Research Article

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Compositions of positive integers with 2*s* and 3*s*

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Abstract: In this article, we consider compositions of positive integers with 2*s* and 3*s*. We see that these compositions lead us to results that involve Padovan numbers, and we give some tiling models of these compositions. Moreover, we examine some tiling models of the compositions related to the Padovan polynomials and prove some identities using the tiling model's method. Next, we obtain various identities of the compositions of positive integers with 2*s* and 3*s* related to the Padovan numbers. The number of palindromic compositions of this type is determined, and some numerical arithmetic functions are defined. Finally, we provide a table that compares all of the results obtained from compositions of positive integers with 2*s* and 3*s*.

Keywords: Fibonacci numbers, Padovan numbers, binomials, Pascal triangle, compositions, palindromic

MSC 2020: 11B37, 11B39, 11B83, 11B65, 11B75, 05A19

1 Introduction

Studies on compositions have a long and rich history. The first publication on composition was published by Machahon (1854–1924) in 1893 entitled "Memoir on the Theory of Compositions of a Number" [1]. A composition of a positive integers n is any $G = (G_1, G_2, \ldots, G_m)$ of positive integers such that $\sum_{i=1}^{m} G_i = n$. Here, the G_is are called the part of the compositions and m denotes the numbers part of compositions. For example, the compositions of 4 are 4, 3 + 1, 1 + 3, 2 + 2, 2 + 1 + 1, 1 + 2 + 1, 1 + 1 + 2 and 1 + 1 + 1 + 1, and their number is 8. The total number of compositions of any positive integer n is well known to be 2^{n-1} [1]. There are restricted and unrestricted types of compositions in the literature. Unrestricted integer compositions are vast. The parts of restricted compositions are in a fixed subset G of positive integers and have also received a fair amount of attention [2,3]. Heubach and Mansour found generating functions for the number of compositions, avoiding a single pattern or a pair of patterns of length three on the alphabet {1, 2} and determining which of them are Wilf-equivalent on compositions. They also derived the number of permutations of a multiset that avoid these same patterns and determined the Wilf equivalence of these patterns on permutations of multisets in [4]. Heubach and Mansour gave more detailed information on the subject in their book titled "Combinatorics of Compositions and Words" in [5], in [6], they also studied the generating functions for several counting problems for compositions, palindromic compositions, Carlitz compositions, and Carlitz palindromic compositions with parts in A, respectively. In [7], Savage and Wilf showed that among the compositions of n into positive parts, the number $g(n)$ that avoid a given pattern π of three letters is independent of π, found the generating function of $\{g(n)\}$, and it shows that the sequence $\{g(n)\}$ is not P-recursive. Alladi and Hoggatt examined palindromic numbers and various identities of the compositions of positive integers with 1*s* and 2*s* in [8]. Banderier and Hitczenko studied pairs and m-tuples of compositions of a positive integer n with parts

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restricted to a subset P of positive integers and obtained some exact enumeration results for the number of tuples of such compositions having the same number of parts as in [9]. Chinn and Heubach counted the number of compositions and the number of palindromes of n that do not contain any occurrence of a particular positive integer k, and also found the total number of occurrences of each positive integer among all the compositions of n without occurrences of k in [10]. Eger proved a simple relationship between extended binomial coefficients, which are natural extensions of the well-known binomial coefficients, and weighted, restricted integer compositions. Moreover, Eger gave a very useful interpretation of extended binomial coefficients as representing distributions of sums of independent discrete random variables in [11]. The earliest papers on restricted compositions are those on compositions with ones and twos, which are given by Alladi and Hoggatt [12,13]. The authors in [12,13] discussed the compositions of integers with 1*s* and 2*s*, and they proved that these compositions lead us the Fibonacci numbers. In [14], Sills provided some commentary about the history of partitions, compositions, and Fibonacci numbers. In [15], Gessel and Li studied formulas expressing Fibonacci numbers as sums over compositions, provided a systematic account of such formulas using free monoids, and showed that the number of compositions of n with parts 1 and 2 is the Fibonacci number F_{n+1}. In [16], Kimberling studied the enumeration of paths, compositions of integers, and Fibonacci numbers. In [17], Knopfmacher and Robbins worked on binary and Fibonacci compositions.

Although there have been many studies on the combinatorial interpretations of the Fibonacci numbers in the mathematical literature, there have not yet been enough studies on the combinatorial studies for the Padovan numbers. In the present work, we investigate the combinatorial interpretations of the Padovan sequence by considering the compositions of positive integers with 2*s* and 3*s*. We introduce some tiling models related to the Padovan numbers, and we prove some results involving various identities for the compositions of positive integers with 2*s* and 3*s*. We also give tiling models and certain binomial sums of the Padovan polynomials. Finally, we explore the palindromic representations of the compositions with 2*s* and 3*s*, and obtain various results. Let us begin the introduction with the definition of Padovan numbers.

The Padovan sequence $\{P_n\}_{n\geq 0}$ is defined by the third-order recurrence:

$$P_{n+3} = P_{n+1} + P_n, \tag{1}$$

with the initial conditions $P_0 = 1$, $P_1 = 0$, and $P_2 = 1$. The Padovan sequence appears as sequence A000931 on the On-Line Encyclopedia of Integer Sequences [18]. Consistency we consider as $P_{-2} = P_{-1} = 0$. The first few values of this sequence are 1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151.

Many authors have studied the sequence of Padovan numbers for decades. However, Padovan [19] first defined it formally, and then Stewart [20] gave them the name Padovan numbers. For details, we refer to works; Cerda-Morales investigated new identities for the Padovan numbers in [21]. Deveci and Karaduman defined the Padovan p-numbers, and then they obtained their miscellaneous properties such as the generating matrix, the Binet formula, the generating function, the exponential representation, the combinatorial representations, the sums and permanental representation, and also studied the Padovan p-numbers modulo m in [22]. İşbilir and Gürses investigated the Padovan (or Cordonnier) and Perrin generalized quaternions, and obtained the new identities for these special quaternions related to matrix forms. They also introduced Binet-like formulae, generating functions, several summation, and binomial properties concerning these quaternions in [23]. Soykan investigated the generalized Padovan sequences and deal with, in detail, four special cases, namely, Padovan, Perrin, Padovan-Perrin, and modified Padovan sequences, and presented Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences in [24]. Yilmaz and Taskara developed the matrix sequences that represent Padovan and Perrin numbers, and by taking into account matrix properties of these new matrix sequences, some behaviors of Padovan and Perrin numbers investigated. Moreover, they presented some important relationships between Padovan and Perrin matrix sequences in [25].

The Fibonacci polynomials are defined in [12] by

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x),$$

$F_0 = 0$, $F_1 = 1$. In [26], the Padovan polynomial sequence $\{P_n(x)\}_{n\geq 0}$ is defined by a third-order recurrence

$$P_{n+3}(x) = xP_{n+1}(x) + P_n(x), \tag{2}$$

with the initial conditions $P_0(x) = 1$, $P_1(x) = 0$, and $P_2(x) = x$. For consistency, we consider as $P_{-2}(x) = P_{-1}(x) = 0$. To simplify notation, take $P_n(x) = \mathcal{P}_n$. The first few values of this sequence are 1, 0, x, 1, x^2, $2x$, $x^3 + 1$, $3x^2$, $x^4 + 3x$.

The recurrence (2) involves the characteristic equation:

$$t^3 - xt - 1 = 0.$$

By α_x, β_x, and γ_x if we denote the roots of the characteristic equation above, we can derive the following equalities:

$$\alpha_x + \beta_x + \gamma_x = 0, \quad \alpha_x\beta_x + \alpha_x\gamma_x + \beta_x\gamma_x = -x, \quad \alpha_x\beta_x\gamma_x = 1.$$

The Binet-like formula for the Padovan polynomials sequence is

$$\mathcal{P}_n = a_x\alpha_x^n + b_x\beta_x^n + c_x\gamma_x^n, \tag{3}$$

where

$$a_x = \frac{\beta_x\gamma_x + x}{(\alpha_x - \beta_x)(\alpha_x - \gamma_x)}, \quad b_x = \frac{\alpha_x\gamma_x + x}{(\beta_x - \alpha_x)(\beta_x - \gamma_x)}, \quad c_x = \frac{\alpha_x\beta_x + x}{(\gamma_x - \alpha_x)(\gamma_x - \beta_x)}.$$

There are two different studies on the compositions of positive integers with 2*s* and 3*s* in the literature, and on the Padovan numbers. First, Tedford also interpreted the Padovan numbers combinatorially by having them count the number of tilings of an n-strip using dominoes and triominoes, and developed a collection of identities satisfied by the sequence of Padovan numbers using this interpretation in [27]. Secondly, Vieira et al. obtained a generalization of the Padovan combinatorial model and introduced the Padovan sequence combinatorial approach in [28]. Now, let us give the relation between Padovan numbers and compositions of positive integers with 2*s* and 3*s*. Similarly, let us show the relation between Fibonacci numbers and compositions of positive integers with 1*s* and 2*s*.

A composition of any positive integer n is a representation of n as a sum of positive integers. For instance, the 16 compositions of 5 are

$$5, 4+1, 1+4, 3+2, 2+3, 3+1+1, 1+3+1, 1+1+3, 2+2+1, 2+1+2, 2+1+1+1, 1+2+2, 1+2+1+1, 1+1+2+1, 1+1+1+2, 1+1+1+1+1.$$

A partition of any positive integer n is a representation of n as a sum of positive integers where the order of the summands is considered irrelevant [14]. For instance, the seven partitions of 5 are

$$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.$$

In 1974, first, Alladi and Hoggatt [8] have considered all the (ordered) representations of a positive integer n as a sum of ones and twos. Denote by C_n for positive integer n, the number of compositions of n using only 1 and 2. For example, 2 has two distinct such compositions, 3 has three, and 4 has five; see Table 1.

Table 1: Some compositions of integers related to the Fibonacci numbers

n	Compositions with 1*s* and 2*s*	C_n
1	1	1
2	1 + 1, 2	2
3	1 + 1 + 1, 1 + 2, 2 + 1	3
4	1 + 1 + 1 + 1, 1 + 1 + 2, 1 + 2 + 1, 2 + 1 + 1, 2 + 2	5
5	1 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 2, 1 + 1 + 2 + 1, 1 + 2 + 1 + 1, 2 + 1 + 1 + 1, 1 + 2 + 2, 2 + 1 + 2, 2 + 2 + 1	8
6	1 + 1 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 2, 1 + 1 + 1 + 2 + 1, 1 + 1 + 2 + 1 + 1, 1 + 2 + 1 + 1 + 1, 2 + 1 + 1 + 1 + 1, 1 + 1 + 2 + 2, 1 + 2 + 1 + 2, 1 + 2 + 2 + 1, 2 + 1 + 2 + 1, 2 + 2 + 1 + 1, 2 + 1 + 1 + 2, 2 + 2 + 2	13

Compositions with ones and twos are related to the second-order recurrence relations, and it is showed [8] that the number of distinct compositions C_n of any positive integer n with ones and twos is the Fibonacci number F_{n+1}, for any $n \in \mathbb{Z}^+$ (for details, see [29,30]).

In the present work, we attempt to throw some new light by discussing compositions that lead to the recurrence relations of the Padovan numbers. We restrict our attention to compositions using only twos and threes. For example, 3 has one distinct composition and 4 has two. Table 2 shows which components correspond to the Padovan number values.

Table 2: The compositions of with 2*s* and 3*s*

n	Compositions with 2*s* and 3*s*	CP_n
1	—	0
2	2	1
3	3	1
4	2 + 2	1
5	2 + 3, 3 + 2	2
6	2 + 2 + 2, 3 + 3	2
7	2 + 2 + 3, 2 + 3 + 2, 3 + 2 + 2	3
8	2 + 2 + 2 + 2, 2 + 3 + 3, 3 + 2 + 3, 3 + 3 + 2	4
9	2 + 2 + 2 + 3, 2 + 2 + 3 + 2, 2 + 3 + 2 + 2, 3 + 2 + 2 + 2, 3 + 3 + 3	5
10	2 + 2 + 2 + 2 + 2, 2 + 2 + 3 + 3, 2 + 3 + 2 + 3, 2 + 3 + 3 + 2, 3 + 3 + 2 + 2, 3 + 2 + 2 + 3, 3 + 2 + 3 + 2	7

The present article is organized as follows: We briefly review the importance of and studies on the composition of positive integers in Section 1. Section 2 is devoted to some results from compositions of positive integers with 2*s* and 3*s*. In Section 3, we give tiling models for these compositions. Section 4 is devoted to a model or certain binomial sums of the Padovan polynomials. In Section 5, we consider the palindromic representations of the compositions with 2*s* and 3*s*, and define an arithmetical function related to these palindromic representations.

2 The results of the compositions with 2*s* and 3*s*, and the Padovan numbers

Theorem 2.1. *The number of distinct compositions* CP_n *of a positive integer* n *with* 2*s and* 3*s is* P_n, *where* $n \geq 1$.

Proof. Let $\mathrm{CP}_n(2)$ and $\mathrm{CP}_n(3)$ denote the number of compositions of n that end in 2 and 3, respectively. So,

$$\begin{aligned}
&\text{For} \quad \mathrm{CP}_1(2) = 0 \quad \text{and} \quad \mathrm{CP}_1(3) = 0, \mathrm{CP}_1 = \mathrm{CP}_1(2) + \mathrm{CP}_1(3) = 0,\\
&\text{For} \quad \mathrm{CP}_2(2) = 1 \quad \text{and} \quad \mathrm{CP}_2(3) = 0, \mathrm{CP}_2 = \mathrm{CP}_2(2) + \mathrm{CP}_2(3) = 1,\\
&\text{For} \quad \mathrm{CP}_3(2) = 0 \quad \text{and} \quad \mathrm{CP}_3(3) = 1, \mathrm{CP}_3 = \mathrm{CP}_3(2) + \mathrm{CP}_3(3) = 1.
\end{aligned}$$

Assume that $n \geq 3$.
Case 1. Pick a composition of n, ending in 2. Deleting this 2 yield a composition of $n - 2$. Thus,

$$\mathrm{CP}_n(2) = \mathrm{CP}_{n-2}. \tag{4}$$

Case 2. Suppose a composition of n, ending in 3. Deleting this 3, we obtain a composition of $n - 3$. Therefore,

$$\mathrm{CP}_n(3) = \mathrm{CP}_{n-3}. \tag{5}$$

Using (4) and (5), $\mathrm{CP}_n = \mathrm{CP}_n(2) + \mathrm{CP}_n(3) = \mathrm{CP}_{n-2} + \mathrm{CP}_{n-3}$. This recurrence relation, with the initial conditions, proves the desired result. □

In Theorem 2.2, by $f(n)$, we indicate the total number of 2*s* in the compositions of n, and by $g(n)$, we indicate the total number of 3*s* in the compositions of n. For example, $f(8) = 7$ and $g(8) = 6$ (Table 2).

Theorem 2.2. *Let* $n \geq 4$. *Then*
(*i*) $f(n) = f(n-2) + f(n-3) + P_{n-2}$,
(*ii*) $g(n) = g(n-2) + g(n-3) + P_{n-3}$.

Proof. *i.* As in the preceding proof, we have

$$\mathrm{CP}_n = \mathrm{CP}_n(2) + \mathrm{CP}_n(3).$$

Since $CP_n(3) = CP_{n-3}$, there are CP_{n-3} compositions of n that end in 3. But CP_{n-3} denotes the number of compositions of $n-3$. By definition, there is total of $f(n-3)2s$ in the compositions of $n-3$. Since $CP_n(2) = CP_{n-2}$, there are CP_{n-2} compositions of n that end in 2. Excluding this 2, they contain $f(n-2)$ 2s. Since each of the CP_{n-2} compositions contains a 2 as the final addend, they contain a total of $f(n-2) + CP_{n-2} = f(n-2) + P_{n-2}$ 2s. Thus, $f(n) = f(n-2) + f(n-3) + P_{n-2}$, where $n \geq 4$.

ii. Similarly, $g(n) = g(n-2) + g(n-3) + P_{n-3}$, where $n \geq 4$. For example,

$$f(5) + f(4) + P_5 = 2 + 2 + 2 = 6 = f(7), \quad g(5) + g(4) + P_4 = 2 + 0 + 1 = 3 = g(7). \qquad \square$$

Theorem 2.3. *For all* $n \geq 1$,

$$f(n) = g(n+1).$$

Proof. We will prove this using the principle of mathematical induction. Its follows from $f(1) = 0 = g(2)$ and $f(2) = 1 = g(3)$ the result is valid for $n = 1, 2$. Assume that the relation is valid for all positive integers less than n. Thus, we obtain $f(n-2) = g(n-1)$ and $f(n-3) = g(n-2)$. By Theorem 2.2, we then have

$$f(n) = f(n-2) + f(n-3) + P_{n-2} = g(n-1) + g(n-2) + P_{n-2} = g(n+1).$$

Hence, the desired is true for n. So, by the principle of mathematical induction, it is valid for all positive integers. $\square$

3 A Padovan tiling model of compositions with 2*s* and 3*s*

Benjamin and Quinn [31] considered many technical properties for the combinatorial interpretation of special numbers. Tedford [27] have interpreted the Padovan numbers by having them count the number of any n-strip using dominoes (1 × 2 tiles) and triominoes (1 × 3 tiles). He proved that the number of the collection of all such tilings for n-strip p_n is the nth Padovan numbers.

In the present work, we give a similar tilings model for the weight of such tilings. Theorem 2.1 shows an interesting combinatorial interpretation. To clarify this, suppose we would like to tile a $1 \times n$ board with 1×2 tiles and 1×3 tiles. By T_n, we denote the number of tiling of a $1 \times n$ board with a tiling of length n. In Figure 1, the possible tilings of $1 \times n$ board, for $1 \leq n \leq 10$, are given. From the figure, it is shown that the number of n-tilings T_n gives for $n \in \mathbb{Z}^+$ the nth Padovan number.

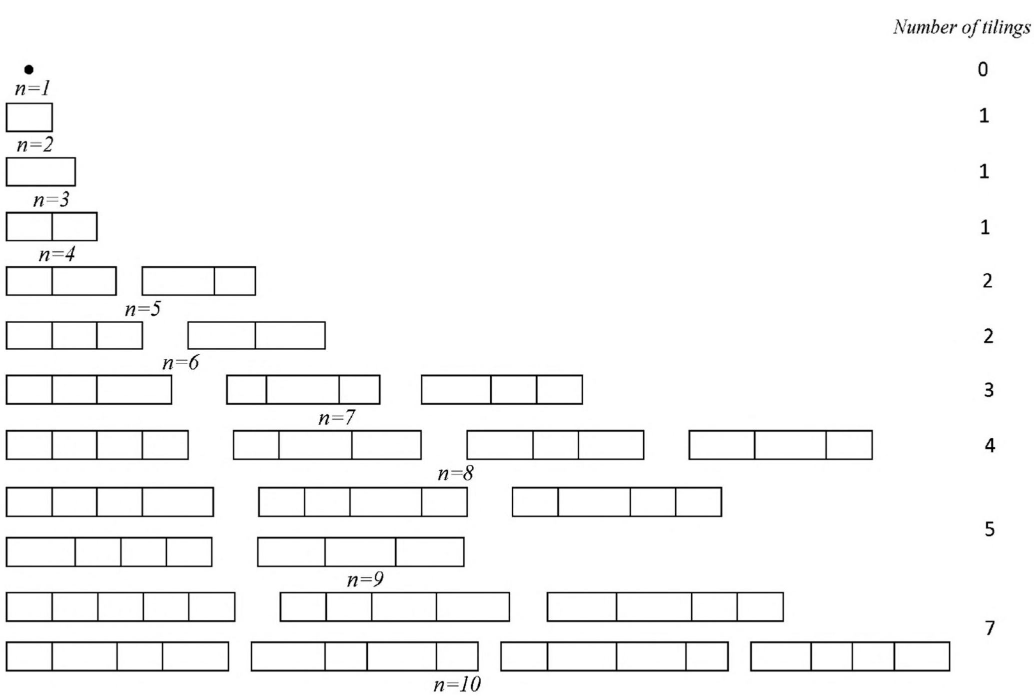

Figure 1: Tilings of a $1 \times n$ board.

So, $T_n = T_{n-2} + T_{n-3}$, where $T_1 = 0$, $T_2 = 1$ and $T_3 = 1$. So, we can prove the following theorem.

Theorem 3.1. [27,28] *The number of tilings of a* $1 \times n$ *board with* 1×2 *tiles and* 1×3 *tiles is* P_n, *where* $n \geq 1$. *In other words,* $T_n = P_n$.

Before proceeding to the proof of the Theorem 3.1, we present Theorem 3.2.

Theorem 3.2. *Let* $n \geq 0$. *Then*

$$P_n = \sum_{j=\left[\frac{n}{3}\right]}^{\left[\frac{n}{2}\right]} \binom{j}{n-2j}.$$

Proof. We will prove the result using the strong induction. Since $\binom{0}{0} = 1 = P_0$, $\binom{0}{1} = 0 = P_1$, and $\binom{0}{2} + \binom{1}{0} = 1 = P_2$, the statement is true when $n = 0, 1, 2$. Now assume it is true for all positive integers ($\leq k$), where $k \geq 0$. By Pascal's identity, we have

$$\begin{aligned}
P_{k+3} &= \sum_{j=\left[\frac{k+3}{3}\right]}^{\left[\frac{k+3}{2}\right]} \binom{j}{k-2j+3} \\
&= \sum_{j=\left[\frac{k+3}{3}\right]}^{\left[\frac{k+3}{2}\right]} \binom{j-1}{k-2j+2} + \sum_{j=\left[\frac{k+3}{3}\right]}^{\left[\frac{k+3}{2}\right]} \binom{j-1}{k-2j+3} \\
&= \sum_{j=\left[\frac{k}{3}\right]}^{\left[\frac{k+1}{2}\right]} \binom{j}{k-2j} + \sum_{j=\left[\frac{k}{3}\right]}^{\left[\frac{k+1}{2}\right]} \binom{j}{k-2j+1} \\
&= \sum_{j=\left[\frac{k+1}{3}\right]}^{\left[\frac{k+1}{2}\right]} \binom{j}{k-2j+1} + \sum_{j=\left[\frac{k}{3}\right]}^{\left[\frac{k}{2}\right]} \binom{j}{k-2j} + \binom{\frac{k}{3}}{\frac{k}{3}+1} + \binom{\frac{k+1}{2}}{-1} \\
&= P_{k+1} + P_k.
\end{aligned}$$

So the formula holds when k is even. Consequently, it is valid when $n = k + 1$. Thus, by the strong induction, the formula is valid for all positive integers n. □

The proof of the Theorem 3.2 is shown according to the Padovan tiling model as follows.

We know that $1 \times n$ board has P_n tilings. We focus on the number of tiles in a tiling to count them. Suppose there are exactly j tiles in an arbitrary tiling. Consequently, the $n - 2j$ tiles can be placed in the j tiling positions in different ways. So, there are $\binom{j}{n-2j}$ tilings, each with exactly j tiles. Since $\left[\frac{n}{3}\right] \leq j \leq \left[\frac{n}{2}\right]$, the total number of tilings equals $\sum_{j=\left[\frac{n}{3}\right]}^{\left[\frac{n}{2}\right]} \binom{j}{n-2j}$. Equating the two counts yields the desired result. Again, we will study a specific case. Let $n = 9$; see Figure 1. There is one tiling with $j = 3$ tiles and three with $j = 4$. So the total number of 9-tilings equals

$$\sum_{j=\left[\frac{9}{3}\right]}^{\left[\frac{9}{2}\right]} \binom{j}{9-2j} = 1 + 4 = 5 = \mathcal{P}_9.$$

4 A model for Padovan polynomials

Now we consider the weight of a tile. Let us define a 1×2 tile to be x: [x]. The product of the weights of all tiles in a tiling gives as the weight of the tiling. For example, the weight of the tiling [x|x|1|x] is x^3, that of the tiling [1|1|1|x|x] is $1 + x^2$, and that of the tiling [x|x|x|x|x] is x^5. We define the weight of the empty tiling to be zero.

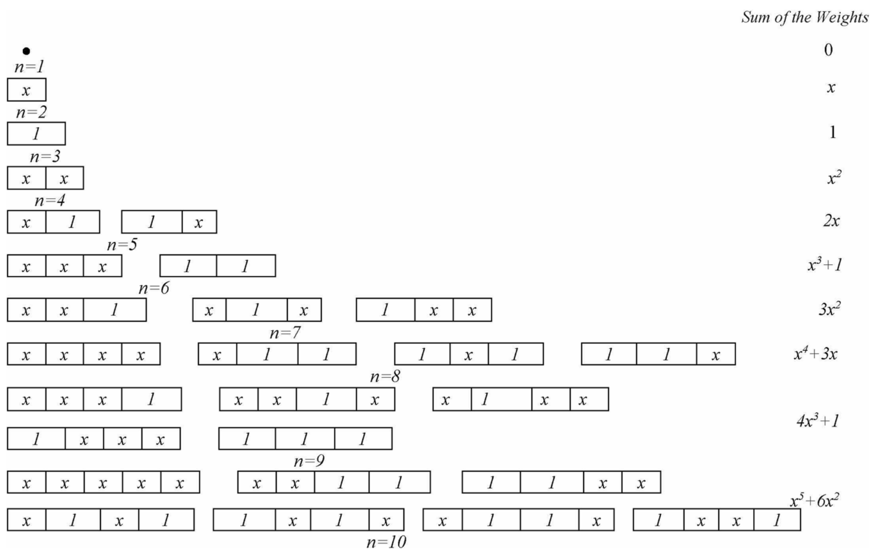

Figure 2: Tilings of a $1 \times n$ board.

Figure 2 shows the tilings and the sum of their weights of a $1 \times n$ board. Combinatorially, we can prove the following theorem.

Theorem 4.1. *The sum of the weights of tilings of a* $1 \times n$ *board with* 1×2 *and* 1×3 *tiles is the nth Padovan polynomial* $\mathcal{P}_n$, *for all* $n \in \mathbb{Z}^+$.

Proof. By $T_n(x)$ we denote the sum of the weights of tiling of a $1 \times n$ board. Clear that $T_1(x) = 0 = \mathcal{P}_1$; $T_2(x) = x = \mathcal{P}_2$. Let us show that $T_n(x)$ holds the Padovan polynomial recurrence relation. Now prove this, consider any tiling of the board. Assume that it ends in 1×2 tiles; subtiling [x]. The sum of the weight of such tiling is equal to $xT_{n-2}(x)$. On the other hand, assume that the tilings end in 1×3 tiles; subtiling [1]. The sum of the weight of such tilings equals $1.\ T_{n-3}(x)$. Hence, the sum of the weights of all tiling of the board equals to $xT_{n-2}(x) + T_{n-3}(x)$ for $n \geq 4$. Thus, $T_n(x) = xT_{n-2}(x) + T_{n-3}(x)$, and that $T_n(x)$ holds the Padovan recurrence relation; so we obtain that $T_n(x) = \mathcal{P}_n$. □

Let us now give the following theorem that can be proved according to the tiling model of Padovan polynomials.

Theorem 4.2. *Let* $n \geq 0$. *Then*

$$\mathcal{P}_n = \sum_{j=\left\lceil \frac{n}{3} \right\rceil}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{j}{n-2j} x^{3j-n}.$$

Proof. We will prove the result using the strong induction. Since $\binom{0}{0} = 1 = \mathcal{P}_0$, $\binom{0}{1} = 0 = \mathcal{P}_1$, $\binom{0}{2} + \binom{1}{0}x = x = \mathcal{P}_2$, the statement is true when $n = 0, 1, 2$. Now assume it is true for all positive integers ($\leq k$), where $k \geq 0$. By Pascal's identity, we have

$$
\begin{aligned}
\mathcal{P}_{k+3} &= \sum_{j=\left[\frac{k+3}{3}\right]}^{\left[\frac{k+3}{2}\right]} \binom{j}{k-2j+3} x^{3j-k-3} \\
&= \sum_{j=\left[\frac{k+3}{3}\right]}^{\left[\frac{k+3}{2}\right]} \binom{j-1}{k-2j+2} x^{3j-k-3} + \sum_{j=\left[\frac{k+3}{3}\right]}^{\left[\frac{k+3}{2}\right]} \binom{j-1}{k-2j+3} x^{3j-k-3} \\
&= \sum_{j=\left[\frac{k}{3}\right]}^{\left[\frac{k+1}{2}\right]} \binom{j}{k-2j} x^{3j-k} + \sum_{j=\left[\frac{k}{3}\right]}^{\left[\frac{k+1}{2}\right]} \binom{j}{k-2j+1} x^{3j-k} \\
&= x \sum_{j=\left[\frac{k+1}{3}\right]}^{\left[\frac{k+1}{2}\right]} \binom{j}{k-2j+1} x^{3j-k-1} + \sum_{j=\left[\frac{k}{3}\right]}^{\left[\frac{k}{2}\right]} \binom{j}{k-2j} x^{3j-k} + \binom{\frac{k}{3}}{\frac{k}{3}+1} + \binom{\frac{k+1}{2}}{-1} x^{\frac{k+3}{2}} \\
&= x\mathcal{P}_{k+1} + \mathcal{P}_{k}.
\end{aligned}
$$

So the formula holds when k is even. Consequently, it is valid when $n = k + 1$. Hence, by the strong induction, the formula is valid for all positive integers n. □

The proof of Theorem 4.2 is shown according to the Padovan polynomials tiling model as follows.

We consider the seven tilings of a 1×10 board; see Figure 2. There is one tiling with no 1×3 tiles; its weight is x^5. There isn't tiling with exactly one 1×3 tile each. There are six tilings with exactly two 1×3 tiles each; the sum of their weights is $6x^2$. So, the sum of the weights of the tilings is $x^5 + 6x^2 = \mathcal{P}_{10}$.

In this study, the elements along the rising diagonals in Pascal's triangle give coefficients of the Padovan polynomials. The first few Padovan polynomials are displayed in Figure 3 as well as the array of their coefficients on the Pascal's triangle.

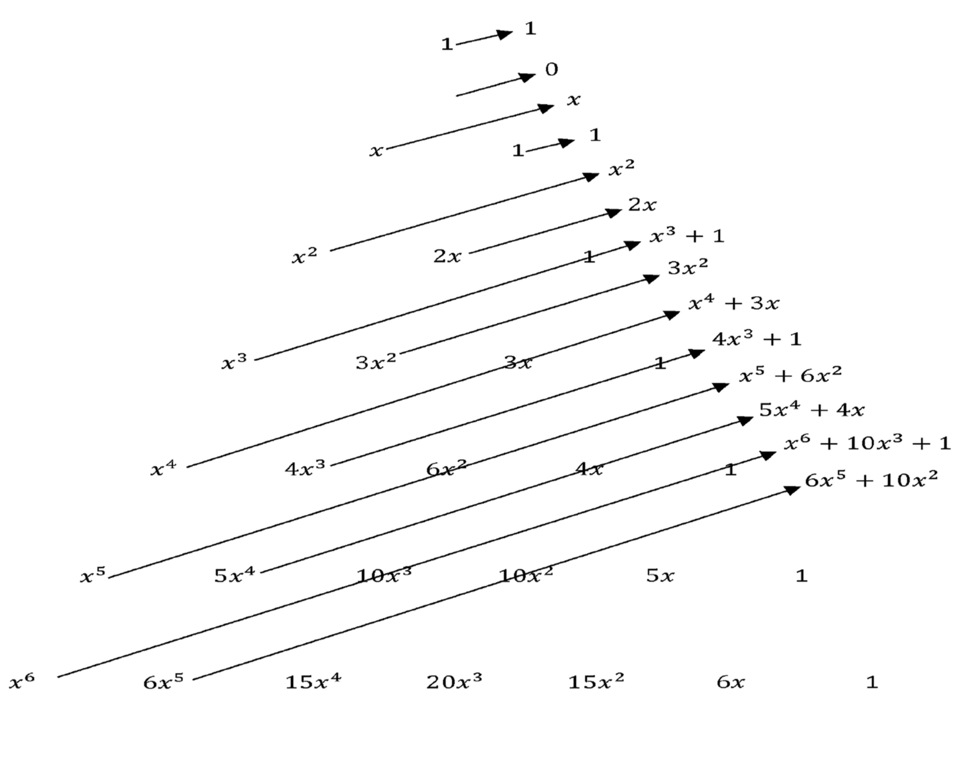

Figure 3: Pascal's triangle of the Padovan polynomial.

Some certain binomial sums for Padovan polynomials are given below:

Theorem 4.3. *Let $n, m \in \mathbb{N}$. Then,*

$$\sum_{n=0}^{m} \binom{m}{n} \mathcal{P}_n x^n = \mathcal{P}_{3m}.$$

Proof. By applying Binet-like formula (6) and combining this with (2), we obtain the identity

$$\begin{aligned}
\sum_{n=0}^{m} \binom{m}{n} \mathcal{P}_n x^n &= \sum_{n=0}^{m} \binom{m}{n} (a_x \alpha_x^n + b_x \beta_x^n + c_x \gamma_x^n) x^n \\
&= a_x \left(\sum_{n=0}^{m} \binom{m}{n} (x\alpha_x)^n 1^{m-n} \right) + b_x \left(\sum_{n=0}^{m} \binom{m}{n} (x\beta_x)^n 1^{m-n} \right) + c_x \left(\sum_{n=0}^{m} \binom{m}{n} (x\gamma_x)^n 1^{m-n} \right) \\
&= a_x (x\alpha_x + 1)^m + b_x (x\beta_x + 1)^m + c_x (x\gamma_x + 1)^m \\
&= a_x \alpha_x^{3m} + b_x \beta_x^{3m} + c_x \gamma_x^{3m}.
\end{aligned}$$

Thus, the proof is completed. □

Theorem 4.4. *Let $n, m \in \mathbb{Z}^+$. Then,*

$$\sum_{k=1}^{m} \binom{m}{k} \mathcal{P}_{n-k} x^{m-k} = \mathcal{P}_{n+2m}.$$

Proof. By applying Binet-like formula (6) and combining this with (2) we obtain the identity

$$\begin{aligned}
\sum_{k=1}^{m} \binom{m}{k} \mathcal{P}_{n-k} x^{m-k} &= \sum_{k=1}^{m} \binom{m}{k} (a_x \alpha_x^{n-k} + b_x \beta_x^{n-k} + c_x \gamma_x^{n-k}) x^{m-k} \\
&= a_x \left(\sum_{n=1}^{m} \binom{m}{n} 1^k (x\alpha_x)^{m-k} \right) (\alpha_x)^{n-m} + b_x \left(\sum_{n=1}^{m} \binom{m}{n} 1^k (x\beta_x)^{m-k} \right) (\beta_x)^{n-m} \\
&\quad + c_x \left(\sum_{n=1}^{m} \binom{m}{n} 1^k (x\gamma_x)^{m-k} \right) (\gamma_x)^{n-m} \\
&= a_x (x\alpha_x + 1)^m (\alpha_x)^{n-m} + b_x (x\beta_x + 1)^m (\beta_x)^{n-m} + c_x (x\gamma_x + 1)^m (\gamma_x)^{n-m} \\
&= a_x \alpha_x^{n+2m} + b_x \beta_x^{n+2m} + c_x \gamma_x^{n+2m}.
\end{aligned}$$

Thus, the proof is completed. □

5 The palindromic representations of the compositions with 2*s* and 3*s*

Now we shift our attention to compositions with special properties. Compositions of n are defined to be "palindromic" are written in reverse order and it remains unchanged. For example, $3 + 2 + 3$ is a palindromic composition of 8, but $3 + 3 + 2$ is not. Let $\psi(n)$ denotes the number of palindromic compositions of n with 2*s* and 3*s*.

Let us investigate some properties of the function $\psi(n)$. Let us prove that the arithmetic function $\psi(n)$ defines a sequence of sixth-order recurrences.

Theorem 5.1. *The arithmetic function $\psi(n)$ defines an integers sequence by the recurrence relation*

$$\psi(n + 6) = \psi(n + 2) + \psi(n).$$

Table 3: The palindromic representations of the compositions with 2*s* and 3*s*

n	**Palindromic compositions**	$\psi(n)$
1	—	0
2	2	1
3	3	1
4	2 + 2	1
5	—	0
6	2 + 2 + 2, 3 + 3	2
7	2 + 3 + 2	1
8	2 + 2 + 2 + 2, 3 + 2 + 3	2
9	3 + 3 + 3	1
10	2 + 2 + 2 + 2 + 2, 2 + 3 + 3 + 2, 3 + 2 + 2 + 3	3
11	2 + 2 + 3 + 2 + 2	1
12	2 + 2 + 2 + 2 + 2 + 2, 3 + 2 + 2 + 2 + 3, 2 + 3 + 2 + 3 + 2, 3 + 3 + 3 + 3	4
13	3 + 2 + 3 + 2 + 3, 2 + 3 + 3 + 3 + 2	2
14	3 + 3 + 2 + 3 + 3, 2 + 2 + 3 + 3 + 2 + 2, 3 + 2 + 2 + 2 + 2 + 3, 2 + 3 + 2 + 2 + 3 + 2, 2 + 2 + 2 + 2 + 2 + 2 + 2	5
15	2 + 2 + 2 + 3 + 2 + 2 + 2, 3 + 3 + 3 + 3 + 3	2

Moreover,

$$\psi(2n) = P_n \quad \textit{and} \quad \psi(2n+1) = P_{n-1}.$$

Proof. By $\psi(n,2)$ and $\psi(n,3)$, we denote the palindromic compositions ending in a 2 and ending in a 3, respectively. So, we can write

$$\psi(n+6) = \psi(n+6,2) + \psi(n+6,3). \tag{6}$$

Since $\psi(n+6,2)$ counts the palindromic compositions ending in a 2, by deleting the 2*s* on the both sides we have a palindromic composition for $n+2$. Hence, we obtain

$$\psi(n+6,2) = \psi(n+2). \tag{7}$$

By the similar way, we obtain

$$\psi(n+6,3) = \psi(n). \tag{8}$$

Now (7), (8), and (6) together from Theorem 5.1. □

From Table 3, the following identity can be easily proved.

Theorem 5.2. *The following identity is valid*:

$$\psi(n+3) = \psi(n+1) + (-1)^{n+1}\psi(n). \tag{9}$$

Now we define counting polynomials $\psi_n(x)$ on the palindromic compositions. For a certain n, $\psi_n(x)$ contains term x for each 2 and term 1 for each 3. Hence, sequence values of $\psi_n(x)$ is 0, x, 1, x^2, 0, x^3+1, x^2, x^4+x, 1, x^5+2x^2, x^4, x^6+2x^3+1, $2x^2$, x^7+3x^4+x, $x^6+1, \ldots$ obeying the recurrence relation

$$\psi_{n+6}(x) = x^2\psi_{n+2}(x) + \psi_n(x),$$

and this is quite obvious, for

$$\psi_n(1) = \psi(n).$$

Now we define counting polynomials $\sigma_n(x)$ on the palindromic compositions. For a certain n, $\sigma_n(x)$ contains the term "ax^b" if there are "a" compositions with "b" d signs. Hence, sequence values of $\sigma_n(x)$ is 0,

$1, 1, x, 0, x^2 + x, x^2, x^3 + x^2, x^2, x^4 + 2x^3, x^4, x^5 + 2x^4 + x^3, 2x^4, x^6 + 3x^5 + x^4, x^6 + x^4, \ldots$ obeying the recurrence relation

$$\sigma_{n+6}(x) = x^2(\sigma_{n+2}(x) + \sigma_n(x)) \tag{10}$$

and this is quite obvious, for

$$\sigma_{n+6}(x) = \psi(n+6).$$

The notations $\psi_d(n)$,$\psi_2(n)$, and $\psi_3(n)$ used in the following theorems denote $\left.\frac{d\sigma_n(x)}{dx}\right|_{x=1}$, the sum of the 2*s* the ending 2 and 3 in n and the sum of the 3*s* the ending 2 and 3 in n, respectively.

Theorem 5.3. *The following identities are valid:*

1. $$\psi_d(n+6) = \psi_d(n+2) + \psi_d(n) + 2\psi(n+6), \tag{11}$$

2. $$\psi_2(n+6) = \psi_2(n+2) + \psi_2(n) + 2\psi(n+2), \tag{12}$$

3. $$\psi_3(n+6) = \psi_3(n+2) + \psi_3(n) + 2\psi(n). \tag{13}$$

Proof.

1. From the definition,

$$\left.\frac{d\sigma_n(x)}{dx}\right|_{x=1} = \psi_d(n).$$

By (10), we have

$$\frac{d\sigma_{n+6}(x)}{dx} = \frac{dx^2(\sigma_{n+2}(x) + \sigma_n(x))}{dx} = x^2\left(\frac{d\sigma_{n+2}(x)}{dx} + \frac{d\sigma_n(x)}{dx}\right) + 2x(\sigma_{n+2}(x) + \sigma_n(x)).$$

Taking $x = 3$, we obtain that

$$\psi_d(n+6) = \psi_d(n+2) + \psi_d(n) + 2\psi(n+6).$$

2. $\psi_2(n+6)$ is equal to the sum of the 2*s* the ending 2 and 3 in $n+6$. That is,

$$\psi_2(n+6) = \psi_2(n+6, 2) + \psi_2(n+6, 3). \tag{14}$$

The number of 2*s* in all compositions ending 3 in $n+6$ is equal to the number of 2*s* in n. That is,

$$\psi_2(n+6, 3) = \psi_2(n).$$

The sum of the 2*s* in all compositions ending 2 in $n+6$ is equal to the sum of the 2*s* in $n+2$ and twice the sum of all the compositions. That is,

$$\psi_2(n+6, 2) = \psi_2(n+2) + 2\psi(n+2).$$

So from (14), this proves

$$\psi_2(n+6) = \psi_2(n+2) + \psi_2(n) + 2\psi(n+2).$$

3. $\psi_2(n+6)$ is equal to the sum of the 3*s* the ending 2 and 3 in $n+6$. That is,

$$\psi_3(n+6) = \psi_3(n+6, 2) + \psi_3(n+6, 3). \tag{15}$$

The number of 3*s* in all compositions ending 2 in $n+6$ is equal to the number of 3*s* in n. That is,

$$\psi_3(n+6, 3) = \psi_3(n).$$

The sum of the 3s in all compositions ending 3 in $n + 6$ is equal to the sum of the 3s in $n + 2$ and twice the sum of all the compositions in n. That is,

$$\psi(n+6,3) = \psi_3(n+2) + 2\psi(n).$$

So from (15), this proves

$$\psi_3(n+6) = \psi_3(n+2)\psi_3(n) + 2\psi(n).$$ □

Theorem 5.4. *The following identities are valid*:

1.
$$\psi_d(n+3) = \psi_d(n+1) + (-1)^{n+1}\psi_d(n) + \psi(n+3), \tag{16}$$

2.
$$\psi_2(n+3) = \psi_2(n+1) + (-1)^{n}\psi_2(n) + \psi(n+2), \tag{17}$$

3.
$$\psi_3(n+3) = \psi_3(n+1) + (-1)^{n-1}\psi_3(n) + \psi(n+1), \tag{18}$$

Proof.

1.
$$\begin{aligned} 2\psi(n+4) &= \psi_d(n+4) - \psi_d(n) - \psi_d(n-2), \\ 2\psi(n+2) &= \psi_d(n+2) - \psi_d(n-2) - \psi_d(n-4), \\ 2\psi(n+1) &= \psi_d(n+1) - \psi_d(n-3) - \psi_d(n-5). \end{aligned}$$

By using (9), we have

$$\psi_d(n+4) - \psi_d(n) - \psi_d(n-2) = \psi_d(n+2) - \psi_d(n-2) - \psi_d(n-4) + (-1)^{n+2}(\psi_d(n+1) - \psi_d(n-3) - \psi_d(n-5)).$$

For $n-3$ and $n-5$ in equation (16), we obtain

$$\psi_d(n+4) = \psi_d(n+2) + (-1)^{n+2}\psi_d(n+1) + \psi(n+4).$$

2.
$$\begin{aligned} 2\psi(n) &= \psi_2(n+4) - \psi_2(n) - \psi_2(n-2), \\ 2\psi(n-2) &= \psi_2(n+2) - \psi_2(n-2) - \psi_2(n-4), \\ 2\psi(n-3) &= \psi_2(n+1) - \psi_2(n-3) - \psi_2(n-5). \end{aligned}$$

By using (9), we have

$$\psi_2(n+4) - \psi_2(n) - \psi_2(n-2) = \psi_2(n+2) - \psi_2(n-2) - \psi_2(n-4) + (-1)^{n-2}(\psi_2(n+1) - \psi_2(n-3) - \psi_2(n-5)).$$

For $n-3$ and $n-5$ in equation (16), we obtain

$$\psi_2(n+4) = \psi_2(n+2) + (-1)^{n+1}\psi_2(n+1) + \psi(n+3).$$

3.
$$\begin{aligned} 2\psi(n) &= \psi_3(n+4) - \psi_3(n) - \psi_3(n-4), \\ 2\psi(n-2) &= \psi_3(n+2) - \psi_3(n-2) - \psi_3(n-6), \\ 2\psi(n-3) &= \psi_3(n+1) - \psi_3(n-3) - \psi_3(n-7). \end{aligned}$$

By using (9), we have

$$\psi_2(n+4) - \psi_3(n) - \psi_3(n-4) = \psi_3(n+2) - \psi_3(n-2) - \psi_3(n-6) + (-1)^{n-2}(\psi_3(n+1) - \psi_3(n-3) - \psi_3(n-7)).$$

For $n-3$ and $n-5$ in equation (16) we obtain

$$\psi_2(n+4) = \psi_2(n+2) + (-1)^n \psi_2(n+1) + \psi(n+2). \qquad \square$$

6 Conclusion

In the present work, the combinatorial interpretations of compositions of positive integers with 2*s* and 3*s* are investigated. The connection between these compositions and the Padovan numbers is established. The numerous identities of these compositions are obtained. Some tiling models are illustrated for the compositions of positive integers with 2*s* and 3*s*. The palindromic representations of the compositions with 2*s* and 3*s* are studied, and an arithmetical function related to these palindromic representations is defined. The numerous identities of these compositions are obtained.

A table that compares the first few values of all the results obtained from compositions of positive integers with 2*s* and 3*s* is provided as in Table 4.

Table 4: All representations of the compositions with 2*s* and 3*s*

n	P_n	CP_n	$f(n)$	$g(n)$	$\sigma_n(x)$	$\psi(n)$	$\psi_d(n)$	$\psi_2(n)$	$\psi_3(n)$
1	0	0	0	0	0	0	0	0	0
2	1	1	1	0	1	1	0	1	0
3	1	1	0	1	1	1	0	0	1
4	1	1	2	0	x	1	1	2	0
5	2	2	2	2	0	0	0	0	0
6	2	2	3	2	x^2+x	2	3	3	2
7	3	3	6	3	x^2	1	2	2	1
8	4	4	7	6	x^3+x^2	2	5	5	2
9	5	5	12	7	x^2	1	2	0	3
10	7	7	17	12	x^4+2x^3	3	10	9	4

In future studies, combinatorial interpretations for the sequences defined by higher-order recurrence relations may be considered. Tiling models may be examined for these sequences.

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References

[1] P. A. MacMahon, *XVII. Memoir on the theory of the compositions of numbers*, Philos. Trans. Roy. Soc. London **184** (1893), 835–901, DOI: https://doi.org/10.1098/rsta.1893.0017.

[2] P. Hitczenko and C. D. Savage, *On the multiplicity of parts in a random composition of a large integer*, SIAM J. Discrete Math. **18** (2004), no.2, 418–435, DOI: https://doi.org/10.1137/S0895480199363155.

[3] P. Hitczenko and G. Stengle, *Expected number of distinct part sizes in a random integer composition*, Comb. Probab. Comput. **9** (2000), no.6, 519–527, DOI: https://doi.org/10.1017/S0963548300004399.
[4] S. Heubach and T. Mansour, *Avoiding patterns of length three in compositions and multiset permutations*, Adv. Appl. Math. **36** (2006), no.2, 156–174, DOI: https://doi.org/10.1016/j.aam.2005.06.005.
[5] S. Heubach and T. Mansour, *Combinatorics of Compositions and Words*, Chapman and Hall/CRC, New York, 2009.
[6] S. Heubach and T. Mansour, *Compositions of n with parts in a set*, Congr. Numerantium **168** (2004), 127.
[7] C. D. Savage and H. S. Wilf, *Pattern avoidance in compositions and multiset permutations*, Adv. Appl. Math. **36** (2006), no. 2, 194–201, DOI: https://doi.org/10.1016/j.aam.2005.06.003.
[8] K. Alladi and V. E. Hoggatt, *Compositions with ones and twos*, Fibonacci Q. **13** (1975), 233–239.
[9] C. Banderier and P. Hitczenko, *Enumeration and asymptotics of restricted compositions having the same number of parts*, Discrete Appl. Math. **160** (2012), no. 18, 2542–2554, DOI: https://doi.org/10.1016/j.dam.2011.12.011.
[10] P. Chinn and S. Heubach, *Compositions of n with no occurence of k*, Congr. Numerantium (2003), **164** 33–52.
[11] S. Eger, *Restricted weighted integer compositions and extended binomial coefficient*, J. Integer. Seq. **16** (2013), 3.
[12] V. E. Hoggatt and M. Bicknell, *Generalized Fibonacci polynomials*, Fibonacci Q. **11** (1973), 457–465.
[13] V. E. Hoggatt and K. Alladi, *Compositions and recurrence relations II*, Fibonacci Q. **15** (1977), 239–245.
[14] A. V. Sills, *Compositions, Partitions, and Fibonacci Numbers*, 2013, arxiv:1312.0693.
[15] I. M. Gessel and J. Li, *Compositions and Fibonacci Identities*, 2013, arxiv:1303.1366.
[16] C. Kimberling, *Enumeration of paths, compositions of integers, and Fibonacci numbers*, Fibonacci Q. **39** (2001), no. 5, 430–434.
[17] A. Knopfmacher and N. Robbins, *On binary and Fibonacci compositions*, Ann. Univ. Sci. Budapest. Sect. Comput. **22** (2003), 193–206.
[18] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A000931.
[19] R. Padovan, *Dom Hans van Der Laan and the plastic number*, Nexus Netw. J. **4** (2002), no. 3, pp. 181–193.
[20] I. Stewart, *Tales of a neglected number*, Sci. Americạn. **274** (1996), no. 6, 102.
[21] G. Cerda-Morales, *New Identities for Padovan Numbers*, 2019, arXiv:1904.05492.
[22] O. Deveci and E. Karaduman, *On the Padovan p-numbers*, Hacet. J. Math. **46** (2017), no. 4, 579–592.
[23] Z. İşbilir and G. Gürses, *Padovan and Perrin generalized quaternions*, Math. Methods Appl. Sci **45** (2022), no. 18, 12060–12076, DOI: https://doi.org/10.1002/mma.7495.
[24] Y. Soykan, *On Generalized Padovan Numbers*, 2011, DOI: https://doi.org/10.20944/preprints202110.0101.v1.
[25] N. Yilmaz and N. Taskara, *Matrix sequences in terms of Padovan and Perrin Numbers*, J. Appl. Math. **2013** (2013), DOI: https://doi.org/10.1155/2013/941673.
[26] N. Gogin and A. Mylläri, *Padovan-like sequences and Bell polynomials*, Math. Comput. Simul. **125** (2016), 168–177, DOI: https://doi.org/10.1016/j.matcom.2015.08.008.
[27] S. J. Tedford, *Combinatorial identities for the Padovan numbers*, Fibonacci Quart. **57** (2019), 291–298.
[28] Vieira, R. P. M., Alves, F. R. V., and Catarino, P. M. M. C., *Combinatorial interpretation of numbers in the generalized Padovan sequence and some of its extensions*. Axioms, **11** (2022), no. 11, 598, DOI: https://doi.org/10.3390/axioms11110598.
[29] T. Koshy, *Fibonacci and Lucas Numbers with Applications 1*, John Wiley & Sons, New York, 2018.
[30] T. Koshy, *Fibonacci and Lucas Numbers with Applications 2*, John Wiley & Sons, New York, 2019.
[31] A. T. Benjamin and J. J. Quinn, *Proofs that Really Count: The Aart of Combinatorial Proof*, vol. 27, American Mathematical Society, Providence-Rhode Island, 2022.