Research Article

Keyu Zhang and Jiafa Xu*

Solvability for a system of Hadamard-type hybrid fractional differential inclusions

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Abstract: In this article, a new system of Hadamard-type hybrid fractional differential inclusions equipped with Dirichlet boundary conditions was constructed. By virtue of a fixed-point theorem due to B. C. Dhage, (*Existence results for neutral functional differential inclusions in Banach algebras*, Nonlinear Anal. **64** (2006), no. 6, 1290–1306, doi: https://doi.org/10.1016/j.na.2005.06.036), the existence results of solutions for the considered problem are derived in a new norm space for multivalued maps. A numerical example is provided to illustrate our main results.

Keywords: fractional differential inclusions, Dirichlet boundary conditions, fixed-point theorem, existence of solutions

MSC 2020: 34A05, 34B18, 26A33

1 Introduction

The purpose of this article is to investigate the following system of Hadamard-type fractional hybrid differential inclusions given by:

$$\begin{cases} D^{\alpha_1}\left(\dfrac{x(t)}{f_1(t,x(t),y(t))}\right) \in F_1(t,x(t),y(t)), \quad 1<t<e,\ 1<\alpha_1\le 2,\\ D^{\alpha_2}\left(\dfrac{y(t)}{f_2(t,x(t),y(t))}\right) \in F_2(t,x(t),y(t)), \quad 1<t<e,\ 1<\alpha_2\le 2,\\ x(1)=x(e)=0,\quad y(1)=y(e)=0, \end{cases} \tag{1}$$

where D^{α_1} and D^{α_2} denote the Hadamard-type fractional derivatives of order α_1 and α_2, $f_1, f_2 \in C([1,e]\times\mathbb{R}^2, \mathbb{R}\backslash\{0\})$ are the continuous functions, $F_1, F_2 : [1,e]\times\mathbb{R}^2 \to \mathfrak{T}(\mathbb{R})$ are the multi-valued maps, and $\mathfrak{T}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

Due to fractional-order derivative generalizes the classical integer-order derivative to an arbitrary-order case. Fractional-order differential equations can more accurately describe various phenomenon than integer-order differential equations in many complex and widespread fields such as physics, mechanics, chemistry, and engineering (see the books [1–5] and references cited therein). There has been a rapid increase in the number of fractional differential equations from both theoretical and applied perspectives (see [6–25]).

Note that most of the works on fractional differential equations involve either Riemann-Liouville-type derivative or Caputo-type derivative (see [8–10,12,15,16,19]). However, there is also another concept of

* **Corresponding author: Jiafa Xu,** School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China, e-mail: 20150028@cqnu.edu.cn
Keyu Zhang: School of Mathematics, Qilu Normal University, Jinan 250013, China, e-mail: 20154381@qlnu.edu.cn

Hadamard-type fractional derivative, which was first introduced by Hadamard in 1892 [26], which contains a logarithmic function of arbitrary exponent in the kernel of integral appearing in its definition. Hadmard-type integrals arise in the formulation of many problems in mechanics such as in fracture analysis. For details and applications of Hadamard-type fractional derivative and integral, see [27–33].

The study of fractional differential inclusions also gained much attraction and interest as these tools of mathematical analysis are found to have a wide range of utility in stochastic modeling and optimal control problems [34]. Some recent achievement to the subject of fractional differential inclusions can be discovered in [35–39] and references cited therein. In particular, the fractional differential inclusion for various types of single fractional differential equations with different boundary conditions is studied systematically in [28].

To our best knowledge, almost all fractional hybrid differential inclusions equipped with initial and boundary-value problem were focused on a single fractional hybrid differential equation in [28]. Is there a similar result for a system of coupled hybrid fractional differential inclusions of Hadamard-type? The main objective of this work is to obtain an existence result for system (1) under Lipschitz and Carathéodory conditions in virtue of a fixed-point theorem by Dhage [40] in a new Banach space. Inspired by the work mentioned earlier, this research (1) is also different from the recent results [35–39]. This means that our work is new in the present configuration and contributes. We overcome some difficulties in proving operator convexity and closed graphs.

2 Preliminaries

In this section, we first recall some useful materials for Hadamard-type fractional derivatives and integrals.

Definition 1. [4,5] The Hadamard-type fractional derivative of order $q > 0$ for an integrable function $g : [1, +\infty) \to \mathbb{R}$ is defined as:

$$D^q g(t) = \frac{1}{\Gamma(n-q)}\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-q-1} \frac{g(s)}{s}\mathrm{d}s, \quad n-1 < q < n,$$

where $n = [q] + 1$, $[q]$ is the smallest integer greater than or equal to q and $\log(\cdot) = \log_e(\cdot)$.

Definition 2. [4,5] The Hadamard-type fractional integral of order $q > 0$ for an integrable function g is defined as:

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{s}\right)^{q-1} \frac{g(s)}{s}\mathrm{d}s,$$

provided that the integral exists.

In what follows, we provide some lemmas that are helpful to the proof of main theorems.

Lemma 1. *Let $\phi_1, \phi_2 \in L^1([1, e], \mathbb{R})$ be continuous functions. Then, the integral solution of the Hadamard-type fractional differential system*

$$\begin{cases} D^{\alpha_1}\left(\dfrac{x(t)}{f_1(t, x(t), y(t))}\right) = \phi_1(t), \quad 1 < t < e,\ 1 < \alpha_1 \le 2, \\ D^{\alpha_2}\left(\dfrac{y(t)}{f_2(t, x(t), y(t))}\right) = \phi_2(t), \quad 1 < t < e,\ 1 < \alpha_2 \le 2, \\ x(1) = x(e) = 0, y(1) = y(e) = 0, \end{cases} \tag{2}$$

is given by:

$$\begin{cases} x(t) = f_1(t, x(t), y(t))\left(\frac{1}{\Gamma(\alpha_1)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{\phi_1(s)}{s}\mathrm{d}s - \frac{(\log t)^{\alpha_1-1}}{\Gamma(\alpha_1)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{\phi_1(s)}{s}\mathrm{d}s\right), \\ y(t) = f_2(t, x(t), y(t))\left(\frac{1}{\Gamma(\alpha_2)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_2-1}\frac{\phi_2(s)}{s}\mathrm{d}s - \frac{(\log t)^{\alpha_2-1}}{\Gamma(\alpha_2)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_2-1}\frac{\phi_2(s)}{s}\mathrm{d}s\right). \end{cases} \tag{3}$$

Proof. As argued in Chapter 9 in [28], the solution for (2) can be written as:

$$\begin{cases} x(t) = f_1(t, x(t), y(t))\left(\frac{1}{\Gamma(\alpha_1)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{\phi_1(s)}{s}\mathrm{d}s + c_1(\log t)^{\alpha_1-1} + c_2(\log t)^{\alpha_1-2}\right), \\ y(t) = f_2(t, x(t), y(t))\left(\frac{1}{\Gamma(\alpha_2)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_2-1}\frac{\phi_2(s)}{s}\mathrm{d}s + d_1(\log t)^{\alpha_2-1} + d_2(\log t)^{\alpha_2-2}\right), \end{cases} \tag{4}$$

where $c_1, c_2, d_1, d_2 \in \mathbb{R}$. Note that $x(1) = x(e) = y(1) = y(e) = 0$ in (2), we have

$$c_2 = 0,\ c_1 = -\frac{1}{\Gamma(\alpha_1)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{\phi_1(s)}{s}\mathrm{d}s,\ d_2 = 0,\ d_1 = \frac{1}{\Gamma(\alpha_2)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_2-1}\frac{\phi_2(s)}{s}\mathrm{d}s.$$

Substituting these values into (4), we can obtain (3). □

Definition 3. A pair of function (x, y) is called the solution of system (1) if there exists a pair of function $(\phi_1, \phi_2) \in L^1([1, e], \mathbb{R}) \times L^1([1, e], \mathbb{R})$ with $\phi_1 \in F_1(t, x(t), y(t))$ and $\phi_2 \in F_2(t, x(t), y(t))$ such that $D^{\alpha_1}\left(\frac{x(t)}{f_1(t, x(t), y(t))}\right) = \phi_1(t)$, $D^{\alpha_2}\left(\frac{x(t)}{f_2(t, x(t), y(t))}\right) = \phi_2(t)$ almost every on $[1, e]$ and $x(1) = x(e) = y(1) = y(e) = 0$.

Next, we introduce some preliminary materials about norm spaces and multi-valued maps. Let $X = C([1, e], \mathbb{R})$ denote a Banach space of continuous functions from $[1, e]$ into $\mathbb{R}$ under the norm $\|x\| = \sup_{t\in[1,e]}|x(t)|$, which becomes a Banach algebra with respect to the multiplication "$\cdot$" defined by $(x, y)(t) = x(t)\cdot y(t)$ for arbitrarily $x, y \in X$. The product space $\Xi = X \times X$ is also a Banach space under the norm $\|(x, y)\| = \|x\| + \|y\|$, which further becomes a Banach algebra with respect to the multiplication "$\cdot$" defined by $((x, y)\cdot(\bar{x}, \bar{y}))(t) = (x, y)(t)\cdot(\bar{x}, \bar{y})(t) = (x(t)\bar{x}(t), y(t)\bar{y}(t))$ for $(x, y), (\bar{x}, \bar{y}) \in X \times X$. For more details about the results concerning algebraic structure of the product space $\Xi = X \times X$, please see [41,42].

As aforementioned, $(\Xi, \|(\cdot,\cdot)\|)$ is a Banach space. Now, we redeclare some basic definitions on multi-valued maps (see [43]):

$$\mathfrak{T}_{cl}(\Xi) = \{\mathfrak{F} \in \mathfrak{T}(\Xi) : \mathfrak{F} \text{ is closed}\}, \quad \mathfrak{T}_b(\Xi) = \{\mathfrak{F} \in \mathfrak{T}(\Xi) : \mathfrak{F} \text{ is bounded}\},$$
$$\mathfrak{T}_{cp}(\Xi) = \{\mathfrak{F} \in \mathfrak{T}(\Xi) : \mathfrak{F} \text{ is compact}\}, \quad \mathfrak{T}_{cp,cv}(\Xi) = \{\mathfrak{F} \in \mathfrak{T}(\Xi) : \mathfrak{F} \text{ is compact and convex}\}.$$

Definition 4. A multi-valued map $G : \Xi \to \mathfrak{T}(\Xi)$ is called convex (closed) valued if $G(x, y)$ is convex (closed) for all $(x, y) \in \Xi$.

Definition 5. The map G is called bounded on bounded sets if $G(\mathbb{B}) = \cup_{(x,y)\in\mathbb{B}} G(x, y)$ is bounded in Ξ for any bounded set $\mathbb{B}$ of Ξ (i.e., $\sup_{(x,y)\in\mathbb{B}}\{\|(u, v)\| : (u, v) \in G(x, y)\} < \infty$).

Definition 6. The map G is called upper semi-continuous (u.s.c.) on Ξ if for each $(x, y) \in \Xi$, the set $G(x, y)$ is a nonempty closed subset of Ξ, and if for each open set $\mathbb{B}$ of Ξ containing $G(x, y)$, there exists an open neighborhood N of (x, y) such that $G(N) \subset \mathbb{B}$.

Definition 7. The map G is called completely continuous if $G(\mathbb{B})$ is relatively compact for every bounded subset $\mathbb{B}$ of Ξ.

Definition 8. A multi-valued map $F : [1, e] \times \mathbb{R}^2 \to \mathfrak{T}(R)$ is called L^1-Carathéodory if
(i) $t \to F(t, x, y)$ is measurable for each $(x, y) \in \mathbb{R} \times \mathbb{R}$,
(ii) $(x, y) \to F(t, x, y)$ is upper semicontinuous for almost all $t \in [1, e]$,
(iii) there exists a function $g_r \in L^1([1, e], \mathbb{R}^+)$ such that

$$\|F(t, x, y)\| = \sup\{|v| : v \in F(t, x, y) \le g_r(t)\},$$

for all $x, y \in \mathbb{R}$ with $\|(x, y)\| \le r$ and for almost every $t \in [1, e]$.

For each $(x, y) \in \Xi$, define the set of selections of $F_{xy} = (F_{1,xy}, F_{2,xy})$ by:

$$F_{1,xy} := \{v_1 \in L^1([1, e], \mathbb{R}) : v_1(t) \in F_1(t, x(t), y(t)), \quad \text{for a.e. } t \in [1, e]\},$$
$$F_{2,xy} := \{v_2 \in L^1([1, e], \mathbb{R}) : v_2(t) \in F_2(t, x(t), y(t)), \quad \text{for a.e. } t \in [1, e]\}.$$

Lemma 2. [40] *Let X be a Banach algebra, $A : X \to X$ a single-valued operator, and $B : X \to \mathfrak{T}_{cp,cv}(X)$ a multi-valued operator satisfying the following conditions*:
(a) *A is a single-valued Lipschitz with Lipschitz constant k,*
(b) *B is compact and upper semicontinuous,*
(c) *$2Mk < 1$, where $M = \|B(X)\|$.*

Then, either:
(i) *the operator inclusion $x \in AxBx$ has a solution,*

or
(ii) *the set $\Phi = \{u \in X | \mu u \in AuBu, \mu > 1\}$ is unbounded.*

Lemma 3. [43] *If $G : X \to \mathfrak{T}_{cl}(Y)$ is upper semi-continuous, then $G_r(G) = \{(x, y) \in X \times Y, y \in G(x)\}$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in N} \subset X$ and $\{y_n\}_{n\in N} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 4. [44] *Let X be a Banach space, $F : [1, e] \times \mathbb{R} \to \mathfrak{T}_{cp,cv}(X)$ an L^1-Carathéodory multi-valued map, and G a linear continuous mapping from $L^1([1, e], X)$ to $C([1, e], X)$. Then, the operator*

$$G \circ F : C([1, e], X) \to \mathfrak{T}_{cp,cv}(C([1, e], X))$$
$$x \mapsto (G \circ F)(x) = G(F(x))$$

is a closed graph operator in $C([1, e], X) \times C([1, e], X)$.

3 Main results

Let $L^1([1, e], \mathbb{R})$ be the Banach space of measurable functions $x : [1, e] \to \mathbb{R}$, which is Lebesgue integrable and normed by $\|x\|_{L^1} = \int_1^e |x(t)| dt$. Now, we are in a position to show the existence of solutions for system (1).

Theorem 1. *Suppose that*
($H1$) *Functions $f_i : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \backslash \{0\}$ are continuous and there exist constants $L_i > 0$ such that*

$$|f_i(t, x, y) - f_i(t, \bar{x}, \bar{y})| \le L_i[|x(t) - \bar{x}(t)| + |y(t) - \bar{y}(t)|], \, i = 1, 2,$$

for almost every $t \in [1, e]$, $\forall x, y, \bar{x}, \bar{y} \in \mathbb{R}$;
($H2$) *operators $F_i : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathfrak{T}(\mathbb{R})$ are L^1-Carathéodory and have nonempty compact and convex values;*

($H3$) *there exists a positive real number r such that*

$$r > \frac{\frac{2F_{10}\|g_{1r}\|_{L^1}}{\Gamma(\alpha_1)} + \frac{2F_{20}\|g_{2r}\|_{L^1}}{\Gamma(\alpha_2)}}{1 - \frac{2k\|g_{1r}\|_{L^1}}{\Gamma(\alpha_1)} - \frac{2k\|g_{2r}\|_{L^1}}{\Gamma(\alpha_2)}},$$

where $\frac{2k\|g_{1r}\|_{L^1}}{\Gamma(\alpha_1)} + \frac{2k\|g_{2r}\|_{L^1}}{\Gamma(\alpha_2)} < \frac{1}{2}$, $k = L_1 + L_2$, $F_{10} = \sup_{t\in[1,e]}|f_1(t, 0, 0)|$, $F_{20} = \sup_{t\in[1,e]}|f_2(t, 0, 0)|$, *here* $g_{1r}(t)$ *and* $g_{2r}(t)$ *have similar approach as Definition* 8.

Then, system (1) has at least one solution on $[1, e] \times [1, e]$.

Proof. Applying Lemma 1, we transform syetem (1) into an equivalent fixed-point problem. Define the operator $T : \Xi \to \mathfrak{T}(\Xi)$ as $T(x, y)(t) = (T_1(x, y)(t), T_2(x, y)(t))$, where

$$\begin{aligned} T_1(x, y)(t) = \Bigg\{ h_1 \in C([1, e], \mathbb{R}) : h_1(t) = f_1(t, x(t), y(t)) \Bigg(\frac{1}{\Gamma(\alpha_1)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1} \frac{v_1(s)}{s} ds \\ - \frac{(\log t)^{\alpha_1-1}}{\Gamma(\alpha_1)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1} \frac{v_1(s)}{s} ds \Bigg), v_1 \in F_{1,xy} \Bigg\} \end{aligned} \tag{5}$$

and

$$\begin{aligned} T_2(x, y)(t) = \Bigg\{ h_2 \in C([1, e], \mathbb{R}) : h_2(t) = f_2(t, x(t), y(t)) \Bigg(\frac{1}{\Gamma(\alpha_2)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha_2-1} \frac{v_2(s)}{s} ds \\ - \frac{(\log t)^{\alpha_2-1}}{\Gamma(\alpha_2)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha_2-1} \frac{v_2(s)}{s} ds \Bigg), v_2 \in F_{2,xy} \Bigg\}. \end{aligned} \tag{6}$$

Next, we consider two operators $A = (A_1, A_2)$ and $B = (B_1, B_2)$, where $A_i : \Xi \to \Xi$ are given by:

$$A_i(x, y)(t) = f_i(t, x(t), y(t)), \quad t \in [1, e], \; i = 1, 2,$$

and $B_i : \Xi \to \mathfrak{T}(\Xi)$ are given by:

$$\begin{aligned} B_i(x, y)(t) = \Bigg\{ h_i \in C([1, e], \mathbb{R}) : h_i(t) = \frac{1}{\Gamma(\alpha_i)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha_i-1} \frac{v_i(s)}{s} ds \\ - \frac{(\log t)^{\alpha_i-1}}{\Gamma(\alpha_i)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha_i-1} \frac{v_i(s)}{s} ds, v_i \in F_{i,xy} \Bigg\}, \quad t \in [1, e], \; i = 1, 2. \end{aligned} \tag{7}$$

Note that $T_i(x, y) = A_i(x, y)B_i(x, y)$, $i = 1, 2$, then $T(x, y) = (A_1(x, y)B_1(x, y), A_2(x, y)B_2(x, y))$. We next show that the operators A and B satisfy all the conditions of Lemma 2. For clarity, we display the proof in several steps.

Step 1. We first show that Lemma 2(a) holds, i.e., the single-valued operator A is a Lipschitz on Ξ. Let $(x, y), (\bar{x}, \bar{y}) \in \Xi$, by (H1), we have

$$\begin{aligned} |A_i(x, y) - A_i(\bar{x}, \bar{y})| &= |f_i(t, x, y) - f_i(t, \bar{x}, \bar{y})| \le L_i[|x(t) - \bar{x}(t)| + |y(t) - \bar{y}(t)|] \\ &\le L_i[\|x - \bar{x}\| + \|y - \bar{y}\|], \quad t \in [1, e], i = 1, 2. \end{aligned} \tag{8}$$

Consequently,

$$\begin{aligned} \|A(x, y) - A(\bar{x}, \bar{y})\| &= \|A_1(x, y) - A_1(\bar{x}, \bar{y})\| + \|A_2(x, y) - A_2(\bar{x}, \bar{y})\| \\ &\le L_1[\|x - \bar{x}\| + \|y - \bar{y}\|] + L_2[\|x - \bar{x}\| + \|y - \bar{y}\|] \\ &\le (L_1 + L_2)\|(x, y) - (\bar{x}, \bar{y})\|. \end{aligned} \tag{9}$$

So, the single-valued operator A is a Lipschitz on Ξ with Lipschitz constant $k = L_1 + L_2$.

Step 2. We show that Lemma 2(b) holds, i.e., the multi-valued operator B is compact and upper semicontinuous on Ξ.

(i) We show that the operator B has convex values. Let $u_{11}, u_{12} \in B_1(x, y)$, $u_{21}, u_{22} \in B_2(x, y)$. Then, there are $v_{11}, v_{12} \in F_{1,xy}$, v_{21}, and $v_{22} \in F_{2,xy}$ such that

$$u_{1j}(t) = \frac{1}{\Gamma(\alpha_1)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{v_{1j}(s)}{s}\mathrm{d}s - \frac{(\log t)^{\alpha_1-1}}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{v_{1j}(s)}{s}\mathrm{d}s, \quad j = 1, 2,\ t \in [1, e], \tag{10}$$

$$u_{2j}(t) = \frac{1}{\Gamma(\alpha_2)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_2-1}\frac{v_{2j}(s)}{s}\mathrm{d}s - \frac{(\log t)^{\alpha_2-1}}{\Gamma(\alpha_2)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_2-1}\frac{v_{2j}(s)}{s}\mathrm{d}s, \quad j = 1, 2,\ t \in [1, e]. \tag{11}$$

For any constant $\lambda \in [0, 1]$, we have

$$\begin{aligned}\lambda u_{11}(t) + (1-\lambda)u_{12}(t) &= \frac{1}{\Gamma(\alpha_1)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{\lambda v_{11}(s) + (1-\lambda)v_{12}(s)}{s}\mathrm{d}s\\ &\quad - \frac{(\log t)^{\alpha_1-1}}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{\lambda v_{11}(s) + (1-\lambda)v_{12}(s)}{s}\mathrm{d}s,\end{aligned} \tag{12}$$

$$\begin{aligned}\lambda u_{21}(t) + (1-\lambda)u_{22}(t) &= \frac{1}{\Gamma(\alpha_2)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_2-1}\frac{\lambda v_{21}(s) + (1-\lambda)v_{22}(s)}{s}\mathrm{d}s\\ &\quad - \frac{(\log t)^{\alpha_2-1}}{\Gamma(\alpha_2)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_2-1}\frac{\lambda v_{21}(s) + (1-\lambda)v_{22}(s)}{s}\mathrm{d}s,\end{aligned} \tag{13}$$

where $\overline{v}_1(t) = \lambda v_{11}(t) + (1-\lambda)v_{12}(t) \in F_{1,xy}$, $\overline{v}_2(t) = \lambda v_{21}(t) + (1-\lambda)v_{22}(t) \in F_{2,xy}$ for all $t \in [1, e]$.

Therefore, we have

$$\begin{aligned}&\lambda u_{11}(t) + (1-\lambda)u_{12}(t) \in B_1(x, y),\ \lambda u_{21}(t) + (1-\lambda)u_{22}(t) \in B_2(x, y),\\ &B(\lambda u_{11}(t) + (1-\lambda)u_{12}(t), \lambda u_{21}(t) + (1-\lambda)u_{22}(t)) = \lambda B(u_{11}(t), u_{21}(t)) + (1-\lambda)B(u_{12}(t), u_{22}(t)) \in B(x, y).\end{aligned}$$

Then, we obtain the operator $B(x, y)$ which is convex for each $(x, y) \in \Xi$. Then, operator B defines a multi-valued operator $B : \Xi \to \mathfrak{T}_{\mathrm{cv}}(\Xi)$.

(ii) We display that the operator B maps bounded sets into bounded sets in Ξ. Let $\Omega = \{(x, y) | \|(x, y)\| < r, (x, y) \in \Xi\}$. Then, for each $h_i \in B_i(x, y)$, $i = 1, 2$, there are $v_i \in F_{i,xy} (i = 1, 2)$ such that

$$h_i(t) = \frac{1}{\Gamma(\alpha_i)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_i-1}\frac{v_i(s)}{s}\mathrm{d}s - \frac{(\log t)^{\alpha_i-1}}{\Gamma(\alpha_i)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_i-1}\frac{v_i(s)}{s}\mathrm{d}s, \quad t \in [1, e]. \tag{14}$$

From (H2), we have

$$\begin{aligned}|B_1(x, y)(t)| &= \left|\frac{1}{\Gamma(\alpha_1)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{v_1(s)}{s}\mathrm{d}s - \frac{(\log t)^{\alpha_1-1}}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{v_1(s)}{s}\mathrm{d}s\right|\\ &\le \frac{1}{\Gamma(\alpha_1)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{g_{1r}(s)}{s}\mathrm{d}s + \frac{(\log t)^{\alpha_1-1}}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{g_{1r}(s)}{s}\mathrm{d}s\\ &\le \frac{2}{\Gamma(\alpha_1+1)}\|g_{1r}\|_{L^1}\end{aligned} \tag{15}$$

and

$$|B_2(x,y)(t)| \le \frac{2}{\Gamma(\alpha_2+1)}\|g_{2r}\|_{L^1}. \tag{16}$$

This implies that

$$\|B(x,y)\| = \|B_1(x,y)\| + \|B_2(x,y)\| \le \frac{2}{\Gamma(\alpha_1+1)}\|g_{1r}\|_{L^1} + \frac{2}{\Gamma(\alpha_2+1)}\|g_{2r}\|_{L^1}. \tag{17}$$

Thus, $B(\Xi)$ is uniformly bounded. Then, B defines a multi-valued operator $B:\Xi\to\mathfrak{T}_{\mathrm{b}}(\Xi)$.

(iii) We show that the operator B maps bounded sets into equicontinuous sets. Let $q_i\in B_i(x,y)(i=1,2)$ for some $(x,y)\in\Omega$, where Ω is given as earlier. So, there exists $u_i\in F_{i,xy}$, such that

$$q_i(t) = \frac{1}{\Gamma(\alpha_i)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_i-1}\frac{u_i(s)}{s}\,\mathrm{d}s - \frac{(\log t)^{\alpha_i-1}}{\Gamma(\alpha_i)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_i-1}\frac{u_i(s)}{s}\,\mathrm{d}s,\quad i=1,2. \tag{18}$$

For any $\tau_1,\tau_2\in[1,e]$ and $\tau_1<\tau_2$, we have

$$\begin{aligned}
|q_1(\tau_1)-q_1(\tau_2)| &\le \frac{\|g_{1r}\|_{L^1}}{\Gamma(\alpha_i)}\left|\int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha_1-1}\frac{1}{s}\,\mathrm{d}s - \int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha_1-1}\frac{1}{s}\,\mathrm{d}s\right| \\
&\quad + \frac{\|g_{1r}\|_{L^1}|(\log\tau_1)^{\alpha_1-1}-(\log\tau_2)^{\alpha_1-1}|}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{1}{s}\,\mathrm{d}s \\
&\le \frac{\|g_{1r}\|_{L^1}}{\Gamma(\alpha_1)}\left|\int_1^{\tau_1}\left[\left(\log\frac{\tau_1}{s}\right)^{\alpha_1-1} - \left(\log\frac{\tau_2}{s}\right)^{\alpha_1-1}\right]\frac{1}{s}\,\mathrm{d}s\right| \\
&\quad + \frac{\|g_{1r}\|_{L^1}}{\Gamma(\alpha_1)}\left|\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha_1-1}\frac{1}{s}\,\mathrm{d}s\right| \\
&\quad + \frac{\|g_{1r}\|_{L^1}|(\log\tau_1)^{\alpha_1-1}-(\log\tau_2)^{\alpha_1-1}|}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{1}{s}\,\mathrm{d}s
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
|q_2(\tau_1)-q_2(\tau_2)| &\le \frac{\|g_{2r}\|_{L^1}}{\Gamma(\alpha_2)}\left|\int_1^{\tau_1}\left[\left(\log\frac{\tau_1}{s}\right)^{\alpha_2-1} - \left(\log\frac{\tau_2}{s}\right)^{\alpha_2-1}\right]\frac{1}{s}\,\mathrm{d}s\right| \\
&\quad + \frac{\|g_{2r}\|_{L^1}}{\Gamma(\alpha_2)}\left|\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha_2-1}\frac{1}{s}\,\mathrm{d}s\right| \\
&\quad + \frac{\|g_{2r}\|_{L^1}|(\log\tau_1)^{\alpha_2-1}-(\log\tau_2)^{\alpha_2-1}|}{\Gamma(\alpha_2)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_2-1}\frac{1}{s}\,\mathrm{d}s.
\end{aligned} \tag{20}$$

Note that the right-hand side of the two inequalities (19) and (20) go to zero for arbitrarily $(x,y)\in\Omega$ as $\tau_2\to\tau_1$. Therefore, B_1 and B_2 are equicontinuous sets. Also, note that $\|B(x,y)\| = \|B_1(x,y)\| + \|B_2(x,y)\|$, so B is equicontinuous sets.

From (ii)–(iii) and the Arzelá-Ascoli theorem, we have $B:\Xi\to\mathfrak{T}(\Xi)$ is completely continuous. Thus, B defines a compact multi-valued operator $B:\Xi\to\mathfrak{T}_{\mathrm{cp}}(\Xi)$.

(iv) We claim that B has a closed graph. Let $(x_n,y_n)\to(x_*,y_*)$, $(h_{1n},h_{2n})\in B(x_n,y_n)$ and $(h_{1n},h_{2n})\to(h_{1*},h_{2*})$. Then, we need to prove that $(h_{1*},h_{2*})\in B(x_*,y_*)$, i.e., $h_{1*}\in B_1(x_*,y_*)$, $h_{2*}\in B_2(x_*,y_*)$. Due to $h_{1n}\in B_1(x_n,y_n)$, $h_{2n}\in B_2(x_n,y_n)$, there are $v_{1n}\in F_{1,xy}$, $v_{2n}\in F_{2,xy}$ such that

$$h_{1n}(t) = \frac{1}{\Gamma(\alpha_1)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{v_{1n}(s)}{s}\mathrm{d}s - \frac{(\log t)^{\alpha_1-1}}{\Gamma(\alpha_1)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{v_{1n}(s)}{s}\mathrm{d}s, \quad t\in[1,e] \tag{21}$$

and

$$h_{2n}(t) = \frac{1}{\Gamma(\alpha_2)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_2-1}\frac{v_{2n}(s)}{s}\mathrm{d}s - \frac{(\log t)^{\alpha_2-1}}{\Gamma(\alpha_2)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_2-1}\frac{v_{2n}(s)}{s}\mathrm{d}s, \quad t\in[1,e]. \tag{22}$$

Thus, it suffices to show that there are $v_{1*}\in F_{1,x_*y_*}$, $v_{2*}\in F_{2,x_*y_*}$ such that for each $t\in[1,e]$,

$$h_{1*} = \frac{1}{\Gamma(\alpha_i)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{v_{i*}(s)}{s}\mathrm{d}s - \frac{(\log t)^{\alpha_1-1}}{\Gamma(\alpha_1)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{v_{1*}(s)}{s}\mathrm{d}s \in B_1(x_*,y_*) \tag{23}$$

and

$$h_{2*} = \frac{1}{\Gamma(\alpha_2)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_2-1}\frac{v_{2*}(s)}{s}\mathrm{d}s - \frac{(\log t)^{\alpha_2-1}}{\Gamma(\alpha_2)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_2-1}\frac{v_{2*}(s)}{s}\mathrm{d}s \in B_2(x_*,y_*). \tag{24}$$

Let us take the linear operator $\Theta=(\Theta_1,\Theta_2)$, where $\Theta_i : L^1([1,e],R)\to C([1,e],\mathbb{R})$ are given by:

$$\Theta_i(v_i)(t) = \frac{1}{\Gamma(\alpha_i)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_i-1}\frac{v_i(s)}{s}\mathrm{d}s - \frac{(\log t)^{\alpha_i-1}}{\Gamma(\alpha_i)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_i-1}\frac{v_i(s)}{s}\mathrm{d}s, \quad i=1,2. \tag{25}$$

On the other hand,

$$\begin{aligned}&\|h_{1n}(t)-h_{1*}(t)\|\\ &= \left\|\frac{1}{\Gamma(\alpha_1)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{v_{1n}(s)-v_{1*}(s)}{s}\mathrm{d}s - \frac{(\log t)^{\alpha_1-1}}{\Gamma(\alpha_1)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{v_{1n}(s)-v_{1*}(s)}{s}\mathrm{d}s\right\| \to 0,\\ &\qquad \text{as } n\to\infty,\end{aligned} \tag{26}$$

and

$$\begin{aligned}&\|h_{2n}(t)-h_{2*}(t)\|\\ &= \left\|\frac{1}{\Gamma(\alpha_2)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_2-1}\frac{v_{2n}(s)-v_{2*}(s)}{s}\mathrm{d}s - \frac{(\log t)^{\alpha_2-1}}{\Gamma(\alpha_2)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_2-1}\frac{v_{2n}(s)-v_{2*}(s)}{s}\mathrm{d}s\right\| \to 0,\\ &\qquad \text{as } n\to\infty.\end{aligned} \tag{27}$$

So it follows from Lemma 4 that $\Theta_i\circ F_{i,xy}$ are the closed graph operators. This means that if $h_{in}(t)\in\Theta_i(F_{i,xy})$, then we obtain $(h_{1*},h_{2*})\in B(x_*,y_*)$ since $(x_n,y_n)\to(x_*,y_*)$.

Note that $B:\Xi\to\mathfrak{T}(\Xi)$ is completely continuous; it follows from Lemma 3 that the operator B is upper semicontinuous operator on Ξ.

Step 3. We show that Lemma 2(c) holds. From (H3), we have $M=\|B(\Xi)\|=\|B(x,y)\|=\|B_1(x,y)\|+\|B_2(x,y)\|\le(2/\Gamma(\alpha_1))\|g_{1r}\|_{L^1}+(2/\Gamma(\alpha_2))\|g_{2r}\|_{L^1}$, and $k=L_1+L_2$ for $(x,y)\in\Omega$.

Up to now, we have proved that all the conditions of Lemma 2 are satisfied and it means that either Lemma 2(i) or Lemma 2(ii) holds. Next, we display that Lemma 2(ii) is not satisfied.

Let $\Phi=\{(u,v)\in\Xi|\mu(u,v)\in(A_1(u,v)B_1(u,v),A_2(u,v)B_2(u,v))\}$ and $(u,v)\in\Phi$ be arbitrary. Then, for $\mu>1$, $\mu(u,v)\in(A_1(u,v)B_1(u,v),A_2(u,v)B_2(u,v))$, there exists $(\omega_1,\omega_2)\in(F_{1,xy},F_{2,xy})$ such that, for any $\mu>1$, we have

$$u(t) = \mu^{-1}f_1(t,u(t),v(t))\left(\frac{1}{\Gamma(\alpha_1)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{\omega_1(s)}{s}\mathrm{d}s - \frac{(\log t)^{\alpha_1-1}}{\Gamma(\alpha_1)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{\omega_1(s)}{s}\mathrm{d}s\right) \tag{28}$$

and

$$v(t) = \mu^{-1} f_2(t, u(t), v(t)) \left(\frac{1}{\Gamma(\alpha_2)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha_2 - 1} \frac{\omega_2(s)}{s} \, \mathrm{d}s - \frac{(\log t)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} \int_1^e \left(\log \frac{e}{s} \right)^{\alpha_2 - 1} \frac{\omega_2(s)}{s} \, \mathrm{d}s \right), \tag{29}$$

for all $t \in [1, e]$. Therefore, we have

$$\begin{aligned} |u(t)| &\le \mu^{-1} |f_1(t, u(t), v(t))| \left(\frac{1}{\Gamma(\alpha_1)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha_1 - 1} \frac{|\omega_1(s)|}{s} \, \mathrm{d}s + \frac{(\log t)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} \int_1^e \left(\log \frac{e}{s} \right)^{\alpha_1 - 1} \frac{|\omega_1(s)|}{s} \, \mathrm{d}s \right) \\ &\le [|f_1(t, u(t), v(t)) - f_1(t, 0, 0)| + |f_1(t, 0, 0)|] \left(\frac{1}{\Gamma(\alpha_1)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha_1 - 1} \frac{\omega_1(s)}{s} \, \mathrm{d}s \right. \\ &\quad \left. + \frac{(\log t)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} \int_1^e \left(\log \frac{e}{s} \right)^{\alpha_1 - 1} \right) \\ &\le [L_1 r + F_{10}] \frac{2}{\Gamma(\alpha_1)} \|g_{1r}\|_{L^1} \end{aligned} \tag{30}$$

and

$$|v(t)| \le [L_2 r + F_{20}] \frac{2}{\Gamma(\alpha_2)} \|g_{2r}\|_{L^1}. \tag{31}$$

Thus, we have

$$\|(u, v)\| \le kr \left(\frac{2}{\Gamma(\alpha_1)} \|g_{1r}\|_{L^1} + \frac{2}{\Gamma(\alpha_2)} \|g_{2r}\|_{L^1} \right) + \frac{2F_{10}}{\Gamma(\alpha_1)} \|g_{1r}\|_{L^1} + \frac{2F_{20}}{\Gamma(\alpha_2)} \|g_{2r}\|_{L^1},$$

where F_{i0} and $g_{ir} (i = 1, 2)$ are defined in (H3). Then, note that $\|(u, v)\| = R$, we have

$$r \le \frac{\frac{2F_{10} \|g_{1r}\|_{L^1}}{\Gamma(\alpha_1)} + \frac{2F_{20} \|g_{2r}\|_{L^1}}{\Gamma(\alpha_2)}}{1 - \frac{2k \|g_{1r}\|_{L^1}}{\Gamma(\alpha_1)} - \frac{2k \|g_{2r}\|_{L^1}}{\Gamma(\alpha_2)}}.$$

Therefore, Lemma 2(ii) is not satisfied by (H3). Then, there exists $(x, y) \in \Xi$ such that

$$(x, y) = (A_1(x, y) B_1(x, y), A_2(x, y) B_2(x, y)),$$

i.e., system (1) at least one solution on $[1, e] \times [1, e]$. This completes the proof. □

Example 1. Consider the following system of Hadamard-type fractional hybrid differential inclusions

$$\begin{cases} D^{1.5} \dfrac{x(t)}{0.1 e^{1-t} (\sin x + \sin y + 2)} \in F_1(t, x(t), y(t)), \ 1 < t < e, \\ D^{1.25} \dfrac{y(t)}{0.15 (\arctan x + \arctan y + 3)} \in F_2(t, x(t), y(t)), \ 1 < t < e, \\ x(1) = x(e) = 0, \ y(1) = y(e) = 0, \end{cases} \tag{32}$$

where $\alpha_1 = 1.5$, $\alpha_2 = 1.25$, $F_i : [1, e] \times \mathbb{R}^2 \to \mathbb{R} (i = 1, 2)$ are multi-valued maps given by:

$$t \mapsto F_1(t, x(t), y(t)) = \left[\frac{|x|^3}{10(|x|^3 + |y|^3 + 3)}, \frac{|\sin x|}{16(|\sin x| + |\sin y| + 1)} + \frac{1}{16} \right]$$

and

$$t \mapsto F_2(t, x(t), y(t)) = \left[\frac{|x|^3}{12(|x|^3 + |y|^3 + 2)} + \frac{1}{12}, \frac{|\sin x|}{10(|\sin x| + |\sin y| + 3)} \right].$$

By (H1), $L_1 = 0.1$, $L_2 = 0.15$ with $k = 0.25$. For $f_1 \in F_1, f_2 \in F_2$, we have

$$|f_1| \leq \max\left\{\frac{|x|^3}{10(|x|^3 + |y|^3 + 3)}, \frac{|\sin x|}{16(|\sin x| + |\sin y| + 1)} + \frac{1}{16}\right\} \leq \frac{1}{8}, \quad (x, y) \in \mathbb{R}^2,$$

and

$$|f_2| \leq \max\left\{\frac{|x|^3}{12(|x|^3 + |y|^3 + 2)} + \frac{1}{12}, \frac{|\sin x|}{10(|\sin x| + |\sin y| + 3)}\right\} \leq \frac{1}{6}, \quad (x, y) \in \mathbb{R}^2.$$

Then,

$$\|F_1(t, x, y)\| = \sup\{|v_1| : v_1 \in F_1(t, x, y)\} \leq \frac{1}{8} = g_{1r}(t), \quad (x, y) \in \mathbb{R}^2,$$
$$\|F_2(t, x, y)\| = \sup\{|v_2| : v_2 \in F_2(t, x, y)\} \leq \frac{1}{6} = g_{2r}(t), \quad (x, y) \in \mathbb{R}^2.$$

Clearly, $\|g_{1r}\|_{L^1} = \frac{e-1}{8}$, $\|g_{2r}\|_{L^1} = \frac{e-1}{6}$, $F_{10} = \frac{1}{16}$, $F_{10} = \frac{1}{12}$. Hence, $\left(\frac{2\|g_{1r}\|_{L^1}}{\Gamma(\alpha_1)} + \frac{2\|g_{2r}\|_{L^1}}{\Gamma(\alpha_2)}\right)k \approx 0.279156 < \frac{1}{2}$ and $r > \left(\frac{2F_{10}\|g_{1r}\|_{L^1}}{\Gamma(\alpha_1)} + \frac{2F_{20}\|g_{2r}\|_{L^1}}{\Gamma(\alpha_2)}\right) \Big/ \left(1 - \frac{2k\|g_{1r}\|_{L^1}}{\Gamma(\alpha_1)} - \frac{2k\|g_{2r}\|_{L^1}}{\Gamma(\alpha_2)}\right) \approx 0.115079$. Consequently, all the conditions of Theorem 1 are satisfied, and system (32) has at least one solution on $[1, e] \times [1, e]$.

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