DE GRUYTER Demonstratio Mathematica 2023; 56: 20220228

Research Article

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On the structure of self-affine Jordan arcs in $\mathbb{R}^2$

https://doi.org/10.1515/dema-2022-0228
received December 20, 2022; accepted April 12, 2023

Abstract: We prove that if a self-affine arc $\gamma \in \mathbb{R}^2$ does not satisfy weak separation condition, then it is a segment of a parabola or a straight line. If a self-affine arc γ is not a segment of a parabola or a line, then it is a component of the attractor of a Jordan multizipper with the same set of generators.

Keywords: self-similar set, self-affine Jordan arc, zipper, weak separation property

MSC 2020: 28A80, 26A27

1 Introduction

The main question the article deals with is what may be a sufficiently general method of defining explicitly self-affine Jordan arcs in the plane and in space.

A Jordan arc γ in $\mathbb{R}^n$ is called *self-affine* (resp. *self-similar, self-conformal*) if it is the attractor of a finite system $\mathcal{S} = \{S_1, \ldots, S_m\}$ of contracting maps of the respective type in $\mathbb{R}^n$. This means [1] that the arc γ is the unique nonempty compact set satisfying the equation $\gamma = S_1(\gamma) \cup \ldots \cup S_m(\gamma)$.

The obvious and natural way to obtain a Jordan self-similar arc γ with endpoints z_0, z_m is to build it up from pieces $S_k(\gamma)$, which follow each other successively and are connected by common endpoints, say, $S_{k-1}(z_m) = S_k(z_0)$. In this case, the maps S_k should send the points z_0, z_m to the points z_{k-1}, z_k, and the relations $S_i(\gamma) \cap S_j(\gamma) = \varnothing$ if $|i - j| > 1$ and $S_i(\gamma) \cap S_{i+1}(\gamma) = \{z_i\}$ should be fulfilled. Such systems of contractions are called Jordan zippers, which were studied in detail by Aseev et al. [2]. This approach is fairly good; however, there exist examples of self-similar Jordan arcs $\gamma = \bigcup_{i=1}^{m} S_i(\gamma)$ for which the overlaps $\gamma_i \cap \gamma_{i+1}$ are non-trivial subarcs of γ. The problem of defining parameters for fractal arcs having non-trivial overlaps is rather complicated.

We also consider a more universal setting that is applicable to each of the three previous cases.

A Jordan arc γ is called *locally self-affine* if, for any open subarc $\gamma' \subset \gamma$, there is a non-degenerate contractive affine mapping S such that $S(\gamma) \subset \gamma'$. Given a locally self-affine arc γ, the semigroup $\mathcal{G}(\gamma) = \{S : S(\gamma) \subset \gamma\}$ of contractive affine maps of γ into γ is infinite, and in general case, it cannot be reduced to a finite number of generators. It is most preferable for us to be able to define such generators explicitly by some finite procedure.

The problem of finite explicit representation of self-similar and self-affine curves is closely related to the rigidity properties of these sets, which we discuss later in this section.

Self-similar curves appeared initially in the works of Peano [3] and von Koch [4] and were studied in detail by Levy [5]. Earliest examples of self-conformal sets were the limit sets of quasi-Fuchsian groups;

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they appeared first in 1897 in the book [6] written by Fricke and Klein. The first studies of self-affine curves were originated by de Rham [7].

These three types of sets manifest various unusual phenomena, which may be (and often are) called rigidity properties. For example, a self-similar arc γ different from a line segment cannot be shifted along itself to a small distance by a similarity close to identity [8]. Certain types of self-similar sets such as Sierpinski gasket or a fractal necklace are topologically rigid, and the only possible continuous injections of such a set K to itself are the maps sending K to some of its pieces $S_{i_1\dots i_n}(K)$. As it was proved by Astala [9], if the boundary of a plane domain contains a self-similar curve, then the domain is conformally rigid in the sense of Thurston [10]. There is one more property: it is well known that, for any subarc γ of a limit set $\Lambda(G)$ of a quasi-Fuchsian group G and any bundle B of parallel lines l intersecting γ, the set $\{l \in B : \#(l \cup \gamma) = 1\}$ is nowhere dense in B; this happens because loxodromic fixed points are dense in $\Lambda(G)$. A similar property takes place for self-similar Jordan arcs in $\mathbb{R}^n$ [11]. If a self-similar Jordan arc $\gamma \subset \mathbb{R}^n$ contains a subarc γ' admitting a bundle of parallel hyperplanes, each of which intersects γ' in a unique point, then the whole γ is a straight line. If a self-similar arc is not a straight line, then it is a Whitney set of some differentiable function [12].

The geometric rigidity properties of self-similar sets were first pointed out in 1982 by Mattila [13]. He proved that if a self-similar set K in $\mathbb{R}^n$ satisfies the open set condition, then there is the following alternative: either the set K lies on an m-dimensional affine subspace or the intersection of K with every m-dimensional C^1-submanifold M of $\mathbb{R}^n$ has zero Hausdorff measure $H^t(E \cap M)$, where t is the Hausdorff dimension of E.

This geometric rigidity property was extended to conformal iterated function system (IFS) in 2001 by Mauldin et al. [14]. They proved that if $\mathcal{S} = \{S_1, \dots, S_m\}$ is a conformal IFS in $\mathbb{R}^n (n \geq 2)$, and its attractor K is a continuum, then either $\dim_H(K) > 1$ or K is a proper compact segment of a geometric circle or a straight line. Further intensive work in this area was done in a cycle of works by Käenmäki [15,16].

As it follows from [13], a self-similar arc in $\mathbb{R}^n$ belongs to class C^1 only if it is a line segment. Conversely, it was proved by Tetenov [8] that if a self-similar Jordan arc γ in $\mathbb{R}^n$ is not a line segment, then it can be represented by some multizipper $\mathcal{Z}$. Moreover [17], if such self-similar arc γ lies in the plane, then it satisfies weak separation condition. The latter statement does not hold for the arcs in $\mathbb{R}^n$, $n \geq 3$ [18].

The situation is quite different for self-affine arcs. There are various non-trivial self-affine arcs in the plane that belong to the class C^1. First, it was shown for the graphs of affine fractal functions in 1989 in [19,20]. Later, the theory of smooth fractal interpolation was developed by many authors [21,22].

Kravchenko (2005) studied smooth self-affine curves in the plane, considering the action of affine transformations on cones in $\mathbb{R}^2$ and found the conditions under which the attractor of a self-affine zipper in the plane is a C^1-smooth curve [23]. In 2009, Bandt and Kravchenko [24] proved that the only C^2-differentiable self-affine arc in $\mathbb{R}^2$ is a segment of a parabola or a straight line.

Another approach to the study of self-affine curves was worked out in 2006 by Protasov [25] in his research of wavelets. He considered these curves as a special case of summable fractal functions $f : [0, 1] \to \mathbb{R}^n$. Any such function is defined as a solution in the space $L^p([0, 1])$ of a system of m equations $f(t) = \tilde{B}_k(mt + k)$; $t \in \left[\frac{k}{m}, \frac{k+1}{m}\right]$, where $k = 0, \dots, m-1$ and $\tilde{B}_k$ is an affine operator with linear part B_k. He found the conditions for smoothness of $\gamma = f[0, 1]$ in terms of eigenvalues of operators B_k and proved that the smoothness class of a self-affine fractal curve in $\mathbb{R}^n$ is either strictly smaller than n or it is infinite. In the latter case, γ is a polynomial curve of order n, which, in fact, is an affine image of a segment of the moment curve $\gamma(t) = (t, t^2, \dots, t^n)$, $t \in \mathbb{R}$.

In the last decade, Polikanova also proved that the moment curves in $\mathbb{R}^n$ are the only non-degenerate C^n-curves, every two oriented arcs of which are affine congruent. Her approach was purely geometrical [26,27].

Finally, in 2017, Feng and Käenmäki [28] proved a very powerful result in this series: an analytic curve in $\mathbb{R}^n$, $n \geq 2$, which cannot be embedded in a hyperplane, contains a non-trivial self-affine set if and only if it is an affine image of a segment of a moment curve in $\mathbb{R}^n$.

We put our questions in a slightly different fashion. Which conditions imply that a self-affine arc should be a parabolic segment? How a self-affine Jordan arc can be constructed, if it is not a parabolic segment? The answers are given by the following two theorems, which we prove in this article.

Theorem 1. *Let $\gamma \subset \mathbb{R}^2$ be a locally self-affine Jordan arc, which does not satisfy weak separation condition. Then, γ is a segment of a parabola or a straight line.*

Theorem 2. *Let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a system of affine contraction maps in $\mathbb{R}^2$, whose attractor is a Jordan arc γ, which is not a segment of a parabola or a straight line. There is a finite affine multizipper $\mathcal{Z} = \{S_{ijk}\}$ such that γ is the attractor of $\mathcal{Z}$. All the maps from the multizipper $\mathcal{Z}$ are elements of the system $\mathcal{S}$.*

The last two theorems extend the rigidity and structural theorems for self-similar Jordan arcs proved in [8] to the self-affine case.

In Section 2, we give all necessary definitions and remind some notions and results from the study by Tetenov and Chelkanova [29].

In Section 3, we consider the ranges of transversal directions to locally self-affine arcs, proving Proposition 18.

In Subsection 3.1, we prove that for any locally self-affine Jordan arc γ, the elements of sufficiently small neighborhood of identity are the affine shifts of γ (Proposition 20) and then prove Theorem 1.

In Section 4, we prove Theorem 2.

2 Preliminaries

Zippers and multizippers. The simplest way to construct a self-similar curve is to take a polygonal line and then make iterations, replacing each of its segments by a smaller copy of the same polygonal line; this construction is called zipper [2].

Definition 3. A system $\mathcal{S} = \{S_1, \dots, S_m\}$ of contraction mappings of $\mathbb{R}^d$ to itself is called a *zipper* with vertices $\{z_0, \dots, z_m\}$ and signature $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$, $\varepsilon_i \in \{0, 1\}$ if, for $i = 1 \dots m$, $S_i(z_0) = z_{i-1+\varepsilon_i}$ and $S_i(z_m) = z_{i-\varepsilon_i}$.

More general approach to the construction of self-similar curves and continua is provided by a graph-directed version of zippers [8], which we called multizippers.

Definition 4. Let $\{X_u, u \in V\}$ be a system of spaces, all isomorphic to $\mathbb{R}^d$. For each X_u, let a finite array of points be given as $\{x_0^{(u)}, \dots, x_{m_u}^{(u)}\}$. Suppose for each $u \in V$ and $0 \le k \le m_u$, we have some $v(u, k) \in V$ and $\varepsilon(u, k) \in \{0, 1\}$ and a map $\mathcal{S}_k^{(u)} : X_v \to X_u$ such that

$S_k^{(u)}(x_0^{(v)}) = x_{k-1}^{(u)}$ or $x_k^{(u)}$ and $S_k^{(u)}(x_{m_v}^{(v)}) = x_k^{(u)}$ or $x_{k-1}^{(u)}$, depending on the signature $\varepsilon(u, r)$.

The graph-directed IFS defined by the maps $S_k^{(u)}$ is called a *multizipper* $\mathcal{Z}$.

The attractor of a multizipper $\mathcal{Z}$ is a system of connected and arcwise connected compact sets $K_u \subset X_u$, satisfying the following system of equations:

$$K_u = \bigcup_{k=1}^{m_u} S_k^{(u)}(K_{v(u,k)}), \quad u \in V.$$

We call the sets K_u the components of the attractor of $\mathcal{Z}$.

In recent years, several authors have considered the multizippers, e.g. [30,31].

The components K_u of the attractor of a multizipper $\mathcal{Z}$ are Jordan arcs if the following conditions are satisfied [8]:

Theorem 5. *Let $\mathcal{Z}_0 = \{S_k^{(u)}\}$ be a multizipper with node points $x_k^{(u)}$ and with a signature $\varepsilon = \{(v(u, k), \varepsilon(u, k)), u \in V, k = 1, \dots, m_u\}$. If for any $u \in V$ and any $i, j \in \{1, 2, \dots, m_u\}$, the set*

$K_{(u,i)} \cap K_{(u,j)} = \varnothing$ *if* $|i-j| > 1$ *and is a singleton if* $|i-j| = 1$, *then any linear parametrization* $\{f_u : I_u \to K_u\}$ *is a homeomorphism and each* K_u *is a Jordan arc with endpoints* $x_0^{(u)}, x_m^{(u)}$.

Let γ be a Jordan arc in $\mathbb{R}^d$. Usually, we denote its endpoints by a and b, and if $x, y \in \gamma$, we denote the subarc of γ with endpoints x, y by $\gamma(x, y)$. We define the orientation on γ by the relation $x \le y$, which is equivalent to $\gamma(a, x) \subset \gamma(a, y)$.

Locally self-affine arcs and weak separation property (WSP). We say γ is *locally self-affine* if, for any open subarc $\gamma' \subset \gamma$, there is a non-degenerate contractive affine mapping S such that $f(\gamma) \subset \gamma'$.

We denote by $\mathcal{G}(\gamma)$ the semigroup of all non-degenerate affine contraction maps of γ into itself. Two maps $f_i, f_j \in \mathcal{G}(\gamma)$ are called γ*-incomparable* if neither $f_i(\gamma) \subset f_j(\gamma)$ nor $f_i(\gamma) \supset f_j(\gamma)$.

We call the family $\mathcal{F}(\gamma) = \{f_i^{-1} f_j : f_i, f_j \in \mathcal{G}(\gamma), f_i, f_j(\gamma)$ are γ-incomparable$\}$ *the associated family* for γ. The idea of the associated family was introduced by Bandt and Graf in [32] as a tool for testing the positiveness of Hausdorff measure of self-similar sets. Our current definition is suited for locally self-affine arcs, so it differs from the one of [32]. Note that γ-incomparability does not permit $\mathcal{F}(\gamma)$ to contain Id. We say $\mathcal{G}(\gamma)$ possesses WSP [33] if Id is not a limit point of $\mathcal{F}(\gamma)$.

Affine maps close to identity and their trajectories. Usually, in some argument, we restrict ourself to affine maps close to identity. We define a neighborhood $\mathcal{U}$ of the identity map Id in the group of non-degenerate affine maps $GA(\mathbb{R}^2)$ by

$$\mathcal{U} = \{f(x) = Ax + b, A \in GL(2, \mathbb{R}), \|A - E\| < 1/2, b \in \mathbb{R}^2, \|b\| < 1\}.$$

Let $A(\mathbb{R}^2)$ be the space of *all* affine maps of the plane equal to $g(y) = Ly + \beta$, where $L \in L(2, \mathbb{R})$ and $\beta \in \mathbb{R}^2$.

We use the following lemma from our article [29].

Lemma 6. *There is a homeomorphism* Ψ *of the set* $\mathcal{U}$ *to a neighborhood* $\mathcal{V}$ *of zero map in* $A(\mathbb{R}^2)$ *such that for any* $f \in \mathcal{U}$, *the map* $g = \Psi(f)$ *satisfies the following condition: for any* $x \in \mathbb{R}^2$, *the solution* $y(t)$ *of the Cauchy problem* $\{\dot{y} = g(y), y(0) = x\}$ *is equal to* $f(x)$ *at* $t = 1$.

The values of L and β for the map $g(y) = Ly + \beta$ are easily found. $L = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(A-E)^n}{n}$ is a matrix logarithm of A and $\beta = \sum_{n=0}^{\infty} (-1)^n \frac{(A-E)^n}{n+1} b$.

The image $g(y) = Ly + \beta$ defines an affine system $\dot{y} = Ly + \beta$, whose evolution operator

$$f^t(x) = e^{tL} x + e^{tL} \int_0^t e^{-sL} ds \cdot \beta \tag{1}$$

assumes the value $f(x)$ at $t = 1$ for any x, so we may write $f^t(x) = y(t)$. Thus, the map f is included to one-parameter multiplicative group of affine maps $\{f^t(x), t \in \mathbb{R}\}$.

This lemma allows us to include each orbit $\{f^k(a), k = 0, \ldots, N\}$ to a trajectory $L_f(x) = \{f^t(x); t \in \mathbb{R}\}$, which is of class C^∞.

Affine shifts and affine displacements of γ. A map $f(x) = Ax + b \in \mathcal{U}$ is called an *affine shift of the arc* γ if $\gamma \ne \gamma \cap f(\gamma) \ne f(\gamma)$ and $\#(\gamma \cap f(\gamma)) > 1$. We will see in Lemma 7 that f has no fixed points on γ. A map $f(x) = Ax + b \in \mathcal{U}$ is called an *affine displacement of the arc* γ if $f(\gamma) \cap \gamma = \varnothing$.

Lemma 7. *If* $g = f_i^{-1} f_j \in \mathcal{F}(\gamma)$ *and* $g(\dot{\gamma}) \cap (\dot{\gamma}) \ne \varnothing$, *and* g *preserves the orientation on* γ, *then* g *has no fixed points on* γ.

Proof. Indeed, let a, b be the endpoints of $\gamma = f_i^{-1} f_j$. Let $x \in \dot{\gamma}$ be a fixed point of g. Then, say, $f_i(a) < f_j(a) < f_i(x) = f_j(x) < f_i(b) < f_j(b)$. So $g(\gamma(a, x)) \subset \gamma(a, x)$ and $g^{-1}(\gamma(x, b)) \subset \gamma(x, b)$. Therefore, $\lim g^n(a) = x$ and $\lim g^{-n}(b) = x$, so x is a saddle point for g. Therefore, $\gamma(a, x)$ and $\gamma(x, b)$ should be non-collinear line segments, which is impossible. □

The first step for the Theorem 1 was the proof in our recent article of the following statement [29, Theorem 1(i)].

Theorem 8. *Let $\gamma = \gamma(a_0, a_1)$ be a locally self-affine Jordan arc with endpoints a_0, a_1 in $\mathbb{R}^2$ such that there is a sequence of affine shifts f_k of γ converging to* Id. *Then γ is a parabolic or a straight line segment.*

We also need the following corollary from this theorem:

Corollary 9. *Let γ be a locally self-affine Jordan arc in $\mathbb{R}^2$. Let $\gamma' \subset \dot{\gamma}$ be its subarc with endpoints x_0, x_1. If there is a sequence of affine maps g_n such that $\gamma' \subset g_n(\gamma) \subset \gamma$, then γ is a segment of a straight line or a parabola.*

Proof. By compactness, we can choose a subsequence, denoted the same way, such that the arcs $g_n(\gamma)$ converge to some arc γ'' with endpoints x_0', x_1' and $\gamma' \subset \gamma'' \subset \gamma$. If there is a sequence of n such that $g_n^{-1}g_{n+1}$ are affine shifts of γ, then γ is a segment of parabola by Theorem 8. Otherwise (by passing to a subsequence), we can assume that for any n, $g_{n+1}(\gamma) \subset g_n(\gamma)$ (or otherwise) and the fixed points y_n of $g_n^{-1}g_{n+1}$ belong to a sufficiently small subarc $\delta \subset \gamma$ so that $S_k(\gamma) \cap \delta = \varnothing$ for $k = 1$ or m. Then, the maps $S_k^{-1}g_n^{-1}g_{n+1}S_k$ form a sequence of affine shifts of γ, converging to identity, which proves the corollary. ☐

3 T-ranges for Jordan arcs in the plane

Let γ be a Jordan arc in the plane. For convenience, we use complex plane notation for our argument, so we consider $\gamma \subset \mathbb{C}$.

Definition 10. We say that $\alpha \in \mathbb{R}/\pi\mathbb{Z}$ defines a *transversal direction to the arc γ*, if for any line l, which intersects real axis in the angle α, $\#(\gamma \cap l) \le 1$.

The set $T(\gamma)$ of all transversal directions α to the arc γ is called the range of transversal directions to γ or *T-range of γ*.

We say that γ has *empty T-range for all subarcs* if, for any non-degenerate subarc $\gamma' \subset \gamma$, its T-range is empty.

In other words, a Jordan arc $\gamma \subset \mathbb{C}$ has empty T-range for all subarcs if, for any subarc $\gamma' \subset \gamma$ and any $\alpha \in \mathbb{R}/\pi\mathbb{Z}$, there is $z \in \gamma'$ and $\rho > 0$ such that $z + \rho e^{i\alpha} \in \gamma'$. Therefore, the complement $CT(\gamma) = \mathbb{R}/\pi\mathbb{Z} \setminus T(\gamma)$ is the set of all directions of non-degenerate chords with endpoints in γ.

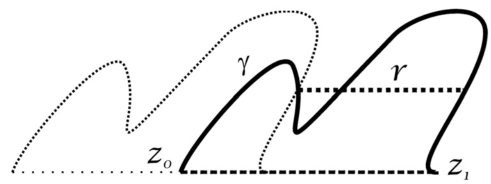
Lemma 11. *If γ is a locally self-affine Jordan arc in the plane, then either $T(\gamma) \neq \varnothing$ or γ has an empty T-range for all its subarcs.*

First, we prove that if γ contains no straight line segment, then $CT(\gamma)$ is an open set.

Lemma 12. *Let γ be a Jordan arc that lies completely in the upper half-plane* $\mathrm{Im} z > 0$ *except for its end points $z_0 < z_1$, which lie on the real axis. Then, for any positive $r < z_1 - z_0$, there are $z_0', z_1' \in \gamma$ such that $z_1' - z_0' = r$.*

Proof. Without loss of generality, we assume for (d1) and (d2) that $z_0 < z_1$.

Take $r < z_1 - z_0$. The points $z_0 + r$ and $z_1 + r$ do not belong to γ; therefore, there is $\delta > 0$ such that the neighborhoods $V_\delta(z_0 + r)$ and $V_\delta(z_1 + r)$ are disjoint from γ. If $0 < \varepsilon < \delta$, then $z_0 + r + i\varepsilon$ lies in the domain D bounded by the arc γ and a segment $[z_0 z_1]$, while the point $z_1 + r + i\varepsilon$ lies in the complement to the closed domain $\bar{D}$.

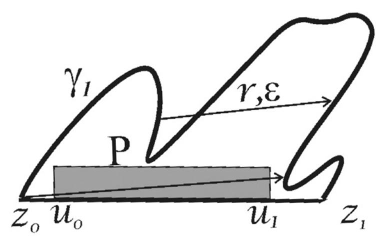


Therefore, the image of γ under the translation by $r + i\varepsilon$ intersects the arc γ. Reducing ε to 0, we obtain that $(\gamma + r) \cap \gamma \neq \varnothing$. This intersection does not contain points with real coordinates, and therefore, it lies in $\gamma \backslash \{z_0, z_1\}$. Take $z_1' \in \gamma \cap (\gamma + r)$. The point $z_0' = z_1' - r$ is also in γ, which proves the lemma. □

Corollary 13. *Let γ be a Jordan arc in $\mathbb{C}$ with endpoints z_0, z_1 such that $\gamma \cap [z_0, z_1] = \{z_0, z_1\}$.*
Then,

(1) *for any positive $r < |z_1 - z_0|$, there are $z_0', z_1' \in \gamma$ such that $|z_1' - z_0'| = r$ and $\operatorname{Arg}(z_1' - z_0') = \operatorname{Arg}(z_1 - z_0)$;*
(2) *there are $h, \varepsilon > 0$ such that for any $r < h$ and any $|\theta| < \varepsilon$, there are $z_0', z_1' \in \gamma$ for which $z_1' - z_0' = re^{i\theta}(z_1 - z_0)$.*

Proof. Consider the intersection of γ and the line l passing through the points $z_0 z_1$. There is at least one pair of points $z_0', z_1' \in l$ such that $[z_0, z_1] \in [z_0', z_1']$ and the subarc $\gamma' \subset \gamma$ with endpoints z_0', z_1' has no other common points with l. Applying Lemma 12 to the arc γ' and the half-plane bounded by l and containing γ', we come to (d1).



Suppose for convenience that $z_0, z_1 \in \mathbb{R}$ and $z_0 < z_1$. Take some segment $[u_0, u_1] \subset (z_0, z_1)$ and put $h = u_1 - u_0$. By the compactness of $[u_0, u_1]$, there is $\lambda > 0$ such that a rectangle $P = [u_0, u_1] \times [0, \lambda]$ does not intersect γ. For any ray l_+ starting at the point z_0 (resp. l_- starting at z_1), which intersects both vertical sides of the rectangle P, the set $l \cap P$ is contained in some interval $(\xi_0(l), \xi_1(l))$ for which $\xi_i(l) \in \gamma_1$, $(\xi_0(l), \xi_1(l)) \cap \gamma_1 = \varnothing$, and $\|\xi_0(l) - \xi_1(l)\| \geq h$. Then, by (d1), for any $r < h$, there are $z_0', z_1' \in \gamma$ such that the interval $[(z_0', z_1')]$ is parallel to l and $|z_1' - z_0'| = r$. Take $\varepsilon > 0$ such that the ray l_+, which forms an angle ε with (z_0, z_1), and the ray $l_- \ni z_1$, which forms an angle $-\varepsilon$ with (z_1, z_0), intersect both vertical sides of the rectangle P. Then, h and ε fit the statement (d2) of the lemma. □

Corollary 14. *Let $\gamma \subset \mathbb{R}^2$ be a Jordan arc $\mathbb{R}^2$ that does not contain a line segment. Then, $T(\gamma)$ is a closed subarc in $\mathbb{R}/\pi\mathbb{Z}$.*

Proof. Suppose $\alpha \in CT(\gamma)$. Then, there is a line l that intersects γ in at least two points. The complement $\gamma \backslash l$ is a disjoint union of subarcs with endpoints on l but no other intersection points with l. Therefore, by the statement (d2) of Corollary 13, there is a neighborhood $(\alpha - \varepsilon, \alpha + \varepsilon) \subset CT(\gamma)$. This shows that $CT(\gamma)$ is open in $\mathbb{R}/\pi\mathbb{Z}$.

Let $\alpha, \beta \in T(\gamma)$. Take a point $x \in \gamma$ and let l_α, l_β be the lines passing through x in directions α and β. These lines divide the plane into four angles. The set $\gamma \backslash \{x\}$ consists of two components, which are contained in two opposite angles of these four angles. As x travels along γ, these four angles remain the same; therefore, all values between α mod π and β mod π that correspond to the remaining two opposite angles belong to $T(\gamma)$. Since $CT(\gamma)$ is open, $T(\gamma)$ is a closed subarc in $\mathbb{R}/\pi\mathbb{Z}$. □

Lemma 15. *Let $\gamma \subset \mathbb{R}^2$ be a Jordan arc that has empty T-range for all subarcs.*
(1) *For any line $l \subset \mathbb{R}^2$, $l \cap \gamma$ is nowhere dense in l and in γ.*
(2) *For any line $l \subset \mathbb{R}^2$ and for any $n \in \mathbb{N}$, the set of those lines l' parallel to l for which $\#(l' \cap \gamma) < n$ is nowhere dense.*

Proof. The first statement is obvious, because the set $\gamma \cap l$ is closed and does not contain any straight line interval. We prove the second statement by induction in n.

Let l be a line and let a be a vector orthogonal to l such that for any $t \in (0, 1)$, $\gamma \cap (l + ta) \neq \varnothing$. We show that the set E_1 of those $t \in (0, 1)$ for which $\#(\gamma \cap (l + ta)) = 1$ is nowhere dense in $(0, 1)$. Suppose that the set E_1 is dense in some interval $(t_1, t_2) \subset (0, 1)$. This interval defines an open strip S, bounded by the lines $l + t_1 a$, $l + t_2 a$. Consider the intersection $\gamma \cap S = \gamma'$. Note that if for some $t \in (t_1, t_2)$, the set $\gamma \cap (l + ta)$ is disconnected, then it is disconnected for any t' in one of the intervals $(t - \varepsilon, t]$, $[t, t + \varepsilon)$ for some $\varepsilon > 0$. Therefore, γ' is a Jordan arc, and for any $t \in (t_1, t_2)$, the intersection $\gamma' \cap (l + ta)$ is either a point or a line segment contained in $l + ta$. The second case is impossible, because each line segment of that kind is a subarc in γ whose T-range is non-empty. If for each t, $\#(\gamma' \cap (l + ta)) = 1$, then $T(\gamma')$ is non-empty.

Therefore, for any Jordan arc γ, which has empty T-range for all subarcs, and for any line l and its orthogonal vector a, the set $\{t : \#(\gamma \cap (l + ta) = 1)\}$ is nowhere dense in $\mathbb{R}$.

Now suppose that for any subarc $\gamma' \subset \gamma$, the set $\{t : 0 < \#(\gamma' \cap (l + ta)) < n\}$ is nowhere dense in $(0, 1)$, whereas the set $\{t : 0 < \#(\gamma \cap (l + ta)) = n\}$ is dense in some interval $(t_1, t_2) \subset (0, 1)$. Let S be the open strip bounded by the lines $l + t_1 a$, $l + t_2 a$. Since γ intersects both these lines, one of the components of $\gamma \cap S$ is a subarc γ_0 with endpoints on different sides of S. Choose an endpoint x of this component so that both components γ_1 and γ_2 of $\gamma \backslash \{x\}$ have nonempty intersection with S. The arc γ_0 is a subarc of one of these components, say, γ_1. Suppose $(t_1', t_2') \subset (t_1, t_2)$ is such that for any $t \in (t_1', t_2')$, $\gamma_2 \cap (l + ta) \neq \varnothing$. Being contained in the intersection of the sets $\{t : 0 < \#(\gamma_1 \cap (l + ta)) < n\}$ and $\{t : 0 < \#(\gamma_2 \cap (l + ta)) < n\}$, which are both nowhere dense in (t_1', t_2'), the set $\{t \in (t_1', t_2') : \#(\gamma \cap (l + ta)) = n\}$ is nowhere dense in (t_1', t_2') too. □

It follows directly from Lemma 15 that if a Jordan arc γ has empty T-range for all subarcs, then for any line $l \subset \mathbb{R}^2$ and any $n \in \mathbb{N}$, there is a line $l' || l$, such that $\#(l' \cap \gamma) \geq n$.

We denote by $C(z, \rho, \alpha)$ an open half-circle $\{\xi = z + re^{i\theta}, 0 < r < \rho, \theta \in (\alpha - \pi/2, \alpha + \pi/2)\}$.

Lemma 16. *Let $\gamma(t) = (\gamma_x(t), \gamma_y(t))$, $t \in [0, 1]$ be a Jordan arc with endpoints $\gamma(0) = (0, 0)$, $\gamma(1) = (0, h)$ whose interior $\gamma((0, 1))$ lies in the open strip $S = \{(x, y) : 0 < y < h\}$ and has empty T-range for all subarcs. Let S_+ be a connected component of $S \backslash \gamma$ whose boundary contains $[0, +\infty)$.*

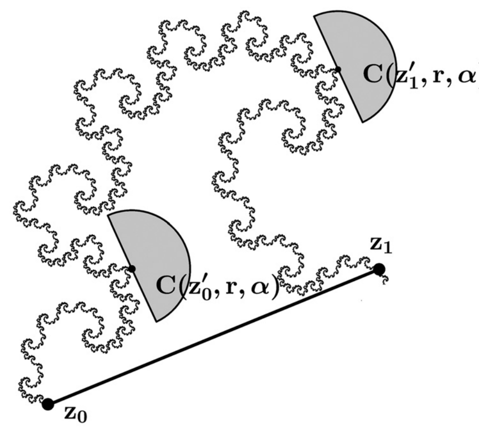
There is a point $a \in \gamma$ and $\rho > 0$ such that $C(a, \rho, 0) \subset S_+$.

Proof. Suppose $\max \gamma_x(t) > 0$, and $\tau \in (0, 1)$ satisfies $\gamma_x(\tau) = \max \gamma_x(t)$. Let $a = \gamma(\tau)$. If $\rho = \min(a_y, 1 - a_y)$, then $C(a, \rho, 0) \subset S_+$.

Suppose for any $x \in (0, 1)$, $\gamma_x(t) \leq 0$. As it follows from Lemma 15, there is a vertical line $l : x = \chi, \chi < 0$, which intersects γ more than three times. Let $\tilde{S}_+$ be an unbounded component of $S_+ \backslash l$. The set $\partial \tilde{S}_+ \backslash l$ is an union of more than two subarcs of the arc γ. Therefore, at least one of these subarcs, say γ', has both its endpoints in l. Let $a \in \gamma'$ be the point at which γ_x' reaches its maximum. Then, there is such ρ, that $C(a, \rho, 0) \subset S_+$. □

Lemma 17. *Let $\gamma(t)$, $t \in [0, 1]$ be a Jordan arc that has empty T-range for all subarcs. Suppose $\gamma(0) = z_0$, $\gamma(1) = z_1$, and $z_1 = z_0 + \mathrm{R}e^{i\alpha}$ and $\gamma((0, 1))$ lies completely in an open half-plane, bounded by the line l containing z_0, z_1. Let D be a domain bounded by γ and $[z_0, z_1]$.*

There are points $z_0', z_1' \in \gamma$ and $r > 0$ such that $C(z_0', r, \alpha) \subset D$ and $C(z_1', r, \alpha) \cap \bar{D} = \varnothing$



Proof. Without loss of generality, we may assume that $z_0 = 0$, $z_1 = 1$, $\alpha = 0$, and the domain D lies in the upper half-space. Let $h_2 = \max \gamma_y(t)$ and $\tau_0 = \min\{\tau : \gamma_y(\tau) = h_2\}$. Denote by γ_1 the subarc in γ with endpoints $(0, 0)$ and $(h_1, h_2) = \gamma(\tau_0)$. By Lemma 16, there exists a point $a \in \gamma_1$ such that $C(a, \rho, 0) \subset S_+$. There is a ball $B(\rho', a)$ such that $B(\rho', a) \cap \gamma \setminus \gamma_1 = \varnothing$. The set D is a component of $S_+ \setminus \gamma$ whose boundary contains γ_1, hence $C(a, \rho', 0) \subset S_+(\gamma_1)$. Let now $\tau_1 = \max\{\tau : \gamma_y(\tau) = h_2\}$ and $h_1' = \gamma_x(\tau_1)$. Denote by γ_2 a subarc in γ with endpoints $(1, 0)$ and (h_1', h_2) and let S_+' be an unbounded component of $S_+ \setminus \gamma_2$. By Lemma 16, there is a point $b \in \gamma_2$, for which there is a sector $C(b, \rho, 0) \subset S_+'$. Since $S_+' \cap D = \varnothing$, the same is true for $C(b, \rho, 0)$. □

Summarizing all previous argument, we come to the following proposition.

Proposition 18. *Let γ be a Jordan arc that has empty T-range for all its subarcs. For any $\alpha \in [0, \pi)$, there is a subarc $\gamma' \subset \gamma$ with endpoints z_1, z_2 such that $\arg(z_2 - z_1) = \alpha$ and $\gamma' \cup [z_1 z_2]$ is a closed Jordan curve bounding a domain D.*

There is $r > 0$ and points $z_0', z_1' \in \gamma'$ such that $C(z_0', r, \alpha) \subset D$ and $C(z_1', r, \alpha) \cap \bar{D} = \varnothing$.

3.1 Arbitrary small displacements of self-affine arcs cannot be neighbor maps

Proposition 19. *Let γ be a locally self-affine arc in $\mathbb{R}^2$. Suppose there is a sequence $\{f_n\}$ of affine displacements of γ that converges to Id. Then, $T(\gamma) \neq \varnothing$.*

Proof. Suppose contrary. Then, γ has empty T-range for all subarcs.

Therefore, we can choose a subsequence $f_n = A_n x + b_n$ (denoted the same way), a point $z_0 \in \gamma$, and a circle $V_0 = B(z_0, R)$, such that $\|A_n - Id\| < 1/2$ and fixed points of the maps or the points of invariant lines of f_n are not contained in B_0. By Lemma 6, we put into the correspondence to each of the maps $f_n(x) = A_n x + b_n$ an affine map $g_n(y) = B_n y + \beta_n$ such that $f_n(x)$ is equal to the value at $t = 1$ of the evolution operator $\varphi_{nt}(x)$ of non-homogeneous linear system $\dot{y} = B_n y + \beta_n$. Put $t_n = \sup\{\|g_n(x)\|, x \in V_0\}$. Let $\hat{g}_n(y) = g_n(y)/t_n$.

This way we obtain a sequence of linear dynamical systems in V_0, whose integral curves coincide with integral curves of respective systems $\dot{x} = g_k(x)$. At the same time, $\sup\{\|\hat{g}_k(x)\|, x \in V_0\} = 1$. Due to the convexity of functions $\|\hat{g}_k(x)\|$, the maximum value of each of these functions is attained on the boundary of the disc V_0.

Due to Arzelà-Ascoli theorem, the sequence $\hat{g}_n$ contains a subsequence, which converges uniformly in V_0 to some affine function g_0 such that $\sup\|g_0(z)\|$ on B_0 is equal to 1. Without loss of generality, we may assume that the initial sequence $\{g_n\}$ was chosen in such way that $\hat{g}_n \rightrightarrows g_0$ on V_0. Since, for any of the functions, its zero value or minimal value is attained outside the disc V_0, the value of $h = g_0(z_0)$ cannot be equal to zero. Let α be the direction of the vector h. There is a disc V_1 with the center z_0 and a number N such that $n > N$ and $z \in V_1$ imply $\|g_0(z) - g_0(z_0)\| < \|h\|/4$. Let γ_1 be a subarc in $\gamma \cap V_1$ such that for any $n > N$, $f_n(\gamma_1) \subset V_1$.

Let $\varphi_n(x, t)$ be the evolution operator for the linear non-homogeneous system $\dot{x} = \hat{g}_n(x)$. Then, for any n and $z \in \mathbb{R}^2$, $f_n(z) = \varphi_n(z, t_n)$.

Since $\varphi_n(z, t_n) - z = \int_0^{t_n} \hat{g}_n(\varphi_n(z, \tau))d\tau$, the inequality

$$\|\hat{g}_n(z) - g_0(z_0)\| \le \|\hat{g}_n(z) - g_0(z)\| + \|g_0(z) - g_0(z_0)\| < \|h\|/2$$

implies that $\|f_n(z) - z - t_n h\| \le t_n\|h\|/2$.

Therefore, for any $z \in \gamma_1$, the angle between vectors h and $f_n(z) - z$ is no greater than $\pi/6$.

Since the arc γ_1 has empty T-range for all its subarcs, there is a line l parallel to h, which intersects γ_1 at least at two points. As it follows from Corollary 13, there are $z_1, z_2 \in l \cap \gamma_1$ such that $\gamma_1(z_1, z_2) \cap l = \{z_1, z_2\}$. Let D be the domain bounded by $\gamma_1(z_1, z_2) \cup [z_1, z_2]$. By Lemma 17, there are the points $z_+, z_- \in \gamma_1(z_1, z_2)$ and $\rho > 0$ such that $C(z_+, r, \alpha) \subset D$, $C(z_-, r, \alpha) \cap \bar{D} = \varnothing$ and $C(z_-, r, \alpha) \cap \gamma = \varnothing$.

If we choose n such that $|f_n(z_0) - z_0| < \rho/2$, then $f_n(z_+) \in C(z_+, r, \alpha) \subset D$ and $f_n(z_-) \in C(z_-, r, \alpha) \subset C\bar{D}$. Therefore, $f_n(\gamma_1) \cap \gamma_1 \ne \varnothing$, which contradicts the condition $f_n(\gamma) \cap \gamma = \varnothing$.

This contradiction shows that there is a line l such that any line l' parallel to l intersects γ in at most one point, so T-range $T(\gamma)$ is non-empty. ☐

Proposition 20. *Let γ be a locally self-affine Jordan arc in $\mathbb{R}^2$, which is not a line segment. For any sequence $g_n \in \mathfrak{F}(\gamma)$, converging to* Id *and for any subarc $\gamma' \subset \gamma$, there is N such that for any $n > N$, $g_n(\gamma') \cap \gamma' \ne \varnothing$.*

Proof. Suppose contrary. Then $T(\gamma) \ne \varnothing$. Take a subarc $\gamma' \subset \gamma$ and let ε be a minimal width of a of a strip S bounded by two parallel lines such that $S \supset \gamma'$. The arc γ' is not a straight line, so $\varepsilon > 0$. Since the sequence g_n converges to Id, there is N such that for any $n > N$ and any $x \in \gamma$, $|g_n(x) - x| < \varepsilon/2$. Take $n > N$ and let $g_n = f_i^{-1}f_j$, where f_i and f_j are affine transformations mapping γ to its subarcs $f_i(\gamma)$ and $f_j(\gamma)$. Note that γ and $g_n(\gamma)$ are the subarcs of an arc $f_i^{-1}(\gamma)$. This arc is affine equivalent to γ, and therefore, $T(f_i^{-1}(\gamma)) \ne \varnothing$ and $T(f_i^{-1}(\gamma)) \subset T(\gamma)$. Take $\alpha \in T(f_i^{-1}(\gamma))$. Consider the minimal strips S_1 and S_2 bounded by pairs of parallel lines that intersect horizontal axis in the angle α and contain γ and $g_n(\gamma)$, respectively. Since $\gamma \cap g_n(\gamma) = \varnothing$, $S_1 \cap S_2 = \varnothing$. Take a point $z \in \gamma$ such that $d(z, S_2) > \varepsilon/2$. Since $g(z) \in S_2$, $|g(z) - z| > \varepsilon/2$. The obtained contradiction proves the lemma. ☐

Proof of Theorem 1. Let γ be a self-similar arc that does not satisfy WSP. Then, there is a sequence $g_n \in \mathcal{F}(\gamma)$ converging to Id. For sufficiently large n, $g_n \in \mathcal{U}$, and by Proposition 20, g_n are affine shifts of γ. Therefore, by Theorem 8, the arc γ is a segment of a straight line or a parabola. ☐

4 Proof of Theorem 2: The partition to elementary subarcs

Let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a system of contractive affine maps in $\mathbb{R}^2$ with Jordan attractor γ.

We use standard notation in this case. The set of indices $\{1, \dots, m\}$ is denoted by I. The subarcs $S_i(\gamma)$ are denoted by γ_i. $\mathcal{G}$ is a semigroup, generated by $\{S_1, \dots, S_m\}$, and the families $\mathcal{F}$ and $\mathcal{F}'$ are defined accordingly.

Let a_0 and a_1 be the endpoints of γ. For any two points $x, y \in \gamma$, we write that $x \le y$, if $\gamma(a_0, x) \subset \gamma(a_0, y)$.

We may suppose that the system $\mathcal{S}$ is irreducible, i.e., for any $k \in I$, $\bigcup_{i \in I \setminus \{k\}} \gamma_i \ne \gamma$. Hence, we can order the maps $S_1, \dots, S_m$ so that $\gamma_i \cap \gamma_j \ne \varnothing$ if and only if $|i - j| = 1$, while $a_0 \in \gamma_1$ and $a_1 \in \gamma_m$.

The idea of the proof of Theorem 2 is to construct a finite set $\mathcal{P} \subset \gamma$, whose points $a_0 = p_0 < p_1 < \dots < p_{N-1} < p_N = a_1$ define a partition of γ to subarcs δ_i, $i = 1, \dots, N$, satisfying the following conditions:

a1. For any δ_i and any $k = 1, \dots, m$, there is δ_j such that $S_k(\delta_i) \subset \delta_j$.

a2. For any $k_1, k_2 = 1, \dots, m$, and for any $\delta_{i_1}, \delta_{i_2}$, $S_{k_1}(\dot{\delta}_{i_1})$ and $S_{k_2}(\dot{\delta}_{i_2})$ are either equal or disjoint.

For each $g \in \mathcal{F}'$, the set $\gamma \cap g(\gamma)$ is a non-degenerate subarc, which we denote by γ_g. The endpoints of γ_g are the points $g(a_i)$ and a_j, $i, j \in \{0, 1\}$.

Let $\mathcal{P}$ be the set consisting of a_0 and a_1 and of points $g(a_i)$, where $i = 0, 1$, $g \in \mathcal{F}'$ and $g(a_i) \in \gamma_g \cap \dot{\gamma}$. Let $\mathcal{P}_i$ be the set of those $p \in \mathcal{P} \cap \dot{\gamma}$, which are the endpoints of subarcs γ_g that do not contain a_{1-i}. Thus, $\mathcal{P} = \{a_0, a_1\} \cup \mathcal{P}_0 \cup \mathcal{P}_1$.

Note two properties of the set $\mathcal{P}$, which directly follow from its definition:

b1. Let $g \in \mathcal{G}$. Then, $\mathcal{P} \cap \dot{g}(\gamma) \subset g(\mathcal{P})$.

b2. Let $g_1, g_2 \in \mathcal{G}$ be two γ-incomparable affine maps such that $g_1(\gamma) \cap g_2(\gamma)$ is a non-degenerate subarc of γ. Then, the endpoint of the subarc $g_1(\gamma)$, contained in $g_2(\dot{\gamma})$, lies in $g_2(\mathcal{P})$, and *vice versa*.

In the case when the maps g belong to $\mathcal{S}$, the conditions **b1** and **b2** become the following ones:

c1. For any $i \in \{1, \dots, m\}$, $\mathcal{P} \cap \dot{\gamma}_i \subset S_i(\mathcal{P})$;

c2. For any $1 \le j \le m-1$, $S_j(\{a_0, a_1\}) \cap \dot{\gamma}_{j+1} \subset g_{j+1}(\mathcal{P})$ and $S_{j+1}(\{a_0, a_1\}) \cap \dot{\gamma}_j \subset g_j(\mathcal{P})$

Lemma 21. *Suppose the system $\mathcal{S} = \{S_1, \dots, S_m\}$ is irreducible. Then,*

d1. the set of limit points of $\mathcal{P}$ is contained in $\{a_0, a_1\}$.

d2. there are such neighborhoods U_i of the points a_i, where $i = 0, 1$, that $P_{1-i} \cap U_i = \varnothing$, and if for some $k \in \{1, m\}$ and $i, j \in \{0, 1\}$, $S_k(a_i) = a_j$, then S_k is a bijection of $U_i \cap \mathcal{P}_i$ to $S_k(U_i) \cap \mathcal{P}_j$.

Proof. First, we show that the set $\mathcal{P}$ has no limit points in $\dot{\gamma}$. Suppose there is a $c \in \dot{\gamma} \cap \bar{\mathcal{P}}$. Then, for one of the endpoints of γ, say, a_0, there is a sequence $g_n \in \mathcal{G}$ such that $g_n(a_0) \to c$. It follows from Corollary 9 that γ is a segment of a parabola, which contradicts the assumptions of the lemma, so **d1** is true. The same argument shows that a_1 cannot be a limit point of $\mathcal{P}_0$ and a_0 cannot be a limit point of $\mathcal{P}_1$. Therefore, there are such neighborhoods U_i of the points a_i such that $\mathcal{P}_{1-i} \cap U_i = \varnothing$. Moreover, we choose U_0, U_1 in such a way that $\gamma \cap U_0 \subset \gamma_1$ and $\gamma \cap U_1 \subset \gamma_m$.

To check **d2**, we first consider the case when $S_1(a_1) = a_0$. If $p \in \mathcal{P}_0 \cap U_0$ and $p = g(a_i)$, then $S_1^{-1} \circ g \in \mathcal{G}$ and $S_1^{-1}(p) \in \mathcal{P}_1 \cap S_1^{-1}(U_0)$. Conversely, if $p \in \mathcal{P}_1 \cap U_1$ and $p = g(a_i)$, then $S_1 \circ g \in \mathcal{G}$ and $S_1(p) \in \mathcal{P}_0 \cap S_1(U_1)$. Therefore, S_1 defines a bijection $\mathcal{P} \cap U_0 \cap S_1(U_1)$ to $\mathcal{P} \cap U_1 \cap S_1^{-1}(U_0)$. Enumerating all possibilities:

1. $S_1(a_0) = a_0$, $S_m(a_1) = a_1$;
2. $S_1(a_0) = a_0$, $S_m(a_1) = a_0$;
3. $S_1(a_0) = a_1$, $S_m(a_1) = a_1$;
4. $S_1(a_0) = a_1$, $S_m(a_1) = a_0$,

we find the desired pairs of neighborhoods for each of the cases. □

Lemma 22. *The set $\mathcal{P}$ contains a finite subset $\mathcal{P}'$, which also satisfies* **c1** *and* **c2**.

Proof. For each of the points $S_k(a_i) \in \dot{\gamma}$, where $k \in I$ and $i = 0, 1$, we denote by $w(k, i)$ the connected component of the set $\gamma_k \backslash \mathcal{P}$, which has $S_k(a_i)$ as its endpoint, whereas for $S_k(a_i) = a_j$, we put $w(k, i) = U_j$. Let $W_i = \bigcap_{k \in I} S_k^{-1}(w(k, i)) \cap U_i$ and let $\mathcal{P}' = \{a_0, a_1\} \cup \mathcal{P} \backslash (W_0 \cup W_1)$.

The set $\mathcal{P}'$ is finite, so we denote its elements by $a_0 = p_0 < p_1 < \dots < P_M = a_1$, and we denote the subarcs $\gamma(p_{k-1}, p_k)$ by δ_k.

For any $j \in I$, $S_j(\mathcal{P}) \subset S_j(W_0 \cup W_1) \cup S_j(\mathcal{P}')$. At the same time, the definition of $\mathcal{P}'$ implies that $S_j(W_0 \cup W_1) \cup S_j(\mathcal{P}') = S_j(\{a_0, a_1\})$. Therefore, $\mathcal{P}' \cap \gamma_j \subset S_j(\mathcal{P}')$. Thus, the set $\mathcal{P}'$ satisfies the condition **c1**. The condition **c2** directly follows from the definition of $\mathcal{P}'$. □

Lemma 23. *Each of the subarcs δ_i, $i = 1, \dots, M$ and γ_i, $i \in I$ is an union of subarcs $S_j(\delta_k)$ for some $j \in I$ and some $k \in \{1, \dots, M\}$ whose interiors are disjoint.*

Proof. The system $\mathcal{S}$ is irreducible, and therefore, each subarc γ_j, $1 < j < m$ intersects two adjacent subarcs $\gamma_{j-1}, \gamma_{j+1}$, so that $\gamma_j \setminus (\gamma_{j-1} \cup \gamma_{j+1}) \neq \varnothing$. For any subarc $\bar{\gamma}_j = \gamma_j \setminus (\dot{\gamma}_{j-1} \cup \dot{\gamma}_{j+1})$, its endpoints by **c2** are contained in $S_j(\mathcal{P}')$; let them be the points $S_j(p_{k_j})$, $S_j(p_{K_j})$. The arc $\bar{\gamma}_j$ has unique representation $\bigcup_{i=k_j}^{K_j-1} S_j(\delta_i)$. For each of the subarcs $\gamma_j \cap \gamma_{j+1}$, there are exactly two partitions: first, to the subarcs $S_j(\delta_i)$ and second, to the subarcs $S_{j+1}(\delta_i)$; choose one of them. Taking the union over all subarcs and renumerating all the points, we obtain the desired partition for the whole γ. By the property **c1**, the partition we obtained is at the same time a partition for each of the subarcs δ_k. □

Proof of the Theorem 2. Now we can construct a Jordan multizipper, for which the components of the attractor will be the subarcs δ_j. Each of the subarcs δ_j, $j = 1, \dots M$, is a finite union of consequent subarcs $S_i(\delta_k)$, which form a partition of δ_j. Therefore, we can create a graph $\tilde{G}$ whose vertices are $u_j = \delta_j$, and an edge e_{ij} is directed from u_i to u_j if there is such S_k, that $S_k(U_j) \subset \delta_i$. □

Acknowledgements: The authors thank the reviewers for their careful reading, constructive comments, and suggestions.

Funding information: The work is supported by the Mathematical Center in Akademgorodok under the agreement no. 075-15-2022-281 with the Ministry of Science and Higher Education of the Russian Federation.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state that there is no conflict of interest.

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