Research Article

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Fekete-Szegö functional for a class of non-Bazilevic functions related to quasi-subordination

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Abstract: In this article, we study the Fekete-Szegö functional associated with a new class of analytic functions related to the class of bounded turning by using the principle of quasi-subordination. We derived the coefficient estimates including the classical Fekete-Szegö inequality for functions belonging to this class. We also improved some existing results.

Keywords: analytic function, bounded turning, principle of quasi-subordination, Fekete-Szegö functional

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1 Introduction

Let $\mathcal{A}$ be the class of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = 0$ and $f'(0) = 1$. We also denote the class P of functions φ analytic in U, such that

$$\varphi(0) = 1 \text{ and } \Re(\varphi(z)) > 0, \quad (z \in U).$$

For two functions f and g, we say that f is subordinate to g written as $f(z) \prec g(z)$, $(z \in U)$, if there exists a Schwarz function w, analytic in U with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. Further, if the function g is univalent in U, then we have the following equivalence relation:

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$$f(z) \prec g(z) \text{ if and only if } f(0) = g(0) \text{ and } g(U) \supset f(U). \tag{2}$$

For two analytic functions f and g, the function f is quasi-subordinate to the function g (written as: $f(z) \prec_q g(z)$), if there exists an analytic function $\psi(z)$ with $|\psi(z)| \le 1$, such that

$$f(z) = \psi(z)(g(w(z))), \quad (z \in U).$$

The concept of quasi-subordination was given by Robertson [1]. It can be observed that if $\psi(z) \equiv 1 (z \in U)$, then quasi-subordination coincides with usual subordination. For the Schwarz function $w(z)$ given by $w(z) = z$, the quasi-subordination becomes the majorization. In this case:

$$f(z) \prec_q g(z) \ \Rightarrow \ f(z) = \psi(z)g(z) \ \Rightarrow \ f(z) \ll g(z), \quad (z \in U). \tag{3}$$

MacGregor [2] introduced the concept of majorization. For $t \ne 1$ and $|t| \le 1$ and for some α $(0 \le \alpha < 1)$, Owa et al. [3] introduced a Sakaguchi-type class $\mathcal{S}^*(\alpha, t)$ of functions f of form (1), which satisfies the condition:

$$\Re\left(\frac{(1-t)zf'(z)}{f(z)-f(tz)}\right) > \alpha, \quad (z \in U). \tag{4}$$

The class of non-Bazilevic functions was introduced by Obradovic [4] as:

$$\Re\left(f'(z)\left(\frac{z}{f(z)}\right)^{1+\rho}\right) > 0, \quad (z \in U; 0 < \rho < 1). \tag{5}$$

For $z \in U$, $1 \ne t \in \mathbb{C}$, $|t| \le 1$, and $\rho \ge 0$, by using the aforementioned concept of quasi-subordination, Sharma and Raina [5] defined as:

$$\varsigma_q^\rho(\varphi, t) \text{ if such that } \left[f'(z)\left(\frac{(1-t)z}{f(z)-f(tz)}\right)^\rho - 1 \prec_q [\varphi(z)-1]\right]. \tag{6}$$

Definition 1. [6] For $1 \ne t \in \mathbb{C}$, $|t| \le 1$, $\rho \ge 0$, $0 \le \alpha \le 1$, $f \in \mathfrak{I}_q^\rho(\varphi, t, \alpha)$ if and only if

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\left(\frac{(1-t)z}{f(z)-f(tz)}\right)^\rho - 1 \prec_q [\varphi(z)-1]. \tag{7}$$

For different choices of parameters involved in definition (1), we have the following classes of analytic functions:

(i) For $\psi(z) \equiv 1$, the class $\mathfrak{I}_q^\rho(\varphi, t, \alpha)$ becomes class $\mathfrak{I}^\rho(\varphi, t, \alpha)$.
(ii) For $\alpha = 1$, the class $\mathfrak{I}_q^\rho(\varphi, t, \alpha)$ is equivalent to class $\varsigma_q^\rho(\varphi, t)$.
(iii) For $\varphi(z) = \frac{1+z}{1-z}(z \in U)$, the class $\mathfrak{I}_q^\rho(\varphi, t, \alpha)$ reduces to class studied by Nunokwa et al. [7] (see [8–13]).

Motivated by the aforementioned work, we define the following class.

Definition 2. For $z \in U$, $1 \ne t \in \mathbb{C}$; $|t| \le 1$; $\rho \ge 0$; $0 \le \alpha \le 1$, and $f \in \mathfrak{I}_q^\rho(\varphi, s, t, \alpha)$ if and only if

$$(1-\alpha)f'(z) + \alpha f'(z)\left(\frac{(s-t)z}{f(sz)-f(tz)}\right)^\rho - 1 \prec_q [\varphi(z)-1]. \tag{8}$$

Special cases

(i) For $s = 1$ and $\alpha = 1$ in $\mathfrak{I}_q^\rho(\varphi, s, t, \alpha)$, we have the class $\varsigma_q^\rho(\varphi, t)$ studied by Sharma and Raina [5].
(ii) For $\alpha = 0$ and $\rho = 0$ in $\mathfrak{I}_q^\rho(\varphi, s, t, \alpha)$, we have the class $\mathcal{R}_q(\varphi)$, $f'(z) - 1 \prec_q [\varphi(z)-1]$.

The study of functional made up of combinations of the coefficients of the original function is a typical problem in geometric function theory. Initially in 1933, Fekete and Sezgö [14] (see also [15–22]) obtained a sharp bound of the functional $|a_3 - \mu a_2^2|$ for univalent functions $f \in \mathcal{A}$ of form (1) with real μ $(0 \leq \mu \leq 1)$. Since then, the problem of finding the sharp bounds for this functional of any compact family of function $f \in \mathcal{A}$ has with any complex μ is generally known as the classical Fekete-Szegö problem. Many authors have studied the Fekete-Szegö problem for several subclasses of analytic functions in detail [23–40].

The primary goal of this article is to determine the coefficient estimates including Fekete-Szegö inequality the functions belonging the class $\mathfrak{I}_q^\rho(\varphi, s, t, \alpha)$. Improving our results, we use the following lemmas:

Lemma 1. [37] *Let the Schwarz function $w(z)$ be given by*:

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \quad (z \in U), \tag{9}$$

then

$$|w_1| \leq 1 \text{ and } |w_2 - \kappa w_1^2| \leq 1 + (|\kappa| - 1)|w_1|^2 \leq \max\{1, |\kappa|\} \quad (\kappa \in \mathbb{C}). \tag{10}$$

Remark 1.

(i) Let the function $f \in \mathcal{A}$ be of form (1), then it can be observed that

$$\frac{f(sz) - f(tz)}{s - t} = z + \sum_{n=2}^{\infty} \delta_n a_n z^n \quad (z \in U), \tag{11}$$

where

$$\delta_n = \frac{s^n - t^n}{s - t} \quad (n \in \mathbb{N}). \tag{12}$$

Therefore, for $\rho \geq 0$, we find that

$$\left(\frac{(s - t)z}{f(sz) - f(tz)}\right)^\rho - 1 = 1 - \rho\delta_2 a_2 z + \left(\rho\frac{(1 + \rho)}{2}\delta_2^2 a_2^2 - \rho\delta_3 a_3\right)z^2 + \cdots. \tag{13}$$

(ii) We also suppose that the function $\varphi(z) \in P$ is of the form:

$$\varphi(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (c_1 > 0,\ z \in U), \tag{14}$$

and the $\psi(z)$, analytic in U, is of the form:

$$\psi(z) = b_0 + b_1 z + b_2 z^2 + \cdots (z \in U). \tag{15}$$

(iii) Throughout this article, we assume that ρ and t are such that

$$\rho\delta_n \neq n \quad (z \in U) \text{ and } \rho\delta_n < n, \quad (t \in \mathbb{R}, \quad n = 2, 3, \ldots). \tag{16}$$

2 Main results

Theorem 1. *Let the function $f \in \mathcal{A}$ be in the class $\mathfrak{I}_q^\rho(\varphi, s, t, \alpha)$. Then,*

$$|a_2| \leq \frac{c_1}{|2 - \alpha\rho\delta_2|}, \tag{17}$$

and for some $\mu \in \mathbb{C}$

$$|a_3 - \mu a_2^2| \leq \frac{c_1}{|3 - \alpha\rho\delta_2|} \max\left\{1, \left|c_1 L - \frac{c_2}{c_1}\right|\right\}, \tag{18}$$

where

$$L = \frac{\mu(3 - \alpha\rho\delta_3)}{(2 - \alpha\rho\delta_2)^2} - \frac{\alpha\rho(4 - (1 + \rho)\delta_2)\delta_2}{2(2 - \alpha\rho\delta_2)^2}, \tag{19}$$

and $\delta_n (n \in \mathbb{N})$ is given by (12).

Proof. Suppose that the function $f \in \mathfrak{I}_q^{\rho}(\varphi, s, t, \alpha)$ is given by (1). Then, by using the concept of quasi-subordination with Schwarz functions $w(z)$ given by (9) and for the analytic function $\psi(z)$ defined in (15), it follows that

$$(1-\alpha)f'(z) + \alpha f'(z)\left(\frac{(s-t)z}{f(sz)-f(tz)}\right)^{\rho} - 1 = \psi(z)[\varphi(w(z)) - 1]. \tag{20}$$

Moreover, in view of (14), it is easy to see that

$$\begin{aligned}\psi(z)[\varphi(z)-1] &= (b_0 + b_1 z + b_2 z^2 + \cdots)(c_1 w_1 z + (c_1 w_2 + c_2 w_1^2)z^2 + \cdots)\\ &= b_0 c_1 w_1 z + [b_0(c_1 w_2 + c_2 w_1^2) + b_1 c_1 w_1]z^2 + \cdots.\end{aligned} \tag{21}$$

Since $f \in \mathcal{A}$ by using (1) together with (13) and after some computation, we obtain

$$\begin{aligned}&(1-\alpha)f'(z) + \alpha f'(z)\left(\frac{(s-t)z}{f(sz)-f(tz)}\right)^{\rho} - 1\\ &= (2a_2 - \alpha\rho\delta_2 a_2)z + \left(3a_3 + \alpha\rho\left(\left(\frac{1+\rho}{2}\right)\delta_2 - 2\right)\delta_2 a_2^2 - \alpha\rho\delta_3 a_3\right)z^2 + \cdots\end{aligned} \tag{22}$$

Now, by putting the values from (21), (22) in (20) and equating the coefficients of z and z^2, we have

$$a_2(2 - \alpha\rho\delta_2) = b_0 c_1 w_1 \tag{23}$$

and

$$a_3(3 - \alpha\rho\delta_3) - \alpha\rho\left[2 - \frac{(1+\rho)}{2}\delta_2\right]\delta_2 a_2^2 = b_0(c_1 w_2 + c_2 w_1^2) + b_1 c_1 w_1. \tag{24}$$

Since $|b_0| \le 1$ and $|c_1| \le 1$, we find from (23) that

$$|a_2| \le \frac{c_1}{|2 - \alpha\rho\delta_2|}. \tag{25}$$

Now, (24) and (25) yield

$$a_3(3 - \alpha\rho\delta_3) = \alpha\rho\left[2 - \frac{(1+\rho)}{2}\delta_2\right]\delta_2 \frac{b_0^2 c_1^2 w_1^2}{(2-\alpha\rho\delta_2)^2} + b_0(c_1 w_2 + c_2 w_1^2) + b_1 c_1 w_1,$$

which implies that

$$a_3 = \frac{c_1}{(3 - \alpha\rho\delta_3)}\left[b_1 w_1 + b_0\left\{w_2 + \left(\frac{c_2}{c_1} + \frac{\alpha\rho(4 - (1+\rho)\delta_2)b_0 c_1 \delta_2}{2(2-\alpha\rho\delta_2)^2}\right)w_1^2\right\}\right]. \tag{26}$$

Also, for some $\mu \in \mathbb{C}$, we find from (25) and (26) that

$$a_3 - \mu a_2^2 = \frac{c_1}{(3 - \alpha\rho\delta_3)}\left[b_1 w_1 + (w_2 + \frac{c_2}{c_1}w_1^2)b_0 - c_1 L w_1^2 b_0^2\right], \tag{27}$$

where L is given by (19). Since the function ψ defined in (19) is analytic and bounded in U, by using a result recorded in Nehari [38], we have

$$|b_0| \le 1 \text{ and } b_1 = (1 - b_0^2)x^2, \tag{28}$$

for some $x(|x| \le 1)$. Thus, upon substituting the values of b_1 from (28) into (27), we obtain

$$a_3 - \mu a_2^2 = \frac{c_1}{(3 - \alpha\rho\delta_3)}\left[x w_1 + \left(w_2 + \frac{c_2}{c_1}w_1^2\right)b_0 - (c_1 L w_1^2 + x w_1)b_0^2\right]. \tag{29}$$

For $b_0 = 0$ in (29), it follows that

$$|a_3 - \mu a_2^2| \le \frac{c_1}{|3 - \alpha\rho\delta_3|}.$$

For the case $b_0 \neq 0$, we consider

$$g(b_0) = xw_1 + \left(w_2 + \frac{c_2}{c_1}w_1^2\right)b_0 - (c_1Lw_1^2 + xw_1)b_0^2,$$

which is a polynomial in b_0 and is, therefore, analytic with $|b_0| \leq 1$, and the maximum of $g(b_0)$ is attained at $b_0 = e^{i\theta}(0 \leq \theta < 2\pi)$. We find that

$$\max_{0\leq\theta<2\pi} |g(b_0)| = |g(1)| \quad (b_0 = \mathrm{e}^{i\theta})$$

and

$$|a_3 - \mu a_2^2| \leq \frac{c_1}{|3 - \alpha\rho\delta_3|}\left|w_2 - \left(c_1L - \frac{c_2}{c_1}\right)w_1^2\right|. \tag{30}$$

Finally, by using Lemma 1, we have assertion (18). □

It is worth mentioning that for specializing the parameters, we obtain the number of important corollaries as:

Corollary 1. [5] *For $\alpha = 1$, $s = 1$, we have*

$$|a_2| \leq \frac{c_1}{|2 - \rho\delta_2|},$$

and for some $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \frac{c_1}{|3 - \rho\delta_3|}\max\left\{1, \left|c_1L - \frac{c_2}{c_1}\right|\right\},$$

where

$$L = \frac{\mu(3 - \alpha\rho\delta_3)}{(2 - \alpha\rho\delta_2)^2} - \frac{\alpha\rho(4 - (1 + \rho)\delta_2)\delta_2}{2(2 - \alpha\rho\delta_2)^2},$$

and $\delta_n(n \in \mathbb{N})$ is given by (12).

Corollary 2. *For $\alpha = 1$, $s = 1$, and $t = 0$, the class $\mathfrak{I}_q^\rho(\varphi, s, t, \alpha)$ reduces to $\mathcal{S}_q^*(\varphi)$, and we have*

$$|a_2| \leq c_1,$$

and for some $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \frac{c_1}{2}\max\left\{1, \left|\frac{c_2}{c_1} + (1 - 2\mu)c_1\right|\right\}.$$

Corollary 3. *For $\alpha = 0$, the class $\mathfrak{I}_q^\rho(\varphi, s, t, \alpha)$ reduces to class $\mathcal{R}_q(\varphi)$, then*

$$|a_2| \leq \frac{c_1}{2},$$

and for some $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \frac{c_1}{3}\max\left\{1, \left|\frac{c_2}{c_1} - \frac{3\mu}{4}c_1\right|\right\}.$$

Corollary 4. *For $\alpha = \frac{1}{2}$ and $s = 1$, we have*

$$|a_2| \leq \frac{2c_1}{|4 - \rho\delta_2|},$$

and for some $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \frac{2c_1}{|6 - \rho\delta_3|} \max\left\{1, \left|c_1 L - \frac{c_2}{c_1}\right|\right\},$$

where

$$L = \frac{\mu(3 - \alpha\rho\delta_3)}{(2 - \alpha\rho\delta_2)^2} - \frac{\alpha\rho(4 - (1 + \rho)\delta_2)\delta_2}{2(2 - \alpha\rho\delta_2)^2}.$$

Corollary 5. *For* $\psi(z) \equiv 1$,

$$|a_2| \leq \frac{c_1}{|2 - \alpha\rho\delta_2|},$$

and for some $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \frac{c_1}{|3 - \alpha\rho\delta_3|} \max\left\{1, \left|c_1 L - \frac{c_2}{c_1}\right|\right\},$$

where

$$L = \frac{\mu(3 - \alpha\rho\delta_3)}{(2 - \alpha\rho\delta_2)^2} - \frac{\alpha\rho(4 - (1 + \rho)\delta_2)\delta_2}{2(2 - \alpha\rho\delta_2)^2}.$$

Remark 2. For $t = 0$, $\alpha \neq 1$, and $\rho = 1$, we obtain the following result:

$$|a_2| \leq \frac{c_1}{|2 - \alpha|},$$

and for some $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \frac{c_1}{|3 - \alpha|} \max\left\{1, \left|\frac{c_2}{c_1} + \left(\frac{\alpha - (3 - \alpha)\mu}{(2 - \alpha)^2}\right)c_1\right|\right\}.$$

Remark 3. For $t = 0$, $\alpha \neq 1$, and $\rho = 0$, we obtain the following result:

$$|a_2| \leq \frac{c_1}{2},$$

and for some $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \frac{c_1}{3} \max\left\{1, \left|\frac{c_2}{c_1} - \frac{3\mu c_1}{4}\right|\right\}.$$

Theorem 2. *Let* $1 \neq b \in \mathbb{C}$, $|b| \leq 1$, *and* $0 \leq \alpha \leq 1$. *If the function* $f \in \mathfrak{I}_q^\rho(\varphi, s, t, \alpha)$ *is of the form given by* (1) *and satisfies the following majorization condition*:

$$(1 - \alpha)f'(z) + \alpha f'(z)\left(\frac{(s - t)z}{f(sz) - f(tz)}\right)^\rho - 1 \ll [\varphi(z) - 1], \tag{31}$$

then

$$|a_2| \leq \frac{c_1}{|2 - \alpha\rho\delta_2|}.$$

Also for some $\mu \in \mathbb{C}$

$$|a_3 - \mu a_2^2| \leq \frac{c_1}{|3 - \alpha\rho\delta_2|} \max\left\{1, \left|c_1 L - \frac{c_2}{c_1}\right|\right\},$$

where L is defined in (19) *and* δ_n *is defined in* (12). *The result is sharp.*

Proof. By using the concept of majorization and Theorem 1, we have $w(z) \equiv z$ in (9). This implies that

$$w_1 = 1 \text{ and } w_n = 0 \quad (n = 2, 3, \ldots).$$

Thus, if we make use of (25) and (27), we obtain

$$|a_2| \leq \frac{c_1}{|2 - \alpha\rho\delta_2|}$$

and

$$a_3 - \mu a_2^2 = \frac{c_1}{(3 - \alpha\rho\delta_3)}\left[b_1 w_1 + (w_2 + \frac{c_2}{c_1}w_1^2)b_0 - c_1 L w_1^2 b_0^2\right]. \tag{32}$$

Substituting the values of b_1 from (28) into (22), we have

$$a_3 - \mu a_2^2 = \frac{c_1}{(3 - \alpha\rho\delta_3)}\left[x + \frac{c_2}{c_1}b_0 - (c_1 L + x)b_0^2\right]. \tag{33}$$

If $b_0 = 0$, then

$$|a_3 - \mu a_2^2| = \frac{c_1}{|3 - \alpha\rho\delta_3|}. \tag{34}$$

Also for the case $b_0 \neq 0$, we consider

$$H(b_0) = x + \frac{c_2}{c_1}b_0 - (c_1 L + x)b_0^2,$$

which is a polynomial in b_0 and is, therefore, analytic in $|b_0| \leq 1$, and the maximum of $|H(b_0)|$ is attained at $b_0 = e^{i\theta}(0 \leq \theta < 2\pi)$. We thus find that

$$\max_{0 \leq \theta < 2\pi} |H(b_0)| = |H(1)|,$$

and therefore, we have

$$|a_3 - \mu a_2^2| \leq \frac{c_1}{|3 - \alpha\rho\delta_3|}\left|c_1 L - \frac{c_2}{c_1}\right|. \tag{35}$$

Now, for inequality (35) together with (34), we have the result asserted by Theorem 2.

We now find the bounds of the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ when μ and t are the real numbers. We first obtain the following result for the class $\Im_q^\rho(\varphi, s, t, \alpha)$. □

Corollary 6. *Let the function $f \in \Im_q^\rho(\varphi, s, t, \alpha)$ be of form* (1). *Then, for real values μ and b,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{c_1}{|3 - \alpha\rho\delta_3|}\left(\dfrac{c_2}{c_1} + \dfrac{c_1(\rho(4\alpha - \alpha(1+\rho)\delta_2)\delta_2 - 2\mu(3 - \alpha\rho\delta_3))}{2(2 - \alpha\rho\delta_2)^2}\right)(\mu \leq \nu) \\ \dfrac{c_1}{|3 - \alpha\rho\delta_3|}(\nu \leq \mu \leq \nu + 2\gamma) \\ \dfrac{c_1}{|3 - \alpha\rho\delta_3|}\left(\dfrac{c_1(2\mu(3 - \alpha\rho\delta_3)) - (\rho(4\alpha - \alpha(1+\rho)\delta_2)\delta_2) -}{2(2 - \alpha\rho\delta_2)^2} - \dfrac{c_2}{c_1}\right)(\gamma + \nu < \mu < \nu + 2\gamma), \end{cases}$$

where

$$\nu = \frac{\alpha\rho((4 - (1+\rho)\delta_2)\delta_2)}{2(3 - \alpha\rho\delta_3)} - \frac{(2 - \alpha\rho\delta_2)^2}{(3 - \alpha\rho\delta_3)}\left(\frac{1}{c_1} - \frac{c_2}{c_1}\right) \tag{36}$$

and

$$\gamma = \frac{(2 - \alpha\rho\delta_2)^2}{c_1(3 - \alpha\rho\delta_3)}. \tag{37}$$

Proof. For real values of μ and b and by working in a similar way in Theorem 1, we obtain the result asserted by Corollary 6 by taking the following three cases.

$$c_1L - \frac{c_2}{c_1} \leq -1, \quad c_1L - \frac{c_2}{c_1} \leq 1, \text{ and } c_1L - \frac{c_2}{c_1} \geq -1.$$ □

Theorem 3. *Let the function $f(z) \in \mathfrak{I}_q^\rho(\varphi, s, t, \alpha)$, then for real values of μ and b,*

$$|a_3 - \mu a_2^2| + (\mu - \nu)|a_2^2| \leq \frac{c_1}{|3 - \alpha\rho\delta_3|} \quad (\nu < \mu \leq \nu + \gamma) \tag{38}$$

and

$$|a_3 - \mu a_2^2| + (\nu + 2\gamma - \mu)|a_2^2| \leq \frac{c_1}{|3 - \alpha\rho\delta_3|} \quad (\gamma + \nu < \mu < \nu + 2\gamma), \tag{39}$$

where ν and γ are given by (36) *and* (37), *respectively, and δ_n is given by* (12).

Proof. Suppose that the function $f \in \mathfrak{I}_q^\rho(\varphi, s, t, \alpha)$ be in the form is given by (1). If $\nu < \mu \leq \nu + \gamma$, then by using (25) and (30), we have

$$\begin{aligned}|a_3 - \mu a_2^2| + (\mu - \nu)|a_2^2| &\leq \frac{c_1}{|3 - \alpha\rho\delta_3|}\left[|w_2| - \frac{c_1(3 - \alpha\rho\delta_3)}{(2 - \alpha\rho\delta_2)^2}(\mu - \nu - \gamma)|w_1^2| + \frac{c_1(3 - \alpha\rho\delta_3)}{(2 - \alpha\rho\delta_2)^2}(\mu - \nu)|w_1^2|\right],\\ &\leq \frac{c_1}{|3 - \alpha\rho\delta_3|}\left(|w_2| + \frac{c_1(3 - \alpha\rho\delta_3)}{(2 - \alpha\rho\delta_2)^2}\gamma|w_1^2|\right),\\ &\leq \frac{c_1}{|3 - \alpha\rho\delta_3|}(|w_2| - (-1)|w_1^2|).\end{aligned}$$

Hence, by applying Lemma 1, we obtain

$$|a_3 - \mu a_2^2| + (\mu - \nu)|a_2^2| \leq \frac{c_1}{|3 - \alpha\rho\delta_3|}.$$

If $\gamma + \nu < \mu < \nu + 2\gamma$, then we again make use of (25) and (30), in conjunction with Lemma 1, thus we obtain

$$\begin{aligned}|a_3 - \mu a_2^2| + (\nu + 2\gamma - \mu)|a_2^2| \leq \frac{c_1}{|3 - \alpha\rho\delta_3|}\Bigg[&|w_2| - \frac{c_1(3 - \alpha\rho\delta_3)}{(2 - \alpha\rho\delta_2)^2}(\mu - \nu - \gamma)|w_1^2|\\ &+ \frac{c_1(3 - \alpha\rho\delta_3)}{(2 - \alpha\rho\delta_2)^2}(\nu + 2\gamma - \mu)|w_1^2|\Bigg],\end{aligned}$$

so that, by using Lemma 1, we obtain

$$|a_3 - \mu a_2^2| + (\nu + 2\gamma - \mu)|a_2^2| \leq \frac{c_1}{|3 - \alpha\rho\delta_3|}.$$

Hence, we have established the result asserted by Theorem 3. □

3 Conclusion

Using the concept of quasi-subordination, we have systematically studied coefficient estimates and Fekete-Szegö functional for a new class of analytic functions related to non-Bazilevic functions and bounded turning. We also pointed out many new and already existing corollaries by assigning by specializing the parameters.

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