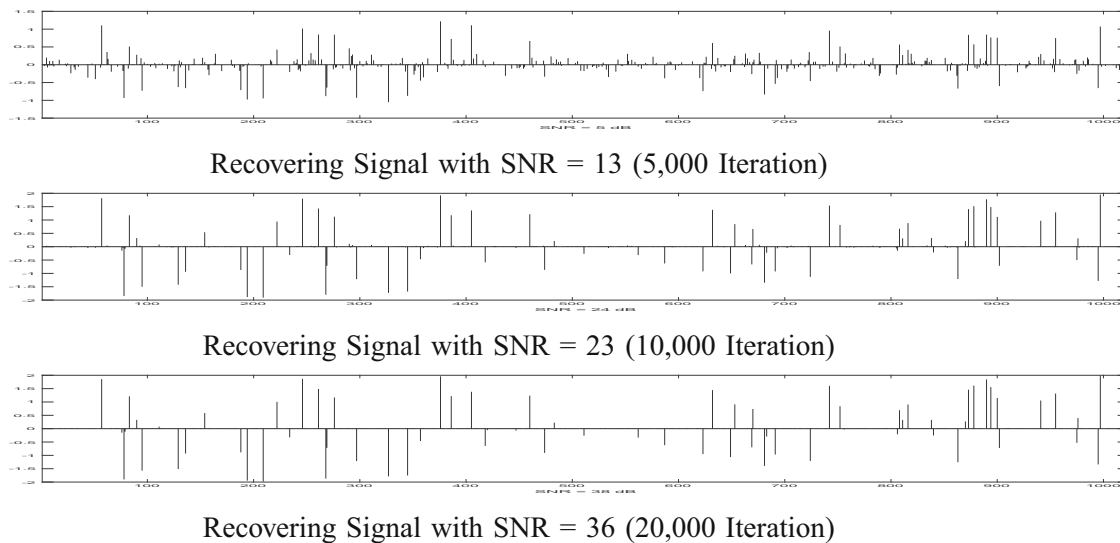

Recovering Signal with SNR = 36 (20,000 Iteration)

Figure 19: Recovering signals based on SNR quality for the degraded signal with operator A_2 and noise v_2.

algorithm gets the quality improvements of the observed signal. Figures 18–20 shows the restored signal by using the proposed algorithms with the group of operator and noise A_i and v_i, $i = 1, 2, 3$.

The improvement of SNR quality for the recovering signals based on 5,000th, 10,000th, and 20,000th number of iterations is also shown in Figures 18–20. It can be seen from these figures that the quality of recovering signal using the proposed algorithms in solving the signal recovering problem gets the quality improvements for the three different types of the degraded signals.

6 Conclusion

In this article, we have proved weak and strong convergence theorems of the M-iteration method for $\mathcal{G}$-nonexpansive mapping in a uniformly convex Banach space with a directed graph. Also, we have proved the weak
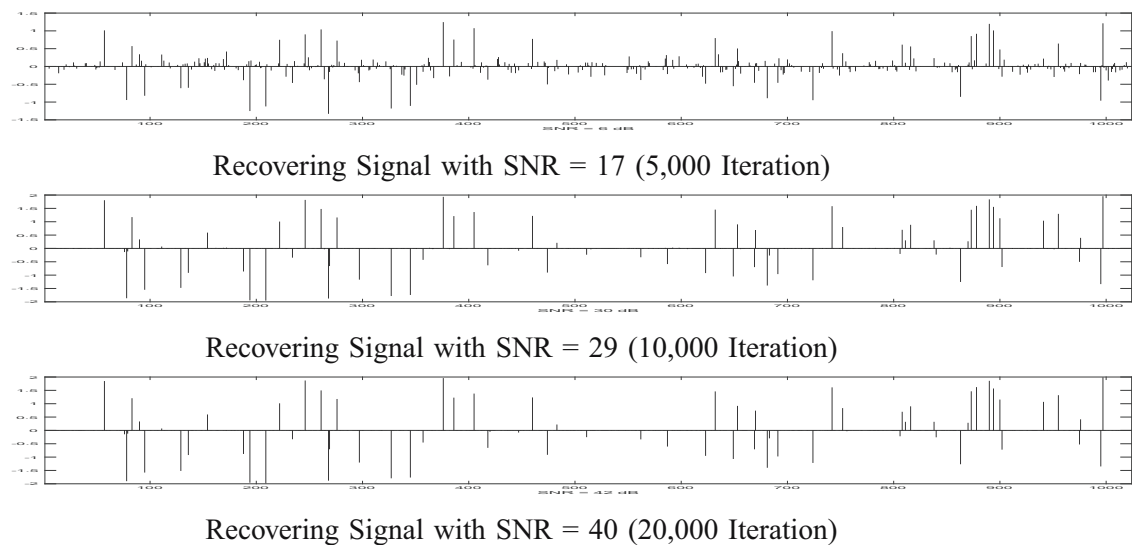
Recovering Signal with SNR = 17 (5,000 Iteration)

Recovering Signal with SNR = 29 (10,000 Iteration)

Recovering Signal with SNR = 40 (20,000 Iteration)

Figure 20: Recovering signals based on SNR quality for the degraded signal with operator A_3 and noise v_3.

convergence theorem without using Opial's condition (see Theorem 3.4). The conditions for convergence of the method are established by systematic proof. The M-iteration algorithm was found to be faster than the Noor and SP iterations for the class $\mathcal{G}$-contraction mappings (see Theorem 4.6). A numerical example illustrating the performance of the suggested algorithm was provided. All numerical experiments for a fixed-point solution by using the M-iteration, the three-step Noor iteration, and the SP-iteration methods with the three operators are shown in Figures 1–4. The M-iteration technique was shown to be more efficient than the three-step Noor iteration and the SP-iteration approaches. As applications, we applied the M-iteration algorithm to solve the image deblurring problems (Figures 7–12). We also apply the M-iteration algorithm for solving signal recovery in situations where the type of noise is unknown (Figures 13–17). We found that the M-iteration algorithm is flexible and has good quality for use with common types of blur and noise effects in image deblurring and signal recovery problems.

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Ethical approval: The conducted research is not related to either human or animal use.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during this study.

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