

Research Article

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New concrete inequalities of the Hermite-Hadamard-Jensen-Mercer type in terms of generalized conformable fractional operators via majorization

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Abstract: The Hermite-Hadamard inequality is regarded as one of the most favorable inequalities from the research point of view. Currently, mathematicians are working on extending, improving, and generalizing this inequality. This article presents concrete inequalities of the Hermite-Hadamard-Jensen-Mercer type in weighted and unweighted forms by using the idea of majorization and convexity together with generalized conformable fractional integral operators. They not only represent continuous and discrete inequalities in compact form but also produce generalized inequalities connecting various fractional operators such as Hadamard, Katugampola, Riemann-Liouville, conformable, and Riemann integrals into one single form. Also, two new integral identities have been investigated pertaining a differentiable function and three tuples. By using these identities and assuming $|f'|$ and $|f'|^q$ ($q > 1$) as convex, we deduce bounds concerning the discrepancy of the terms of the main inequalities.

Keywords: Jensen inequality, Mercer inequality, Hermite-Hadamard inequality, Hölder inequality, majorization theory

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1 Introduction

In mathematics, inequalities are very helpful, especially when working with the quantities about which we are not sure what exactly they are equal to. Sometimes, rather than writing down the exact solution, one can solve a mathematical problem by approximating the solution [1–3]. Nowadays, inequalities are recognized and taught as some of the most practical areas of mathematics because they have successfully extended their validity to the fields of engineering [4], economics [5], mathematical statistics [6,7], and information theory [8].

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Inequalities are deeply connected to convexity theory. Convexity theory can be used to produce numerous concepts concerning mathematical inequalities and their uses in different branches of science [9–13]. Convex functions are one of the fundamental concepts in convexity theory. This concept has led to the discovery of new inequalities, among which the Hermite-Hadamard [14], Jensen's [15], the Jensen-Mercer [16], Ostrowski [17], and Fejér [18] inequalities are notable. In recent years, the Hermite-Hadamard inequality, which is considered to be the most attractive inequality in the literature, has drawn a lot of attention. The definition of this inequality is given as follows:

Let $f : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ be a convex function. Then

$$f\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} f(u) du \leq \frac{f(\alpha_1) + f(\alpha_2)}{2}. \quad (1)$$

If the function being applied is concave, the inequality stated earlier can be obtained in the opposite direction. Researchers have also used several classes of convex functions, including s -convex function [19], coordinate convex function [20], strongly convex function [21], and η -convex function [22] to derive this inequality.

The fractional calculus is another field that has drawn the attention of many researchers working in different fields. The impact of this field seems to have increased significantly over the past years in both pure and applied branches of science and engineering [23–29]. Currently, researchers are using convexity and the concepts from this subject to produce new findings [30–33]. There are various known versions of the fractional integral operators used in fractional calculus, among which the two operators, i.e., Riemann-Liouville [34] and Hadamard [35] integrals have been widely studied due to their extensive applications [34–36]. These operators are given as follows:

Definition 1. [34] (Riemann-Liouville fractional integral operators). Let $f : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ be a function, $0 < \nu$ and $\Gamma(\cdot)$ represent the gamma function. Then, the following fractional integrals are known as Riemann-Liouville fractional integral operators.

$$J_{\alpha_1^+}^\nu f(\zeta) = \frac{1}{\Gamma(\nu)} \int_{\alpha_1}^{\zeta} (\zeta - u)^{\nu-1} f(u) du, \quad \zeta > \alpha_1 \quad (\text{left-sided integral})$$

and

$$J_{\alpha_2^-}^\nu f(\zeta) = \frac{1}{\Gamma(\nu)} \int_{\zeta}^{\alpha_2} (u - \zeta)^{\nu-1} f(u) du, \quad \zeta < \alpha_2 \quad (\text{right-sided integral}).$$

Another, fractional integral operator was introduced by Hadamard [35] and is expressed as follows:

Definition 2. [35] (**Hadamard fractional integral operators**) Let $f : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ be a function, $0 < \nu$ and $\Gamma(\cdot)$ represent the gamma function. Then, the following fractional integrals are known as Hadamard fractional integral operators.

$$H_{\alpha_1^+}^\nu f(\zeta) = \frac{1}{\Gamma(\nu)} \int_{\alpha_1}^{\zeta} \left(\ln \frac{\zeta}{u} \right)^{\nu-1} \frac{f(u)}{u} du, \quad \zeta > \alpha_1 \quad (\text{left-sided integral}) \quad (2)$$

and

$$H_{\alpha_2^-}^\nu f(\zeta) = \frac{1}{\Gamma(\nu)} \int_{\zeta}^{\alpha_2} \left(\ln \frac{u}{\zeta} \right)^{\nu-1} \frac{f(u)}{u} du, \quad \zeta < \alpha_2 \quad (\text{right-sided integral}).$$

In [30], Ögülmüş and Sarikaya presented the Hermite-Hadamard inequalities by using Riemann-Liouville fractional integrals along with the convexity of the function. This inequality is expressed in the form of the following theorem.

Theorem 1. Let $f : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ be a convex function with $\alpha_2 > \alpha_1 \geq 0$, $\nu > 0$ and $f \in L_1[\alpha_1, \alpha_2]$. Then

$$f\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{\Gamma(\nu + 1)}{2(\alpha_2 - \alpha_1)^\nu} [J_{\alpha_1^+}^\nu f(\alpha_2) + J_{\alpha_2^-}^\nu f(\alpha_1)] \leq \frac{f(\alpha_1) + f(\alpha_2)}{2}. \quad (3)$$

In this article, the authors also obtained an integral identity, and then by employing that identity, they obtained bounds for the difference of terms associated to (3). Also, the inequality (3) implies inequality (1) for a specific value of the parameter.

Katugampola, discovered a generalized fractional integral operator that generalizes the Riemann-Liouville and Hadamard fractional integral operators. It is defined as follows:

Definition 3. [37] (**Generalized fractional integral operator**). Let $f : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ be a function such that $f \in X_c^p(\alpha_1, \alpha_2)$ ($c \in \mathbb{R}$, $1 \leq p \leq \infty$), $0 < \nu$, $\tau > 0$ and $\Gamma(\cdot)$ represent the gamma function. Then, the following fractional integrals are known as generalized fractional integral operators.

$${}^\tau I_{\alpha_1^+}^\nu f(\zeta) = \frac{1}{\Gamma(\nu)} \int_{\alpha_1}^{\zeta} \left(\frac{\zeta^\tau - w^\tau}{\tau} \right)^{\nu-1} w^{\tau-1} f(w) dw, \quad \zeta > \alpha_1 \quad (\text{left-sided integral}) \quad (4)$$

and

$${}^\tau I_{\alpha_2^-}^\nu f(\zeta) = \frac{1}{\Gamma(\nu)} \int_{\zeta}^{\alpha_2} \left(\frac{w^\tau - \zeta^\tau}{\tau} \right)^{\nu-1} w^{\tau-1} f(w) dw, \quad \alpha_2 > \zeta \quad (\text{right-sided integral}). \quad (5)$$

For $\tau \rightarrow 1$ in Definition 3, we obtain Riemann-Liouville fractional integral operator, and for $\tau \rightarrow 0$, we obtain Hadamard fractional integrals. In [38], Jleli et al. introduced the following Hermite-Hadamard type inequality using generalized fractional integrals.

Theorem 2. [38] Let $f : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ be a convex function with $0 < \alpha_1 < \alpha_2 < \infty$, $\nu > 0$, and $\tau > 0$. Then

$$f\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{\tau^\nu \Gamma(\nu + 1)}{4(\alpha_2^\tau - \alpha_1^\tau)^\nu} [{}^\tau I_{\alpha_1^+}^\nu F(\alpha_2) + {}^\tau I_{\alpha_2^-}^\nu F(\alpha_1)] \leq \frac{f(\alpha_1) + f(\alpha_2)}{2}, \quad (6)$$

where $F(z) = f(z) + \tilde{f}(z)$ and $\tilde{f}(z) = f(\alpha_1 + \alpha_2 - z)$ for $z \in [\alpha_1, \alpha_2]$.

In [39], Khan and Adil Khan introduced new conformable fractional integral operators that are a generalized version of some more significant integral operators. These operators are defined as follows:

Definition 4. [39] (**Generalized conformable fractional operators**). Let f be a conformable integrable function on $[\alpha_1, \alpha_2]$ with $0 \leq \alpha_1 < \alpha_2 < \infty$, $\nu > 0$, $\theta \in (0, 1]$, $\tau \in \mathbb{R}$, and $\theta + \tau \neq 0$. Then the following fractional integrals are known as generalized conformable fractional integral operators.

$${}^\tau K_{\alpha_1^+}^\nu f(\zeta) = \frac{1}{\Gamma(\nu)} \int_{\alpha_1}^{\zeta} \left(\frac{\zeta^{\tau+\theta} - w^{\tau+\theta}}{\tau + \theta} \right)^{\nu-1} f(w) w^\tau d_\theta w, \quad \zeta > \alpha_1 \quad (\text{left-sided integral}) \quad (7)$$

and

$${}^\tau K_{\alpha_2^-}^\nu f(\zeta) = \frac{1}{\Gamma(\nu)} \int_{\zeta}^{\alpha_2} \left(\frac{w^{\tau+\theta} - \zeta^{\tau+\theta}}{\tau + \theta} \right)^{\nu-1} f(w) w^\tau d_\theta w, \quad \alpha_2 > \zeta \quad (\text{right-sided integral}). \quad (8)$$

Remark 1.

- (1) Definition 4 reduces to Katugampola fractional integrals for $\theta = 1$ [37];
- (2) If we take $\tau = 0$ and $\theta \rightarrow 0$ in Definition 4, we obtain Hadamard fractional integrals [35];

- (3) By taking $\tau = 0$ and $\theta = 1$, we obtain the famous Riemann-Liouville fractional integrals [34];
 (4) Definition 4 reduces to the conformable fractional integrals and the classical Riemann integrals by choosing $\nu = 1$, $\tau = 0$, and $\nu = \theta = 1$, $\tau = 0$, respectively.

In [40], Khan and Adil Khan also utilized these operators to obtain the Hermite-Hadamard inequality which is expressed as follows:

Theorem 3. [40] *Let $f : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ be a convex function such that f is conformal integrable on $[\alpha_1, \alpha_2]$ with $0 \leq \alpha_1 < \alpha_2 < \infty$, $\nu > 0$, and $\theta \in (0, 1]$, $\tau \in \mathbb{R}$, and $\tau + \theta > 0$. Then*

$$f\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{(\tau + \theta)^\nu \Gamma(\nu + 1)}{4(\alpha_2^{\tau+\theta} - \alpha_1^{\tau+\theta})^\nu} [\theta^\tau I_{\alpha_1}^\nu F(\alpha_2) + \tau^\tau K_{\alpha_2}^\nu F(\alpha_1)] \leq \frac{f(\alpha_1) + f(\alpha_2)}{2}, \quad (9)$$

where $F(z) = f(z) + \tilde{f}(z)$ and $\tilde{f}(z) = f(\alpha_1 + \alpha_2 - z)$ for $z \in [\alpha_1, \alpha_2]$.

We now describe the concept of majorization, which will be used to express our new findings.

Definition 5. [41] (**Majorization**). Let $\theta = (\theta_1, \theta_2, \dots, \theta_\varepsilon)$ and $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_\varepsilon)$ be two tuples arranged in the order $\theta_{[\varepsilon]} \leq \theta_{[\varepsilon-1]} \leq \dots \leq \theta_{[1]}$ and $\vartheta_{[\varepsilon]} \leq \vartheta_{[\varepsilon-1]} \leq \dots \leq \vartheta_{[1]}$. Then the tuple θ is said to majorize the tuple ϑ (in symbols $\vartheta < \theta$) if:

$$\sum_{\varsigma=1}^k \vartheta_{[\varsigma]} \leq \sum_{\varsigma=1}^k \theta_{[\varsigma]} \quad \text{for } k = 1, 2, \dots, \varepsilon - 1, \quad (10)$$

and

$$\sum_{\varsigma=1}^{\varepsilon} \theta_{\varsigma} = \sum_{\varsigma=1}^{\varepsilon} \vartheta_{\varsigma}. \quad (11)$$

By using the majorization concept, Neizgoda [42] presented an extended version of the Jensen-Mercer inequality which is stated as follows:

Theorem 4. [42] (**Majorized discrete Jensen-Mercer inequality**). *Let f be a function that is defined to be convex on the interval I of real numbers, and let $(x_{i\varsigma})$ be an $n \times \varepsilon$ matrix such that $x_{i\varsigma} \in I$ for all $i = 1, 2, \dots, n$, $\varsigma = 1, 2, \dots, \varepsilon$. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\varepsilon)$ be a tuple with $\alpha_\varsigma \in I$ for $\varsigma = 1, 2, \dots, \varepsilon$ and $\sigma_i \geq 0$ for $i = 1, 2, \dots, n$ with $\sum_{i=1}^n \sigma_i = 1$. Also, suppose that each row of the matrix $(x_{i\varsigma})$ is majorized by α , then*

$$f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \sum_{i=1}^n \sigma_i x_{i\varsigma}\right) \leq \sum_{\varsigma=1}^{\varepsilon} f(\alpha_{\varsigma}) - \sum_{\varsigma=1}^{\varepsilon-1} \sum_{i=1}^n \sigma_i f(x_{i\varsigma}). \quad (12)$$

From the overall discussion about the inequalities, we come to know that the researchers have constructed generalized inequalities in discrete or continuous form [43,44] by using integral operators, convexity or both [45–47]. However, it is essential to develop combined inequalities that can act as both continuous and discrete inequalities simultaneously. This requirement can be fulfilled by using the majorization approach discussed earlier. In [48], Faisal et al. utilized Riemann-Liouville fractional integral operators along with the concept of convexity and majorization to establish “conticrete” fractional Hermite-Hadamard-Jensen-Mercer inequalities. The authors have introduced “conticrete” name for these inequalities due to the fact that they represent both discrete and continuous inequalities in a combined form. The authors also obtained integral identities by using which bounds for the major results have been obtained. For some more conticrete inequalities via majorization in terms of Caputo fractional derivative operators and Riemann integrals, see [49,50].

This article is divided into four sections. Section 2 is devoted to the derivation of new conticrete inequalities by using majorization and generalized conformable fractional operators. These results have been expressed in the form of Theorems 5 and 6. Remark 2 demonstrates that our key findings generalize the previously known inequalities in the literature for different fractional operators. They also present the conticrete versions of the Hermite-Hadamard-Jensen-Mercer inequalities for various fractional operators and classical integrals. In Section 3, we derive weighted forms of our main results by first considering decreasing tuples and then taking those tuples, which show the same monotonic behavior. These results have been obtained on the basis of two previously proved lemmas and expressed as Theorems 7 and 8. In Section 4, two new integral identities are investigated by considering three tuples and a differentiable function. These identities are then used to establish bounds for the main results. The conclusion of overall work is presented in the last section.

2 Main results

We present our new results in the context of generalized conformable fractional operators in the following way.

Theorem 5. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\varepsilon)$, $\beta = (\beta_1, \beta_2, \dots, \beta_\varepsilon)$, and $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_\varepsilon)$ be three tuples, where $\beta_\varepsilon > \gamma_\varepsilon$ and $\alpha_\varsigma, \beta_\varsigma, \gamma_\varsigma \in I = [\gamma_\varepsilon, \beta_\varepsilon]$, for all $\varsigma = 1, 2, \dots, \varepsilon$. Let $\nu > 0$ and $\tau \in \mathbb{R}$, $\theta \in (0, 1]$ such that $\tau + \theta > 0$. Let $f: I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a function such that $f \in L_\theta(I)$ and $F(z) = f(z) + \tilde{f}(z)$, $\tilde{f}(z) = f(\beta_\varepsilon + \gamma_\varepsilon - z)$. Also, suppose that f is convex function on I with $\beta < \alpha$ and $\gamma < \alpha$, then

$$\begin{aligned} f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) &\leq \frac{(\tau + \theta)^\nu \Gamma(\nu + 1)}{4(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} [\tilde{\mathcal{I}}_\theta^\nu K_{\gamma_\varepsilon}^\nu F(\beta_\varepsilon) + \tilde{\mathcal{I}}_\theta^\nu K_{\beta_\varepsilon}^\nu F(\gamma_\varepsilon)] \\ &\leq \frac{f(\beta_\varepsilon) + f(\gamma_\varepsilon)}{2} \\ &\leq \sum_{\varsigma=1}^{\varepsilon} f(\alpha_{\varsigma}) - \frac{1}{2} \left[\sum_{\varsigma=1}^{\varepsilon-1} f(\beta_{\varsigma}) + \sum_{\varsigma=1}^{\varepsilon-1} f(\gamma_{\varsigma}) \right]. \end{aligned} \quad (13)$$

Proof. By the fact that $\beta < \alpha$ and $\gamma < \alpha$, we have $\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} = \beta_\varepsilon$ and $\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} = \gamma_\varepsilon$. Now, we take start by considering that

$$f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) = f\left[\frac{1}{2}(\beta_\varepsilon + \gamma_\varepsilon)\right] = f\left[\frac{1}{2}(\eta\beta_\varepsilon + (1-\eta)\gamma_\varepsilon + \eta\gamma_\varepsilon + (1-\eta)\beta_\varepsilon)\right]. \quad (14)$$

By using convexity of f in (14), we have

$$f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) \leq \frac{1}{2} [f(\eta\beta_\varepsilon + (1-\eta)\gamma_\varepsilon) + f(\eta\gamma_\varepsilon + (1-\eta)\beta_\varepsilon)]. \quad (15)$$

By multiplying both sides of (15) by

$$\frac{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma})(\tau + \theta)^{1-\nu} ((1-\eta)\gamma_\varepsilon + \eta\beta_\varepsilon)^{\tau+\theta-1}}{\Gamma(\nu)(\beta_\varepsilon^{\tau+\theta} - ((1-\eta)\gamma_\varepsilon + \eta\beta_\varepsilon)^{\tau+\theta})^{1-\nu}}$$

and integrating with respect to η , we obtain

$$\begin{aligned}
& f\left(\sum_{\varsigma=1}^{\varepsilon}\alpha_{\varsigma}-\sum_{\varsigma=1}^{\varepsilon-1}\left(\frac{\beta_{\varsigma}+\gamma_{\varsigma}}{2}\right)\right)\frac{1}{\Gamma(\nu)}\sum_{\varsigma=1}^{\varepsilon-1}(\gamma_{\varsigma}-\beta_{\varsigma})(\tau+\theta)^{1-\nu}\int_0^1((1-\eta)\gamma_{\varepsilon}+\eta\beta_{\varepsilon})^{\tau+\theta-1}(\beta_{\varepsilon}^{\tau+\theta}-((1-\eta)\gamma_{\varepsilon}+\eta\beta_{\varepsilon})^{\tau+\theta})^{\nu-1}d\eta \\
& \leq \frac{1}{2}\left[\frac{1}{\Gamma(\nu)}\sum_{\varsigma=1}^{\varepsilon-1}(\gamma_{\varsigma}-\beta_{\varsigma})(\tau+\theta)^{1-\nu}\int_0^1((1-\eta)\gamma_{\varepsilon}+\eta\beta_{\varepsilon})^{\tau+\theta-1}(\beta_{\varepsilon}^{\tau+\theta}-((1-\eta)\gamma_{\varepsilon}+\eta\beta_{\varepsilon})^{\tau+\theta})^{\nu-1}\right. \\
& \quad \times f(\eta\beta_{\varepsilon}+(1-\eta)\gamma_{\varepsilon})d\eta + \frac{1}{\Gamma(\nu)}\sum_{\varsigma=1}^{\varepsilon-1}(\gamma_{\varsigma}-\beta_{\varsigma})(\tau+\theta)^{1-\nu}\int_0^1((1-\eta)\gamma_{\varepsilon}+\eta\beta_{\varepsilon})^{\tau+\theta-1} \\
& \quad \times (\beta_{\varepsilon}^{\tau+\theta}-((1-\eta)\gamma_{\varepsilon}+\eta\beta_{\varepsilon})^{\tau+\theta})^{\nu-1}f(\eta\gamma_{\varepsilon}+(1-\eta)\beta_{\varepsilon})d\eta] \\
& \quad f\left(\sum_{\varsigma=1}^{\varepsilon}\alpha_{\varsigma}-\sum_{\varsigma=1}^{\varepsilon-1}\left(\frac{\beta_{\varsigma}+\gamma_{\varsigma}}{2}\right)\right)\frac{1}{\Gamma(\nu)}\sum_{\varsigma=1}^{\varepsilon-1}(\gamma_{\varsigma}-\beta_{\varsigma})(\tau+\theta)^{1-\nu}I_1 \\
& \leq \frac{1}{2}\left[\frac{1}{\Gamma(\nu)}\sum_{\varsigma=1}^{\varepsilon-1}(\gamma_{\varsigma}-\beta_{\varsigma})(\tau+\theta)^{1-\nu}I_2 + \frac{1}{\Gamma(\nu)}\sum_{\varsigma=1}^{\varepsilon-1}(\gamma_{\varsigma}-\beta_{\varsigma})(\tau+\theta)^{1-\nu}I_3\right]. \tag{16}
\end{aligned}$$

Now finding I_1 , I_2 , and I_3 , we have

$$\frac{1}{\Gamma(\nu)}\sum_{\varsigma=1}^{\varepsilon-1}(\gamma_{\varsigma}-\beta_{\varsigma})(\tau+\theta)^{1-\nu}I_1 = \frac{1}{\Gamma(\nu+1)(\tau+\theta)^{\nu}}(\beta_{\varepsilon}^{\tau+\theta}-\gamma_{\varepsilon}^{\tau+\theta})^{\nu}. \tag{17}$$

Now, by substituting $w = (1-\eta)\gamma_{\varepsilon} + \eta\beta_{\varepsilon}$, we obtain

$$\frac{1}{\Gamma(\nu)}\sum_{\varsigma=1}^{\varepsilon-1}(\gamma_{\varsigma}-\beta_{\varsigma})(\tau+\theta)^{1-\nu}I_2 = {}_{\theta}^{\tau}K_{\gamma_{\varepsilon}}^{\nu}f(\beta_{\varepsilon}) \tag{18}$$

Also, by using the following identity

$$\tilde{f}((1-\eta)\gamma_{\varepsilon} + \eta\beta_{\varepsilon}) = f((1-\eta)\beta_{\varepsilon} + \eta\gamma_{\varepsilon})$$

and by substituting the aforementioned, we obtain I_3 as follows:

$$\frac{1}{\Gamma(\nu)}\sum_{\varsigma=1}^{\varepsilon-1}(\gamma_{\varsigma}-\beta_{\varsigma})(\tau+\theta)^{1-\nu}I_3 = {}_{\theta}^{\tau}K_{\gamma_{\varepsilon}}^{\nu}\tilde{f}(\beta_{\varepsilon}) \tag{19}$$

By using (17), (18), and (19) in (16), we deduce

$$\frac{1}{(\tau+\theta)^{\nu}\Gamma(\nu+1)}(\beta_{\varepsilon}^{\tau+\theta}-\gamma_{\varepsilon}^{\tau+\theta})^{\nu}f\left(\sum_{\varsigma=1}^{\varepsilon}\alpha_{\varsigma}-\sum_{\varsigma=1}^{\varepsilon-1}\left(\frac{\beta_{\varsigma}+\gamma_{\varsigma}}{2}\right)\right) \leq \frac{1}{2}[{}_{\theta}^{\tau}K_{\gamma_{\varepsilon}}^{\nu}F(\beta_{\varepsilon})]. \tag{20}$$

Again multiplying (15) by

$$\frac{\sum_{\varsigma=1}^{\varepsilon-1}(\gamma_{\varsigma}-\beta_{\varsigma})(\tau+\theta)^{1-\nu}((1-\eta)\gamma_{\varepsilon}+\eta\beta_{\varepsilon})^{\tau+\theta-1}}{\Gamma(\nu)((1-\eta)\gamma_{\varepsilon}+\eta\beta_{\varepsilon})^{\tau+\theta}-\gamma_{\varepsilon}^{\tau+\theta})^{1-\nu}}$$

and then integrating with respect to η and using the same procedure as earlier, we have

$$\frac{1}{(\tau+\theta)^{\nu}\Gamma(\nu+1)}(\beta_{\varepsilon}^{\tau+\theta}-\gamma_{\varepsilon}^{\tau+\theta})^{\nu}f\left(\sum_{\varsigma=1}^{\varepsilon}\alpha_{\varsigma}-\sum_{\varsigma=1}^{\varepsilon-1}\left(\frac{\beta_{\varsigma}+\gamma_{\varsigma}}{2}\right)\right) \leq \frac{1}{2}[{}_{\theta}^{\tau}K_{\beta_{\varepsilon}}^{\nu}F(\gamma_{\varepsilon})]. \tag{21}$$

By adding (20) and (21), we obtain

$$f\left(\sum_{\varsigma=1}^{\varepsilon}\alpha_{\varsigma}-\sum_{\varsigma=1}^{\varepsilon-1}\left(\frac{\beta_{\varsigma}+\gamma_{\varsigma}}{2}\right)\right) \leq \frac{(\tau+\theta)^{\nu}\Gamma(\nu+1)}{4(\beta_{\varepsilon}^{\tau+\theta}-\gamma_{\varepsilon}^{\tau+\theta})^{\nu}}[{}_{\theta}^{\tau}K_{\gamma_{\varepsilon}}^{\nu}F(\beta_{\varepsilon}) + {}_{\theta}^{\tau}K_{\beta_{\varepsilon}}^{\nu}F(\gamma_{\varepsilon})].$$

Hence, the first part of the inequality (13) is completed.

To prove the second part of the inequality (13), we utilize the convexity of f as follows:

$$f(\eta\beta_\varepsilon + (1 - \eta)\gamma_\varepsilon) \leq \eta f(\beta_\varepsilon) + (1 - \eta)f(\gamma_\varepsilon) \quad (22)$$

and

$$f(\eta\gamma_\varepsilon + (1 - \eta)\beta_\varepsilon) \leq \eta f(\gamma_\varepsilon) + (1 - \eta)f(\beta_\varepsilon). \quad (23)$$

Adding (22) and (23) and then applying Theorem 4 by taking $n = 1$ and $\sigma_1 = 1$, we have

$$f(\eta\beta_\varepsilon + (1 - \eta)\gamma_\varepsilon) + f(\eta\gamma_\varepsilon + (1 - \eta)\beta_\varepsilon) \leq f(\beta_\varepsilon) + f(\gamma_\varepsilon) \leq 2 \sum_{\varsigma=1}^{\varepsilon} f(\alpha_\varsigma) - \left(\sum_{\varsigma=1}^{\varepsilon-1} f(\beta_\varsigma) + \sum_{\varsigma=1}^{\varepsilon-1} f(\gamma_\varsigma) \right) \quad (24)$$

Multiplying both sides of (24) by

$$\frac{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma)(\tau + \theta)^{1-\nu} ((1 - \eta)\gamma_\varepsilon + \eta\beta_\varepsilon)^{\tau+\theta-1}}{\Gamma(\nu)(\beta_\varepsilon^{\tau+\theta} - ((1 - \eta)\gamma_\varepsilon + \eta\beta_\varepsilon)^{\tau+\theta})^{1-\nu}}$$

and integrating with respect to η , we obtain

$$\frac{(\tau + \theta)^\nu \Gamma(\nu + 1)}{2(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} {}^\tau K_{\gamma_\varepsilon^+}^\nu F(\beta_\varepsilon) \leq \frac{1}{2} [f(\beta_\varepsilon) + f(\gamma_\varepsilon)] \leq \sum_{\varsigma=1}^{\varepsilon} f(\alpha_\varsigma) - \frac{1}{2} \left[\sum_{\varsigma=1}^{\varepsilon-1} f(\beta_\varsigma) + \sum_{\varsigma=1}^{\varepsilon-1} f(\gamma_\varsigma) \right]. \quad (25)$$

Again multiplying (24) by

$$\frac{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma)(\tau + \theta)^{1-\nu} ((1 - \eta)\gamma_\varepsilon + \eta\beta_\varepsilon)^{\tau+\theta-1}}{\Gamma(\nu)((1 - \eta)\gamma_\varepsilon + \eta\beta_\varepsilon)^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^{1-\nu}}$$

and then integrating with respect to η and using the same procedure as earlier, we have

$$\frac{(\tau + \theta)^\nu \Gamma(\nu + 1)}{2(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} {}^\tau K_{\beta_\varepsilon^-}^\nu F(\gamma_\varepsilon) \leq \frac{1}{2} [f(\beta_\varepsilon) + f(\gamma_\varepsilon)] \leq \sum_{\varsigma=1}^{\varepsilon} f(\alpha_\varsigma) - \frac{1}{2} \left[\sum_{\varsigma=1}^{\varepsilon-1} f(\beta_\varsigma) + \sum_{\varsigma=1}^{\varepsilon-1} f(\gamma_\varsigma) \right]. \quad (26)$$

By adding (25) and (26), we obtain the second and third part of the inequality (13). $\square$

Remark 2.

- (1) By taking $\varepsilon = 2$, $\beta_1 = \alpha_1$, and $\gamma_1 = \alpha_2$ in inequality (13), we obtain inequality (9), which is proved in [40].
- (2) By substituting $\theta = 1$ in (13), we acquire the following inequality of the Hermite-Hadamard-Jensen-Mercer type in terms of Katugampola fractional operators:

$$\begin{aligned} f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2}\right)\right) &\leq \frac{(\tau + 1)^\nu \Gamma(\nu + 1)}{4(\beta_\varepsilon^{\tau+1} - \gamma_\varepsilon^{\tau+1})^\nu} [{}_1^\tau K_{\gamma_\varepsilon^+}^\nu F(\beta_\varepsilon) + {}_1^\tau K_{\beta_\varepsilon^-}^\nu F(\gamma_\varepsilon)] \\ &\leq \frac{1}{2} [f(\beta_\varepsilon) + f(\gamma_\varepsilon)] \\ &\leq \sum_{\varsigma=1}^{\varepsilon} f(\alpha_\varsigma) - \frac{1}{2} \left[\sum_{\varsigma=1}^{\varepsilon-1} f(\beta_\varsigma) + \sum_{\varsigma=1}^{\varepsilon-1} f(\gamma_\varsigma) \right]. \end{aligned} \quad (27)$$

- (3) By taking $\theta = 1$, $\varepsilon = 2$, $\beta_1 = \alpha_1$, and $\gamma_1 = \alpha_2$ in (13), we obtain inequality (6), which is proved in [38].

- (4) For $\tau + \theta \rightarrow 0$ in (13) and using L' Hospital rule and the relation $f \in [\gamma_\varepsilon, \beta_\varepsilon]$, $\int_{\gamma_\varepsilon}^{\beta_\varepsilon} f(w) d_\theta w =$

$\int_{\gamma_\varepsilon}^{\beta_\varepsilon} f(w) w^{\theta-1} dw$, where $\int_{\gamma_\varepsilon}^{\beta_\varepsilon} d_\theta w$ represents conformable integration, we obtain the following Hermite-Hadamard-Jensen-Mercer type inequality for Hadamard fractional integral operators:

$$\begin{aligned}
f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) &\leq \frac{\Gamma(\nu+1)}{2\left(\ln\left(\frac{\beta_{\varepsilon}}{\gamma_{\varepsilon}}\right)\right)^{\nu}} [{}_0^0 K_{\gamma_{\varepsilon}}^{\nu} f(\beta_{\varepsilon}) + {}_0^0 K_{\beta_{\varepsilon}}^{\nu} f(\gamma_{\varepsilon})] \\
&\leq \frac{1}{2} [f(\beta_{\varepsilon}) + f(\gamma_{\varepsilon})] \\
&\leq \sum_{\varsigma=1}^{\varepsilon} f(\alpha_{\varsigma}) - \frac{1}{2} \left[\sum_{\varsigma=1}^{\varepsilon-1} f(\beta_{\varsigma}) + \sum_{\varsigma=1}^{\varepsilon-1} f(\gamma_{\varsigma}) \right].
\end{aligned} \tag{28}$$

(5) By inserting $\tau + \theta = 1$, in (13), we obtain the following Hermite-Hadamard-Jensen-Mercer type inequality for Riemann-Liouville fractional integrals [48]:

$$f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) \leq \frac{\Gamma(\nu+1)}{2\left(\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma})\right)^{\nu}} [{}_1^0 K_{\gamma_{\varepsilon}}^{\nu} f(\beta_{\varepsilon}) + {}_1^0 K_{\beta_{\varepsilon}}^{\nu} f(\gamma_{\varepsilon})].$$

(6) For $\tau + \theta = 1$, $\varepsilon = 2$, $\beta_1 = \alpha_1$, $\gamma_1 = \alpha_2$, the inequality (13) produces the inequality (3).

(7) For $\nu = 1$ and $\tau = 0$ in (13), we obtain the Hermite-Hadamard-Jensen-Mercer type inequality in terms of conformable fractional integrals as follows:

$$\begin{aligned}
f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) &\leq \frac{1}{2(\beta_{\varepsilon}^{\theta} - \gamma_{\varepsilon}^{\theta})^{\nu}} \int_{\gamma_{\varepsilon}}^{\beta_{\varepsilon}} F(w) d_{\theta} w \\
&\leq \frac{1}{2} [f(\gamma_{\varepsilon}) + f(\beta_{\varepsilon})] \\
&\leq \sum_{\varsigma=1}^{\varepsilon} f(\alpha_{\varsigma}) - \frac{1}{2} \left[\sum_{\varsigma=1}^{\varepsilon-1} f(\beta_{\varsigma}) + \sum_{\varsigma=1}^{\varepsilon-1} f(\gamma_{\varsigma}) \right].
\end{aligned} \tag{29}$$

(8) For $\theta = \nu = 1$, $\tau = 0$, we deduce the following inequality of the Hermite-Hadamard-Jensen-Mercer type in terms of classical Riemann integrals [49]:

$$f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) \leq \frac{1}{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma})} \int_{\gamma_{\varepsilon}}^{\beta_{\varepsilon}} f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - u\right) du \leq \sum_{\varsigma=1}^{\varepsilon} f(\alpha_{\varsigma}) - \frac{1}{2} \left[\sum_{\varsigma=1}^{\varepsilon-1} f(\beta_{\varsigma}) + \sum_{\varsigma=1}^{\varepsilon-1} f(\gamma_{\varsigma}) \right]. \tag{30}$$

Now, we present another result of Hermite-Hadamard-Jensen-Mercer type by using generalized conformable fractional operators as follows:

Theorem 6. Let all the conditions in the hypotheses of Theorem 5 hold true, then

$$\begin{aligned}
&f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) \\
&\leq \frac{\Gamma(\nu+1)(\tau+\theta)^{\nu}}{2Q_{\tau+\theta}^{\nu}(\beta_{\varepsilon}, \gamma_{\varepsilon})} \left[{}_{\theta}^{\tau} K_{\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right)}^{\nu} F(\beta_{\varepsilon}) + {}_{\theta}^{\tau} K_{\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right)}^{\nu} F(\gamma_{\varepsilon}) \right] \\
&\leq \sum_{\varsigma=1}^{\varepsilon} f(\alpha_{\varsigma}) - \frac{1}{2} \left[\sum_{\varsigma=1}^{\varepsilon-1} f(\beta_{\varsigma}) + \sum_{\varsigma=1}^{\varepsilon-1} f(\gamma_{\varsigma}) \right],
\end{aligned} \tag{31}$$

where

$$Q_{\tau+\theta}^{\nu}(\beta_{\varepsilon}, \gamma_{\varepsilon}) = \left(\beta_{\varepsilon}^{\tau+\theta} - \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \frac{1}{2} \sum_{\varsigma=1}^{\varepsilon-1} (\beta_{\varsigma} + \gamma_{\varsigma}) \right)^{\tau+\theta} \right)^{\nu} + \left(\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \frac{1}{2} \sum_{\varsigma=1}^{\varepsilon-1} (\beta_{\varsigma} + \gamma_{\varsigma}) \right)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta} \right)^{\nu}.$$

Proof. For $\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} = \beta_{\varepsilon}$, $\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} = \gamma_{\varepsilon}$, and $\eta \in [0, 1]$, it may be written that

$$\begin{aligned} f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) &= f\left[\frac{1}{2}(\beta_{\varepsilon} + \gamma_{\varepsilon})\right] \\ &= f\left[\frac{1}{2}\left(\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma}\right)\right)\right.\right. \\ &\quad \left.\left. + \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma}\right)\right)\right)\right]. \end{aligned} \quad (32)$$

By using convexity of f in (32), we have

$$f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) \leq \frac{1}{2} \left[f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma}\right)\right) + f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma}\right)\right) \right]. \quad (33)$$

Multiplying both sides of (33) by

$$\frac{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma})(\tau + \theta)^{1-\nu} \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma}\right)\right)^{\tau+\theta-1}}{2\Gamma(\nu) \left(\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma}\right)\right)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta}\right)^{1-\nu}},$$

and integrating with respect to η , we obtain

$$\begin{aligned} &f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) \frac{1}{2\Gamma(\nu)} \sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma})(\tau + \theta)^{1-\nu} \int_0^1 \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma}\right)\right)^{\tau+\theta-1} \\ &\quad \times \left(\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma}\right)\right)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta}\right)^{\nu-1} d\eta \\ &\leq \frac{1}{2} \left[\frac{1}{2\Gamma(\nu)} \sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma})(\tau + \theta)^{1-\nu} \int_0^1 \left(\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma}\right)\right)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta}\right)^{\nu-1} \right. \\ &\quad \times \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma}\right)\right)^{\tau+\theta-1} f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma}\right)\right) d\eta \\ &\quad + \frac{1}{2\Gamma(\nu)} \sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma})(\tau + \theta)^{1-\nu} \int_0^1 \left(\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma}\right)\right)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta}\right)^{\nu-1} \\ &\quad \times \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma}\right)\right)^{\tau+\theta-1} f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma}\right)\right) d\eta \Big]. \\ &f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) \frac{1}{2\Gamma(\nu)} \sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma})(\tau + \theta)^{1-\nu} I_4 \\ &\leq \frac{1}{2} \left[\frac{1}{2\Gamma(\nu)} \sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma})(\tau + \theta)^{1-\nu} I_5 + \frac{1}{2\Gamma(\nu)} \sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma})(\tau + \theta)^{1-\nu} I_6 \right]. \end{aligned} \quad (34)$$

Now, finding I_4 , I_5 , and I_6 , we have

$$\frac{1}{2\Gamma(\nu)} \sum_{\zeta=1}^{\varepsilon-1} (\gamma_{\zeta} - \beta_{\zeta})(\tau + \theta)^{1-\nu} I_4 = \frac{1}{\Gamma(\nu+1)(\tau + \theta)^{\nu}} \left(\left(\sum_{\zeta=1}^{\varepsilon} \alpha_{\zeta} - \frac{1}{2} \sum_{\zeta=1}^{\varepsilon-1} (\beta_{\zeta} + \gamma_{\zeta}) \right)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta} \right)^{\nu}. \quad (35)$$

Now by substituting $w = \sum_{\zeta=1}^{\varepsilon} \alpha_{\zeta} - \frac{\eta}{2} \sum_{\zeta=1}^{\varepsilon-1} \beta_{\zeta} - \frac{2-\eta}{2} \sum_{\zeta=1}^{\varepsilon-1} \gamma_{\zeta}$, we obtain

$$\frac{1}{2\Gamma(\nu)} \sum_{\zeta=1}^{\varepsilon-1} (\gamma_{\zeta} - \beta_{\zeta})(\tau + \theta)^{1-\nu} I_5 = {}_{\theta}^{\tau} K^{\nu} \left(\sum_{\zeta=1}^{\varepsilon} \alpha_{\zeta} - \sum_{\zeta=1}^{\varepsilon-1} \left(\frac{\beta_{\zeta} + \gamma_{\zeta}}{2} \right) \right)^{\nu} f(\gamma_{\varepsilon}). \quad (36)$$

Also, by using the identity

$$\tilde{f} \left(\sum_{\zeta=1}^{\varepsilon} \alpha_{\zeta} - \frac{\eta}{2} \sum_{\zeta=1}^{\varepsilon-1} \gamma_{\zeta} - \frac{2-\eta}{2} \sum_{\zeta=1}^{\varepsilon-1} \beta_{\zeta} \right) = f \left(\sum_{\zeta=1}^{\varepsilon} \alpha_{\zeta} - \frac{\eta}{2} \sum_{\zeta=1}^{\varepsilon-1} \beta_{\zeta} - \frac{2-\eta}{2} \sum_{\zeta=1}^{\varepsilon-1} \gamma_{\zeta} \right)$$

and the aforementioned substitution, we have

$$\frac{1}{2\Gamma(\nu)} \sum_{\zeta=1}^{\varepsilon-1} (\gamma_{\zeta} - \beta_{\zeta})(\tau + \theta)^{1-\nu} I_6 = {}_{\theta}^{\tau} K^{\nu} \left(\sum_{\zeta=1}^{\varepsilon} \alpha_{\zeta} - \sum_{\zeta=1}^{\varepsilon-1} \left(\frac{\beta_{\zeta} + \gamma_{\zeta}}{2} \right) \right)^{\nu} \tilde{f}(\gamma_{\varepsilon}). \quad (37)$$

By using (35), (36), and (37) in (34), we deduce

$$\begin{aligned} & \frac{1}{\Gamma(\nu+1)(\tau + \theta)^{\nu}} \left(\left(\sum_{\zeta=1}^{\varepsilon} \alpha_{\zeta} - \frac{1}{2} \sum_{\zeta=1}^{\varepsilon-1} (\beta_{\zeta} + \gamma_{\zeta}) \right)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta} \right)^{\nu} f \left(\sum_{\zeta=1}^{\varepsilon} \alpha_{\zeta} - \sum_{\zeta=1}^{\varepsilon-1} \left(\frac{\beta_{\zeta} + \gamma_{\zeta}}{2} \right) \right) \\ & \leq \frac{1}{2} \left[{}_{\theta}^{\tau} K^{\nu} \left(\sum_{\zeta=1}^{\varepsilon} \alpha_{\zeta} - \sum_{\zeta=1}^{\varepsilon-1} \left(\frac{\beta_{\zeta} + \gamma_{\zeta}}{2} \right) \right)^{\nu} F(\gamma_{\varepsilon}) \right]. \end{aligned} \quad (38)$$

Similarly, multiplying both sides of (33) by

$$\frac{\sum_{\zeta=1}^{\varepsilon-1} (\gamma_{\zeta} - \beta_{\zeta})(\tau + \theta)^{1-\nu} \left(\sum_{\zeta=1}^{\varepsilon} \alpha_{\zeta} - \left(\frac{\eta}{2} \sum_{\zeta=1}^{\varepsilon-1} \gamma_{\zeta} + \frac{2-\eta}{2} \sum_{\zeta=1}^{\varepsilon-1} \beta_{\zeta} \right) \right)^{\tau+\theta-1}}{2\Gamma(\nu) \left(\beta_{\varepsilon}^{\tau+\theta} - \left(\sum_{\zeta=1}^{\varepsilon} \alpha_{\zeta} - \left(\frac{\eta}{2} \sum_{\zeta=1}^{\varepsilon-1} \gamma_{\zeta} + \frac{2-\eta}{2} \sum_{\zeta=1}^{\varepsilon-1} \beta_{\zeta} \right) \right)^{\tau+\theta} \right)^{1-\nu}},$$

and then integrating with respect to η , we finally obtain

$$\begin{aligned} & \frac{1}{\Gamma(\nu+1)(\tau + \theta)^{\nu}} \left(\beta_{\varepsilon}^{\tau+\theta} - \left(\sum_{\zeta=1}^{\varepsilon} \alpha_{\zeta} - \frac{1}{2} \sum_{\zeta=1}^{\varepsilon-1} (\beta_{\zeta} + \gamma_{\zeta}) \right)^{\tau+\theta} \right)^{\nu} f \left(\sum_{\zeta=1}^{\varepsilon} \alpha_{\zeta} - \sum_{\zeta=1}^{\varepsilon-1} \left(\frac{\beta_{\zeta} + \gamma_{\zeta}}{2} \right) \right) \\ & \leq \frac{1}{2} \left[{}_{\theta}^{\tau} K^{\nu} \left(\sum_{\zeta=1}^{\varepsilon} \alpha_{\zeta} - \sum_{\zeta=1}^{\varepsilon-1} \left(\frac{\beta_{\zeta} + \gamma_{\zeta}}{2} \right) \right)^{\nu} F(\beta_{\varepsilon}) \right]. \end{aligned} \quad (39)$$

By adding (38) and (39), we have

$$\begin{aligned} & f \left(\sum_{\zeta=1}^{\varepsilon} \alpha_{\zeta} - \sum_{\zeta=1}^{\varepsilon-1} \left(\frac{\beta_{\zeta} + \gamma_{\zeta}}{2} \right) \right) \\ & \leq \frac{\Gamma(\nu+1)(\tau + \theta)^{\nu}}{2Q_{\tau+\theta}^{\nu}(\beta_{\varepsilon}, \gamma_{\varepsilon})} \left[{}_{\theta}^{\tau} K^{\nu} \left(\sum_{\zeta=1}^{\varepsilon} \alpha_{\zeta} - \sum_{\zeta=1}^{\varepsilon-1} \left(\frac{\beta_{\zeta} + \gamma_{\zeta}}{2} \right) \right)^{\nu} F(\beta_{\varepsilon}) + {}_{\theta}^{\tau} K^{\nu} \left(\sum_{\zeta=1}^{\varepsilon} \alpha_{\zeta} - \sum_{\zeta=1}^{\varepsilon-1} \left(\frac{\beta_{\zeta} + \gamma_{\zeta}}{2} \right) \right)^{\nu} F(\gamma_{\varepsilon}) \right]. \end{aligned} \quad (40)$$

Thus, the first part of the inequality (31) is proved. To prove the second part of the inequality (31), we proceed as follows:

As a consequence of convexity of f , we use Theorem 4 for the values $n = 2$, $\sigma_1 = \frac{\eta}{2}$, and $\sigma_2 = \frac{2-\eta}{2}$, to obtain

$$f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma}\right)\right) \leq \sum_{\varsigma=1}^{\varepsilon} f(\alpha_{\varsigma}) - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} f(\beta_{\varsigma}) + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} f(\gamma_{\varsigma})\right) \quad (41)$$

and

$$f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma}\right)\right) \leq \sum_{\varsigma=1}^{\varepsilon} f(\alpha_{\varsigma}) - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} f(\gamma_{\varsigma}) + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} f(\beta_{\varsigma})\right). \quad (42)$$

By adding (41) and (42), we have

$$\begin{aligned} & f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma}\right)\right) + f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma}\right)\right) \\ & \leq 2 \sum_{\varsigma=1}^{\varepsilon} f(\alpha_{\varsigma}) - \left(\sum_{\varsigma=1}^{\varepsilon-1} f(\beta_{\varsigma}) + \sum_{\varsigma=1}^{\varepsilon-1} f(\gamma_{\varsigma})\right). \end{aligned} \quad (43)$$

By multiplying

$$\frac{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma})(\tau + \theta)^{1-\nu} \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma}\right)\right)^{\tau+\theta-1}}{2\Gamma(\nu) \left(\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma}\right)\right)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta}\right)^{1-\nu}},$$

to both sides of (43), we obtain

$$\begin{aligned} & {}_{\theta}^{\tau} K^{\nu} \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right)^{-F(\gamma_{\varepsilon})} \\ & \leq \frac{1}{\Gamma(\nu+1)(\tau+\theta)^{\nu}} \left[2 \sum_{\varsigma=1}^{\varepsilon} f(\alpha_{\varsigma}) - \left(\sum_{\varsigma=1}^{\varepsilon-1} f(\beta_{\varsigma}) + \sum_{\varsigma=1}^{\varepsilon-1} f(\gamma_{\varsigma})\right) \right] \left[\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \frac{1}{2} \sum_{\varsigma=1}^{\varepsilon-1} (\beta_{\varsigma} + \gamma_{\varsigma})\right)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta} \right]^{\nu}. \end{aligned} \quad (44)$$

Also, by multiplying

$$\frac{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma})(\tau + \theta)^{1-\nu} \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma}\right)\right)^{\tau+\theta-1}}{2\Gamma(\nu) \left(\beta_{\varepsilon}^{\tau+\theta} - \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma}\right)\right)^{\tau+\theta}\right)^{1-\nu}}$$

to both sides of (43), we obtain

$$\begin{aligned} & {}_{\theta}^{\tau} K^{\nu} \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right)^{+F(\beta_{\varepsilon})} \\ & \leq \frac{1}{\Gamma(\nu+1)(\tau+\theta)^{\nu}} \left[2 \sum_{\varsigma=1}^{\varepsilon} f(\alpha_{\varsigma}) - \left(\sum_{\varsigma=1}^{\varepsilon-1} f(\beta_{\varsigma}) + \sum_{\varsigma=1}^{\varepsilon-1} f(\gamma_{\varsigma})\right) \right] \left[\beta_{\varepsilon}^{\tau+\theta} - \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \frac{1}{2} \sum_{\varsigma=1}^{\varepsilon-1} (\beta_{\varsigma} + \gamma_{\varsigma})\right)^{\tau+\theta} \right]^{\nu}. \end{aligned} \quad (45)$$

Now, by adding (44) and (45), we obtain the second part of the inequality (31). $\square$

Remark 3.

(1) By substituting $\tau + \theta = 1$, in (31), we obtain the following Hermite-Hadamard-Jensen-Mercer type inequality for Riemann-Liouville fractional integrals [48]:

$$\begin{aligned} & f\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) \\ & \leq \frac{2^{1-\nu}\Gamma(\nu+1)}{\left(\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma})\right)^{\nu}} \left[{}^0K^{\nu}_{\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)} f(\beta_{\varepsilon}) + {}^0K^{\nu}_{\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)} f(\gamma_{\varepsilon}) \right] \\ & \leq \sum_{\varsigma=1}^{\varepsilon} f(\alpha_{\varsigma}) - \frac{1}{2} \left[\sum_{\varsigma=1}^{\varepsilon-1} f(\beta_{\varsigma}) + \sum_{\varsigma=1}^{\varepsilon-1} f(\gamma_{\varsigma}) \right]. \end{aligned} \quad (46)$$

(2) For $\tau + \theta = 1$ and $\varepsilon = 2$, the inequality (31) reduces to the following inequality [30]:

$$\begin{aligned} & f\left(\alpha_1 + \alpha_2 - \frac{\beta_1 + \gamma_1}{2}\right) \\ & \leq \frac{2^{1-\nu}\Gamma(\nu+1)}{(\gamma_1 - \beta_1)^{\nu}} \left[{}^0K^{\nu}_{\left(\alpha_1 + \alpha_2 - \frac{\beta_1 + \gamma_1}{2}\right)} f(\alpha_1 + \alpha_2 - \beta_1) + {}^0K^{\nu}_{\left(\alpha_1 + \alpha_2 - \frac{\beta_1 + \gamma_1}{2}\right)} f(\alpha_1 + \alpha_2 - \gamma_1) \right] \\ & \leq f(\alpha_1) + f(\alpha_2) - \frac{1}{2} [f(\beta_1) + f(\gamma_1)]. \end{aligned} \quad (47)$$

(3) For $\tau + \theta = 1$, $\varepsilon = 2$, and $\nu = 1$ then the inequality (31) reduces to the inequality (2.2) in [51].

3 Weighted forms of the main results

We generate weighted versions of our first primary result in this section. The similar process can be used to produce weighted versions of the second main outcome. They are acquired by utilizing lemmas that have already been proved. We first recall these lemmas [49].

Lemma 1. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\varepsilon})$, $\mathbf{r} = (r_1, r_2, \dots, r_{\varepsilon})$ be two tuples and $(x_{i\varepsilon})$ be a real $n \times \varepsilon$ matrix such that α_{ε} , $x_{i\varepsilon} \in I$, $r_{\varsigma} \geq 0$ with $r_{\varepsilon} \neq 0$, $\varepsilon = \frac{1}{r_{\varepsilon}}$ for all $i = 1, 2, \dots, n$, $\varsigma = 1, 2, \dots, \varepsilon$, and f be a convex function defined on I . Also, let $\sigma_i \geq 0$ for $i = 1, 2, \dots, n$ with $\sum_{i=1}^n \sigma_i = 1$. If for each $i = 1, 2, \dots, n$, $(x_{i1}, x_{i2}, \dots, x_{i\varepsilon})$ is a decreasing tuple and satisfying

$$\sum_{\varsigma=1}^k r_{\varsigma} x_{i\varsigma} \leq \sum_{\varsigma=1}^k r_{\varsigma} \alpha_{\varsigma} \text{ for } k = 1, 2, \dots, \varepsilon - 1, \quad \sum_{\varsigma=1}^{\varepsilon} r_{\varsigma} \alpha_{\varsigma} = \sum_{\varsigma=1}^{\varepsilon} r_{\varsigma} x_{i\varsigma},$$

then

$$f\left(\sum_{\varsigma=1}^{\varepsilon} \varepsilon r_{\varsigma} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \sum_{i=1}^n \varepsilon \sigma_i r_{\varsigma} x_{i\varsigma}\right) \leq \sum_{\varsigma=1}^{\varepsilon} \varepsilon r_{\varsigma} f(\alpha_{\varsigma}) - \sum_{\varsigma=1}^{\varepsilon-1} \sum_{i=1}^n \varepsilon \sigma_i r_{\varsigma} f(x_{i\varsigma}).$$

Lemma 2. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\varepsilon})$, $\mathbf{r} = (r_1, r_2, \dots, r_{\varepsilon})$ be two tuples and $(x_{i\varepsilon})$ be a real $n \times \varepsilon$ matrix such that α_{ε} , $x_{i\varepsilon} \in I$, $r_{\varsigma} \geq 0$ with $r_{\varepsilon} \neq 0$, $\varepsilon = \frac{1}{r_{\varepsilon}}$ for all $i = 1, 2, \dots, n$, $\varsigma = 1, 2, \dots, \varepsilon$, and f be a convex function defined on I . Also, let $\sigma_i \geq 0$ for $i = 1, 2, \dots, n$ with $\sum_{i=1}^n \sigma_i = 1$.

If for each $i = 1, 2, \dots, n$, $(\alpha_{\varsigma} - x_{i\varsigma})$ and $x_{i\varsigma}$ are monotonically in the same sense and

$$\sum_{\varsigma=1}^{\varepsilon} r_{\varsigma} \alpha_{\varsigma} = \sum_{\varsigma=1}^{\varepsilon} r_{\varsigma} x_{i_{\varsigma}},$$

then

$$f\left(\sum_{\varsigma=1}^{\varepsilon} \varepsilon r_{\varsigma} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \sum_{i=1}^n \varepsilon \sigma_i r_{\varsigma} x_{i_{\varsigma}}\right) \leq \sum_{\varsigma=1}^{\varepsilon} \varepsilon r_{\varsigma} f(\alpha_{\varsigma}) - \sum_{\varsigma=1}^{\varepsilon-1} \sum_{i=1}^n \varepsilon \sigma_i r_{\varsigma} f(x_{i_{\varsigma}}).$$

Now, we present weighted form of Theorem 5 on the basis of Lemma 1 as follows:

Theorem 7. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\varepsilon})$, $\beta = (\beta_1, \beta_2, \dots, \beta_{\varepsilon})$, $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_{\varepsilon})$, and $\mathbf{r} = (r_1, r_2, \dots, r_{\varepsilon})$ be four tuples, where $\beta_{\varepsilon} > \gamma_{\varepsilon}$ and $\alpha_{\varsigma}, \beta_{\varsigma}, \gamma_{\varsigma} \in I = [\gamma_{\varepsilon}, \beta_{\varepsilon}]$, $r_{\varsigma} \geq 0$ with $r_{\varepsilon} \neq 0$ for all $\varsigma = 1, 2, \dots, \varepsilon$, $\varepsilon = \frac{1}{r_{\varepsilon}}$. Let $\nu > 0$ and $\tau \in \mathbb{R}$, $\theta \in (0, 1]$ such that $\tau + \theta > 0$. Let $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a function such that $f \in L_{\theta}(I)$ and $F(z) = f(z) + \tilde{f}(z)$, $\tilde{f}(z) = f(\beta_{\varepsilon} + \gamma_{\varepsilon} - z)$. Also, suppose that f is convex function on I , β , and γ are decreasing tuples and

$$\begin{aligned} \sum_{\varsigma=1}^k r_{\varsigma} \beta_{\varsigma} &\leq \sum_{\varsigma=1}^k r_{\varsigma} \alpha_{\varsigma}, & \sum_{\varsigma=1}^k r_{\varsigma} \gamma_{\varsigma} &\leq \sum_{\varsigma=1}^k r_{\varsigma} \alpha_{\varsigma} \quad \text{for } k = 1, \dots, \varepsilon - 1, \\ \sum_{\varsigma=1}^{\varepsilon} r_{\varsigma} \alpha_{\varsigma} &= \sum_{\varsigma=1}^{\varepsilon} r_{\varsigma} \beta_{\varsigma}, & \sum_{\varsigma=1}^{\varepsilon} r_{\varsigma} \alpha_{\varsigma} &= \sum_{\varsigma=1}^{\varepsilon} r_{\varsigma} \gamma_{\varsigma}, \end{aligned}$$

then

$$\begin{aligned} f\left(\sum_{\varsigma=1}^{\varepsilon} \varepsilon r_{\varsigma} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \varepsilon r_{\varsigma} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) &\leq \frac{(\tau + \theta)^{\nu} \Gamma(\nu + 1)}{4(\beta_{\varepsilon}^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta})^{\nu}} [\tau K_{\gamma_{\varepsilon}}^{\nu} F(\beta_{\varepsilon}) + \tau K_{\beta_{\varepsilon}}^{\nu} F(\gamma_{\varepsilon})] \\ &\leq \frac{1}{2} [f(\beta_{\varepsilon}) + f(\gamma_{\varepsilon})] \\ &\leq \sum_{\varsigma=1}^{\varepsilon} \varepsilon r_{\varsigma} f(\alpha_{\varsigma}) - \frac{1}{2} \left[\sum_{\varsigma=1}^{\varepsilon-1} \varepsilon r_{\varsigma} f(\beta_{\varsigma}) + \sum_{\varsigma=1}^{\varepsilon-1} \varepsilon r_{\varsigma} f(\gamma_{\varsigma}) \right]. \end{aligned} \quad (48)$$

Proof. From the fact that

$$\sum_{\varsigma=1}^{\varepsilon} r_{\varsigma} \alpha_{\varsigma} = \sum_{\varsigma=1}^{\varepsilon} r_{\varsigma} \beta_{\varsigma} \quad \text{and} \quad \sum_{\varsigma=1}^{\varepsilon} r_{\varsigma} \alpha_{\varsigma} = \sum_{\varsigma=1}^{\varepsilon} r_{\varsigma} \gamma_{\varsigma},$$

we arrive at

$$\sum_{\varsigma=1}^{\varepsilon} \varepsilon r_{\varsigma} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \varepsilon r_{\varsigma} \beta_{\varsigma} = \beta_{\varepsilon} \quad \text{and} \quad \sum_{\varsigma=1}^{\varepsilon} \varepsilon r_{\varsigma} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \varepsilon r_{\varsigma} \gamma_{\varsigma} = \gamma_{\varepsilon}.$$

Now, for $\eta \in [0, 1]$, we have

$$f\left(\sum_{\varsigma=1}^{\varepsilon} \varepsilon r_{\varsigma} \alpha_{\varsigma} - \varepsilon \sum_{\varsigma=1}^{\varepsilon-1} r_{\varsigma} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) = f\left[\left(\frac{\beta_{\varepsilon} + \gamma_{\varepsilon}}{2}\right)\right] = f\left[\frac{1}{2}(\eta \beta_{\varepsilon} + (1 - \eta) \gamma_{\varepsilon}) + \eta \gamma_{\varepsilon} + (1 - \eta) \beta_{\varepsilon}\right]. \quad (49)$$

By using convexity of f in (49), we have

$$f\left(\sum_{\varsigma=1}^{\varepsilon} \varepsilon r_{\varsigma} \alpha_{\varsigma} - \varepsilon \sum_{\varsigma=1}^{\varepsilon-1} r_{\varsigma} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) \leq \frac{1}{2} [f(\eta \beta_{\varepsilon} + (1 - \eta) \gamma_{\varepsilon}) + f(\eta \gamma_{\varepsilon} + (1 - \eta) \beta_{\varepsilon})]. \quad (50)$$

Multiplying both sides of (50) by

$$\frac{\left(\sum_{\varsigma=1}^{\varepsilon-1}(\varepsilon r_{\varsigma}\gamma_{\varsigma} - \varepsilon r_{\varsigma}\beta_{\varsigma})\right)(\tau + \theta)^{1-\nu}((1-\eta)\gamma_{\varepsilon} + \eta\beta_{\varepsilon})^{\tau+\theta-1}}{\Gamma(\nu)(\beta_{\varepsilon}^{\tau+\theta} - ((1-\eta)\gamma_{\varepsilon} + \eta\beta_{\varepsilon})^{\tau+\theta})^{1-\nu}}$$

and integrating with respect to η , we obtain

$$\begin{aligned} & f\left(\sum_{\varsigma=1}^{\varepsilon}\varepsilon r_{\varsigma}\alpha_{\varsigma} - \varepsilon\sum_{\varsigma=1}^{\varepsilon-1}r_{\varsigma}\left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right)\frac{1}{\Gamma(\nu)}\left(\sum_{\varsigma=1}^{\varepsilon-1}(\varepsilon r_{\varsigma}\gamma_{\varsigma} - \varepsilon r_{\varsigma}\beta_{\varsigma})\right)(\tau + \theta)^{1-\nu} \\ & \quad \times \int_0^1 ((1-\eta)\gamma_{\varepsilon} + \eta\beta_{\varepsilon})^{\tau+\theta-1}(\beta_{\varepsilon}^{\tau+\theta} - ((1-\eta)\gamma_{\varepsilon} + \eta\beta_{\varepsilon})^{\tau+\theta})^{\nu-1}d\eta \\ & \leq \frac{1}{2}\left[\frac{1}{\Gamma(\nu)}\left(\sum_{\varsigma=1}^{\varepsilon-1}(\varepsilon r_{\varsigma}\gamma_{\varsigma} - \varepsilon r_{\varsigma}\beta_{\varsigma})\right)(\tau + \theta)^{1-\nu}\int_0^1 ((1-\eta)\gamma_{\varepsilon} + \eta\beta_{\varepsilon})^{\tau+\theta-1}(\beta_{\varepsilon}^{\tau+\theta} - ((1-\eta)\gamma_{\varepsilon} + \eta\beta_{\varepsilon})^{\tau+\theta})^{\nu-1} \right. \\ & \quad \times f(\eta\beta_{\varepsilon} + (1-\eta)\gamma_{\varepsilon})d\eta + \frac{1}{\Gamma(\nu)}\left(\sum_{\varsigma=1}^{\varepsilon-1}(\varepsilon r_{\varsigma}\gamma_{\varsigma} - \varepsilon r_{\varsigma}\beta_{\varsigma})\right)(\tau + \theta)^{1-\nu}\int_0^1 ((1-\eta)\gamma_{\varepsilon} + \eta\beta_{\varepsilon})^{\tau+\theta-1}(\beta_{\varepsilon}^{\tau+\theta} \\ & \quad \left. - ((1-\eta)\gamma_{\varepsilon} + \eta\beta_{\varepsilon})^{\tau+\theta})^{\nu-1}f(\eta\gamma_{\varepsilon} + (1-\eta)\beta_{\varepsilon})d\eta\right] \\ & f\left(\sum_{\varsigma=1}^{\varepsilon}\varepsilon r_{\varsigma}\alpha_{\varsigma} - \varepsilon\sum_{\varsigma=1}^{\varepsilon-1}r_{\varsigma}\left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right)\frac{1}{\Gamma(\nu)}\left(\sum_{\varsigma=1}^{\varepsilon-1}(\varepsilon r_{\varsigma}\gamma_{\varsigma} - \varepsilon r_{\varsigma}\beta_{\varsigma})\right)(\tau + \theta)^{1-\nu}I_7 \\ & \leq \frac{1}{2}\left[\frac{1}{\Gamma(\nu)}\left(\sum_{\varsigma=1}^{\varepsilon-1}(\varepsilon r_{\varsigma}\gamma_{\varsigma} - \varepsilon r_{\varsigma}\beta_{\varsigma})\right)(\tau + \theta)^{1-\nu}I_8 + \frac{1}{\Gamma(\nu)}\left(\sum_{\varsigma=1}^{\varepsilon-1}(\varepsilon r_{\varsigma}\gamma_{\varsigma} - \varepsilon r_{\varsigma}\beta_{\varsigma})\right)(\tau + \theta)^{1-\nu}I_9\right]. \end{aligned} \quad (51)$$

Now, finding I_7 , I_8 , and I_9 in the following way:

$$\frac{1}{\Gamma(\nu)}\left(\sum_{\varsigma=1}^{\varepsilon-1}(\varepsilon r_{\varsigma}\gamma_{\varsigma} - \varepsilon r_{\varsigma}\beta_{\varsigma})\right)(\tau + \theta)^{1-\nu}I_7 = \frac{1}{\Gamma(\nu+1)(\tau + \theta)^{\nu}}(\beta_{\varepsilon}^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta})^{\nu}. \quad (52)$$

Now, by substituting $w = (1-\eta)\gamma_{\varepsilon} + \eta\beta_{\varepsilon}$, we obtain

$$\frac{1}{\Gamma(\nu)}\left(\sum_{\varsigma=1}^{\varepsilon-1}(\varepsilon r_{\varsigma}\gamma_{\varsigma} - \varepsilon r_{\varsigma}\beta_{\varsigma})\right)(\tau + \theta)^{1-\nu}I_8 = {}^{\tau}K_{\gamma_{\varepsilon}}^{\nu}f(\beta_{\varepsilon}) \quad (53)$$

Also, by using the following identity

$$\tilde{f}((1-\eta)\gamma_{\varepsilon} + \eta\beta_{\varepsilon}) = f((1-\eta)\beta_{\varepsilon} + \eta\gamma_{\varepsilon})$$

and making the aforementioned substitution, we obtain I_9 as follows:

$$\frac{1}{\Gamma(\nu)}\left(\sum_{\varsigma=1}^{\varepsilon-1}(\varepsilon r_{\varsigma}\gamma_{\varsigma} - \varepsilon r_{\varsigma}\beta_{\varsigma})\right)(\tau + \theta)^{1-\nu}I_9 = {}^{\tau}K_{\gamma_{\varepsilon}}^{\nu}\tilde{f}(\beta_{\varepsilon}). \quad (54)$$

By using (52), (53), and (54) in (51), we have

$$\frac{1}{\Gamma(\nu+1)(\tau + \theta)^{\nu}}(\beta_{\varepsilon}^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta})^{\nu}f\left(\sum_{\varsigma=1}^{\varepsilon}\varepsilon r_{\varsigma}\alpha_{\varsigma} - \varepsilon\sum_{\varsigma=1}^{\varepsilon-1}r_{\varsigma}\left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) \leq \frac{1}{2}[{}^{\tau}K_{\gamma_{\varepsilon}}^{\nu}F(\beta_{\varepsilon})]. \quad (55)$$

Again by multiplying both sides of (50) by

$$\frac{\left(\sum_{\varsigma=1}^{\varepsilon-1}(\varepsilon r_{\varsigma}\gamma_{\varsigma} - \varepsilon r_{\varsigma}\beta_{\varsigma})\right)(\tau + \theta)^{1-\nu}((1-\eta)\gamma_{\varepsilon} + \eta\beta_{\varepsilon})^{\tau+\theta-1}}{\Gamma(\nu)((1-\eta)\gamma_{\varepsilon} + \eta\beta_{\varepsilon})^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta})^{1-\nu}}$$

and integrating with respect to η and using the same procedure as earlier, we have

$$\frac{1}{\Gamma(\nu+1)(\tau+\theta)^\nu}(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu f\left(\sum_{\varsigma=1}^{\varepsilon} \varepsilon r_\varsigma \alpha_\varsigma - \varepsilon \sum_{\varsigma=1}^{\varepsilon-1} r_\varsigma \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2}\right)\right) \leq \frac{1}{2} [{}^\tau_\theta K_{\beta_\varepsilon}^\nu F(\gamma_\varepsilon)]. \quad (56)$$

By adding (55) and (56), we obtain the first part of the inequality (48).

To prove the second part of the inequality (48), we utilize the convexity of f as follows:

$$f(\eta\beta_\varepsilon + (1-\eta)\gamma_\varepsilon) \leq \eta f(\beta_\varepsilon) + (1-\eta)f(\gamma_\varepsilon) \quad (57)$$

and

$$f(\eta\gamma_\varepsilon + (1-\eta)\beta_\varepsilon) \leq \eta f(\gamma_\varepsilon) + (1-\eta)f(\beta_\varepsilon). \quad (58)$$

Adding (57) and (58) and then using Lemma 1 for $n = 2$, $\sigma_1 = \eta$, and $\sigma_2 = 1 - \eta$, we obtain

$$f(\eta\beta_\varepsilon + (1-\eta)\gamma_\varepsilon) + f(\eta\gamma_\varepsilon + (1-\eta)\beta_\varepsilon) \leq f(\beta_\varepsilon) + f(\gamma_\varepsilon) \leq 2 \sum_{\varsigma=1}^{\varepsilon} \varepsilon r_\varsigma f(\alpha_\varsigma) - \left(\sum_{\varsigma=1}^{\varepsilon-1} \varepsilon r_\varsigma f(\beta_\varsigma) + \sum_{\varsigma=1}^{\varepsilon-1} \varepsilon r_\varsigma f(\gamma_\varsigma) \right). \quad (59)$$

Multiplying both sides of (59) by

$$\frac{\left(\sum_{\varsigma=1}^{\varepsilon-1} (\varepsilon r_\varsigma \gamma_\varsigma - \varepsilon r_\varsigma \beta_\varsigma)\right)(\tau+\theta)^{1-\nu}((1-\eta)\gamma_\varepsilon + \eta\beta_\varepsilon)^{\tau+\theta-1}}{\Gamma(\nu)(\beta_\varepsilon^{\tau+\theta} - ((1-\eta)\gamma_\varepsilon + \eta\beta_\varepsilon)^{\tau+\theta})^{1-\nu}}$$

and integrating with respect to η , we obtain

$$\frac{(\tau+\theta)^\nu \Gamma(\nu+1)}{2(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} {}^\tau K_{\gamma_\varepsilon}^\nu F(\beta_\varepsilon) \leq \frac{1}{2} [f(\beta_\varepsilon) + f(\gamma_\varepsilon)] \leq \sum_{\varsigma=1}^{\varepsilon} \varepsilon r_\varsigma f(\alpha_\varsigma) - \frac{1}{2} \left[\sum_{\varsigma=1}^{\varepsilon-1} \varepsilon r_\varsigma f(\beta_\varsigma) + \sum_{\varsigma=1}^{\varepsilon-1} \varepsilon r_\varsigma f(\gamma_\varsigma) \right]. \quad (60)$$

Again by multiplying both sides of (59) by

$$\frac{\left(\sum_{\varsigma=1}^{\varepsilon-1} (\varepsilon r_\varsigma \gamma_\varsigma - \varepsilon r_\varsigma \beta_\varsigma)\right)(\tau+\theta)^{1-\nu}((1-\eta)\gamma_\varepsilon + \eta\beta_\varepsilon)^{\tau+\theta-1}}{\Gamma(\nu)((1-\eta)\gamma_\varepsilon + \eta\beta_\varepsilon)^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^{1-\nu}}$$

and integrating with respect to η , we have

$$\frac{(\tau+\theta)^\nu \Gamma(\nu+1)}{2(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} {}^\tau K_{\beta_\varepsilon}^\nu F(\gamma_\varepsilon) \leq \frac{1}{2} [f(\beta_\varepsilon) + f(\gamma_\varepsilon)] \leq \sum_{\varsigma=1}^{\varepsilon} \varepsilon r_\varsigma f(\alpha_\varsigma) - \frac{1}{2} \left[\sum_{\varsigma=1}^{\varepsilon-1} \varepsilon r_\varsigma f(\beta_\varsigma) + \sum_{\varsigma=1}^{\varepsilon-1} \varepsilon r_\varsigma f(\gamma_\varsigma) \right]. \quad (61)$$

By adding (60) and (61), we deduce the second and third part of inequality (48). $\square$

Another, weighted form of Theorem 5 on the basis of Lemma 2 is obtained as follows:

Theorem 8. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\varepsilon)$, $\beta = (\beta_1, \beta_2, \dots, \beta_\varepsilon)$, $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_\varepsilon)$, and $\mathbf{r} = (r_1, r_2, \dots, r_\varepsilon)$ be four tuples, where $\beta_\varepsilon > \gamma_\varepsilon$ and $\alpha_\varsigma, \beta_\varsigma, \gamma_\varsigma \in I = [\gamma_\varepsilon, \beta_\varepsilon]$, $r_\varsigma \geq 0$ with $r_\varepsilon \neq 0$ for all $\varsigma = 1, 2, \dots, \varepsilon$, and $\varepsilon = \frac{1}{r_\varepsilon}$. Let $\nu > 0$ and $\tau \in \mathbb{R}$, $\theta \in (0, 1]$ such that $\tau + \theta > 0$. Let $f: I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a function such that $f \in L_\theta(I)$ and $F(z) = f(z) + \tilde{f}(z)$, $\tilde{f}(z) = f(\beta_\varepsilon + \gamma_\varepsilon - z)$. Also, suppose that f is convex function on I and $\alpha - \beta$, β , $\alpha - \gamma$, and γ are monotonically in the same sense and

$$\sum_{\varsigma=1}^{\varepsilon} r_\varsigma \alpha_\varsigma = \sum_{\varsigma=1}^{\varepsilon} r_\varsigma \beta_\varsigma, \quad \sum_{\varsigma=1}^{\varepsilon} r_\varsigma \alpha_\varsigma = \sum_{\varsigma=1}^{\varepsilon} r_\varsigma \gamma_\varsigma,$$

then

$$\begin{aligned}
f\left(\sum_{\varsigma=1}^{\varepsilon} \varepsilon r_{\varsigma} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \varepsilon r_{\varsigma} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2}\right)\right) &\leq \frac{(\tau + \theta)^{\nu} \Gamma(\nu + 1)}{4(\beta_{\varepsilon}^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta})^{\nu}} [\tau K_{\gamma_{\varepsilon}}^{\nu} F(\beta_{\varepsilon}) + \tau K_{\beta_{\varepsilon}}^{\nu} F(\gamma_{\varepsilon})] \\
&\leq \frac{1}{2} [f(\beta_{\varepsilon}) + f(\gamma_{\varepsilon})] \\
&\leq \sum_{\varsigma=1}^{\varepsilon} \varepsilon r_{\varsigma} f(\alpha_{\varsigma}) - \frac{1}{2} \left[\sum_{\varsigma=1}^{\varepsilon-1} \varepsilon r_{\varsigma} f(\beta_{\varsigma}) + \sum_{\varsigma=1}^{\varepsilon-1} \varepsilon r_{\varsigma} f(\gamma_{\varsigma}) \right].
\end{aligned} \tag{62}$$

Proof. By using Lemma 2, we can derive (62) by following the steps given in the proof of Theorem 7. $\square$

Remark 4. Weighted versions of Theorem 6 can also be established in a similar manner by utilizing Lemmas 1 and 2.

4 Derivation of new integral identities and bounds for the main results

First, in this section, we construct two new integral identities connected to the right and left sides of our major inequalities. Then, by using these identities, we bound the absolute difference between the terms on the right and left of the main results.

Lemma 3. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\varepsilon})$, $\beta = (\beta_1, \beta_2, \dots, \beta_{\varepsilon})$, and $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_{\varepsilon})$ be three tuples, where $\beta_{\varepsilon} > \gamma_{\varepsilon}$ and $\alpha_{\varsigma}, \beta_{\varsigma}, \gamma_{\varsigma} \in I = [\gamma_{\varepsilon}, \beta_{\varepsilon}]$, for all $\varsigma = 1, 2, \dots, \varepsilon$, and $\nu > 0$, $\eta \in [0, 1]$, $\tau \in \mathbb{R}$, $\theta \in (0, 1]$ such that $\tau + \theta > 0$. Also, let $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a differentiable function such that $f' \in L_{\theta}(I)$, then

$$\begin{aligned}
&\frac{1}{2} [f(\beta_{\varepsilon}) + f(\gamma_{\varepsilon})] - \frac{(\tau + \theta)^{\nu} \Gamma(\nu + 1)}{4(\beta_{\varepsilon}^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta})^{\nu}} [\tau K_{\gamma_{\varepsilon}}^{\nu} F(\beta_{\varepsilon}) + \tau K_{\beta_{\varepsilon}}^{\nu} F(\gamma_{\varepsilon})] \\
&= \frac{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma})}{4(\beta_{\varepsilon}^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta})^{\nu}} \int_0^1 \nabla_{\tau+\theta}^{\nu}(\eta) f' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} (\eta \gamma_{\varsigma} + (1-\eta) \beta_{\varsigma}) \right) d\eta,
\end{aligned} \tag{63}$$

where

$$\begin{aligned}
\nabla_{\tau+\theta}^{\nu}(\eta) &= \left(\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} - (1-\eta) \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} \right)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta} \right)^{\nu} - \left(\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} - (1-\eta) \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} \right)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta} \right)^{\nu} \\
&+ \left(\beta_{\varepsilon}^{\tau+\theta} - \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} - (1-\eta) \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} \right)^{\tau+\theta} \right)^{\nu} - \left(\beta_{\varepsilon}^{\tau+\theta} - \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} - (1-\eta) \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} \right)^{\tau+\theta} \right)^{\nu}.
\end{aligned}$$

Proof. By using integration by parts, we obtain

$$\begin{aligned}
{}^{\tau}K_{\gamma_{\varepsilon}}^{\nu}F(\beta_{\varepsilon}) &= \frac{(\beta_{\varepsilon}^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta})^{\nu}}{(\tau + \theta)^{\nu}\Gamma(\nu + 1)}F(\gamma_{\varepsilon}) + \frac{\sum_{\varsigma=1}^{\varepsilon-1}(\gamma_{\varsigma} - \beta_{\varsigma})}{(\tau + \theta)^{\nu}\Gamma(\nu + 1)} \\
&\times \int_0^1 \left(\beta_{\varepsilon}^{\tau+\theta} - \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} - (1 - \eta) \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} \right)^{\tau+\theta} \right)^{\nu} \\
&\times F' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} - (1 - \eta) \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} \right) d\eta.
\end{aligned} \tag{64}$$

Also, we obtain

$$\begin{aligned}
{}^{\tau}K_{\beta_{\varepsilon}}^{\nu}F(\gamma_{\varepsilon}) &= \frac{(\beta_{\varepsilon}^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta})^{\nu}}{(\tau + \theta)^{\nu}\Gamma(\nu + 1)}F(\beta_{\varepsilon}) - \frac{\sum_{\varsigma=1}^{\varepsilon-1}(\gamma_{\varsigma} - \beta_{\varsigma})}{(\tau + \theta)^{\nu}\Gamma(\nu + 1)} \\
&\times \int_0^1 \left(\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} - (1 - \eta) \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} \right)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta} \right)^{\nu} \\
&\times F' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} - (1 - \eta) \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} \right) d\eta.
\end{aligned} \tag{65}$$

By adding (64) and (65), we obtain

$$\begin{aligned}
&\frac{f(\beta_{\varepsilon}) + f(\gamma_{\varepsilon})}{2} - \frac{(\tau + \theta)^{\nu}\Gamma(\nu + 1)}{4(\beta_{\varepsilon}^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta})^{\nu}} [{}^{\tau}K_{\gamma_{\varepsilon}}^{\nu}F(\beta_{\varepsilon}) + {}^{\tau}K_{\beta_{\varepsilon}}^{\nu}F(\gamma_{\varepsilon})] \\
&= \frac{\sum_{\varsigma=1}^{\varepsilon-1}(\gamma_{\varsigma} - \beta_{\varsigma})}{4(\beta_{\varepsilon}^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta})^{\nu}} \int_0^1 \left[\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} - (1 - \eta) \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} \right)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta} \right)^{\nu} \\
&\quad - \left(\beta_{\varepsilon}^{\tau+\theta} - \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} - (1 - \eta) \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} \right)^{\tau+\theta} \right)^{\nu} \right] F' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} - (1 - \eta) \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} \right) d\eta.
\end{aligned} \tag{66}$$

Also, we have

$$\begin{aligned}
&F' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} - (1 - \eta) \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} \right) \\
&= f' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} - (1 - \eta) \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} \right) - f' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} - (1 - \eta) \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} \right).
\end{aligned} \tag{67}$$

And

$$\begin{aligned}
&\int_0^1 \left(\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} - (1 - \eta) \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} \right)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta} \right)^{\nu} F' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} - (1 - \eta) \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} \right) d\eta \\
&= \int_0^1 \left(\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} - (1 - \eta) \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} \right)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta} \right)^{\nu} f' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} - (1 - \eta) \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} \right) d\eta \\
&\quad - \int_0^1 \left(\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} - (1 - \eta) \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} \right)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta} \right)^{\nu} f' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \eta \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} - (1 - \eta) \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} \right) d\eta.
\end{aligned} \tag{68}$$

Also,

$$\begin{aligned}
& \int_0^1 \left(\beta_\varepsilon^{\tau+\theta} - \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \eta \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma - (1-\eta) \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma \right)^{\tau+\theta} \right)^v F' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \eta \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma - (1-\eta) \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma \right) d\eta \\
&= \int_0^1 \left(\beta_\varepsilon^{\tau+\theta} - \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \eta \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma - (1-\eta) \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma \right)^{\tau+\theta} \right)^v f' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \eta \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma - (1-\eta) \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma \right) d\eta \\
&\quad - \int_0^1 \left(\beta_\varepsilon^{\tau+\theta} - \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \eta \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma - (1-\eta) \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma \right)^{\tau+\theta} \right)^v f' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \eta \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma - (1-\eta) \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma \right) d\eta.
\end{aligned} \tag{69}$$

Now by using (68) and (69) in (66), we obtain (63). $\square$

Remark 5.

- (1) For $\tau + \theta = 1$, Lemma 3 reduces to Lemma 3 given in [48].
- (2) For $\tau + \theta = 1$, $\nu = 1$, $\varepsilon = 2$, $\beta_1 = \alpha_1$, and $\gamma_1 = \alpha_2$ in (63), we obtain the following identity:

$$\frac{f(\alpha_1) + f(\alpha_2)}{2} - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} f(u) du = \frac{\alpha_2 - \alpha_1}{2} \int_0^1 (2\eta - 1) f'(\eta \alpha_2 + (1 - \eta) \alpha_1) d\eta. \tag{70}$$

The equality (70) has been proved by Dragomir and Agarwal in [14].

Lemma 4. Let all the conditions in the hypotheses of Lemma 3 hold true, then

$$\begin{aligned}
& \frac{\Gamma(\nu + 1)(\tau + \theta)^\nu}{2Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \left[{}^\tau K^\nu \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right) \right)^+ F(\beta_\varepsilon) + {}^\tau K^\nu \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right) \right)^- F(\gamma_\varepsilon) \right] - f \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right) \right) \\
&= \frac{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma)}{4Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \int_0^1 P_{\tau+\theta}^\nu(\eta) \left[f' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma \right) \right) \right. \\
&\quad \left. - f' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma \right) \right) \right] d\eta,
\end{aligned} \tag{71}$$

where

$$Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) = \left(\beta_\varepsilon^{\tau+\theta} - \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \frac{1}{2} \sum_{\varsigma=1}^{\varepsilon-1} (\beta_\varsigma + \gamma_\varsigma) \right)^{\tau+\theta} \right)^\nu + \left(\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \frac{1}{2} \sum_{\varsigma=1}^{\varepsilon-1} (\beta_\varsigma + \gamma_\varsigma) \right)^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta} \right)^\nu$$

and

$$P_{\tau+\theta}^\nu(\eta) = \left(\beta_\varepsilon^{\tau+\theta} - \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma \right) \right)^{\tau+\theta} \right)^\nu + \left(\left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma \right) \right)^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta} \right)^\nu.$$

Proof. We can easily prove (71), by following the same procedure as adopted in the proof of Lemma 3. $\square$

To prove some further results, we first give the following definition.

Definition 6. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\varepsilon)$, $\beta = (\beta_1, \beta_2, \dots, \beta_\varepsilon)$, and $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_\varepsilon)$ be three tuples, where $\beta_\varepsilon > \gamma_\varepsilon$ and $\alpha_\varsigma, \beta_\varsigma, \gamma_\varsigma \in I = [\gamma_\varepsilon, \beta_\varepsilon] \subseteq [0, \infty)$, for all $\varsigma = 1, 2, \dots, \varepsilon$, $\nu > 0$, $\tau + \theta > 0$. If $\xi, z \in I$, then we define the operators:

$$\Lambda_1^\nu(z, \tau + \theta) = \int_{\gamma_\varepsilon}^{\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)} |z^{\tau+\theta} - w^{\tau+\theta}|^\nu dw - \int_{\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)}^{\beta_\varepsilon} |z^{\tau+\theta} - w^{\tau+\theta}|^\nu dw. \quad (72)$$

$$\Lambda_2^\nu(z, \tau + \theta) = \int_{\gamma_\varepsilon}^{\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)} |w^{\tau+\theta} - z^{\tau+\theta}|^\nu dw - \int_{\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)}^{\beta_\varepsilon} |w^{\tau+\theta} - z^{\tau+\theta}|^\nu dw. \quad (73)$$

$$\Upsilon_1^\nu(\xi, z, \tau + \theta) = \int_{\gamma_\varepsilon}^{\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)} |\xi - w| |z^{\tau+\theta} - w^{\tau+\theta}|^\nu dw - \int_{\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)}^{\beta_\varepsilon} |\xi - w| |z^{\tau+\theta} - w^{\tau+\theta}|^\nu dw. \quad (74)$$

$$\Upsilon_2^\nu(\xi, z, \tau + \theta) = \int_{\gamma_\varepsilon}^{\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)} |\xi - w| |w^{\tau+\theta} - z^{\tau+\theta}|^\nu dw - \int_{\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)}^{\beta_\varepsilon} |\xi - w| |w^{\tau+\theta} - z^{\tau+\theta}|^\nu dw. \quad (75)$$

Theorem 9. Let f be a differentiable function defined on I and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\varepsilon)$, $\beta = (\beta_1, \beta_2, \dots, \beta_\varepsilon)$, and $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_\varepsilon)$ be three tuples, where $\beta_\varepsilon > \gamma_\varepsilon$ and $\alpha_\varsigma, \beta_\varsigma, \gamma_\varsigma \in I = [\gamma_\varepsilon, \beta_\varepsilon]$, for all $\varsigma = 1, 2, \dots, \varepsilon$. Let $\nu > 0$ and $\tau \in \mathbb{R}$, $\theta \in (0, 1]$ such that $\tau + \theta > 0$. If $\beta < \alpha$, $\gamma < \alpha$ and $|f'|$ is convex on I , then

$$\left| \frac{f(\gamma_\varepsilon) + f(\beta_\varepsilon)}{2} - \frac{(\tau + \theta)^\nu \Gamma(\nu + 1)}{4(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} [\bar{I}_\theta^\nu K_{\gamma_\varepsilon}^\nu F(\beta_\varepsilon) + \bar{I}_\theta^\nu K_{\beta_\varepsilon}^\nu F(\gamma_\varepsilon)] \right| \leq \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma|}{4(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} \left[2M_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) \sum_{\varsigma=1}^\varepsilon |f'(\alpha_\varsigma)| - N_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) \left(\sum_{\varsigma=1}^{\varepsilon-1} |f'(\beta_\varsigma)| + \sum_{\varsigma=1}^{\varepsilon-1} |f'(\gamma_\varsigma)| \right) \right], \quad (76)$$

where

$$M_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) = \frac{\Lambda_1^\nu(\beta_\varepsilon, \tau + \theta) - \Lambda_2^\nu(\gamma_\varepsilon, \tau + \theta)}{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma)}$$

and

$$N_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) = \frac{1}{\left(\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma) \right)^2} [\Upsilon_1^\nu(\beta_\varepsilon, \beta_\varepsilon, \tau + \theta) - \Upsilon_1^\nu(\gamma_\varepsilon, \beta_\varepsilon, \tau + \theta) + \Upsilon_2^\nu(\gamma_\varepsilon, \gamma_\varepsilon, \tau + \theta) - \Upsilon_2^\nu(\beta_\varepsilon, \gamma_\varepsilon, \tau + \theta)].$$

Proof. By using Lemma 3, we have

$$\begin{aligned}
& \left| \frac{f(\gamma_\varepsilon) + f(\beta_\varepsilon)}{2} - \frac{(\tau + \theta)^\nu \Gamma(\nu + 1)}{4(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} [{}^\tau K_{\gamma_\varepsilon}^\nu F(\beta_\varepsilon) + {}^\tau K_{\beta_\varepsilon}^\nu F(\gamma_\varepsilon)] \right| \\
&= \left| \frac{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma)}{4(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} \int_0^1 \nabla_{\tau+\theta}^\nu(\eta) f' \left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} (\eta \gamma_\varsigma + (1-\eta) \beta_\varsigma) \right) d\eta \right| \\
&\leq \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma|}{4(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} \int_0^1 |\nabla_{\tau+\theta}^\nu(\eta)| \left| f' \left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} (\eta \gamma_\varsigma + (1-\eta) \beta_\varsigma) \right) \right| d\eta.
\end{aligned} \tag{77}$$

By taking $\sigma_1 = \eta$, $\sigma_2 = 1 - \eta$, and $n = 2$ and applying Theorem 4 in (77), we have

$$\begin{aligned}
&\leq \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma|}{4(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} \int_0^1 |\nabla_{\tau+\theta}^\nu(\eta)| \left[\sum_{\varsigma=1}^\varepsilon |f'(\alpha_\varsigma)| - \eta \sum_{\varsigma=1}^{\varepsilon-1} |f'(\gamma_\varsigma)| - (1-\eta) \sum_{\varsigma=1}^{\varepsilon-1} |f'(\beta_\varsigma)| \right] d\eta \\
&= \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma|}{4(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} \left[\sum_{\varsigma=1}^\varepsilon |f'(\alpha_\varsigma)| \int_0^1 |\nabla_{\tau+\theta}^\nu(\eta)| d\eta - \sum_{\varsigma=1}^{\varepsilon-1} |f'(\gamma_\varsigma)| \int_0^1 \eta |\nabla_{\tau+\theta}^\nu(\eta)| d\eta - \sum_{\varsigma=1}^{\varepsilon-1} |f'(\beta_\varsigma)| \int_0^1 (1-\eta) |\nabla_{\tau+\theta}^\nu(\eta)| d\eta \right].
\end{aligned} \tag{78}$$

By substituting $u = \sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \eta \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma - (1-\eta) \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma$, the function $\nabla_{\tau+\theta}^\nu(\eta)$ becomes

$$\psi(u) = (u^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu - ((\beta_\varepsilon) + (\gamma_\varepsilon) - u)^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta} + (\beta_\varepsilon^{\tau+\theta} - ((\beta_\varepsilon) + (\gamma_\varepsilon) - u)^{\tau+\theta})^\nu - (\beta_\varepsilon^{\tau+\theta} - u^{\tau+\theta})^\nu. \tag{79}$$

We see that

$$\psi(\gamma_\varepsilon) = -2(\beta_\varepsilon^{\tau+\theta} - (\gamma_\varepsilon)^{\tau+\theta})^\nu,$$

$$\psi(\beta_\varepsilon) = 2(\beta_\varepsilon^{\tau+\theta} - (\gamma_\varepsilon)^{\tau+\theta})^\nu,$$

and

$$\psi \left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right) \right) = 0.$$

Therefore, we have

$$\begin{cases} \psi(u) \leq 0, & \text{if } \gamma_\varepsilon \leq u \leq \sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right) \\ \psi(u) > 0, & \text{if } \sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right) < u \leq \beta_\varepsilon. \end{cases} \tag{80}$$

Now, we find $\int_0^1 |\nabla_{\tau+\theta}^\nu(\eta)| d\eta$, $\int_0^1 \eta |\nabla_{\tau+\theta}^\nu(\eta)| d\eta$, and $\int_0^1 (1-\eta) |\nabla_{\tau+\theta}^\nu(\eta)| d\eta$ as follows:

$$\begin{aligned}
\int_0^1 |\nabla_{\tau+\theta}^\nu(\eta)| d\eta &= \frac{1}{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma)} \int_{\gamma_\varepsilon}^{\beta_\varepsilon} |\psi(u)| du \\
&= \frac{1}{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma)} \left[- \int_{\gamma_\varepsilon}^{\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)} \psi(u) du + \int_{\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)}^{\beta_\varepsilon} \psi(u) du \right] \\
&= \frac{1}{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma)} [L_1 + L_2 + L_3 + L_4].
\end{aligned} \tag{81}$$

Here,

$$L_1 = - \int_{\gamma_\varepsilon}^{\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)} (u^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu du + \int_{\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)}^{\beta_\varepsilon} (u^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu du = -\Lambda_2^\nu(\gamma_\varepsilon, \tau + \theta), \quad (82)$$

$$L_2 = \int_{\gamma_\varepsilon}^{\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)} ((\beta_\varepsilon + \gamma_\varepsilon - u)^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu du - \int_{\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)}^{\beta_\varepsilon} ((\beta_\varepsilon + \gamma_\varepsilon - u)^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu du, \quad (83)$$

$$L_3 = - \int_{\gamma_\varepsilon}^{\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)} (\beta_\varepsilon^{\tau+\theta} - (\beta_\varepsilon + \gamma_\varepsilon - u)^{\tau+\theta})^\nu du + \int_{\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)}^{\beta_\varepsilon} (\beta_\varepsilon^{\tau+\theta} - (\beta_\varepsilon + \gamma_\varepsilon - u)^{\tau+\theta})^\nu du. \quad (84)$$

By the change of variables $v = \beta_\varepsilon + \gamma_\varepsilon - u$, we have

$$L_2 = -\Lambda_2^\nu(\gamma_\varepsilon, \tau + \theta), \quad L_3 = \Lambda_1^\nu(\beta_\varepsilon, \tau + \theta), \quad (85)$$

$$L_4 = \int_{\gamma_\varepsilon}^{\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)} (\beta_\varepsilon^{\tau+\theta} - u^{\tau+\theta})^\nu du - \int_{\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)}^{\beta_\varepsilon} (\beta_\varepsilon^{\tau+\theta} - u^{\tau+\theta})^\nu du = \Lambda_1^\nu(\beta_\varepsilon, \tau + \theta). \quad (86)$$

By using (82), (85), and (86) in (81), we obtain

$$\int_0^1 |\nabla_{\tau+\theta}^\nu(\eta)| d\eta = \frac{2}{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma)} [\Lambda_1^\nu(\beta_\varepsilon, \tau + \theta) - \Lambda_2^\nu(\gamma_\varepsilon, \tau + \theta)]. \quad (87)$$

And

$$\begin{aligned} \int_0^1 \eta |\nabla_{\tau+\theta}^\nu(\eta)| d\eta &= \frac{1}{\left(\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma) \right)^2} \int_{\gamma_\varepsilon}^{\beta_\varepsilon} (\beta_\varepsilon - u) |\psi(u)| du \\ &= \frac{1}{\left(\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma) \right)^2} \left[- \int_{\gamma_\varepsilon}^{\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)} (\beta_\varepsilon - u) \psi(u) du \right. \\ &\quad \left. + \int_{\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)}^{\beta_\varepsilon} (\beta_\varepsilon - u) \psi(u) du \right] \\ &= \frac{1}{\left(\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma) \right)^2} [A_1 + A_2 + A_3 + A_4]. \end{aligned} \quad (88)$$

Here,

$$\begin{aligned}
A_1 &= - \int_{\gamma_\varepsilon}^{\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2} \right)} (\beta_\varepsilon - u)(u^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu du + \int_{\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2} \right)}^{\beta_\varepsilon} (\beta_\varepsilon - u)(u^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu du \\
&= -Y_2^\nu(\beta_\varepsilon, \gamma_\varepsilon, \tau + \theta) \\
A_2 &= \int_{\gamma_\varepsilon}^{\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2} \right)} (\beta_\varepsilon - u)((\beta_\varepsilon + \gamma_\varepsilon - u)^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu du \\
&\quad + \int_{\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2} \right)}^{\beta_\varepsilon} (\beta_\varepsilon - u)((\beta_\varepsilon + \gamma_\varepsilon - u)^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu du, \\
A_3 &= - \int_{\gamma_\varepsilon}^{\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2} \right)} (\beta_\varepsilon - u)(\beta_\varepsilon^{\tau+\theta} - (\beta_\varepsilon + \gamma_\varepsilon - u)^{\tau+\theta})^\nu du \\
&\quad + \int_{\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2} \right)}^{\beta_\varepsilon} (\beta_\varepsilon - u)(\beta_\varepsilon^{\tau+\theta} - (\beta_\varepsilon + \gamma_\varepsilon - u)^{\tau+\theta})^\nu du.
\end{aligned} \tag{89}$$

By the same change of variables as adopted earlier, we obtain

$$A_2 = Y_2^\nu(\gamma_\varepsilon, \gamma_\varepsilon, \tau + \theta), \quad A_3 = -Y_1^\nu(\gamma_\varepsilon, \beta_\varepsilon, \tau + \theta), \tag{90}$$

$$\begin{aligned}
A_4 &= \int_{\gamma_\varepsilon}^{\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2} \right)} (\beta_\varepsilon - u)(\beta_\varepsilon^{\tau+\theta} - u^{\tau+\theta})^\nu du + \int_{\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2} \right)}^{\beta_\varepsilon} (\beta_\varepsilon - u)(\beta_\varepsilon^{\tau+\theta} - u^{\tau+\theta})^\nu du \\
&= Y_1^\nu(\beta_\varepsilon, \beta_\varepsilon, \tau + \theta).
\end{aligned} \tag{91}$$

By using (89), (90), and (91) in (88), we deduce

$$\begin{aligned}
\int_0^1 \eta |\nabla_{\tau+\theta}^\nu(\eta)| d\eta &= \frac{1}{\left(\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma) \right)^2} [Y_1^\nu(\beta_\varepsilon, \beta_\varepsilon, \tau + \theta) - Y_1^\nu(\gamma_\varepsilon, \beta_\varepsilon, \tau + \theta) + Y_2^\nu(\gamma_\varepsilon, \gamma_\varepsilon, \tau + \theta) \\
&\quad - Y_2^\nu(\beta_\varepsilon, \gamma_\varepsilon, \tau + \theta)].
\end{aligned} \tag{92}$$

Similarly,

$$\begin{aligned}
\int_0^1 (1 - \eta) |\nabla_{\tau+\theta}^\nu(\eta)| d\eta &= \frac{1}{\left(\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma) \right)^2} [Y_1^\nu(\beta_\varepsilon, \beta_\varepsilon, \tau + \theta) - Y_1^\nu(\gamma_\varepsilon, \beta_\varepsilon, \tau + \theta) + Y_2^\nu(\gamma_\varepsilon, \gamma_\varepsilon, \tau + \theta) \\
&\quad - Y_2^\nu(\beta_\varepsilon, \gamma_\varepsilon, \tau + \theta)].
\end{aligned} \tag{93}$$

By using (87), (92), and (93) in (78), we obtain (76). $\square$

Remark 6.

- (1) For $\tau + \theta = 1$, the inequality (76) reduces to inequality (36) in [48];
- (2) For $\tau + \theta = 1$ and $\varepsilon = 2$, the inequality (76) reduces to inequality (3.4) in [30];
- (3) For $\tau + \theta = 1$, $\varepsilon = 2$, $\beta_1 = \alpha_1$, and $\gamma_1 = \alpha_2$, the inequality (76) reduces to inequality (3.5) in [31].

Theorem 10. Let f be a differentiable function defined on I and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\varepsilon)$, $\beta = (\beta_1, \beta_2, \dots, \beta_\varepsilon)$, and $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_\varepsilon)$ be three tuples, where $\beta_\varepsilon > \gamma_\varepsilon$ and $\alpha_\varsigma, \beta_\varsigma, \gamma_\varsigma \in I = [\gamma_\varepsilon, \beta_\varepsilon]$, for all $\varsigma = 1, 2, \dots, \varepsilon$. Let $\nu > 0$ and $\tau \in \mathbb{R}$, $\theta \in (0, 1]$ such that $\tau + \theta > 0$. If $q > 1$, $\beta < \alpha$, $\gamma < \alpha$ and $|f'|^q$ is convex on I , then

$$\begin{aligned} & \left| \frac{f(\gamma_\varepsilon) + f(\beta_\varepsilon)}{2} - \frac{(\tau + \theta)^\nu \Gamma(\nu + 1)}{4(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} [\tau K_{\gamma_\varepsilon}^\nu F(\beta_\varepsilon) + \theta K_{\beta_\varepsilon}^\nu F(\gamma_\varepsilon)] \right| \\ & \leq \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma|}{2(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} (M_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)) \left[\sum_{\varsigma=1}^{\varepsilon} |f'(\alpha_\varsigma)|^q - \frac{N_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)}{2M_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \left(\sum_{\varsigma=1}^{\varepsilon-1} |f'(\beta_\varsigma)|^q + \sum_{\varsigma=1}^{\varepsilon-1} |f'(\gamma_\varsigma)|^q \right) \right]^{\frac{1}{q}}, \end{aligned} \quad (94)$$

where

$$M_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) = \frac{\Lambda_1^\nu(\beta_\varepsilon, \tau + \theta) - \Lambda_2^\nu(\gamma_\varepsilon, \tau + \theta)}{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma)}$$

and

$$N_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) = \frac{1}{\left(\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma) \right)^2} [\Upsilon_1^\nu(\beta_\varepsilon, \beta_\varepsilon, \tau + \theta) - \Upsilon_1^\nu(\gamma_\varepsilon, \beta_\varepsilon, \tau + \theta) + \Upsilon_2^\nu(\gamma_\varepsilon, \gamma_\varepsilon, \tau + \theta) - \Upsilon_2^\nu(\beta_\varepsilon, \gamma_\varepsilon, \tau + \theta)].$$

Proof. By using Lemma 3, we have

$$\begin{aligned} & \left| \frac{f(\beta_\varepsilon) + f(\gamma_\varepsilon)}{2} - \frac{(\tau + \theta)^\nu \Gamma(\nu + 1)}{4(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} [\tau K_{\gamma_\varepsilon}^\nu F(\beta_\varepsilon) + \theta K_{\beta_\varepsilon}^\nu F(\gamma_\varepsilon)] \right| \\ & = \left| \frac{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma)}{4(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} \int_0^1 \nabla_{\tau+\theta}^\nu(\eta) f' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} (\eta \gamma_\varsigma + (1-\eta) \beta_\varsigma) \right) d\eta \right| \\ & \leq \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma|}{4(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} \int_0^1 |\nabla_{\tau+\theta}^\nu(\eta)| \left| f' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} (\eta \gamma_\varsigma + (1-\eta) \beta_\varsigma) \right) \right| d\eta. \end{aligned} \quad (95)$$

By applying power mean inequality, we deduce

$$\leq \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma|}{4(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} \left(\int_0^1 |\nabla_{\tau+\theta}^\nu(\eta)| d\eta \right)^{1-\frac{1}{q}} \left(\int_0^1 |\nabla_{\tau+\theta}^\nu(\eta)| \left| f' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} (\eta \gamma_\varsigma + (1-\eta) \beta_\varsigma) \right) \right|^q d\eta \right)^{\frac{1}{q}}. \quad (96)$$

Since $|f'|^q$ is convex. Therefore, we apply Theorem 4 for the values $\sigma_1 = \eta$, $\sigma_2 = 1 - \eta$, and $n = 2$ in (96) to obtain

$$\begin{aligned} & \leq \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma|}{4(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} \left(\int_0^1 |\nabla_{\tau+\theta}^\nu(\eta)| d\eta \right)^{1-\frac{1}{q}} \\ & \quad \times \left(\int_0^1 |\nabla_{\tau+\theta}^\nu(\eta)| \left(\sum_{\varsigma=1}^{\varepsilon} |f'(\alpha_\varsigma)|^q - \eta \sum_{\varsigma=1}^{\varepsilon-1} |f'(\gamma_\varsigma)|^q - (1-\eta) \sum_{\varsigma=1}^{\varepsilon-1} |f'(\beta_\varsigma)|^q \right) d\eta \right)^{\frac{1}{q}} \\ & = \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma|}{4(\beta_\varepsilon^{\tau+\theta} - \gamma_\varepsilon^{\tau+\theta})^\nu} \left(\int_0^1 |\nabla_{\tau+\theta}^\nu(\eta)| d\eta \right)^{1-\frac{1}{q}} \left(\sum_{\varsigma=1}^{\varepsilon} |f'(\alpha_\varsigma)|^q \int_0^1 |\nabla_{\tau+\theta}^\nu(\eta)| d\eta \right. \\ & \quad \left. - \sum_{\varsigma=1}^{\varepsilon-1} |f'(\gamma_\varsigma)|^q \int_0^1 \eta |\nabla_{\tau+\theta}^\nu(\eta)| d\eta - \sum_{\varsigma=1}^{\varepsilon-1} |f'(\beta_\varsigma)|^q \int_0^1 (1-\eta) |\nabla_{\tau+\theta}^\nu(\eta)| d\eta \right)^{\frac{1}{q}}. \end{aligned} \quad (97)$$

Now again by substituting (87), (92), and (93) into (97), we obtain (94). $\square$

To obtain more new results, we give another definition as follows:

Definition 7. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\varepsilon)$, $\beta = (\beta_1, \beta_2, \dots, \beta_\varepsilon)$, and $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_\varepsilon)$ be three tuples, where $\beta_\varepsilon > \gamma_\varepsilon$ and $\alpha_\varsigma, \beta_\varsigma, \gamma_\varsigma \in I = [\gamma_\varepsilon, \beta_\varepsilon] \subseteq [0, \infty)$, for all $\varsigma = 1, 2, \dots, \varepsilon$, $\nu > 0$, $\tau + \theta > 0$. If $\xi_1, \xi_2 \in I$, then we define the operators:

$$\Lambda_3^\nu(\xi_1, \xi_2, \tau + \theta) = \int_{\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)}^{\beta_\varepsilon} |\xi_1^{\tau+\theta} - w^{\tau+\theta}|^\nu dw + \int_{\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)}^{\beta_\varepsilon} |(\xi_1 + \xi_2 - w)^{\tau+\theta} - \xi_2^{\tau+\theta}|^\nu dw. \quad (98)$$

$$\begin{aligned} \Lambda_4^\nu(\xi_1, \xi_2, \tau + \theta) = & \int_{\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)}^{\beta_\varepsilon} |\xi_1 - w| |\xi_1^{\tau+\theta} - w^{\tau+\theta}|^\nu dw \\ & + \int_{\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)}^{\beta_\varepsilon} |\xi_1 - w| |(\xi_1 + \xi_2 - w)^{\tau+\theta} - \xi_2^{\tau+\theta}|^\nu dw. \end{aligned} \quad (99)$$

$$\begin{aligned} \Lambda_5^\nu(\xi_1, \xi_2, \tau + \theta) = & \int_{\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)}^{\beta_\varepsilon} |w - \xi_2| |\xi_1^{\tau+\theta} - w^{\tau+\theta}|^\nu dw \\ & + \int_{\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right)}^{\beta_\varepsilon} |w - \xi_2| |(\xi_1 + \xi_2 - w)^{\tau+\theta} - \xi_2^{\tau+\theta}|^\nu dw. \end{aligned} \quad (100)$$

On the basis of Lemma 4, we derive the following results.

Theorem 11. Let all the conditions in the hypotheses of Theorem 9 hold true, then

$$\begin{aligned} & \frac{\Gamma(\nu + 1)(\tau + \theta)^\nu}{2Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \left[{}^\tau K^\nu \left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right) \right) {}^\tau F(\beta_\varepsilon) + {}^\tau K^\nu \left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right) \right) {}^\tau F(\gamma_\varepsilon) \right] - f \left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right) \right) \\ & \leq \frac{1}{Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma| \left[R_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) \sum_{\varsigma=1}^\varepsilon |f'(\alpha_\varsigma)| - \frac{1}{2} S_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) \left(\sum_{\varsigma=1}^{\varepsilon-1} |f'(\beta_\varsigma)| + \sum_{\varsigma=1}^{\varepsilon-1} |f'(\gamma_\varsigma)| \right) \right], \end{aligned} \quad (101)$$

where

$$R_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) = \frac{1}{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma)} \Lambda_3^\nu(\beta_\varepsilon, \gamma_\varepsilon, \tau + \theta)$$

and

$$S_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) = \frac{1}{\left(\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma) \right)^2} [\Lambda_4^\nu(\beta_\varepsilon, \gamma_\varepsilon, \tau + \theta) + \Lambda_5^\nu(\beta_\varepsilon, \gamma_\varepsilon, \tau + \theta)].$$

Proof. By using Lemma 4, we have

$$\begin{aligned}
& \left| \frac{\Gamma(\nu+1)(\tau+\theta)^\nu}{2Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \left[\frac{{}_\tau I_{\tau+\theta}^\nu}{\left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right) \right)} {}^+ F(\beta_\varepsilon) + \frac{{}_\theta I_{\tau+\theta}^\nu}{\left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right) \right)} {}^- F(\gamma_\varepsilon) \right] \right. \\
& \quad \left. - f \left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2} \right) \right) \right| \\
&= \left| \frac{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma)}{4Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \int_0^1 P_{\tau+\theta}^\nu(\eta) \left[f' \left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma \right) \right) \right. \right. \\
& \quad \left. \left. - f' \left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma \right) \right) \right] d\eta \right|, \tag{102} \\
&\leq \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma|}{4Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \left[\int_0^1 |P_{\tau+\theta}^\nu(\eta)| \left| f' \left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma \right) \right) \right| d\eta \right. \\
& \quad \left. + \int_0^1 |P_{\tau+\theta}^\nu(\eta)| \left| f' \left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma \right) \right) \right| d\eta \right] \\
&= \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma|}{4Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \left[\int_0^1 P_{\tau+\theta}^\nu(\eta) \left| f' \left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma \right) \right) \right| d\eta \right. \\
& \quad \left. + \int_0^1 P_{\tau+\theta}^\nu(\eta) \left| f' \left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma \right) \right) \right| d\eta \right].
\end{aligned}$$

By using Theorem 5 for the values $n = 2$, $\sigma_1 = \frac{\eta}{2}$, and $\sigma_2 = \frac{2-\eta}{2}$ in (102), we have

$$\begin{aligned}
&\leq \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma|}{4Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \left[\int_0^1 P_{\tau+\theta}^\nu(\eta) \left(\sum_{\varsigma=1}^\varepsilon |f'(\alpha_\varsigma)| - \frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} |f'(\gamma_\varsigma)| - \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} |f'(\beta_\varsigma)| \right) d\eta \right. \\
& \quad \left. + \int_0^1 P_{\tau+\theta}^\nu(\eta) \left(\sum_{\varsigma=1}^\varepsilon |f'(\alpha_\varsigma)| - \frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} |f'(\beta_\varsigma)| - \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} |f'(\gamma_\varsigma)| \right) d\eta \right] \\
&= \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma|}{4Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \left[2 \sum_{\varsigma=1}^\varepsilon |f'(\alpha_\varsigma)| \int_0^1 P_{\tau+\theta}^\nu(\eta) d\eta - \frac{1}{2} \left(\sum_{\varsigma=1}^{\varepsilon-1} |f'(\beta_\varsigma)| + \sum_{\varsigma=1}^{\varepsilon-1} |f'(\gamma_\varsigma)| \right) \int_0^1 \eta P_{\tau+\theta}^\nu(\eta) d\eta \right. \\
& \quad \left. - \frac{1}{2} \left(\sum_{\varsigma=1}^{\varepsilon-1} |f'(\beta_\varsigma)| + \sum_{\varsigma=1}^{\varepsilon-1} |f'(\gamma_\varsigma)| \right) \int_0^1 (2-\eta) P_{\tau+\theta}^\nu(\eta) d\eta \right] \\
&= \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma|}{4Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \left[2 \sum_{\varsigma=1}^\varepsilon |f'(\alpha_\varsigma)| \int_0^1 P_{\tau+\theta}^\nu(\eta) d\eta - \frac{1}{2} \left(\sum_{\varsigma=1}^{\varepsilon-1} |f'(\beta_\varsigma)| + \sum_{\varsigma=1}^{\varepsilon-1} |f'(\gamma_\varsigma)| \right) \right. \\
& \quad \left. \times \left[\int_0^1 \eta P_{\tau+\theta}^\nu(\eta) d\eta + \int_0^1 (2-\eta) P_{\tau+\theta}^\nu(\eta) d\eta \right] \right]. \tag{103}
\end{aligned}$$

By using the substitution $u = \sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} \right)$, we have

$$\begin{aligned} \int_0^1 P_{\tau+\theta}^{\nu}(\eta) d\eta &= \frac{2}{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma})} \left[\int_{\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2} \right)}^{\beta_{\varepsilon}} (\beta_{\varepsilon}^{\tau+\theta} - u^{\tau+\theta})^{\nu} du \right. \\ &\quad \left. + \int_{\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2} \right)}^{\beta_{\varepsilon}} ((\beta_{\varepsilon} + \gamma_{\varepsilon} - u)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta})^{\nu} du \right] \\ &= \frac{2}{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma})} \Lambda_3^{\nu}(\beta_{\varepsilon}, \gamma_{\varepsilon}, \tau + \theta), \end{aligned} \quad (104)$$

$$\begin{aligned} \int_0^1 \eta P_{\tau+\theta}^{\nu}(\eta) d\eta &= \frac{4}{\left(\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma}) \right)^2} \left[\int_{\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2} \right)}^{\beta_{\varepsilon}} (\beta_{\varepsilon} - u)(\beta_{\varepsilon}^{\tau+\theta} - u^{\tau+\theta})^{\nu} du \right. \\ &\quad \left. + \int_{\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2} \right)}^{\beta_{\varepsilon}} (\beta_{\varepsilon} - u)[(\beta_{\varepsilon} + \gamma_{\varepsilon} - u)^{\tau+\theta} - (\gamma_{\varepsilon})^{\tau+\theta}]^{\nu} du \right] \\ &= \frac{4}{\left(\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma}) \right)^2} \Lambda_4^{\nu}(\beta_{\varepsilon}, \gamma_{\varepsilon}, \tau + \theta), \end{aligned} \quad (105)$$

and

$$\begin{aligned} \int_0^1 (2 - \eta) P_{\tau+\theta}^{\nu}(\eta) d\eta &= \frac{4}{\left(\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma}) \right)^2} \left[\int_{\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2} \right)}^{\beta_{\varepsilon}} (u - \gamma_{\varepsilon})(\beta_{\varepsilon}^{\tau+\theta} - u^{\tau+\theta})^{\nu} du \right. \\ &\quad \left. + \int_{\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_{\varsigma} + \gamma_{\varsigma}}{2} \right)}^{\beta_{\varepsilon}} (u - \beta_{\varepsilon})(\beta_{\varepsilon} + \gamma_{\varepsilon} - u)^{\tau+\theta} - \gamma_{\varepsilon}^{\tau+\theta})^{\nu} du \right] \\ &= \frac{4}{\left(\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_{\varsigma} - \beta_{\varsigma}) \right)^2} \Lambda_5^{\nu}(\beta_{\varepsilon}, \gamma_{\varepsilon}, \tau + \theta). \end{aligned} \quad (106)$$

By substituting (104), (105), and (106) in (103), we obtain (101). $\square$

Theorem 12. *Let all the conditions in the hypotheses of Theorem 10 hold true, then*

$$\begin{aligned}
& \frac{\Gamma(\nu+1)(\tau+\theta)^\nu}{2Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \left[\frac{\tau K^\nu}{\left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2}\right)\right)} F(\beta_\varepsilon) + \frac{\tau K^\nu}{\left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2}\right)\right)} F(\gamma_\varepsilon) \right] - f\left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2}\right)\right) \\
& \leq \frac{1}{2Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma| R_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) \left[\left(\sum_{\varsigma=1}^\varepsilon |f'(\alpha_\varsigma)|^q - \sum_{\varsigma=1}^{\varepsilon-1} |f'(\gamma_\varsigma)|^q B_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) \right. \right. \\
& \quad \left. \left. - \sum_{\varsigma=1}^{\varepsilon-1} |f'(\beta_\varsigma)|^q C_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) \right)^{\frac{1}{q}} + \left(\sum_{\varsigma=1}^\varepsilon |f'(\alpha_\varsigma)|^q - \sum_{\varsigma=1}^{\varepsilon-1} |f'(\beta_\varsigma)|^q B_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) - \sum_{\varsigma=1}^{\varepsilon-1} |f'(\gamma_\varsigma)|^q C_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) \right)^{\frac{1}{q}} \right], \quad (107)
\end{aligned}$$

where

$$\begin{aligned}
R_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) &= \frac{1}{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma)} \Lambda_3^\nu(\beta_\varepsilon, \gamma_\varepsilon, \tau + \theta) \\
B_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) &= \frac{\Lambda_4^\nu(\beta_\varepsilon, \gamma_\varepsilon, \tau + \theta)}{\Lambda_3^\nu(\beta_\varepsilon, \gamma_\varepsilon, \tau + \theta) \left(\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma) \right)} \\
C_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon) &= \frac{\Lambda_5^\nu(\beta_\varepsilon, \gamma_\varepsilon, \tau + \theta)}{\Lambda_3^\nu(\beta_\varepsilon, \gamma_\varepsilon, \tau + \theta) \left(\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma) \right)}.
\end{aligned}$$

Proof. By using Lemma 4, we have

$$\begin{aligned}
& \left| \frac{\Gamma(\nu+1)(\tau+\theta)^\nu}{2Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \left[\frac{\tau K^\nu}{\left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2}\right)\right)} F(\beta_\varepsilon) + \frac{\tau K^\nu}{\left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2}\right)\right)} F(\gamma_\varepsilon) \right] \right. \\
& \quad \left. - f\left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \sum_{\varsigma=1}^{\varepsilon-1} \left(\frac{\beta_\varsigma + \gamma_\varsigma}{2}\right)\right) \right| \\
& = \left| \frac{\sum_{\varsigma=1}^{\varepsilon-1} (\gamma_\varsigma - \beta_\varsigma)}{4Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \int_0^1 P_{\tau+\theta}^\nu(\eta) \left[f'\left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma\right)\right) \right. \right. \\
& \quad \left. \left. - f'\left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma\right)\right) \right] d\eta \right|, \quad (108) \\
& \leq \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma|}{4Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \left[\int_0^1 |P_{\tau+\theta}^\nu(\eta)| \left| f'\left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma\right)\right) \right| d\eta \right. \\
& \quad \left. + \int_0^1 |P_{\tau+\theta}^\nu(\eta)| \left| f'\left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma\right)\right) \right| d\eta \right] \\
& = \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_\varsigma - \beta_\varsigma|}{4Q_{\tau+\theta}^\nu(\beta_\varepsilon, \gamma_\varepsilon)} \left[\int_0^1 P_{\tau+\theta}^\nu(\eta) \left| f'\left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma\right)\right) \right| d\eta \right. \\
& \quad \left. + \int_0^1 P_{\tau+\theta}^\nu(\eta) \left| f'\left(\sum_{\varsigma=1}^\varepsilon \alpha_\varsigma - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_\varsigma + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_\varsigma\right)\right) \right| d\eta \right].
\end{aligned}$$

By using power mean inequality, we have

$$\begin{aligned}
 &\leq \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_{\varsigma} - \beta_{\varsigma}|}{4Q_{\tau+\theta}^{\nu}(\beta_{\varepsilon}, \gamma_{\varepsilon})} \left[\left(\int_0^1 P_{\tau+\theta}^{\nu}(\eta) \right)^{1-\frac{1}{q}} \left(\int_0^1 P_{\tau+\theta}^{\nu}(\eta) \left| f' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} \right) \right|^q d\eta \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + \left(\int_0^1 P_{\tau+\theta}^{\nu}(\eta) \right)^{1-\frac{1}{q}} \left(\int_0^1 P_{\tau+\theta}^{\nu}(\eta) \left| f' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} \right) \right|^q d\eta \right)^{\frac{1}{q}} \right] \\
 &\leq \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_{\varsigma} - \beta_{\varsigma}|}{4Q_{\tau+\theta}^{\nu}(\beta_{\varepsilon}, \gamma_{\varepsilon})} \left(\int_0^1 P_{\tau+\theta}^{\nu}(\eta) \right)^{1-\frac{1}{q}} \left[\left(\int_0^1 P_{\tau+\theta}^{\nu}(\eta) \left| f' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} \right) \right|^q d\eta \right)^{\frac{1}{q}} \right. \right. \\
 &\quad \left. \left. + \left(\int_0^1 P_{\tau+\theta}^{\nu}(\eta) \left| f' \left(\sum_{\varsigma=1}^{\varepsilon} \alpha_{\varsigma} - \left(\frac{\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \beta_{\varsigma} + \frac{2-\eta}{2} \sum_{\varsigma=1}^{\varepsilon-1} \gamma_{\varsigma} \right) \right|^q d\eta \right)^{\frac{1}{q}} \right] \right]. \tag{109}
 \end{aligned}$$

By applying Theorem 5 for the values $n = 2$, $\sigma_1 = \frac{\eta}{2}$, and $\sigma_2 = \frac{2-\eta}{2}$ in (109), we have

$$\begin{aligned}
 &\leq \frac{\sum_{\varsigma=1}^{\varepsilon-1} |\gamma_{\varsigma} - \beta_{\varsigma}|}{4Q_{\tau+\theta}^{\nu}(\beta_{\varepsilon}, \gamma_{\varepsilon})} \left(\int_0^1 P_{\tau+\theta}^{\nu}(\eta) \right)^{1-\frac{1}{q}} \\
 &\quad \times \left[\left(\sum_{\varsigma=1}^{\varepsilon} |f'(\alpha_{\varsigma})|^q \int_0^1 P_{\tau+\theta}^{\nu}(\eta) d\eta - \frac{1}{2} \sum_{\varsigma=1}^{\varepsilon-1} |f'(\gamma_{\varsigma})|^q \int_0^1 \eta P_{\tau+\theta}^{\nu}(\eta) d\eta - \frac{1}{2} \sum_{\varsigma=1}^{\varepsilon-1} |f'(\beta_{\varsigma})|^q \int_0^1 (2-\eta) P_{\tau+\theta}^{\nu}(\eta) d\eta \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + \left(\sum_{\varsigma=1}^{\varepsilon} |f'(\alpha_{\varsigma})|^q \int_0^1 P_{\tau+\theta}^{\nu}(\eta) d\eta - \frac{1}{2} \sum_{\varsigma=1}^{\varepsilon-1} |f'(\beta_{\varsigma})|^q \int_0^1 \eta P_{\tau+\theta}^{\nu}(\eta) d\eta - \frac{1}{2} \sum_{\varsigma=1}^{\varepsilon-1} |f'(\gamma_{\varsigma})|^q \int_0^1 (2-\eta) P_{\tau+\theta}^{\nu}(\eta) d\eta \right)^{\frac{1}{q}} \right]. \tag{110}
 \end{aligned}$$

By substituting (104), (105), and (106) in (110), we obtain (107). $\square$

5 Conclusion

New concrete versions of the Hermite-Hadamard-Jensen-Mercer inequalities were obtained for generalized conformable fractional integral operators. These new inequalities were established by using majorized tuples and convexity of the function along with fractional operators. As particular cases, new and old versions of the Hermite-Hadamard-Jensen-Mercer inequalities were also presented for various fractional operators such as Katugampola, Hadamard, Riemann-Liouville, conformable, and Riemann integrals. The obtained results adopted a more beautiful look when they were expressed in a weighted form. These weighted versions were acquired for two types of tuples, i.e., for decreasing tuples and the tuples which show same monotonic behavior. Furthermore, two new identities were investigated by using a differentiable function and three tuples. Then by applying these newly obtained identities and assuming the convexity of $|f'|$ and $|f'|^q$ ($q > 1$), we constructed bounds for the absolute difference of terms involved in the main results. The current work can also be seen as application of majorization theory and produces new set of inequalities when combined with convex theory and fractional operators.

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