

## Research Article

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# Some identities on generalized harmonic numbers and generalized harmonic functions

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**Abstract:** The harmonic numbers and generalized harmonic numbers appear frequently in many diverse areas such as combinatorial problems, many expressions involving special functions in analytic number theory, and analysis of algorithms. The aim of this article is to derive some identities involving generalized harmonic numbers and generalized harmonic functions from the beta functions  $F_n(x) = B(x+1, n+1)$ , ( $n = 0, 1, 2, \dots$ ) using elementary methods. For instance, we show that the Hurwitz zeta function  $\zeta(x+1, r)$  and  $r!$  are expressed in terms of those numbers and functions, for every  $r = 2, 3, 4, 5$ .

**Keywords:** generalized harmonic numbers of order  $\alpha$ , generalized harmonic function

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## 1 Introduction

The harmonic numbers and generalized harmonic numbers have been studied in connection with combinatorial problems, special functions in analytic number theory, and analysis of algorithms [1–14]. As a generalization of harmonic numbers in another direction, the hyperharmonic numbers were introduced by Conway and Guy in 1996 [4,5]. Recently, the study of degenerate versions of many special numbers and special polynomials regained interests of some mathematicians, which began with the pioneering work of Carlitz on the degenerate Bernoulli and degenerate Euler numbers (see [6,7] and the references therein). In this line of study, the degenerate harmonic numbers and the degenerate hyperharmonic numbers were investigated in [8,9].

The aim of this article is to derive some identities relating to generalized harmonic numbers  $H_n^{(\alpha)}$  (see (7)) and generalized harmonic functions  $H_n(x, \alpha)$  (see (11)) by using elementary means like integration, differentiation, and binomial theorem. In more detail, by making use of the beta functions  $F_n(x) = B(x+1, n+1)$ , ( $n = 0, 1, 2, \dots$ ), we show that, for every  $r = 2, 3, 4, 5$ ,

$$\sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{(k+x+1)^r}, \quad \zeta(x+1, r), \quad r!$$

are all expressed in terms of the generalized harmonic functions and the generalized harmonic numbers. For example, we show that

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$$\begin{aligned} \frac{2H_n(x, 3) + 3H_n(x, 1)H_n(x, 2) + (H_n(x, 1))^3}{3! \binom{x+n+1}{n} (x+1)} &= \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{(x+k+1)^4}, \\ \frac{1}{3!} \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{\binom{n}{k} (-1)^k}{\binom{x+k+1}{k} (x+1)} (2H_k(x, 3) + 3H_k(x, 1)H_k(x, 2) + (H_k(x, 1))^3) &= \zeta(x+1, 4), \\ \sum_{n=1}^{\infty} \frac{2H_{n+1}^{(3)} + 3H_{n+1}H_{n+1}^{(2)} + (H_{n+1})^3}{n(n+1)} &= 4!, \end{aligned}$$

where  $\zeta(x, s)$  is the Hurwitz zeta function (see (4)).

In principle, the methods employed in this article could be continued further for  $r = 6, 7, 8, \dots$ . However, this requires determination of an explicit expression for  $F_n^{(r-1)}(x)$  in terms of the generalized harmonic functions.

Similar method to the present article is exploited in [10] to find summation formulas involving harmonic numbers and the generalized harmonic numbers (called harmonic numbers of order  $r$  there). However, the beta functions  $G_n(x) = \frac{1}{2}B(n+1, \frac{x+1}{2})$ , ( $n = 0, 1, 2, \dots$ ) are used in [10] instead of  $F_n(x) = B(x+1, n+1)$ , ( $n = 0, 1, 2, \dots$ ) in this article. In addition, the generalized harmonic functions are not introduced in [10], whereas they play an important role in this article.

The motivation of this article is very simple. We note that  $G_n(x) = \int_0^1 (1-t^2)^n t^x dt$  and  $F_n(x) = \int_0^1 (1-t)^n t^x dt$ . So it is natural to ask what if we replace  $(1-t^2)^n$  by  $(1-t)^n$  in the integrand of  $G_n(x)$ . This question led us to carry out the present research and the introduction of the generalized harmonic functions.

Interested readers can refer to [11,12] for the existing literature on the same topic. Indeed, some identities are obtained about certain finite or infinite series involving harmonic numbers and the generalized harmonic numbers by applying an algorithmic method to a known summation formula for the hypergeometric function  ${}_5F_4(1)$  in [11] and by applying a differential operator to a known identity in [12]. For the rest of this section, we recall the facts that are needed throughout this article.

For  $s \in \mathbb{C}$ , the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (\operatorname{Re}(s) > 1), \quad (\text{see [13,14,15]}).$$

As is well known, we have

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n}}{2(2n)!} (2\pi)^{2n}, \quad (n \in \mathbb{N}), \quad (\text{see [15]}), \quad (1)$$

where  $B_n$  are the Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = 1 - \frac{1}{2}t + \frac{1}{6} \frac{t^2}{2!} - \frac{1}{30} \frac{t^4}{4!} + \frac{1}{42} \frac{t^6}{6!} - \frac{1}{30} \frac{t^8}{8!} + \dots, \quad (\text{see [1-3,5-15]}). \quad (2)$$

Thus, we see from (1) and (2) that

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9,450}, \dots \quad (3)$$

More generally, the Hurwitz zeta function is defined by

$$\zeta(x, s) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad (\operatorname{Re}(s) > 1, x > 0), \quad (\text{see [15]}). \quad (4)$$

The harmonic numbers are defined by

$$H_0 = 0, \quad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}, \quad (n \in \mathbb{N}), \quad (\text{see } [3,5,7-9]). \quad (5)$$

The generating function of harmonic numbers is given by

$$-\frac{1}{1-x} \log(1-x) = \sum_{n=1}^{\infty} H_n x^n, \quad (\text{see } [3,5,8,9]). \quad (6)$$

For  $\alpha \in \mathbb{N}$ , the generalized harmonic numbers of order  $\alpha$  are defined by

$$H_0^{(\alpha)} = 0, \quad H_n^{(\alpha)} = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \cdots + \frac{1}{n^\alpha}, \quad (\text{see } [2,9,11,12]). \quad (7)$$

For  $s \in \mathbb{C}$  with  $\operatorname{Re}(s) > 0$ , the gamma function is given by

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt, \quad (\text{see } [6,14,15]), \quad (8)$$

with  $\Gamma(s+1) = s\Gamma(s)$  and  $\Gamma(1) = 1$ . For  $\operatorname{Re}(\alpha) > 0$  and  $\operatorname{Re}(\beta) > 0$ , the beta function is given by

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt, \quad (\text{see } [6,14,15]). \quad (9)$$

From (8) and (9), we have

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad (\text{see } [15]). \quad (10)$$

Finally, the following binomial inversion theorem is well known.

**Theorem 1.1.** For any integer  $n \geq 0$ , we have

$$b_n = \sum_{k=0}^n \binom{n}{k} (-1)^k a_k \Leftrightarrow a_n = \sum_{k=0}^n \binom{n}{k} (-1)^k b_k.$$

In this article, we derive some expressions for  $\sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{(k+x+1)^r}$ ,  $\zeta(x+1, r)$ , and  $r!$  in terms of the generalized harmonic functions and the generalized harmonic numbers using elementary methods like the binomial inversion, differentiation, and integration.

## 2 Some identities on generalized harmonic numbers and generalized harmonic functions

For every  $\alpha \in \mathbb{N}$ , we consider the generalized harmonic function  $H_n(x, \alpha)$  defined by

$$H_n(x, \alpha) = \sum_{k=0}^n \frac{1}{(k+x+1)^\alpha}, \quad (x \geq 0), \quad (\text{see } [2,11,12,16,17]). \quad (11)$$

Note that

$$\frac{d}{dx} H_n(x, \alpha) = -\alpha H_n(x, \alpha+1). \quad (12)$$

In particular, for  $x = 0$ , we have  $H_n(0, \alpha) = H_{n+1}^{(\alpha)}$ .

For  $x > -1$  and  $n = 0, 1, 2, \dots$ , we let

$$\begin{aligned}
 F_n(x) &= \int_0^1 (1-t)^n t^x dt = B(x+1, n+1) = \frac{\Gamma(x+1)\Gamma(n+1)}{\Gamma(x+n+2)} \\
 &= \frac{n!}{(x+n+1)(x+n)\cdots(x+1)} = \frac{1}{\binom{x+n+1}{n}(x+1)}.
 \end{aligned} \tag{13}$$

By the binomial theorem, (13) is also equal to

$$F_n(x) = \int_0^1 (1-t)^n t^x dt = \sum_{k=0}^n \binom{n}{k} (-1)^k \int_0^1 t^{k+x} dt = \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{x+k+1}. \tag{14}$$

We will use the following lemma repeatedly throughout this article.

**Lemma 2.1.** *Let  $F_n(x) = B(x+1, n+1)$ . Then, for any positive integer  $r$ , we have*

$$\begin{aligned}
 (a) \quad F_n^{(r)}(x) &= \int_0^1 (1-t)^n (\log t)^r t^x dt = r! \sum_{k=0}^n \binom{n}{k} (-1)^{k-r} \frac{1}{(x+k+1)^{r+1}}, \\
 (b) \quad F_n'(x) &= -H_n(x, 1)F_n(x), \\
 (c) \quad \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{(k+1)^r} &= r.
 \end{aligned}$$

**Proof.**

(a) Follows from (14).

(b) From (13), we obtain

$$\begin{aligned}
 F_n'(x) &= \frac{d}{dx} F_n(x) = \frac{d}{dx} \left( \frac{n!}{(x+n+1)(x+n)\cdots(x+1)} \right) \\
 &= - \sum_{k=0}^n \frac{1}{k+x+1} \cdot \frac{n!}{(x+n+1)(x+n)\cdots(x+1)} \\
 &= -H_n(x, 1)F_n(x).
 \end{aligned}$$

(c) The left-hand side of (c) is equal to

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 \cdots \int_0^1 (1-x_1 x_2 \cdots x_r)^n dx_1 dx_2 \cdots dx_r = - \int_0^1 \cdots \int_0^1 (\log x_1 + \cdots + \log x_r) dx_1 dx_2 \cdots dx_r = r,$$

where the integrals are  $r$ -multiple ones. □

From Lemma 2.1, (a), (b), and (13), we have

$$\sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{(k+x+1)^2} = H_n(x, 1)F_n(x) = \frac{H_n(x, 1)}{\binom{x+n+1}{n}(x+1)}. \tag{15}$$

In particular, for  $x = 0$ , we obtain

$$\sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{(k+1)^2} = \frac{1}{n+1} H_{n+1}, \quad (n \geq 0), \tag{16}$$

which, by Lemma 2.1 (c), yields

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} H_{n+1} = 2.$$

Therefore, from (15) and (16), by binomial inversion, we obtain the following theorem.

**Theorem 2.2.** For  $n \geq 0$ , we have

$$\sum_{k=0}^n \binom{n}{k} (-1)^k \frac{H_k(x, 1)}{\binom{x+k+1}{k} (x+1)} = \frac{1}{(n+x+1)^2}, \quad \text{and} \quad \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{H_{k+1}}{k+1} = \frac{1}{(n+1)^2}.$$

We note from Theorem 2.2, (3) and (4) that

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{H_k(x, 1)}{\binom{x+k+1}{k} (x+1)} &= \sum_{n=0}^{\infty} \frac{1}{(n+x+1)^2} = \zeta(x+1, 2), \\ \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{H_{k+1}}{k+1} &= \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6}. \end{aligned}$$

From Lemma 2.1 (b), (12) and (13), we note that

$$\begin{aligned} F_n^{(2)}(x) &= \frac{d^2}{dx^2} F_n(x) = (-1) \frac{d}{dx} (H_n(x, 1) F_n(x)) \\ &= (-1)^2 H_n(x, 2) F_n(x) + (-1)^2 (H_n(x, 1))^2 F_n(x) \\ &= (-1)^2 (H_n(x, 2) + (H_n(x, 1))^2) \frac{1}{\binom{x+n+1}{n} (x+1)} \end{aligned} \quad (17)$$

and that

$$\begin{aligned} F_n^{(3)}(x) &= \frac{d^3}{dx^3} F_n(x) = \frac{d}{dx} ((-1)^2 H_n(x, 2) F_n(x) + (-1)^2 (H_n(x, 1))^2 F_n(x)) \\ &= 2(-1)^3 H_n(x, 3) F_n(x) + (-1)^3 H_n(x, 2) H_n(x, 1) F_n(x) + 2(-1)^3 H_n(x, 1) H_n(x, 2) F_n(x) \\ &\quad + (-1)^3 (H_n(x, 1))^3 F_n(x) \\ &= 2(-1)^3 H_n(x, 3) F_n(x) + 3(-1)^3 H_n(x, 1) H_n(x, 2) F_n(x) + (-1)^3 (H_n(x, 1))^3 F_n(x) \\ &= (-1)^3 (2H_n(x, 3) + 3H_n(x, 1) H_n(x, 2) + (H_n(x, 1))^3) \frac{1}{\binom{x+n+1}{n} (x+1)}. \end{aligned} \quad (18)$$

By Lemma 2.1 (a), we obtain

$$F_n^{(2)}(x) = 2! \sum_{k=0}^n \binom{n}{k} (-1)^{k-2} \frac{1}{(x+k+1)^3}, \quad F_n^{(3)}(x) = 3! \sum_{k=0}^n \binom{n}{k} (-1)^{k-3} \frac{1}{(x+k+1)^4}. \quad (19)$$

Therefore, by (17)–(19), we obtain the following theorem.

**Theorem 2.3.** For  $n \geq 0$ , we have

$$\frac{H_n(x, 2) + (H_n(x, 1))^2}{\binom{x+n+1}{n} (x+1)} = 2! \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{(x+k+1)^3}$$

and

$$\frac{2H_n(x, 3) + 3H_n(x, 1)H_n(x, 2) + (H_n(x, 1))^3}{\binom{x+n+1}{n} (x+1)} = 3! \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{(x+k+1)^4}.$$

For  $x = 0$ , we also have

$$2! \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{(k+1)^3} = \frac{H_{n+1}^{(2)} + (H_{n+1})^2}{n+1}$$

and

$$3! \sum_{k=0}^n \frac{\binom{n}{k} (-1)^k}{(k+1)^4} = \frac{2H_{n+1}^{(3)} + 3H_{n+1}H_{n+1}^{(2)} + (H_{n+1})^3}{n+1}.$$

From Theorem 2.3, by binomial inversion, we obtain

$$\frac{1}{2!} \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{H_k(x, 2) + (H_k(x, 1))^2}{\binom{x+k+1}{k} (x+1)} = \frac{1}{(x+n+1)^3}, \quad (20)$$

$$\frac{1}{3!} \sum_{k=0}^n \binom{n}{k} (-1)^k \left( \frac{2H_k(x, 3) + 3H_k(x, 1)H_k(x, 2) + (H_k(x, 1))^3}{\binom{x+k+1}{k} (x+1)} \right) = \frac{1}{(x+n+1)^4}. \quad (21)$$

By (20) and (4), we obtain

$$\frac{1}{2!} \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{\binom{n}{k} (-1)^k}{\binom{x+k+1}{k} (x+1)} (H_k(x, 2) + (H_k(x, 1))^2) = \zeta(x+1, 3).$$

For  $x = 0$ , we have

$$\frac{1}{2!} \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{\binom{n}{k} (-1)^k}{k+1} (H_{k+1}^{(2)} + (H_{k+1})^2) = \zeta(3).$$

In addition, from (21) and (4), we also have

$$\frac{1}{3!} \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{\binom{n}{k} (-1)^k}{\binom{x+k+1}{k} (x+1)} (2H_k(x, 3) + 3H_k(x, 1)H_k(x, 2) + (H_k(x, 1))^3) = \zeta(x+1, 4).$$

For  $x = 0$ , from (3), we obtain

$$\frac{1}{3!} \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{\binom{n}{k} (-1)^k}{k+1} (2H_{k+1}^{(3)} + 3H_{k+1}H_{k+1}^{(2)} + (H_{k+1})^3) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^4} = \frac{\pi^4}{90}.$$

From Theorem 2.3 and Lemma 2.1 (c), we note that

$$\frac{1}{2!} \sum_{n=1}^{\infty} \frac{H_{n+1}^{(2)} + (H_{n+1})^2}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{(k+1)^3} = 3 \quad (22)$$

and that

$$\frac{1}{3!} \sum_{n=1}^{\infty} \frac{2H_{n+1}^{(3)} + 3H_{n+1}H_{n+1}^{(2)} + (H_{n+1})^3}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{(k+1)^4} = 4. \quad (23)$$

Therefore, by (22) and (23), we obtain the following corollary.

**Corollary 2.4.** *For  $n \geq 1$ , we have*

$$\sum_{n=1}^{\infty} \frac{H_{n+1}^{(2)} + (H_{n+1})^2}{n(n+1)} = 3!, \quad \sum_{n=1}^{\infty} \frac{2H_{n+1}^{(3)} + 3H_{n+1}H_{n+1}^{(2)} + (H_{n+1})^3}{n(n+1)} = 4!.$$

From (21), Lemma 2.1 (b), (12), and (13), we have

$$F_n^{(4)}(x) = \frac{d}{dx} (2(-1)^3 H_n(x, 3) F_n(x) + 3(-1)^3 H_n(x, 1) H_n(x, 2) F_n(x) + (-1)^3 (H_n(x, 1))^3 F_n(x)) \quad (24)$$

$$\begin{aligned}
&= 6(-1)^4 H_n(x, 4) F_n(x) + 2(-1)^4 H_n(x, 3) H_n(x, 1) F_n(x) + 3(-1)^4 H_n(x, 2) H_n(x, 2) F_n(x) \\
&\quad + 6(-1)^4 H_n(x, 1) H_n(x, 3) F_n(x) + 3(-1)^4 (H_n(x, 1))^2 H_n(x, 2) F_n(x) \\
&\quad + 3(-1)^4 (H_n(x, 1))^2 H_n(x, 2) F_n(x) + (-1)^4 (H_n(x, 1))^4 F_n(x) \\
&= (-1)^4 (6H_n(x, 4) + 8H_n(x, 3)H_n(x, 1) + 3(H_n(x, 2))^2 + 6(H_n(x, 1))^2 H_n(x, 2) \\
&\quad + (H_n(x, 1))^4) F_n(x) \\
&= (-1)^4 \frac{6H_n(x, 4) + 8H_n(x, 3)H_n(x, 1) + 3(H_n(x, 2))^2 + 6(H_n(x, 1))^2 H_n(x, 2) + (H_n(x, 1))^4}{\binom{x+n+1}{n}(x+1)}
\end{aligned}$$

and

$$F_n^{(4)}(0) = (-1)^4 \frac{6H_{n+1}^{(4)} + 8H_{n+1}^{(3)}H_{n+1} + 3(H_{n+1}^{(2)})^2 + 6(H_{n+1})^2 H_{n+1}^{(2)} + (H_{n+1})^4}{n+1}. \quad (25)$$

On the other hand, by Lemma 2.1 (a), we obtain

$$F_n^{(4)}(x) = 4! \sum_{k=0}^n \binom{n}{k} (-1)^{k-4} \frac{1}{(k+x+1)^5} = \int_0^1 (1-t)^n (\log t)^4 t^x dt \quad (26)$$

and

$$F_n^{(4)}(0) = 4! \sum_{k=0}^n \binom{n}{k} (-1)^{k-4} \frac{1}{(k+1)^5} = \int_0^1 (1-t)^n (\log t)^4 dt. \quad (27)$$

By (24)–(27), we obtain

$$\begin{aligned}
&\sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{(k+x+1)^5} = \frac{1}{4!} \int_0^1 (1-t)^n (\log t)^4 t^x dt \\
&= \frac{6H_n(x, 4) + 8H_n(x, 3)H_n(x, 1) + 3(H_n(x, 2))^2 + 6(H_n(x, 1))^2 H_n(x, 2) + (H_n(x, 1))^4}{4! \binom{x+n+1}{n}(x+1)}
\end{aligned} \quad (28)$$

and

$$\begin{aligned}
&\sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{(k+1)^5} = \frac{1}{4!} \int_0^1 (1-t)^n (\log t)^4 dt \\
&= \frac{6H_{n+1}^{(4)} + 8H_{n+1}^{(3)}H_{n+1} + 3(H_{n+1}^{(2)})^2 + 6(H_{n+1})^2 H_{n+1}^{(2)} + (H_{n+1})^4}{4!(n+1)}.
\end{aligned} \quad (29)$$

Therefore, from (28) and (29), by binomial inversion, we obtain the following theorem.

**Theorem 2.5.** For  $n \geq 0$ , we have

$$\sum_{k=0}^n \binom{n}{k} (-1)^k \frac{6H_k(x, 4) + 8H_k(x, 3)H_k(x, 1) + 3(H_k(x, 2))^2 + 6(H_k(x, 1))^2 H_k(x, 2) + (H_k(x, 1))^4}{4! \binom{x+k+1}{k}(x+1)} = \frac{1}{(n+x+1)^5}$$

and

$$\frac{1}{4!} \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{6H_{k+1}^{(4)} + 8H_{k+1}^{(3)}H_{k+1} + 3(H_{k+1}^{(2)})^2 + 6(H_{k+1})^2 H_{k+1}^{(2)} + (H_{k+1})^4}{k+1} = \frac{1}{(n+1)^5}.$$

By Theorem 2.5 and (4), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{6H_k(x, 4) + 8H_k(x, 3)H_k(x, 1) + 3(H_k(x, 2))^2 + 6(H_k(x, 1))^2 H_k(x, 2) + (H_k(x, 1))^4}{4! \binom{x+k+1}{k} (x+1)} \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+x+1)^5} = \zeta(x+1, 5). \end{aligned}$$

For  $x = 0$ , we also have

$$\frac{1}{4!} \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{6H_{k+1}^{(4)} + 8H_{k+1}^{(3)}H_{k+1} + 3(H_{k+1}^{(2)})^2 + 6(H_{k+1})^2 H_{k+1}^{(2)} + (H_{k+1})^4}{k+1} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^5} = \zeta(5).$$

From (29) and Lemma 2.1 (c), we note that

$$\frac{1}{4!} \sum_{n=1}^{\infty} \frac{6H_{n+1}^{(4)} + 8H_{n+1}^{(3)}H_{n+1} + 3(H_{n+1}^{(2)})^2 + 6(H_{n+1})^2 H_{n+1}^{(2)} + (H_{n+1})^4}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{(k+1)^5} = 5.$$

Hence, we have

$$\sum_{n=1}^{\infty} \frac{6H_{n+1}^{(4)} + 8H_{n+1}^{(3)}H_{n+1} + 3(H_{n+1}^{(2)})^2 + 6(H_{n+1})^2 H_{n+1}^{(2)} + (H_{n+1})^4}{n(n+1)} = 5!.$$

From Lemma 2.1 (a), (b), and (12), we note that

$$\begin{aligned} (r+1)! \sum_{k=0}^n \binom{n}{k} (-1)^{k-r} \frac{1}{(k+x+1)^{r+2}} &= \frac{d^r}{dx^r} (H_n(x, 1) F_n(x)) \\ &= \sum_{l=0}^r \binom{r}{l} \left( \frac{d^l}{dx^l} H_n(x, 1) \right) \left( \frac{d^{r-l}}{dx^{r-l}} F_n(x) \right) \\ &= \sum_{l=0}^r \binom{r}{l} l! (-1)^l H_n(x, l+1) \int_0^1 (1-t)^n (\log t)^{r-l} t^x dt, \end{aligned}$$

which gives us the following:

$$\sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{(k+x+1)^{r+2}} = \frac{(-1)^r}{(r+1)!} \sum_{l=0}^r \binom{r}{l} l! (-1)^l H_n(x, l+1) \int_0^1 (1-t)^n (\log t)^{r-l} t^x dt. \quad (30)$$

From (30), by binomial inversion and summing over  $n$ , we obtain

$$\begin{aligned} & \frac{(-1)^r}{(r+1)!} \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} (-1)^k \sum_{l=0}^r \binom{r}{l} l! (-1)^l H_k(x, l+1) \int_0^1 (1-t)^k (\log t)^{r-l} t^x dt \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+x+1)^{r+2}} = \zeta(x+1, r+2), \end{aligned} \quad (31)$$

which, for  $x = 0$ , yields

$$\begin{aligned} & \frac{(-1)^r}{(r+1)!} \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} (-1)^k \sum_{l=0}^r \binom{r}{l} l! (-1)^l H_{k+1}^{(l+1)} \int_0^1 (1-t)^k (\log t)^{r-l} dt \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)^{r+2}} = \zeta(r+2). \end{aligned}$$

By (30) with  $x = 0$  and Lemma 2.1 (c), we have

$$\frac{(-1)^r}{(r+1)!} \sum_{l=0}^r \binom{r}{l} l! (-1)^l \sum_{n=1}^{\infty} \frac{1}{n} H_{n+1}^{(l+1)} \int_0^1 (1-t)^n (\log t)^{r-l} dt = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{(k+1)^{r+2}} = r+2. \quad (32)$$

Thus, from (30)–(32), we obtain the following theorem.

**Theorem 2.6.** For  $n, r \geq 0$ , we have the following identities:

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{(k+x+1)^{r+2}} &= \frac{(-1)^r}{(r+1)!} \sum_{l=0}^r \binom{r}{l} l! (-1)^l H_n(x, l+1) \int_0^1 (1-t)^n (\log t)^{r-l} t^x dt, \\ \frac{(-1)^r}{(r+1)!} \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} (-1)^k \sum_{l=0}^r \binom{r}{l} l! (-1)^l H_k(x, l+1) \int_0^1 (1-t)^k (\log t)^{r-l} t^x dt &= \zeta(x+1, r+2), \end{aligned}$$

and

$$\sum_{l=0}^r \binom{r}{l} l! (-1)^l \sum_{n=1}^{\infty} \frac{1}{n} H_{n+1}^{(l+1)} \int_0^1 (1-t)^n (\log t)^{r-l} dt = (-1)^r (r+2)!.$$

### 3 Conclusion

In this article, we used elementary methods to derive some identities involving the generalized harmonic numbers and the generalized harmonic functions from the beta functions  $F_n(x) = B(x+1, n+1)$ , ( $n = 0, 1, 2, \dots$ ). In more detail, for every  $r = 2, 3, 4, 5$ , we showed that

$$\sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{(k+x+1)^r}, \quad \zeta(x+1, r), \quad r! \quad (33)$$

are all expressed in terms of the generalized harmonic functions and the generalized harmonic numbers. The methods employed in this article could be continued further for  $r = 6, 7, 8, \dots$ , which requires to find an explicit expression for  $F_n^{(r-1)}(x)$  in terms of the generalized harmonic functions. In Theorem 2.6, for any  $r = 2, 3, 4, \dots$ , we found expressions for (33) without deriving the explicit expressions for  $F_n^{(r-l)}(x) = \int_0^1 (1-t)^n (\log t)^{r-l} dt$  in terms of the generalized harmonic functions.

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