Research Article

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Analytical and numerical analysis of damped harmonic oscillator model with nonlocal operators

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Abstract: Nonlocal operators with different kernels were used here to obtain more general harmonic oscillator models. Power law, exponential decay, and the generalized Mittag-Leffler kernels with Delta-Dirac property have been utilized in this process. The aim of this study was to introduce into the damped harmonic oscillator model nonlocalities associated with these mentioned kernels and see the effect of each one of them when computing the Bode diagram obtained from the Laplace and the Sumudu transform. For each case, we applied both the Laplace and the Sumudu transform to obtain a solution in a complex space. For each case, we obtained the Bode diagram and the phase diagram for different values of fractional orders. We presented a detailed analysis of uniqueness and an exact solution and used numerical approximation to obtain a numerical solution.

Keywords: damped harmonic oscillator, nonlocal operators, Bode diagram, existence and uniqueness of the solutions, numerical methods

MSC 2020: 42B37, 34A12, 26A33, 65R10

1 Introduction

Generally, in applied physics, oscillatory movement can be interpreted as a rhythmic periodic response; there are several types of these movements in nature. In particular, oscillation with damping or damped oscillation is one that fades away with time. Several types have been studied in the last few decades by prominent researchers, and some classifications have been done [1,2]. For example, the harmonic oscillator has been recognized as a subclass of oscillatory movement with application in several real-world problems. We shall note that this class has been studied in particular in a classical mechanic where it is known to be a system that, when displaced from its steady state, recovered with a restoring force that is proportional to the displacement made. More importantly, when a frictional force proportionate to the velocity is involved in the model, the system is viewed as a simple harmonic oscillator [3,4]. There are two subclasses, including undamped, which is the case where the system oscillates with a lower frequency with a decrease in amplitude with time, and the case where the system decay to a steady state without oscillations; this case is referred to as overdamped. There are several physical problems displaying these behaviors, for example, a swinging pendulum; several investigations have been done to better understand this system and the resistor, inductor, and capacitor circuit

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system where the resistor, inductor, and capacitor are put in series [5–7]. These behaviors are described using the concept of differentiation. Within the framework of mathematics and applied mathematics, several concepts of differentiation have been suggested including classical differentiation based on the rate of change, fractional differentiation based on the convolution of the rate of change, exponential decay kernel with Delta-Dirac properties, differentiation based on convolution between the generalized Mittag-Leffler function and the classical derivative, and finally the differentiation based on power law kernel [8–12]. The last three were introduced to capture nonlocal behaviors. In simple cases, analytical methods can be used to derive their exact solutions, and one of the most used methods is the one based on integral transformation like Laplace, Sumudu transform, and others. The first integral transformation is comparable; however, it was documented that the Sumudu transformation has more important properties than the Laplace transformation. An important note is that a solution obtained by the Laplace transform in complex space can be used to obtain the Bode diagram, which has been recognized as a powerful tool for identifying whether the system is a high-pass filter or a low-pass filter [13–16]. It was suggested in an innovative work that such analysis could be done using the Sumudu transformation. In this article, we shall undertake different analyses. One of the most well-known issues in mechanics is the damped harmonic oscillator. It illustrates how a mechanical oscillator, such as a spring pendulum, moves when friction and a restoring force are present. While this issue has received attention from several authors, efforts have also been made in the context of nonlocal operators [17,18]. Noting that because of the characteristics of the kernels, these operators frequently contain nonlocal processes in mathematical equations. The power law-based differential operator case, the exponential decay function case, and the generalized Mittag-Leffler function case will all be taken into consideration. They were used because the model will incorporate processes like power laws, memory fading, and transitions from exponential decay to power laws.

2 Damped harmonic oscillation with power law process

We start this work by considering a model of damped harmonic oscillation where the time differential operator is replaced by the well-known Caputo fractional derivative. The aim of this is to include in the mathematical model the effect of the power law that will probably have a great impact on the Bode diagram.

$$ {}_0^C D_t^{\alpha} x(t) + 2\zeta\omega_0 {}_0^C D_t^{\beta} x(t) + \omega_0^2 x(t) = 0, \tag{1}$$

where

$$ {}_0^C D_t^{\alpha} x(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t x''(\tau)(t-\tau)^{1-\alpha}\mathrm{d}\tau, \tag{2}$$

$$ {}_0^C D_t^{\beta} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t x'(\tau)(t-\tau)^{-\alpha}\mathrm{d}\tau. \tag{3}$$

We first aim to obtain the transfer function associated with the above equation using the Sumudu and the Laplace transform.

We recall that

$$ \mathcal{L}({}_0^C D_t^{\alpha} x(t)) = \mathcal{L}\left(\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} \times \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right), \tag{4}$$

$$ \mathcal{L}({}_0^C D_t^{\alpha} x(t)) = \mathcal{L}\left(\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t}\right) \cdot \mathcal{L}\left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right), \tag{5}$$

$$ \mathcal{L}({}_0^C D_t^{\alpha} x(t)) = s^{\alpha}\widetilde{x}(s) - x(0)s^{\alpha-1} - s^{\alpha}x'(0^+), \tag{6}$$

$$ \mathcal{L}({}_0^C D_t^{\beta} x(t)) = s^{\beta}\widetilde{x}(s) - x(0)s^{\beta-1}. \tag{7}$$

Replacing the above into our equation yields

$$s^{\alpha}\tilde{x}(s) - s^{\alpha}x'(0^{+}) - x(0^{+})s^{\alpha-1} + 2\zeta\omega_0(\Delta^{\beta}\tilde{x}(S) - s^{\beta-1}x(0^{+})) + \omega_0^2\tilde{x}(s) = 0. \tag{8}$$

The above equation is rearranged as follows:

$$\tilde{x}(s)\{s^{\alpha} + 2\zeta\omega_0 s^{\beta} + \omega_0^2\} = s^{\alpha}x'(0) + x(0)s^{\alpha-1} + 2\zeta\omega_0 s^{\alpha-1}x(0^{+}), \tag{9}$$

$$\tilde{x}(s) = \frac{s^2x'(0) + S^{\alpha+1}x(0) + 2\zeta\omega_0 s^{\beta-1}x(0)}{s^{\alpha} + 2\zeta\omega_0 s^{\beta} + \omega_0^2}. \tag{10}$$

In the same line of idea, we have

$$S_u({}_0^C D_t^{\alpha}x(t)) = S_u\left(\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}\right)S_u\left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right), \tag{11}$$

$$S_u({}_0^C D_t^{\beta}x(t)) = S_u\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)S_u\left(\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right). \tag{12}$$

Replacing the above into our original equation yields

$$\tilde{x}(u) = \frac{u^{1-\alpha}x'(0) + x(0)(u^{-\alpha} + 2\zeta\omega_0 u^{-\beta})}{u^{-\alpha} + 2\zeta\omega_0 u^{-\beta} + \omega_0^2}. \tag{13}$$

We now have two solutions in frequency space and aim to determine the Bode diagram. We should note, however, that the Laplace transform and the Sumudu transform of a function $f(t)$ that satisfies all the requirements of the Laplace and the Sumudu transform are given (Figure 1).

The Laplace transform of a function f is given as follows:

$$\mathcal{L}(f)(s) = \int_0^{\infty} f(t)\exp(-st)\mathrm{d}t.$$

The Sumudu transform is given as follows:

$$S(f)(u) = \frac{1}{u}\int_0^{\infty} f(t)\exp\left(-\frac{t}{u}\right)\mathrm{d}t.$$

Figure 1 shows that the Bode and the phase diagram depend on the fractional order, and this is correct because each fractional order describes a specific memory. In Figure 2, we present the Bode and phase diagrams obtained from the Sumudu transform.

It is worth mentioning that the transfer function obtained from Sumudu provides a different Bode and phase diagram. At this state, we will be unable to provide a clear conclusion as to what transform present a better Bode and phase diagram. A clear conclusion will be drawn only through the experimental process.

While it is possible after serious manipulations to obtain the inverse Laplace transform of the $\tilde{x}(s)$ and the inverse Sumudu transform of $\tilde{x}(u)$, we will, however, avoid this road by choosing an accurate numerical scheme to provide a numerical solution to our equation. We will first show that the equation has a unique solution.

We defined the following mapping:

$$\Gamma x(t) = -\frac{1}{\omega_0^2}\{{}_0^C D_t^{\alpha}x(t) + 2\zeta\omega_0{}_0^C D_t^{\beta}x(t)\}. \tag{14}$$

We defined the following norm:

$$\|\varphi\|_{\infty} = \sup_{t\in D\varphi}|\varphi(t)|. \tag{15}$$

We aim to verify that

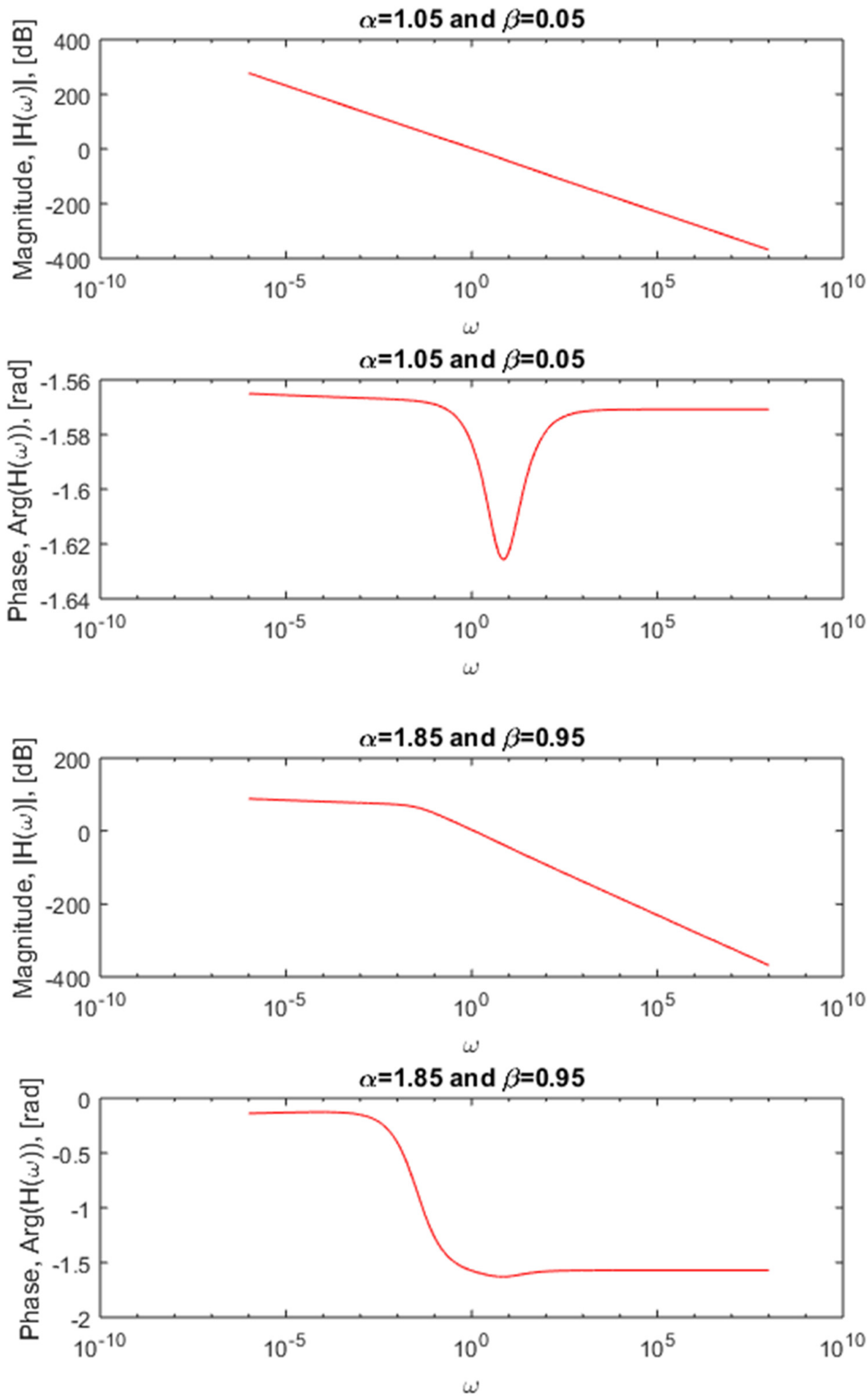

Figure 1: Diagrams for different alpha and beta.

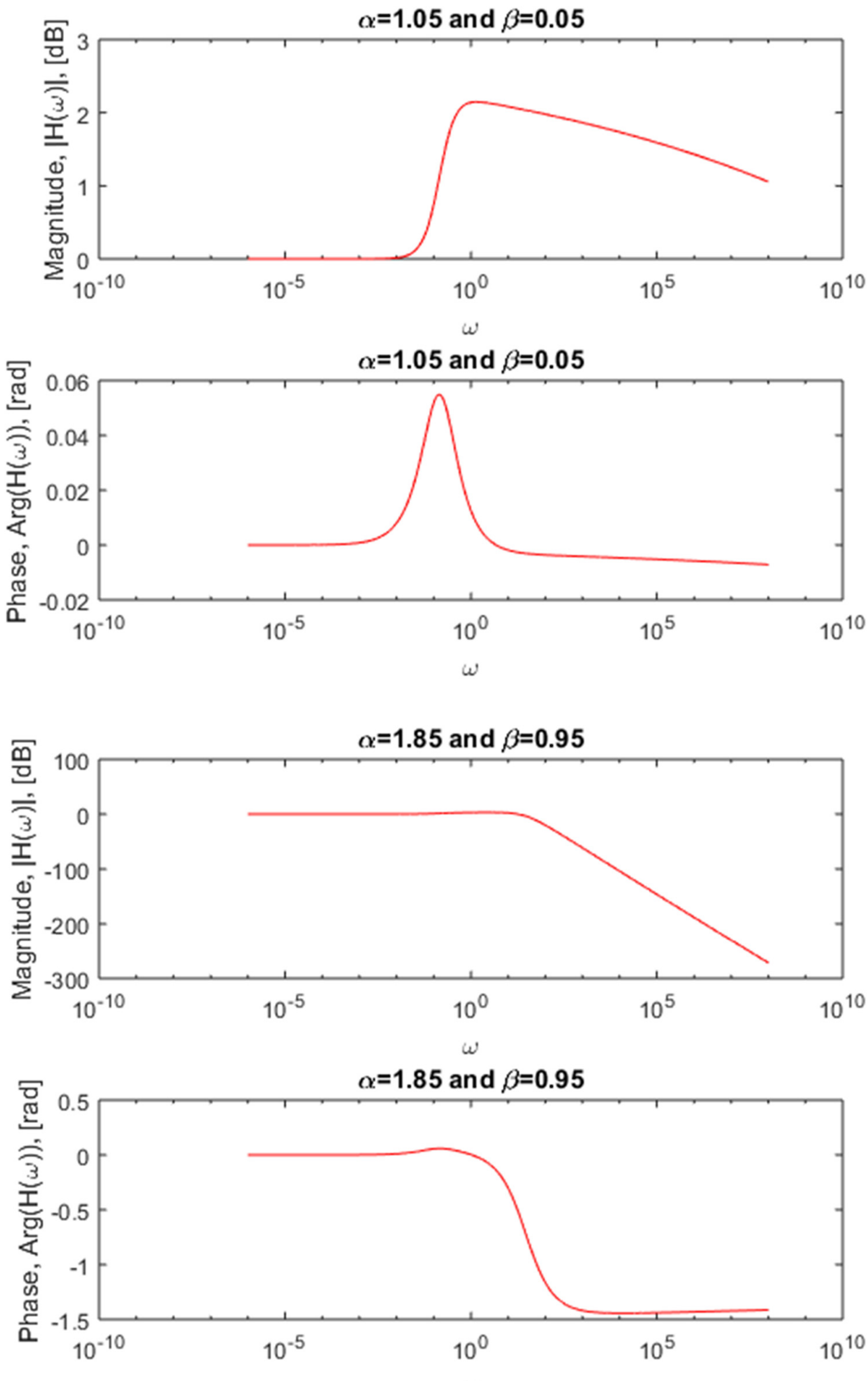

Figure 2: The Bode and phase diagrams obtained from the Sumudu transform.

$$||\Gamma x(t)||_\infty^2 < (1 + ||x||_\infty^2)K, \tag{16}$$

$$||\Gamma x(t) - \Gamma y(t)||_\infty^2 < \bar{K}||x - y||_\infty^2, \tag{17}$$

$$\begin{aligned} |\Gamma x(t)|^2 &= \frac{1}{\omega_0^4}|\{{}_0^C D_t^\alpha x(t) + 2\zeta\omega_0{}_0^C D_t^\beta x(t)\}|^2 < \frac{1}{\omega_0^4}\{2|{}_0^C D_t^\alpha x(t)|^2 + 4\zeta^2\omega_0^2|{}_0^C D_t^\beta x(t)|^2 \\ &< \frac{1}{\omega_0^4}\left\{2\left|\frac{1}{\Gamma(2-\alpha)}\int_0^t x''(\tau)(t-\tau)^{1-\alpha}\mathrm{d}\tau\right|^2 + 4\zeta^2\omega_0^2\left|\frac{1}{\Gamma(1-\beta)}\int_0^t x'(\tau)(t-\tau)^{-\beta}\mathrm{d}\tau\right|^2.\right. \end{aligned} \tag{18}$$

We shall use the Hölder inequality to proceed as follows:

$$\begin{aligned} |\Gamma x(t)|^2 &< \frac{1}{\omega_0^4}\left\{\frac{2t^{4-2\alpha}}{(2-\alpha)\Gamma(2-\alpha)}\left|\int_0^t |x''(\tau)|^2\mathrm{d}\tau\right|^2 + \delta\zeta^2\omega_0^2\frac{t^{2-2\beta}}{\Gamma(1-\beta)(1-\beta)}\left|\int_0^t x'(\tau)\mathrm{d}\tau\right|^2\right. \\ &< \frac{1}{\omega_0^4}\left\{\frac{2T^{4-2\alpha}}{\Gamma(3-\alpha)}\right\{|x'(t)-x'(0)|^2 + \delta\zeta^2\omega_0^2\frac{T^{2-2\beta}}{\Gamma(2-\beta)}|x(t)-x(0)|^2. \end{aligned} \tag{19}$$

Since $x'(t)$ is differentiable, there exists $c \in [0, t]$ such that by the mean value theorem

$$x''(c)(t-0) = x'(t) - x'(0). \tag{20}$$

Therefore,

$$|\Gamma(x(t))^2| < \frac{1}{\omega_0^4}\left\{\frac{2T^{4-2\alpha}}{\Gamma(3-\alpha)}|x''(c)|^2 t + \delta\zeta^2\omega_0^2\frac{T^{2-2\beta}}{\Gamma(2-\beta)}\{2|x(t)|^2 + 2|x(0)|^2\}\right\}, \tag{21}$$

$$\sup_{t\in(0,T)}|\Gamma(x(t))^2| < \frac{1}{\omega_0^4}\left\{\frac{2T^{5-2\alpha}}{\Gamma(3-\alpha)}|x''(c)|^2 + \frac{16\zeta^2\omega_0^2 T^{2-2\beta}}{\omega_0^2\Gamma(2-\beta)}|x(0)|^2\right\} + \frac{16\zeta^2\omega_0^2 T^{2-2\beta}}{\omega_0^4\Gamma(2-\beta)}\sup_{t\in(0,T)}|x(t)|^2. \tag{22}$$

Therefore,

$$||\Gamma(x(t))||_\infty^2 < K(1 + ||x||_\infty^2). \tag{23}$$

Since

$$||\Gamma(x(t))||_\infty^2 < \frac{1}{\omega_0^4}\left\{\frac{2T^{5-2\alpha}}{\Gamma(3-\alpha)}||x''(c)||_\infty^2 + \frac{16\zeta^2\omega_0^2 T^{2-2\beta}}{\omega_0^2\Gamma(2-\beta)}|x(0)|^2\right\}\left\{1 + \frac{\frac{16\zeta^2 T^{2-2\beta}}{\omega_0^2\Gamma(2-\beta)}||x||_\infty^2}{\frac{1}{\omega_0^4}\left\{\frac{2T^{5-2\alpha}}{\Gamma(3-\alpha)}||x''(c)||_\infty^2 + \frac{16\zeta^2\omega_0^2 T^{2-2\beta}}{\Gamma(2-\beta)}|x(0)|^2\right\}}\right\}. \tag{24}$$

If

$$\frac{\frac{16\zeta^2 T^{2-2\beta}}{\omega_0^2\Gamma(2-\beta)}}{\frac{1}{\omega_0^4}\left\{\frac{2T^{3-\alpha}}{\Gamma(3-\alpha)}||x''(c)||_\infty^2 + \frac{16\zeta^2\omega_0^2 T^{2-2\beta}}{\Gamma(2-\beta)}|x(0)|^2\right\}} < 1, \tag{25}$$

then

$$K = \frac{1}{\omega_0^4}\left\{\frac{2T^{5-2\alpha}}{\Gamma(3-\alpha)}||x''(c)||_\infty^2 + \frac{16\zeta^2\omega_0^2 T^{2-2\beta}}{\Gamma(2-\beta)}|x(0)|^2.\right. \tag{26}$$

We shall now evaluate

$$|\Gamma x(t) - \Gamma y(t)|^2 = \left|\frac{1}{\omega_0^2}\{{}_0^C D_t^\alpha x(t) + 2\zeta\omega_0{}_0^C D_t^\beta x(t) - {}_0^C D_t^\alpha y(t) - 2\zeta\omega_0{}_0^C D_t^\beta y(t)\}\right|^2. \tag{27}$$

The linearity property of the Caputo derivative leads us to

$$|\Gamma(x(t)) - \Gamma(y(t))|^2 = \frac{1}{\omega_0^4}|{}_0^C D_t^\alpha (x-y)(t) + 2\zeta\omega_0 {}_0^C D_t^\beta (x-y)(t)|^2 < \frac{2}{\omega_0^4}\{|{}_0^C D_t^\alpha (x-y)(t)|^2 + 2\zeta^2\omega_0^2|{}_0^C D_t^\beta (x-y)(t)|^2\} < \frac{2}{\omega_0^4}\left\{\left|\frac{1}{\Gamma(2-\alpha)}\int_0^t (x-y)''(\tau)(t-\tau)^{1-\alpha}\mathrm{d}\tau\right|^2 + 2\zeta^2\omega_0^2\left|\frac{1}{\Gamma(1-\alpha)}\int_0^t (x-y)'(\tau)(t-\tau)^{-\alpha}\mathrm{d}\tau\right|^2\right\} < \frac{2}{\omega_0^4}\left\{\frac{T^{4-2\alpha}}{\Gamma(3-\alpha)}|x'-y'|^2 + \frac{4\zeta^2\omega_0^2 T^{2-2\beta}}{\Gamma(2-\beta)}|x-y|^2.\right. \tag{28}$$

We have, of course, removed the contribution of initial conditions. We will need the following condition:

$$\|x'\|_\infty^2 < M_1 \quad \text{and} \quad \|y'\|_\infty^2 < M_2. \tag{29}$$

Then

$$\sup_{t\in D_y n D_x} |\Gamma(x(t)) - \Gamma(y(t))|^2 < \frac{4T^{4-2\alpha}}{\omega_0^4\Gamma(3-\alpha)}\frac{(M_1+M_2)}{\|x-y\|_\infty^2}\|x-y\|_\infty^2 + \frac{4\zeta^2\omega_0^2T^{2-2\beta}}{\Gamma(2-\beta)}\|x-y\|_\infty^2\|\Gamma(x)-\Gamma(y)\|_\infty^2 < \left\{\frac{4T^{4-2\alpha}}{\omega_0^4\Gamma(3-\alpha)}\frac{(M_1+M_2)}{\|x-y\|_\infty^2} + \frac{4\zeta^2\omega_0^2T^{2-2\beta}}{\Gamma(2-\beta)}\right\}\|x-y\|_\infty^2 < \bar{K}\|x-y\|_\infty^2, \tag{30}$$

where

$$\bar{K} = \frac{4T^{4-2\alpha}}{\omega_0^4\Gamma(3-\alpha)}\frac{(M_1+M_2)}{\|x-y\|_\infty^2} + \frac{4\zeta^2\omega_0^2T^{2-2\beta}}{\Gamma(2-\beta)}. \tag{31}$$

Under these conditions, the equation admits a unique solution.

We can now provide a numerical solution to our equation.

We have that

$$x(t) = -\frac{1}{\omega_0^2}\{{}_0^C D_t^\alpha x(t) + 2\zeta\omega_0 {}_0^C D_t^\beta x(t)\}. \tag{32}$$

We consider $[0, T]$ and $0 < t_1 < t_2 < \ldots < t_{n+1}$.

At $t_{n+1} = t$, we have

$$x(t_{n+1}) = -\frac{1}{\omega_0^2}\left\{\frac{1}{\Gamma(2-\alpha)}\int_0^{t_{n+1}}\frac{d^2x(\tau)}{d\tau}(t_{n+1}-\tau)^{1-\alpha}d\tau + \frac{2\zeta\omega_0}{\Gamma(1-\beta)}\int_0^{t_{n+1}}\frac{dx(\tau)}{d\tau}(t_{n+1}-\tau)^{-\beta}\mathrm{d}\tau\right\}, \tag{33}$$

$$x(t_{n+1}) \approx x_{n+1} = -\frac{1}{\omega_0^2}\left\{\frac{1}{\Gamma(2-\alpha)}\sum_{j=1}^{n}\int_{t_j}^{t_{j+1}}(x_{j+1}-2x_j+x_{j-1})\frac{(t_{n+1}-\tau)^{1-\alpha}}{(\Delta t)^2}\mathrm{d}\tau + \frac{2\zeta\omega_0}{\Gamma(1-\beta)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}(x_{j+1}-x_j)\frac{(t_{n+1}-\tau)^{-\beta}}{(\Delta t)}\mathrm{d}\tau,\right. \tag{34}$$

$$x_{n+1} = -\frac{1}{\omega_0^2}\left\{\frac{1}{\Gamma(2-\alpha)}\sum_{j=1}^{n}\frac{(x_{j+1}-2x_j+x_{j-1})}{(\Delta t)^2}\left\{\frac{(t_{n+1}-t_j)^{2-\alpha}}{2-\alpha} - \frac{(t_{n+1}-t_{j+1})^{2-\alpha}}{2-\alpha}\right\} + \frac{2\zeta\omega_0}{\Gamma(1-\beta)}\sum_{j=0}^{n}\frac{x_{j+1}-x_j}{\Delta t}\left\{\frac{(t_{n+1}-t_j)^{1-\beta}}{1-\beta} - \frac{(t_{n+1}-t_{j+1})^{1-\beta}}{1-\beta}\right\},\right. \tag{35}$$

$$x_{n+1} = -\frac{1}{\omega_0^2}\left\{\frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=1}^{n}(x_{j+1} - 2x_j + x_{j-1})\delta_{1,n}^{\alpha} + \frac{2\zeta\omega_0\,\Delta t^{-\beta}}{\Gamma(2-\beta)}\sum_{j=0}^{n}(x_{j+1} - x_j)\delta_{2,n}^{\beta}\right\}, \tag{36}$$

where

$$\delta_{1,n}^{\alpha} = (n-j+1)^{2-\alpha} - (n-j)^{2-\alpha}, \tag{37}$$

$$\delta_{2,n}^{\beta} = (n-j+1)^{1-\beta} - (n-j)^{1-\beta}. \tag{38}$$

The above iteration will be used for simulation purposes.

3 Damped harmonic oscillation with Caputo-Fabrizio derivative

In this section, we provide a detailed analysis of the same model, where the time classical derivative is replaced by the Caputo-Fabrizio derivative.

$${}_0^{\mathrm{CF}}D_t^{\alpha}x(t) + 2\zeta\omega_0\,{}_0^{\mathrm{CF}}D_t^{\beta}x(t) + \omega_0^2 x(t) = 0, \tag{39}$$

where

$${}_0^{\mathrm{CF}}D_t^{\alpha}x(t) = \frac{1}{2-\alpha}\int_0^t x''(\tau)\exp\left[-\frac{\alpha}{2-\alpha}(t-\tau)\right]\mathrm{d}\tau, \tag{40}$$

$${}_0^{\mathrm{CF}}D_t^{\beta}x(t) = \frac{1}{1-\beta}\int_0^t x'(\tau)\exp\left[-\frac{\beta}{1-\beta}(t-\tau)\right]\mathrm{d}\tau. \tag{41}$$

Replacing into the original equation and then applying the Laplace transform yield

$$\mathcal{L}({}_0^{\mathrm{CF}}D_t^{\alpha}x(t)) = \frac{1}{2-\alpha}\mathcal{L}(x''(\tau))\mathcal{L}\left(\exp\left[-\frac{\alpha}{2-\alpha}(t-\tau)\right]\right), \tag{42}$$

$$\mathcal{L}({}_0^{\mathrm{CF}}D_t^{\alpha}x(t)) = \frac{1}{2-\alpha}\cdot\frac{2-\alpha}{\alpha + s(2-\alpha)}(\tilde{x}(s)s^2 - sx(0)x'(0)), \tag{43}$$

$$\mathcal{L}({}_0^{\mathrm{CF}}D_t^{\alpha}x(t)) = \frac{s^2\tilde{x}(\Delta)}{\alpha + s(2-\alpha)} - \frac{sx(0)}{\alpha + s(2-\alpha)} - \frac{x'(0)}{\alpha + s(2-\alpha)}, \tag{44}$$

$$\mathcal{L}({}_0^{\mathrm{CF}}D_t^{\beta}x(t)) = \frac{1}{1-\beta}(s\tilde{x}(s) - x(0)) \quad . \frac{1-\beta}{\beta + s(1-\beta)}, \tag{45}$$

Replacing the original equation and rearranging, we obtain

$$\tilde{x}(s) = \frac{x(0)\left(\frac{2\zeta\omega_0}{\beta + s(1-\beta)} + \frac{s}{\alpha(2-\alpha)+\alpha} + \frac{x'(0)}{\alpha + s(2-\alpha)}\right)}{\frac{s^2}{\alpha + s(2-\alpha)} + \frac{2\zeta\omega_0 . s}{\beta + s(1-\beta)} + \omega_0^2}. \tag{46}$$

We apply the same technique with the Sumudu transform and obtain

$$\tilde{x}(u) = \left\{x(0)\left\{\frac{2\zeta\omega_0}{\beta u + 1 - \beta} + \frac{1}{u(2-\alpha+\alpha u)}\right\} + \frac{x'(0)}{2-\alpha+\alpha u}\right\}\frac{1}{\frac{1}{u(2-\alpha+\alpha u)} + \frac{2\zeta\omega_0}{\beta u + 1 - \beta} + \omega_0^2}. \tag{47}$$

We present below the Bode and the phase diagram obtained from the Laplace transform (Figures 3 and 4). We present the conditions under which this model has a unique solution.
To achieve this, we transform the model into an integral equation as

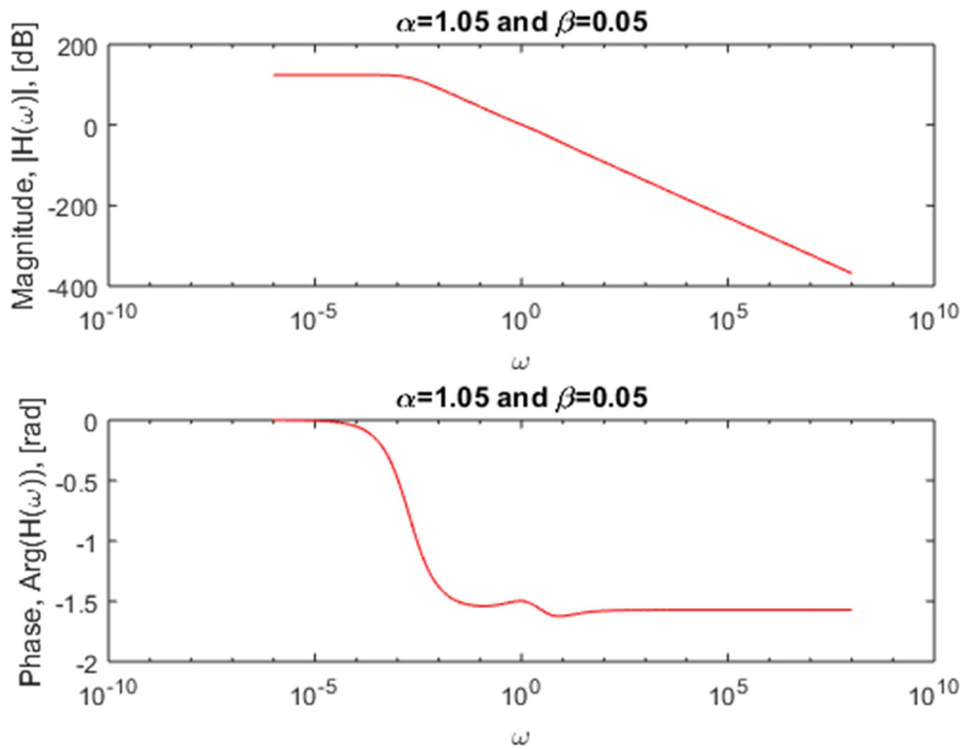

Figure 3: Bode and phase diagrams with Caputo-Fabrizio derivative.

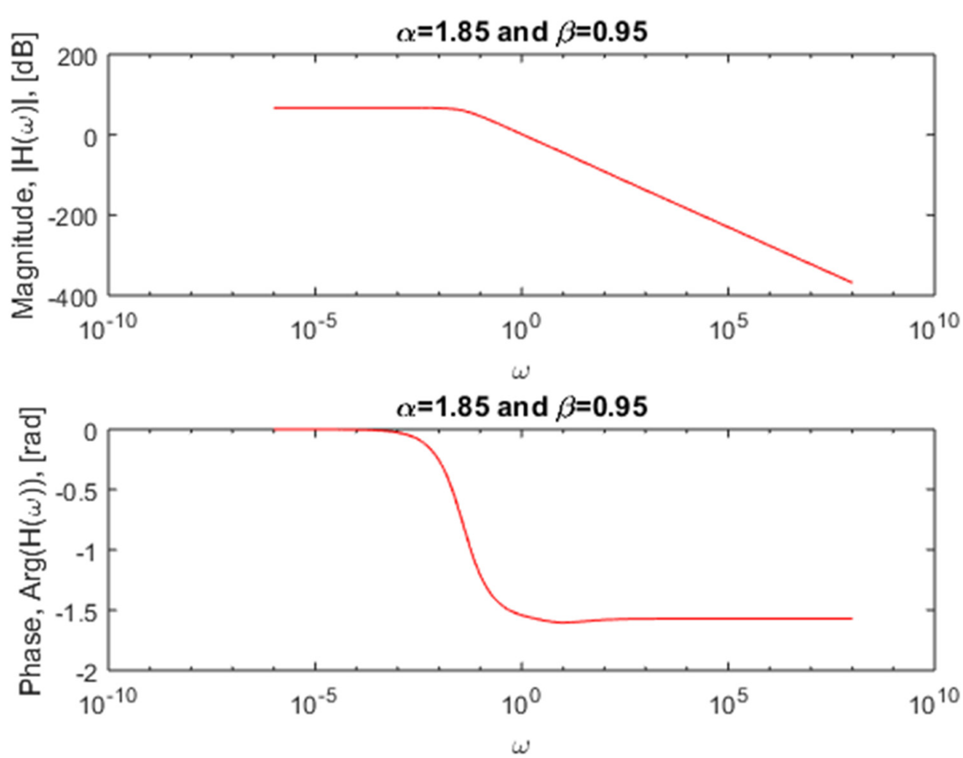

Figure 4: Bode and phase diagrams with Caputo-Fabrizio derivative.

$$x(t) = -\frac{1}{\omega_0^2}\left\{\frac{1}{2-\alpha}\int_0^t x''(\tau)\exp\left[-\frac{\alpha}{2-\alpha}(t-\tau)\right]\mathrm{d}\tau + \frac{2\zeta\omega_0}{1-\beta}\int_0^t x'(\tau)\exp\left[-\frac{\beta}{1-\beta}(t-\tau)\right]\mathrm{d}\tau\right\}. \tag{48}$$

We can now define

$$\Lambda x(t) = -\frac{1}{\omega_0^2}\left\{\frac{1}{2-\alpha}\int_0^t x''(\tau)\exp\left[-\frac{\alpha}{2-\alpha}(t-\tau)\right]\mathrm{d}\tau + \frac{2\zeta\omega_0}{1-\beta}\int_0^t x'(\tau)\exp\left[-\frac{\beta}{1-\beta}(t-\tau)\right]\mathrm{d}\tau\right\}. \tag{49}$$

We aim to evaluate

$$|\Lambda x(t)|^2 \quad \text{and} \quad |\Lambda x(t) - \Lambda y(t)|^2. \tag{50}$$

$$|\Lambda x(t)|^2 = \frac{1}{\omega_0^4}\left|\frac{1}{2-\alpha}\int_0^t x''(\tau)\exp\left[-\frac{\alpha}{2-\alpha}(t-\tau)\right]\mathrm{d}\tau + \frac{2\zeta\omega_0}{1-\beta}\int_0^t x'(\tau)\exp\left[-\frac{\beta}{1-\beta}(t-\tau)\right]\mathrm{d}\tau\right|^2. \tag{51}$$

Due to the decline of the function $\exp\left[-\frac{\alpha}{2-\alpha}(t-\tau)\right]$, we have that

$$\forall_0 < \tau \le t, \quad \exp\left[-\frac{\alpha}{2-\alpha}(t-\tau)\right] \le 1.$$

Therefore,

$$\begin{aligned}
|\Lambda x(t)|^2 &< \frac{1}{\omega_0^4}\left|\frac{1}{(2-\alpha)}\int_0^t x''(\tau)\mathrm{d}\tau + \frac{2\zeta\omega_0}{1-\beta}\int_0^t x'(\tau)\mathrm{d}\tau\right|^2 < \frac{1}{\omega_0^4}\left|\frac{1}{2-\alpha}(x'(t)-x(0)) + \frac{2\zeta\omega_0}{1-\beta}(x(t)) - x(0)\right|^2 \\
&< \frac{1}{\omega_0^4}\left|\frac{1}{2-\alpha}x''(c)t + \frac{2\zeta\omega_0}{1-\beta}x(t) - \frac{2\zeta\omega_0 x(0)}{1-\beta}\right|^2 < \frac{1}{\omega_0^4}\left\{\frac{3T|x''(c)|^2}{(2-\alpha)^2} + \frac{6\zeta^2\omega_0^2|x(0)|^2}{(1-\beta)^2} + \frac{6\zeta^2\omega_0^2|x(t)|^2}{1-\beta}\right. \\
&< \frac{1}{\omega_0^4}\left\{-\frac{2}{(2-\alpha)^2}\frac{2-\alpha}{2\alpha}\exp\left[-\frac{2\alpha}{2-\alpha}(t-\tau)\right]\right|_0^t\{|x''(c)|^2 t \\
&< \frac{1}{\omega_0^4}\left[\frac{2}{(2-\alpha)^2}\frac{2-\alpha}{2\alpha}\left(\exp\left[-\frac{2\alpha}{2-\alpha}t\right]+1\right)\right]|x(c)|_\infty^2 T \\
&< \frac{1}{\omega_0^4}\frac{1}{(2-\alpha)\alpha}\left(1-\exp\left[-\frac{2\alpha}{2-\alpha}t\right]\right)|x''(c)|_\infty^2 T.
\end{aligned} \tag{52}$$

$$\sup_{0\le t\le T}|\Lambda x(t)|^2 < \frac{3T|x''(c)|^2}{\omega_0^4(2-\alpha)^2} + \frac{6\zeta^2|x(0)|^2}{\omega_0^2(1-\beta)^2} + \frac{6\zeta^2\sup_{t\in D_{x(t)}}|x(t)|^2}{\omega_0^2(1-\beta)^2}, \tag{53}$$

$$||\Lambda x(t)||_\infty^2 < \left(\frac{3T|x''(c)|^2}{\omega_0^4(2-\alpha)^2} + \frac{6\zeta^2|x(0)|^2}{\omega_0^2(1-\beta)^2}\right)\left(1 + \frac{6\zeta^2}{\omega_0^2(1-\beta)^2}\cdot \frac{||x||_\infty^2}{\frac{3T|x''(c)|^2}{\omega_0^2(2-\alpha)^2} + \frac{6\zeta^2|x(0)|^2}{\omega_0^2(1-\beta)^2}}\right). \tag{54}$$

We need

$$\frac{6\zeta^2}{\omega_0^2(1-\beta)^2} \cdot \frac{1}{\frac{3T|x''(c)|^2}{\omega_0^2(2-\alpha)^2} + \frac{6\zeta^2|x(0)|^2}{\omega_0^2(1-\beta)^2}} < 1. \tag{55}$$

Therefore,

$$||\Lambda x||_\infty^2 < K(1 + ||x||_\infty^2), \tag{56}$$

where

$$K = \frac{3T|x''(c)|^2}{\omega_0^2(2-\alpha)^2} + \frac{6\zeta^2|x(0)|^2}{\omega_0^2(1-\beta)^2}, \tag{57}$$

$$
\begin{aligned}
|\Lambda x(t)-\Lambda y(t)|^2 &= \frac{1}{\omega_0^4}|{}_0^{\mathrm{CF}}D_t^{\alpha}(x-y)(t)+2\zeta\omega_0{}_0^{\mathrm{CF}}D_t^{\beta}(x-y)(t)|^2\\
&< \frac{1}{\omega_0^4}\{2|{}_0^{\mathrm{CF}}D_t^{\alpha}(x-y)(t)|^2+4\zeta^2\omega_0^2|{}_0^{\mathrm{CF}}D_t^{\beta}(x-y)(t)|^2\}\\
&< \frac{1}{\omega_0^4}\left\{\frac{2}{(2-\beta)^2}\left|\int_0^t (x-y)''(\tau)\mathrm{d}\tau\right|^2+4\zeta^2\omega_0^2\left|\int_0^t (x-y)'(\tau)\mathrm{d}\tau\right|^2\right\}\\
&< \frac{1}{\omega_0^4}\left\{\frac{2}{(2-\beta)^2}\{|(x'-y')(t)-x'(0)+y'(0)|^2+4\zeta^2\omega_0^2|(x-y)(t)-x(0)+y(0)|^2\right\}.
\end{aligned}
\tag{58}
$$

By the mean value theorem, we have

$$
|\Lambda x(t)-\Lambda y(t)|^2 < \frac{2}{(2-\beta)^2\omega_0^2}|(x-y)''(c)|^2T^2+\frac{8\zeta^2}{\omega_0^2}|x-y|^2+\frac{8\zeta^2}{\omega_0^2}|x(0)-y(0)|^2,
\tag{59}
$$

$$
\begin{aligned}
\sup_{t\in D_y n D_x}|\Lambda x(t)-\Lambda y(t)|^2 &< \frac{2}{(2-\beta)^2\omega_0^4}|(x-y)''(c)|^2T^2+\frac{8\zeta^2}{\omega_0^2}\sup_{t\in D_y n D_x}|(x-y)(t)|^2+\frac{8\zeta^2}{\omega_0^2}|(x-y)(0)|^2\\
&< \left\{\frac{\frac{2}{(2-\beta)^2\omega_0^4}|(x-y)''(c)|^2T^2+\frac{8\zeta^2}{\omega_0^2}|(x-y)(0)|^2}{\|x-y\|_\infty^2}\right\}\|x-y\|_\infty^2+\frac{8\zeta^2}{\omega_0^2}\|x-y\|_\infty^2\\
&< \bar{K}\|x-y\|_\infty^2.
\end{aligned}
\tag{60}
$$

That is,

$$
\|\Lambda x-\Lambda y\|_\infty^2 < K\|x-y\|_\infty^2,
\tag{61}
$$

where

$$
\bar{K}=\frac{\frac{2}{(2-\beta)^2\omega_0^2}|(x-y)''(c)|^2T^2+\frac{8\zeta^2}{\omega_0^2}|(x-y)(0)|^2}{\|x-y\|_\infty^2}+\frac{8\zeta^2}{\omega_0^2}.
\tag{62}
$$

Under these conditions, we can conclude that the model has a unique solution. We now present a numerical solution to the model.

$$
x(t)=-\frac{1}{\omega_0^2}\{{}_0^{\mathrm{CF}}D_t^{\alpha}(x)(t)+2\zeta\omega_0{}_0^{\mathrm{CF}}D_t^{\beta}(x)(t)\}.
\tag{63}
$$

At $t=t_{n+1}$,

$$
x(t_{n+1})=-\frac{1}{\omega_0^2}\{{}_0^{\mathrm{CF}}D_{t_{n+1}}^{\alpha}(x)(t)+2\zeta\omega_0{}_0^{\mathrm{CF}}D_{t_{n+1}}^{\beta}(x)(t)\},
\tag{64}
$$

$$
=-\frac{1}{\omega_0^2}\left\{\frac{1}{2-\alpha}\int_0^{t_{n+1}}x''(\tau)\exp\left[-\frac{\alpha}{2-\alpha}(t_{n+1}-\tau)\right]\mathrm{d}\tau+\frac{2\zeta\omega_0}{1-\beta}\int_0^{t_{n+1}}x'(\tau)\exp\left[-\frac{\beta}{1-\beta}(t_{n+1}-\tau)\right]\mathrm{d}\tau\right\},
\tag{65}
$$

$$
\begin{aligned}
\approx -\frac{1}{\omega_0^2}&\left\{\frac{1}{2-\alpha}\sum_{j=1}^{n}\frac{x_{j+1}-2x_j+x_{j-1}}{\Delta t^2}\int_{t_j}^{t_{j+1}}\exp\left[-\frac{\alpha}{2-\alpha}(t_{n+1}-\tau)\right]\mathrm{d}\tau\right.\\
&\left.+\frac{2\zeta\omega_0}{1-\beta}\sum_{j=0}^{n}\frac{x_{j+1}-x_j}{\Delta t}\int_{t_j}^{t_{j+1}}\exp\left[-\frac{\beta}{1-\beta}(t_{n+1}-\tau)\right]\mathrm{d}\tau\right\}.
\end{aligned}
\tag{66}
$$

$$
x(t_{n+1})\approx x_{n+1}=-\frac{1}{\omega_0^2}\left\{\frac{1}{\alpha}\sum_{j=1}^{n}\frac{x_{j+1}-2x_j+x_{j-1}}{\Delta t^2}\delta_{1,n}^{j,\alpha}+2\zeta\omega_0\frac{1}{\beta}\sum_{j=0}^{n}\frac{x_{j+1}-x_j}{\Delta t}\delta_{2,n}^{j,\beta}\right\},
\tag{67}
$$

where

$$\delta_{1,n}^{j,\alpha} = \exp\left[-\frac{\alpha}{2-\alpha}(t_{n+1}-t_{j+1})\right] - \exp\left[-\frac{\alpha}{2-\alpha}(t_{n+1}-t_j)\right], \tag{68}$$

$$\delta_{2,n}^{j,\beta} = \exp\left[-\frac{\beta}{1-\beta}(t_{n+1}-t_{j+1})\right] - \exp\left[-\frac{\beta}{1-\beta}(t_{n+1}-t_j)\right]. \tag{69}$$

The above iterative formula will be used for simulation purposes.

4 Damped harmonic oscillation with ABC derivative

Finally, we consider a damped harmonic oscillation model where the classical derivative is replaced by ABC fractional derivative.

$$^{ABC}_{\ \ 0}D_t^{\alpha}x(t) + 2\zeta\omega_0\, {}^{ABC}_{\ \ 0}D_t^{\beta}x(t) + \omega_0^2 x(t) = 0, \tag{70}$$

where

$$^{ABC}_{\ \ 0}D_t^{\alpha}x(t) = \frac{1}{2-\alpha}\int_0^t x''(\tau)E_\alpha\left[-\frac{\alpha}{2-\alpha}(t-\tau)^{\alpha}\right]\mathrm{d}\tau, \tag{71}$$

$$^{ABC}_{\ \ 0}D_t^{\beta}x(t) = \frac{1}{1-\beta}\int_0^t x'(\tau)E_\beta\left[-\frac{\beta}{1-\beta}(t-\tau)^{\beta}\right]\mathrm{d}\tau. \tag{72}$$

As done previously, we present the Laplace transform of the equation:

$$\mathcal{L}({}^{ABC}_{\ \ 0}D_t^{\alpha}x(t)) = \frac{s^{\alpha-1}}{\alpha+(2-\alpha)s^{\alpha}}[s^2\tilde{x}(s) - sx(0) - x'(0)], \tag{73}$$

$$\mathcal{L}({}^{ABC}_{\ \ 0}D_t^{\beta}x(t)) = \frac{s^{\beta-1}}{\beta+(1-\beta)s^{\beta}}(s\tilde{x}(s) - x(0)), \tag{74}$$

$$\frac{s^{\alpha+1}\tilde{x}(s)}{\alpha+(2-\alpha)s^{\alpha}} + \frac{s^{\alpha}x(0)}{\alpha+(2-\alpha)s^{\alpha}} - \frac{s^{\alpha-1}x'(0)}{\alpha+(2-\alpha)s^{\alpha}} + \frac{2\zeta\omega_0 s^{\beta-1}\tilde{x}(s)}{s^{\beta}(1-\beta)+\beta} - \frac{2\zeta\omega_0 s^{\beta-1}x(0)}{s^{\beta}(1-\beta)+\beta} + \omega_0^2 x'(s) = 0, \tag{75}$$

$$\tilde{x}(s)\left\{\frac{s^{\alpha+1}}{\alpha+(2-\alpha)s^{\alpha}} + \frac{2\zeta\omega_0 s^{\beta}}{s^{\beta}(1-\beta)+\beta} + \omega_0^2\right\} = x(0)\left\{\frac{s^{\alpha}}{\alpha+(2-\alpha)s^{\alpha}} + \frac{2\zeta\omega_0 s^{\beta-1}}{s^{\beta}(1-\beta)+\beta}\right\} + \frac{s^{\alpha-1}}{\alpha+(2-\alpha)s^{\alpha}}x'(0), \tag{76}$$

$$\tilde{x}(s) = \frac{\frac{s^{\alpha}}{\alpha+(2-\alpha)s^{\alpha}} + \frac{2\zeta\omega_0 s^{\beta-1}}{s^{\beta}(1-\beta)+\beta}x(0)}{\frac{s^{\alpha+1}}{\alpha+(2-\alpha)s^{\alpha}} + \frac{2\zeta\omega_0 s^{\beta-1}}{s^{\beta}(1-\beta)+\beta} + \omega_0^2} + \frac{\frac{s^{\alpha-1}x'(0)}{\alpha+(2-\alpha)s^{\alpha}}}{\frac{s^{\alpha+1}}{\alpha+(2-\alpha)s^{\alpha}} + \frac{2\zeta\omega_0 s^{\beta-1}}{s^{\beta}(1-\beta)+\beta} + \omega_0^2}. \tag{77}$$

We apply the Sumudu transform to obtain

$$S[{}^{ABC}_{\ \ 0}D_t^{\alpha}x(t)] + 2\zeta\omega_0 S[{}^{ABC}_{\ \ 0}D_t^{\beta}x(t)] + \omega_0^2 S[x(t)] = 0, \tag{78}$$

$$\frac{1}{2-\alpha+\alpha u^{\alpha}}\left(\frac{\tilde{x}(u)-x(0)}{u^2} - \frac{x'(0)}{u}\right) + \frac{2\zeta\omega_0}{1-\beta+\beta u^{\beta}}\left(\frac{\tilde{x}(u)-x(0)}{u}\right) + \omega_0^2\tilde{x}(u) = 0, \tag{79}$$

$$\tilde{x}(u)\left\{\frac{1}{(2-\alpha+\alpha u^{\alpha})u^2} + \omega_0^2 + \frac{2\zeta\omega_0}{1-\beta+\beta u^{\beta}}\right\} = \frac{x(0)}{(2-\alpha+\alpha u^{\alpha})u^2} + \frac{x'(0)}{(2-\alpha+\alpha u^{\alpha})u} + \frac{2\zeta\omega_0 x(0)}{(1-\beta+\beta u^{\beta})u}, \tag{80}$$

$$\tilde{x}(u) = \frac{\frac{x(0)}{(2-\alpha+\alpha u^{\alpha})u^2} + \frac{x'(0)}{(2-\alpha+\alpha u^{\alpha})u^2} + \frac{2\zeta\omega_0 x(0)}{(1-\beta+\beta u^{\beta})u}}{\frac{1}{(2-\alpha+\alpha u^{\alpha})u^2} + \omega_0^2 + \frac{2\zeta\omega_0}{1-\beta+\beta u^{\beta}}}. \tag{81}$$

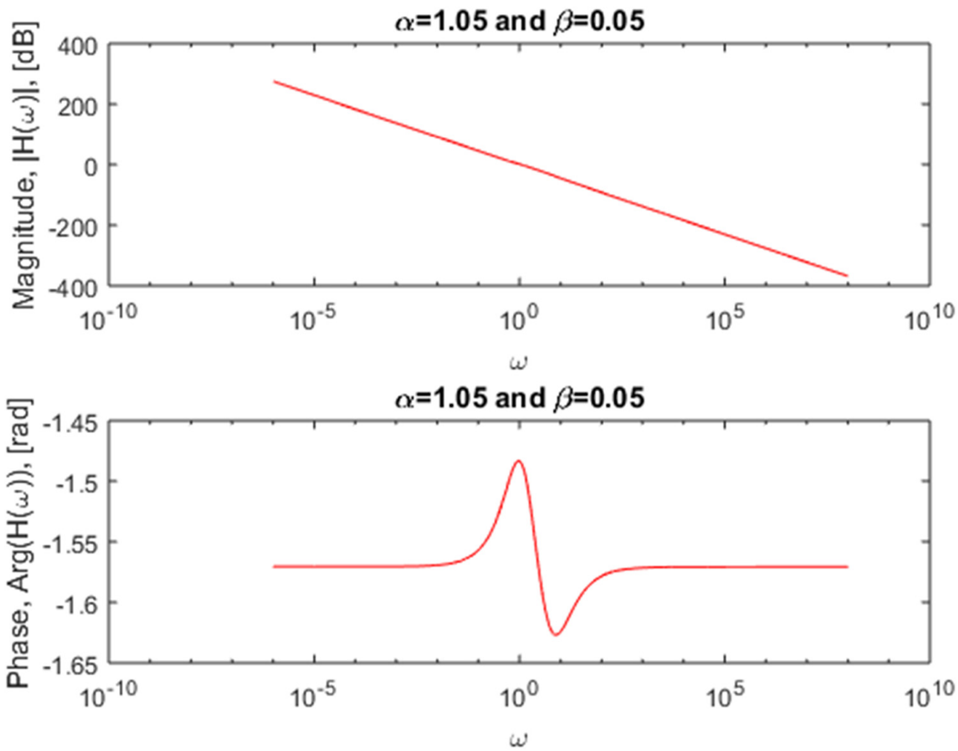

Figure 5: Bode and phase diagrams for ABC derivative.

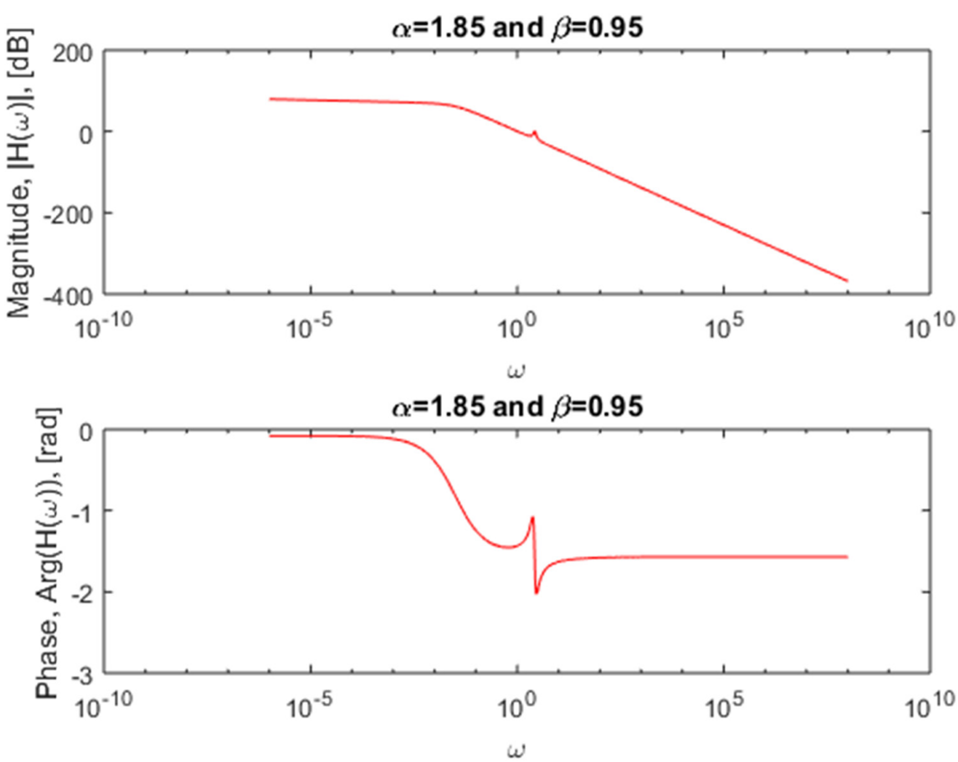

Figure 6: Bode and phase diagrams for ABC derivative.

The Bode diagrams of the Laplace and Sumudu transforms are presented in Figures 5 and 6 for different values of fractional orders.

The existence and uniqueness condition can be achieved by using the same routine presented in the case of the Caputo-Fabrizio derivative. We will proceed with the numerical solution of the model.

$$x(t_{n+1}) = -\frac{1}{\omega_0^2}\{{}^{ABC}_{0}D^{\alpha}_{t_{n+1}}(x)(t) + 2\zeta\omega_0\,{}^{ABC}_{0}D^{\beta}_{t_{n+1}}(x)(t)\}, \tag{82}$$

$$= -\frac{1}{\omega_0^2}\left\{\frac{1}{2-\beta}\int_0^{t_{n+1}}\frac{\mathrm{d}^2x(\tau)}{\mathrm{d}\tau^2}E_\alpha\left[-\frac{\alpha}{2-\alpha}(t_{n+1}-\tau)^\alpha\right]\mathrm{d}\tau + \frac{1}{1-\beta}\int_0^{t_{n+1}}\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau}E_\beta\left[-\frac{\beta}{1-\beta}(t_{n+1}-\tau)^\alpha\right]\mathrm{d}\tau\right\}, \tag{83}$$

$$\begin{aligned} &= -\frac{1}{\omega_0^2}\left\{\frac{1}{2-\alpha}\sum_{j=1}^{n}\frac{x_{j+1}-2x_j+x_{j-1}}{\Delta t^2}\int_{t_j}^{t_{j+1}}E_\alpha\left[-\frac{\alpha}{2-\alpha}(t_{n+1}-\tau)^\alpha\right]\mathrm{d}\tau\right.\\ &\left.+\frac{2\zeta\omega_0}{1-\beta}\sum_{j=1}^{n}\frac{x_{j+1}-x_j}{\Delta t}\int_{t_j}^{t_{j+1}}E_\beta\left[-\frac{\beta}{1-\beta}(t_{n+1}-\tau)^\beta\right]\mathrm{d}\tau\right\}, \end{aligned} \tag{84}$$

$$= -\frac{1}{\omega_0^2}\left\{\frac{1}{2-\alpha}\sum_{j=1}^{n}\frac{x_{j+1}-2x_j+x_{j-1}}{\Delta t^2}\pi_{1,j}^{n,\alpha}+\frac{2\zeta\omega_0}{1-\beta}\sum_{j=0}^{n}\frac{x_{j+1}-x_j}{\Delta t}\pi_{2,j}^{n,\beta}\right\}. \tag{85}$$

5 Conclusion

Caputo, Caputo-Fabrizio, and the Atangana-Baleanu fractional differential operators have been used in this work to modify the damped harmonic oscillator equation with the aim to introduce some nonlocal behaviors into the model. In particular, power law helps introduce the power law process, while exponential decay helps introduce fading memory, and the generalized Mittag-Leffler function helps introduce the change from fading to the power law. Two integral transformation has been used to obtain an exact solution in complex space for each case. For each case, we presented their corresponding Bode diagram with Laplace and Sumudu. The growth and the Lipschitz conditions have been used to derive conditions under which the equations have unique solutions, and, finally, numerical solutions were derived.

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