Research Article

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Approximation spaces inspired by subset rough neighborhoods with applications

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Abstract: In this manuscript, we first generate topological structures by subset neighborhoods and ideals and apply to establish some generalized rough-set models. Then, we present other types of generalized rough-set models directly defined by the concepts of subset neighborhoods and ideals. We explore the main characterizations of the proposed approximation spaces and compare them in terms of approximation operators and accuracy measures. The obtained results and given examples show that the second type of the proposed approximation spaces is better than the first one in cases of u and $\langle u \rangle$, whereas the relationships between the rest of the six cases are posted as an open question. Moreover, we demonstrate the advantages of the current models to decrease the upper approximation and increase the lower approximation compared to the existing approaches in published literature. Algorithms and a flow chart are given to illustrate how the exact and rough sets are determined for each approach. Finally, we analyze the information system of dengue fever to confirm the efficiency of our approaches to maximize the value of accuracy and shrink the boundary regions.

Keywords: rough set, approximation spaces, subset neighborhoods, topology, ideal, dengue fever

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1 Introduction

Rough-set theory is a novel mathematical approach originated by Pawlak [1] in the 1980s to manage inexplicit and uncertain data that cannot be addressed by the classical set theory. The key idea in this approach is the approximation space (AS) which comprises an equivalence relation $\mathcal{R}$ on a nonempty set $\mathcal{U}$ of objects. By Pawlak's approach, each subset of data can be approximated using approximation operators called lower approximation and upper approximation, which are defined by the equivalence classes induced by $\mathcal{R}$. These operators categorize the knowledge obtained from the data into three main regions: positive, negative, and boundary.

In many real-life issues that humans deal with in computer networks, economics, medical sciences, engineering, etc., the condition of an equivalence relation does not appear as a description for the relationship between the objects, which abolishes the ability of Pawlak's rough-set theory to deal with these problems [2]. To overcome this obstacle various frames of rough set theory defined with respect to non-equivalence relations, known as generalized rough-set theory or generalized AS, have been proposed.

The first generalized rough-set model constructed by a non-equivalence relation was introduced by Yao [3] in 1996. He defined the concepts of "right neighborhood N_r" and "left neighborhood N_l" of each object

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under arbitrary relation as alternatives to the equivalence class. That is, the granules or blocks that are used to approximate the knowledge obtained from the subset of data are these types of neighborhoods. Then, researchers have established other kinds of generalized ASs under specific relations like tolerance [2], similarity [4,5], quasiorder [6,7] and dominance [8,9]. It has been introduced that many generalized ASs are produced by specific kinds of neighborhood systems; for example, Dai et al. [10] scrutinized some models of ASs using the maximal right neighborhoods defined over a similarity relation. Al-shami [11] completed studying the other kinds of maximal neighborhoods under any arbitrary relation and showed how they applied to classify patients suspected of infection with COVID-19.

To improve the approximation operators by adding objects to the lower approximation and/or removing objects from the upper approximation, the concepts of core neighborhoods and remote neighborhoods were presented by Mareay [12] and Sun et al. [13], respectively. Also, Abu-Donia [14] adopted a new line of rough-set models depending on a finite family of arbitrary relations instead of one relation. Recently, Al-shami with his co-authors have displayed novel sorts of neighborhood systems and their generalized rough paradigms inspired by some relationships between N_ρ-neighborhoods, such as C_ρ-neighborhoods [15], S_ρ-neighborhoods [16], and E_ρ-neighborhoods [17].

Topology is another interesting orientation for studying rough-sets. The possibility of replacing rough-set concepts with their topological counterparts follows from the similar behaviors of topological and rough-set concepts. Investigation of this link was started by Skowron [18] and Wiweger [19]. This domain attracted many scholars and researchers to initiate rough-set notions via their topological counterparts; for instance, Lashin et al. [20] suggested a family N_ρ-neighborhood of each element as a subbase for topology, and then they coped with the notions of rough-set theory as topological concepts. Salama [21] debated how the missing attribute value problem is solved topologically. Al-shami [22,23] benefited from somewhere dense and somewhat open subsets of topological spaces to present various types of approximation operators and accuracy measures. Al-shami and Alshammari [24] successfully applied the structure of supra topology, one of the generalizations of topology, to study generalized rough ASs. To complete this line of research and enhance the role of generalizations of topology to describe the main concepts of rough sets, Al-shami and Mhemdi [25] investigated the rough approximation operators via the frame of infra topology, and the authors of [26,27] discussed these operators via minimal structures. Many ideas and relationships that associated rough-set models with topological counterparts have been elucidated and reveled in [28–33].

In 2013, Kandi et al. [34] provided a novel method to construct ASs depending on the structure of an ideal. They aimed to improve approximation operators and increase accuracy measures. Then, Hosny [35] introduced new rough-set models induced from topological and ideal structures. Recently, some types of neighborhoods with ideal structures have been applied to obtain rid of uncertainty via information systems in [36–40].

The major motivations for writing this article are, first, to dispense an equivalence relation that limits the applications of Pawlak rough-set theory. Second, to keep the greatest number of properties of Pawlak approximation operators that are missing in some existing approaches generated by topological approaches or otherwise. Third, to maximize the accuracy measures and minimize the boundary regions of subsets compared to the previous approaches introduced in [11,40] under arbitrary relation and those introduced in [17,35,41] under similarity relation.

The rest of this article is designed as follows. In Section 2, we review the main concepts of rough sets and topology required to understand this work and shed light on the inducements that led to these contributions. In Section 3, we provide a method to build some topologies by using subset neighborhoods and ideals with respect to any arbitrary relation. Then, we establish new generalized rough-set models by making use of these topologies and discuss their fundamental characterizations. In Section 4, we construct the counterparts of the previous generalized rough set and elucidate their advantages to develop the approximation operators. In addition, we provide an algorithm illustrating how to determine exact sets. We analyze the information system of dengue fever disease in Section 5 to demonstrate the effectiveness and robustness of the followed approach to maximize accuracy values and shrink boundary regions.

Finally, we summarize the main contributions and give some thoughts that can be applied to expand the scope of this manuscript in Section 6.

2 Preliminaries

To make the exposition self-contained, we mention in this section some basic concepts and results of rough-set theory and topology used in the sequel. In this work, the order pair $(\mathcal{U}, \mathcal{R})$ will denote an AS, where $\mathcal{U}$ is a nonempty finite set and $\mathcal{R}$ is an arbitrary binary relation on $\mathcal{U}$. If $\mathcal{R}$ is an equivalence relation (reflexive, symmetric, and transitive), then we call $(\mathcal{U}, \mathcal{R})$ a Pawlak AS.

Definition 2.1. [1] Let $(\mathcal{U}, \mathcal{R})$ be a Pawlak AS and $[v]$ be the equivalence class of $v \in \mathcal{U}$ induced from $\mathcal{R}$. We associate every subset V of $\mathcal{U}$ with two sets called lower approximation $\underline{H}(V)$ and upper approximation $\overline{H}(V)$. They are defined as follows:

$$\underline{H}(V) = \{v \in \mathcal{U} : [v] \subseteq V\} \quad \text{and} \quad \overline{H}(V) = \{v \in \mathcal{U} : [v] \cap V \neq \phi\}.$$

The next proposition outlines the essential properties of these approximation operators, which is the key point of rough-set theory.

Proposition 2.2. [1] *Let V and W be subsets of a Pawlak AS $(\mathcal{U}, \mathcal{R})$. Then, we have next properties.*

$(L1)$	$\underline{H}(V) \subseteq V$	$(U1)$	$X \subseteq \overline{H}(V)$
$(L2)$	$\underline{H}(\phi) = \phi$	$(U2)$	$\overline{H}(\phi) = \phi$
$(L3)$	$\underline{H}(\mathcal{U}) = \mathcal{U}$	$(U3)$	$\overline{H}(\mathcal{U}) = \mathcal{U}$
$(L4)$	If $V \subseteq W$, *then* $\underline{H}(V) \subseteq \underline{H}(W)$	$(U4)$	If $V \subseteq W$, *then* $\overline{H}(V) \subseteq \overline{H}(W)$
$(L5)$	$\underline{H}(V \cap W) = \underline{H}(V) \cap \underline{H}(W)$	$(U5)$	$\overline{H}(V \cap W) \subseteq \overline{H}(V) \cap \overline{H}(W)$
$(L6)$	$\underline{H}(V) \cup \underline{H}(W) \subseteq \underline{H}(V \cup W)$	$(U6)$	$\overline{H}(V \cup W) = \overline{H}(V) \cup \overline{H}(W)$
$(L7)$	$\underline{H}(V^c) = (\overline{H}(V))^c$	$(U7)$	$\overline{H}(V^c) = (\underline{H}(V))^c$
$(L8)$	$\underline{H}(\underline{H}(V)) = \underline{H}(V)$	$(U8)$	$\overline{H}(\overline{H}(V)) = \overline{H}(V)$
$(L9)$	$\underline{H}((\underline{H}(V))^c) = (\overline{H}(V))^c$	$(U9)$	$\overline{H}((\overline{H}(V))^c) = (\overline{H}(V))^c$

Every subset of data is divided into three regions using approximation operators, aiming to discover the knowledge obtained from a subset and its structure.

Definition 2.3. [1] We associate every subset V of a Pawlak AS $(\mathcal{U}, \mathcal{R})$ with three regions defined as follows:

$$H^{+}(V) = \underline{H}(V) \quad \text{(positive region)},$$
$$B(V) = \overline{H}(V) \setminus \underline{H}(V) \quad \text{(boundary region)},$$
$$H^{-}(V) = \mathcal{U} \setminus \overline{H}(V) \quad \text{(negative region)}.$$

The measure (or completeness degree) of knowledge obtained from a nonempty subset V is given as follows:

$$\mathcal{H}(V) = \frac{|\underline{H}(V)|}{|\overline{H}(V)|}.$$

To expand the applications of rough-set theory, Yao [3,42] replaced the equivalence relation with arbitrary relation. In this situation, we need a counterpart for the equivalence classes as a granule for computing. So, it was defined as "right and left neighborhoods," which play a role in equivalence classes in Pawlak AS.

Definition 2.4. [3,42] Let $(\mathcal{U}, \mathcal{R})$ be an AS. Then, the right neighborhood N_r and left neighborhood N_l of $v \in \mathcal{U}$ are, respectively, given as follows:

$$N_r(v) = \{\mu \in \mathcal{U} : (v, \mu) \in \mathcal{R}\} \quad \text{and} \quad N_l(v) = \{\mu \in \mathcal{U} : (\mu, v) \in \mathcal{R}\}.$$

The approximation operators were formulated in view of right and left neighborhoods as follows.

Definition 2.5. [3,42] It was introduced the $N\rho$-lower and $N\rho$-upper approximations of a subset V of an AS $(\mathcal{U}, \mathcal{R})$ for $\rho \in \{r, l\}$ as follows:

$$\underline{H}_{N\rho}(V) = \{v \in \mathcal{U} : N_\rho(v) \subseteq V\} \quad \text{and} \quad \overline{H}_{N\rho}(V) = \{v \in \mathcal{U} : N_\rho(v) \cap V \neq \phi\}.$$

Subsequently, researchers and scholars have investigated various forms of generalized rough sets inspired by new neighborhood systems, aiming to improve the approximations and increase the accuracy measures of rough subsets. In what follows, we list some of them.

Definition 2.6. [4,43,44] The ρ-neighborhoods of each $v \in \mathcal{U}$, denoted by $N_\rho(v)$, induced from an AS $(\mathcal{U}, \mathcal{R})$ are given for $\rho \in \{\langle r\rangle, \langle l\rangle, i, u, \langle i\rangle, \langle u\rangle\}$ as follows:

(i)

$$N_{\langle r\rangle}(v) = \begin{cases} \bigcap_{v \in N_r(\mu)} N_r(\mu): & \text{there exists } N_r(\mu) \text{ including } v \\ \phi & \text{otherwise} \end{cases}$$

(ii)

$$N_{\langle l\rangle}(v) = \begin{cases} \bigcap_{v \in N_l(\mu)} N_l(\mu): & \text{there exists } N_l(\mu) \text{ including } v \\ \phi & \text{otherwise} \end{cases}$$

(iii) $N_i(v) = N_r(v) \cap N_l(v)$.
(iv) $N_u(v) = N_r(v) \cup N_l(v)$.
(v) $N_{\langle i\rangle}(v) = N_{\langle r\rangle}(v) \cap N_{\langle l\rangle}(v)$.
(vi) $N_{\langle u\rangle}(v) = N_{\langle r\rangle}(v) \cup N_{\langle l\rangle}(v)$.

Following a similar technique of Definition 2.5, the above neighborhoods were employed to present new sorts of approximation operators.

But using the formula of Pawlak accuracy measures leads sometimes to obtaining values greater than one, which is illogical. To remove this failure, the definition of accuracy measures was adjusted as follows.

Definition 2.7. [3,4,42–44] The accuracy measures of a nonempty subset V in an AS $(\mathcal{U}, \mathcal{R})$ is given as follows:

$$\mathcal{H}_{N\rho}(V) = \frac{|\underline{H}_{N\rho}(V) \cap V|}{|\overline{H}_{N\rho}(V) \cup V|}.$$

Two of the celebrated types of rough neighborhoods are E_ρ-neighborhoods and S_ρ-neighborhoods. They were defined as follows.

Definition 2.8. [17] The E_ρ-neighborhoods of each $v \in \mathcal{U}$, denoted by $E_\rho(v)$, induced from an AS $(\mathcal{U}, \mathcal{R})$ are given for $\rho \in \{r, l, \langle r\rangle, \langle l\rangle, i, u, \langle i\rangle, \langle u\rangle\}$ as follows:

(i) $E_r(v) = \{\mu \in \mathcal{U} : N_r(\mu) \cap N_r(v) \neq \phi\}$.
(ii) $E_l(v) = \{\mu \in \mathcal{U} : N_l(\mu) \cap N_l(v) \neq \phi\}$.
(iii) $E_i(v) = E_r(v) \cap E_l(v)$.
(iv) $E_u(v) = E_r(v) \cup E_l(v)$.

(v) $E_{\langle r\rangle}(\nu) = \{\mu \in \mathcal{U} : N_{\langle r\rangle}(\mu) \cap N_{\langle r\rangle}(\nu) \neq \phi\}$.
(vii) $E_{\langle l\rangle}(\nu) = \{\mu \in \mathcal{U} : N_{\langle l\rangle}(\mu) \cap N_{\langle l\rangle}(\nu) \neq \phi\}$.
(viii) $E_{\langle i\rangle}(\nu) = E_{\langle r\rangle}(\nu) \cap E_{\langle l\rangle}(\nu)$.
(ix) $E_{\langle u\rangle}(\nu) = E_{\langle r\rangle}(\nu) \cup E_{\langle l\rangle}(\nu)$.

Definition 2.9. [16] The S_ρ-neighborhoods of each $\nu \in \mathcal{U}$, denoted by $S_\rho(\nu)$, induced from an AS $(\mathcal{U}, \mathcal{R})$ are given for $\rho \in \{r, l, \langle r\rangle, \langle l\rangle, i, u, \langle i\rangle, \langle u\rangle\}$ as follows:
(i) $S_r(\nu) = \{\mu \in \mathcal{U} : N_r(\nu) \subseteq N_r(\mu)\}$.
(ii) $S_l(\nu) = \{\mu \in \mathcal{U} : N_l(\nu) \subseteq N_l(\mu)\}$.
(iii) $S_i(\nu) = S_r(\nu) \cap S_l(\nu)$.
(iv) $S_u(\nu) = S_r(\nu) \cup S_l(\nu)$.
(v) $S_{\langle r\rangle}(\nu) = \{\mu \in \mathcal{U} : N_{\langle r\rangle}(\nu) \subseteq N_{\langle r\rangle}(\mu)\}$.
(vi) $S_{\langle l\rangle}(\nu) = \{\mu \in \mathcal{U} : N_{\langle l\rangle}(\nu) \subseteq N_{\langle l\rangle}(\mu)\}$.
(vii) $S_{\langle i\rangle}(\nu) = S_{\langle r\rangle}(\nu) \cap S_{\langle l\rangle}(\nu)$.
(viii) $S_{\langle u\rangle}(\nu) = S_{\langle r\rangle}(\nu) \cup S_{\langle l\rangle}(\nu)$.

Lemma 2.10. [16] *Let $(\mathcal{U}, \mathcal{R})$ be an AS such that $\nu \in \mathcal{U}$.*
(i) *If $\mathcal{R}$ is reflexive, then $S_\rho(\nu) \subseteq E_\rho(\nu)$ for each ρ.*
(ii) *If $\mathcal{R}$ is similarity, then $S_\rho(\nu) \subseteq N_\rho(\nu) \subseteq E_\rho(\nu)$ for each $\rho \in \{r, l, i, u\}$.*

Definition 2.11. [16,17] The lower and upper approximations and accuracy measure of a subset V of an AS $(\mathcal{U}, \mathcal{R})$ for each ρ are defined with respect to E_ρ-neighborhoods and S_ρ-neighborhoods as follows:

$$\underline{H}_{E\rho}(V) = \{\nu \in \mathcal{U} : E_\rho(\nu) \subseteq V\} \qquad \underline{H}_{S\rho}(V) = \{\nu \in \mathcal{U} : S_\rho(\nu) \subseteq V\}$$
$$\overline{H}_{E\rho}(V) = \{\nu \in \mathcal{U} : E_\rho(\nu) \cap V \neq \phi\} \qquad \overline{H}_{S\rho}(V) = \{\nu \in \mathcal{U} : S_\rho(\nu) \cap V \neq \phi\}$$
$$\mathcal{H}_{E\rho}(V) = \frac{|\underline{H}_{E\rho}(V) \cap V|}{|\overline{H}_{E\rho}(V) \cup V|} \qquad \mathcal{H}_{S\rho}(V) = \frac{|\underline{H}_{S\rho}(V) \cap V|}{|\overline{H}_{S\rho}(V) \cup V|}$$

To minimize the vagueness of the data by decreasing the upper approximation and increasing the lower approximation, the approximation operators were constructed from an ideal structure. We first mention the definition of ideal and then present how this idea was exploited to produce new operators of approximation.

Definition 2.12. We call a non-empty family $\mathcal{K}$ of $2^{\mathcal{U}}$ an ideal on $\mathcal{U}$ if it is closed under subset and finite union. That is, it satisfies the following axioms:
(i) If $V \in \mathcal{K}$ and $W \in \mathcal{K}$, then $V \cup W \in \mathcal{K}$.
(ii) If $V \in \mathcal{K}$ and $W \subseteq V$, then $W \in \mathcal{K}$.

Henceforth, we call the triplet $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ an ideal approximation space (IAS).

Definition 2.13. [34] The lower and upper approximations and accuracy measure of a subset V of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ for each ρ are defined with respect to N_ρ-neighborhoods as follows:

$$\underline{H}^{\mathcal{K}}_{N\rho}(V) = \{\nu \in \mathcal{U} : N_\rho(\nu) - V \in \mathcal{K}\},$$
$$\overline{H}^{\mathcal{K}}_{N\rho}(V) = \{\nu \in \mathcal{U} : N_\rho(\nu) \cap V \notin \mathcal{K}\}, \quad \text{and}$$
$$\mathcal{H}^{\mathcal{K}}_{N\rho}(V) = \frac{|\underline{H}^{\mathcal{K}}_{N\rho}(V) \cap V|}{|\overline{H}^{\mathcal{K}}_{N\rho}(V) \cup V|}, \quad \text{where } V \neq \phi.$$

Definition 2.14. [41] The lower and upper approximations and accuracy measure of a subset V of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ for each ρ are defined with respect to E_ρ-neighborhoods as follows:

$$\underline{H}^{\mathcal{K}}_{E\rho}(V) = \{v \in \mathcal{U} : E_\rho(v) - V \in \mathcal{K}\},$$
$$\overline{H}^{\mathcal{K}}_{E\rho}(V) = \{v \in \mathcal{U} : E_\rho(v) \cap V \notin \mathcal{K}\}, \quad \text{and}$$
$$\mathcal{H}^{\mathcal{K}}_{E\rho}(V) = \frac{|\underline{H}^{\mathcal{K}}_{E\rho}(V) \cap V|}{|\overline{H}^{\mathcal{K}}_{E\rho}(V) \cup V|}, \quad \text{where } V \neq \phi.$$

Definition 2.15. A subfamily Ω of $P(\mathcal{U})$ is called a topology on $\mathcal{U}$ if $\phi, \mathcal{U} \in \Omega$, and it is closed under arbitrary union and finite intersection. We call an order pair $(\mathcal{U}, \Omega)$ a topological space. We call a member of Ω an open set and call the complement of an open set a closed set.

For any subset V of $\mathcal{U}$, the interior points of V, denoted by $\mathrm{int}(V)$, is the union of all open sets that are contained in V, and the closure points of V, denoted by $\mathrm{cl}(V)$, is the intersection of all closed sets containing V.

The rough-set paradigms have been studied topologically in several published literature. The followed methods to link neighborhoods systems and topological structures are proved in the following results.

Theorem 2.16. *Let $(\mathcal{U}, \mathcal{R})$ be an AS. Then, each one of the following families is a topology on $\mathcal{U}$ for each ρ:*
(*i*) $\Omega_{N\rho} = \{V \subseteq \mathcal{U} : N_\rho(v) \subseteq V \text{ for each } v \in V\}$ [43].
(*ii*) $\Omega_{E\rho} = \{V \subseteq \mathcal{U} : E_\rho(v) \subseteq V \text{ for each } v \in V\}$ [17].
(*iii*) $\Omega_{S\rho} = \{V \subseteq \mathcal{U} : S_\rho(v) \subseteq V \text{ for each } v \in V\}$ [40].

The aforementioned topological spaces have been employed to construct novel types of ASs.

Definition 2.17. [17,40,43] Let $(\mathcal{U}, \mathcal{R})$ be an AS. Then, some types of lower and upper approximations and accuracy measures of a subset $V \subseteq \mathcal{U}$ induced from topological spaces Ω_{Nj} and Ω_{Nj} are, respectively, defined as follows:

$$\underline{N}_\rho(V) = \mathrm{int}_{N\rho}(V) \quad \overline{N}_\rho(V) = \mathrm{cl}_{N\rho}(V) \quad \lambda_{N\rho}(V) = \frac{|\underline{N}_\rho(V)|}{|\overline{N}_\rho(V)|}$$
$$\underline{E}_\rho(V) = \mathrm{int}_{E\rho}(V) \quad \overline{E}_\rho(V) = \mathrm{cl}_{E\rho}(V) \quad \lambda_{E\rho}(V) = \frac{|\underline{E}_\rho(V)|}{|\overline{E}_\rho(V)|}$$
$$\underline{S}_\rho(V) = \mathrm{int}_{S\rho}(V) \quad \overline{S}_\rho(V) = \mathrm{cl}_{S\rho}(V) \quad \lambda_{S\rho}(V) = \frac{|\underline{S}_\rho(V)|}{|\overline{S}_\rho(V)|}.$$

To improve the approximation operators and increase the accuracy measure of a set, the topological structures given in Theorem 2.16 were enlarged by inserting ideals as illustrated in the next theorem.

Theorem 2.18. *Let $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ be an IAS. Then, each one of the following families is a topology on $\mathcal{U}$ for each ρ:*
(*i*) $\Omega^{\mathcal{K}}_{N\rho} = \{V \subseteq \mathcal{U} : N_\rho(v) - V \in \mathcal{K} \text{ for each } v \in V\}$ [35].
(*ii*) $\Omega^{\mathcal{K}}_{E\rho} = \{V \subseteq \mathcal{U} : E_\rho(v) - V \in \mathcal{K} \text{ for each } v \in V\}$ [41].

Definition 2.19. [35,41] Let $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ be an IAS. Then, some types of lower and upper approximations and accuracy measures of a subset $V \subseteq \mathcal{U}$ induced from topological spaces $\Omega^{\mathcal{K}}_{Nj}$ and $\Omega^{\mathcal{K}}_{Ej}$ are, respectively, defined as follows:

$$\underline{N}^{\mathcal{K}}_\rho(V) = \mathrm{int}^{\mathcal{K}}_{N\rho}(V) \quad \overline{N}^{\mathcal{K}}_\rho(V) = \mathrm{cl}^{\mathcal{K}}_{N\rho}(V) \quad \lambda^{\mathcal{K}}_{N\rho}(V) = \frac{|\underline{N}^{\mathcal{K}}_\rho(V)|}{|\overline{N}^{\mathcal{K}}_\rho(V)|}$$
$$\underline{E}^{\mathcal{K}}_\rho(V) = \mathrm{int}^{\mathcal{K}}_{E\rho}(V) \quad \overline{E}^{\mathcal{K}}_\rho(V) = \mathrm{cl}^{\mathcal{K}}_{E\rho}(V) \quad \lambda^{\mathcal{K}}_{E\rho}(V) = \frac{|\underline{E}^{\mathcal{K}}_\rho(V)|}{|\overline{E}^{\mathcal{K}}_\rho(V)|}.$$

3 ASs generated by subset topologies and ideal

In this section, we provide a new method to propose a topological approach to construct ASs inspired by the ideas of subset neighborhoods and ideals. First, we illustrate the relationships between them and explore their main characterizations. Then, we confirm the good performance of the proposed approach in terms of improving the accuracy measures and approximation operators compared to some previous methods introduced in [16] for $\rho \in \{r, l, i\langle r\rangle, \langle l\rangle, \langle i\rangle\}$ and [40] for each ρ under any arbitrary relation. Furthermore, the current approach is more accurate than that of Hosny's [35] under a similarity relation and Yildirim's [41] under a reflexive relation for each ρ. Finally, we give an algorithm and flow chart to illustrate how the exact and rough sets are determined.

Theorem 3.1. *Let* $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ *be an IAS. Then, the family* $\Omega^{\mathcal{K}}_{S\rho} = \{V \in 2^{\mathcal{U}} : (S_\rho(v) - V) \in \mathcal{K} \text{ for each } v \in V\}$ *is a topology on* $\mathcal{U}$ *for each* ρ.

Proof. It is clear that $S_\rho(v) - \mathcal{U} = \phi \in \mathcal{K}$ for each $v \in \mathcal{U}$ and $S_\rho(\phi) - \phi = \phi \in \mathcal{K}$. So, $\mathcal{U}, \phi \in \Omega^{\mathcal{K}}_{S\rho}$. Now, consider $X_i \in \Omega^{\mathcal{K}}_{S\rho}$, where $i \in I$. Then, $(S_\rho(v) - X_i) \in \mathcal{K}$ for each $v \in X_i$. Obviously, $(S_\rho(v) - \bigcup_i X_i) \subseteq (S_\rho(v) - X_i)$. According to the hereditary property of $\mathcal{K}$, we obtain $(S_\rho(v) - \bigcup_i X_i) \in \mathcal{K}$ for each $v \in \bigcup_i X_i$. Thus, $\bigcup_i X_i \in \Omega^{\mathcal{K}}_{S\rho}$. It remains to prove that $X_1 \cap X_2 \in \Omega^{\mathcal{K}}_{S\rho}$ for any $X_1, X_2 \in \Omega^{\mathcal{K}}_{S\rho}$. To do this, let $v \in X_1 \cap X_2$. Then, $v \in X_1$ and $v \in X_2$. By assumption, $S_\rho(v) - X_1 \in \mathcal{K}$ and $S_\rho(v) - X_2 \in \mathcal{K}$. Since $S_\rho(v) - (X_1 \cap X_2) = (S_\rho(v) - X_1) \cup (S_\rho(v) - X_2)$, it follows from the union condition of $\mathcal{K}$ that $X_1 \cap X_2 \in \Omega^{\mathcal{K}}_{S\rho}$. Hence, the proof is complete. □

The following result elaborates on the relationships between these topologies.

Theorem 3.2. *The next results hold true.*
(1) $\Omega^{\mathcal{K}}_{Su} \subseteq \Omega^{\mathcal{K}}_{Sr} \subseteq \Omega^{\mathcal{K}}_{Si}$.
(2) $\Omega^{\mathcal{K}}_{Su} \subseteq \Omega^{\mathcal{K}}_{Sl} \subseteq \Omega^{\mathcal{K}}_{Si}$.
(3) $\Omega^{\mathcal{K}}_{S\langle u\rangle} \subseteq \Omega^{\mathcal{K}}_{S\langle r\rangle} \subseteq \Omega^{\mathcal{K}}_{S\langle i\rangle}$.
(4) $\Omega^{\mathcal{K}}_{S\langle u\rangle} \subseteq \Omega^{\mathcal{K}}_{S\langle l\rangle} \subseteq \Omega^{\mathcal{K}}_{S\langle i\rangle}$.
(5) *If* $\mathcal{R}$ *is symmetric, then*

$$\Omega^{\mathcal{K}}_{Sr} = \Omega^{\mathcal{K}}_{Sl} = \Omega^{\mathcal{K}}_{Si} = \Omega^{\mathcal{K}}_{Su} \quad \text{and} \quad \Omega^{\mathcal{K}}_{S\langle r\rangle} = \Omega^{\mathcal{K}}_{S\langle l\rangle} = \Omega^{\mathcal{K}}_{S\langle i\rangle} = \Omega^{\mathcal{K}}_{S\langle u\rangle}.$$

(6) *If* $\mathcal{R}$ *is an equivalence, then*

$$\Omega^{\mathcal{K}}_{Sr} = \Omega^{\mathcal{K}}_{Sl} = \Omega^{\mathcal{K}}_{Si} = \Omega^{\mathcal{K}}_{Su} = \Omega^{\mathcal{K}}_{S\langle r\rangle} = \Omega^{\mathcal{K}}_{S\langle l\rangle} = \Omega^{\mathcal{K}}_{S\langle i\rangle} = \Omega^{\mathcal{K}}_{S\langle u\rangle}.$$

(7) *If* $\mathcal{R}$ *is a quasiorder* (*reflexive and transitive*), *then* $\Omega^{\mathcal{K}}_{S\rho} = \Omega^{\mathcal{K}}_{S\langle\rho\rangle}$.

Proof. The proofs of items 1-4 follows from the below relationships and the hereditary property of the ideal.

$$S_i(v) \subseteq S_r(v) \subseteq S_u(v); \quad S_i(v) \subseteq S_l(v) \subseteq S_u(v);$$
$$S_{\langle i\rangle}(v) \subseteq S_{\langle r\rangle}(v) \subseteq S_{\langle u\rangle} \quad \text{and} \quad S_{\langle i\rangle}(v) \subseteq S_{\langle l\rangle}(v) \subseteq S_{\langle u\rangle}.$$

Since the equalities $S_r(v) = S_l(v) = S_i(v) = S_u(v)$ and $S_{\langle r\rangle}(v) = S_{\langle l\rangle}(v) = S_{\langle i\rangle}(v) = S_{\langle u\rangle}(v)$ hold true under a symmetric relation and the equality $S_r(v) = S_l(v) = S_i(v) = S_u(v) = S_{\langle r\rangle}(v) = S_{\langle l\rangle}(v) = S_{\langle i\rangle}(v) = S_{\langle u\rangle}(v)$ hold true under an equivalence relation, we obtain the proofs of the results (5) and (6).

The proof of result (7) follows from the fact that $N_\rho(v) = N_{\langle\rho\rangle}(v)$ for each element v under a quasiorder relation. □

Now, we benefit from the topological structures constructed in Theorem 3.1 to initiate novel types of ASs.

Definition 3.3. we call the triplet $(\mathcal{U}, \mathcal{R}, \Omega^{\mathcal{K}}_{S\rho})$ an ideal subset topological approximation space (ISTAS), where $\Omega^{\mathcal{K}}_{S\rho}$ is the topological space obtained from Theorem 3.1. A subset V of an ISTAS $(\mathcal{U}, \mathcal{R}, \Omega^{\mathcal{K}}_{S\rho})$ is said to be an $S^{\mathcal{K}}_{\rho}$-open set if $X \in \Omega^{\mathcal{K}}_{S\rho}$, and the complement of an $S^{\mathcal{K}}_{\rho}$-open set is said to be an $S^{\mathcal{K}}_{\rho}$-closed set. The family $\Upsilon_{S\rho}$ of all $S^{\mathcal{K}}_{\rho}$-closed sets is given as follows: $\Upsilon^{\mathcal{K}}_{S\rho} = \{W \subseteq \mathcal{U} : W^c \in \Omega^{\mathcal{K}}_{S\rho}\}$.

Definition 3.4. Let $(\mathcal{U}, \mathcal{R}, \Omega^{\mathcal{K}}_{S\rho})$ be an ISTAS. The $S^{\mathcal{K}}_{\rho}$-interior and $S^{\mathcal{K}}_{\rho}$-closure of a subset V of $\mathcal{U}$ are, respectively, defined as follows:

$$\mathrm{int}^{\mathcal{K}}_{S\rho}(V) = \cup\{G \in \Omega^{\mathcal{K}}_{S\rho} : G \subseteq V\}$$
$$\mathrm{cl}^{\mathcal{K}}_{S\rho}(V) = \cap\{H \in \Upsilon^{\mathcal{K}}_{S\rho} : V \subseteq H\}.$$

The main concepts of ASs are defined with respect to an ISTAS $(\mathcal{U}, \mathcal{R}, \Omega^{\mathcal{K}}_{S\rho})$ as follows.

Definition 3.5. Let $(\mathcal{U}, \mathcal{R}, \Omega^{\mathcal{K}}_{S\rho})$ be an ISTAS. The lower approximation $\underline{S}^{\mathcal{K}}_{\rho}$, upper approximation $\overline{S}^{\mathcal{K}}_{\rho}$, boundary region $B^{\mathcal{K}}_{S\rho}$, positive region $O^{\mathcal{K}+}_{S\rho}$, negative region $O^{\mathcal{K}-}_{S\rho}$, and accuracy measure $\lambda^{\mathcal{K}}_{S\rho}$ of a subset V are, respectively, given as follows:
- $\underline{S}^{\mathcal{K}}_{\rho}(V) = \mathrm{int}^{\mathcal{K}}_{S\rho}(V)$.
- $\overline{S}^{\mathcal{K}}_{\rho}(V) = \mathrm{cl}^{\mathcal{K}}_{S\rho}(V)$.
- $B^{\mathcal{K}}_{S\rho}(V) = \overline{S}^{\mathcal{K}}_{\rho}(V) - \underline{S}^{\mathcal{K}}_{\rho}(V)$
- $O^{\mathcal{K}+}_{S\rho}(V) = \underline{S}^{\mathcal{K}}_{\rho}(V)$.
- $O^{\mathcal{K}-}_{S\rho}(V) = \mathcal{U} - \overline{S}^{\mathcal{K}}_{\rho}(V)$.
- $\lambda^{\mathcal{K}}_{S\rho}(V) = \frac{|\underline{S}^{\mathcal{K}}_{\rho}(V)|}{|\overline{S}^{\mathcal{K}}_{\rho}(V)|}$, where $V \neq \phi$.

The following theorem states the properties of the lower approximation $\underline{S}^{\mathcal{K}}_{\rho}$ and upper approximation $\overline{S}^{\mathcal{K}}_{\rho}$.

Theorem 3.6. *Consider $(\mathcal{U}, \mathcal{R}, \Omega^{\mathcal{K}}_{S\rho})$ as an ISTAS and let V and W be subsets of $\mathcal{U}$. Then,*
(1) $\underline{S}^{\mathcal{K}}_{\rho}(V) \subseteq V \subseteq \overline{S}^{\mathcal{K}}_{\rho}(V)$.
(2) $\underline{S}^{\mathcal{K}}_{\rho}(\phi) = \overline{S}^{\mathcal{K}}_{\rho}(\phi) = \phi$.
(3) $\underline{S}^{\mathcal{K}}_{\rho}(\mathcal{U}) = \overline{S}^{\mathcal{K}}_{\rho}(\mathcal{U}) = \mathcal{U}$.
(4) *If $V \subseteq W$, then $\underline{S}^{\mathcal{K}}_{\rho}(V) \subseteq \underline{S}^{\mathcal{K}}_{\rho}(W)$ and $\overline{S}^{\mathcal{K}}_{\rho}(V) \subseteq \overline{S}^{\mathcal{K}}_{\rho}(W)$.*
(5) $\underline{S}^{\mathcal{K}}_{\rho}(V \cap W) = \underline{S}^{\mathcal{K}}_{\rho}(V) \cap \underline{S}^{\mathcal{K}}_{\rho}(W)$ *and* $\overline{S}^{\mathcal{K}}_{\rho}(V \cup W) = \overline{S}^{\mathcal{K}}_{\rho}(V) \cup \overline{S}^{\mathcal{K}}_{\rho}(W)$.
(6) $\underline{S}^{\mathcal{K}}_{\rho}(V) \cup \underline{S}^{\mathcal{K}}_{\rho}(W) \subseteq \underline{S}^{\mathcal{K}}_{\rho}(V \cup W)$ *and* $\overline{S}^{\mathcal{K}}_{\rho}(V \cap W) \subseteq \overline{S}^{\mathcal{K}}_{\rho}(V) \cap \overline{S}^{\mathcal{K}}_{\rho}(W)$.
(7) $\underline{S}^{\mathcal{K}}_{\rho}[\underline{S}^{\mathcal{K}}_{\rho}(V)] = \underline{S}^{\mathcal{K}}_{\rho}(V)$ *and* $\overline{S}^{\mathcal{K}}_{\rho}[\overline{S}^{\mathcal{K}}_{\rho}(V)] = \overline{S}^{\mathcal{K}}_{\rho}(V)$.
(8) $\underline{S}^{\mathcal{K}}_{\rho}(V^c) = [\overline{S}^{\mathcal{K}}_{\rho}(V)]^c$ *and* $\overline{S}^{\mathcal{K}}_{\rho}(V^c) = [\underline{S}^{\mathcal{K}}_{\rho}(V)]^c$.

Proof. The proof follows from the properties of interior and closure topological operators. □

Proposition 3.7. *Let $(\mathcal{U}, \mathcal{R}, \Omega^{\mathcal{K}}_{S\rho})$ be an ISTAS and $V \subseteq \mathcal{U}$. Then,*
(i) $\underline{S}^{\mathcal{K}}_{u}(V) \subseteq \underline{S}^{\mathcal{K}}_{r}(V) \subseteq \underline{S}^{\mathcal{K}}_{i}(V)$.
(ii) $\underline{S}^{\mathcal{K}}_{u}(V) \subseteq \underline{S}^{\mathcal{K}}_{l}(V) \subseteq \underline{S}^{\mathcal{K}}_{i}(V)$.
(iii) $\underline{S}^{\mathcal{K}}_{\langle u\rangle}(V) \subseteq \underline{S}^{\mathcal{K}}_{\langle r\rangle}(V) \subseteq \underline{S}^{\mathcal{K}}_{\langle i\rangle}(V)$.
(iv) $\underline{S}^{\mathcal{K}}_{\langle u\rangle}(V) \subseteq \underline{S}^{\mathcal{K}}_{\langle l\rangle}(V) \subseteq \underline{S}^{\mathcal{K}}_{\langle i\rangle}(V)$.

(v) $\overline{S}_i^{\mathcal{K}}(V) \subseteq \overline{S}_r^{\mathcal{K}}(V) \subseteq \overline{S}_u^{\mathcal{K}}(V)$.
(vi) $\overline{S}_i^{\mathcal{K}}(V) \subseteq \overline{S}_l^{\mathcal{K}}(V) \subseteq \overline{S}_u^{\mathcal{K}}(V)$.
(vii) $\overline{S}_{\langle i\rangle}^{\mathcal{K}}(V) \subseteq \overline{S}_{\langle r\rangle}^{\mathcal{K}}(V) \subseteq \overline{S}_{\langle u\rangle}^{\mathcal{K}}(V)$.
(viii) $\overline{S}_{\langle i\rangle}^{\mathcal{K}}(V) \subseteq \overline{S}_{\langle l\rangle}^{\mathcal{K}}(V) \subseteq \overline{S}_{\langle u\rangle}^{\mathcal{K}}(V)$.

Proof. We suffice by proving (i), and the other cases can be proved following a similar technique. Let $v \in \underline{S}_u^{\mathcal{K}}(V)$. That is, $v \in \text{int}_{Su}^{\mathcal{K}}(V)$. Then, it follows from Theorem 3.2 that $v \in \text{int}_{Sr}^{\mathcal{K}}(V)$. This means that $v \in \underline{S}_r^{\mathcal{K}}(V)$. Therefore, $\underline{S}_u^{\mathcal{K}}(V) \subseteq \underline{S}_r^{\mathcal{K}}(V)$. In a similar way, we obtain $\underline{S}_r^{\mathcal{K}}(V) \subseteq \underline{S}_i^{\mathcal{K}}(V)$. Hence, the proof is complete. □

Corollary 3.8. *Let V be a nonempty subset of an ISTAS $(\mathcal{U}, \mathcal{R}, \Omega_{S\rho}^{\mathcal{K}})$. Then,*
(i) $\lambda_u^{\mathcal{K}}(V) \leq \lambda_r^{\mathcal{K}}(V) \leq \lambda_i^{\mathcal{K}}(V)$.
(ii) $\lambda_u^{\mathcal{K}}(V) \leq \lambda_l^{\mathcal{K}}(V) \leq \lambda_i^{\mathcal{K}}(V)$.
(iii) $\lambda_{\langle u\rangle}^{\mathcal{K}}(V) \leq \lambda_{\langle r\rangle}^{\mathcal{K}}(V) \leq \lambda_{\langle i\rangle}^{\mathcal{K}}(V)$.
(iv) $\lambda_{\langle u\rangle}^{\mathcal{K}}(V) \leq \lambda_{\langle l\rangle}^{\mathcal{K}}(V) \leq \lambda_{\langle i\rangle}^{\mathcal{K}}(V)$.

Proof. We suffice by proving (i), and the other cases can be proved following a similar technique. To do this, note that $\underline{S}_u^{\mathcal{K}}(V) \subseteq \underline{S}_r^{\mathcal{K}}(V) \subseteq \underline{S}_i^{\mathcal{K}}(V)$. This automatically leads to the next equality

$$|\underline{S}_u^{\mathcal{K}}(V)| \leq |\underline{S}_r^{\mathcal{K}}(V)| \leq |\underline{S}_i^{\mathcal{K}}(V)|. \tag{1}$$

In addition, note that $\overline{S}_i^{\mathcal{K}}(V) \subseteq \overline{S}_r^{\mathcal{K}}(V) \subseteq \overline{S}_u^{\mathcal{K}}(V)$, which automatically leads to the following equality:

$$\frac{1}{|\overline{S}_u^{\mathcal{K}}(V)|} \leq \frac{1}{|\overline{S}_r^{\mathcal{K}}(V)|} \leq \frac{1}{|\overline{S}_i^{\mathcal{K}}(V)|}. \tag{2}$$

By (1) and (2), we obtain

$$\frac{|\underline{S}_u^{\mathcal{K}}(V)|}{|\overline{S}_u^{\mathcal{K}}(V)|} \leq \frac{|\underline{S}_r^{\mathcal{K}}(V)|}{|\overline{S}_r^{\mathcal{K}}(V)|} \leq \frac{|\underline{S}_i^{\mathcal{K}}(V)|}{|\overline{S}_i^{\mathcal{K}}(V)|},$$

which ends the proof. □

In the following proposition and example, we show that the the current ISTASs are more efficient at removing vagueness than their counterparts given in [40].

Proposition 3.9. *Let V be a subset of an ISTAS $(\mathcal{U}, \mathcal{R}, \Omega_{S\rho}^{\mathcal{K}})$. Then, $\lambda_{S\rho}(V) \leq \lambda_{S\rho}^{\mathcal{K}}(V)$ for each ρ.*

Proof. Let V be a subset of $\mathcal{U}$ such that $V \in \Omega_{S\rho}$. By (iii) of Theorem 2.16 we obtain $S_\rho(v) \subseteq V$ for each $v \in V$. This implies that $S_\rho(v) - V = \varnothing$ for each $v \in V$, which automatically means $S_\rho(V) - V \in \mathcal{K}$. According to Theorem 3.1, we obtain $\Omega_{S\rho} \subseteq \Omega_{S\rho}^{\mathcal{K}}$. Therefore, $\text{int}_{S\rho}(V) \subseteq \text{int}_{S\rho}^{\mathcal{K}}(V)$ and $\text{cl}_{S\rho}^{\mathcal{K}}(V) \subseteq \text{cl}_{S\rho}(V)$ for each subset V of $\mathcal{U}$. This automatically leads to the following equalities:

$$|\underline{S}_\rho(V)| \leq |\underline{S}_\rho^{\mathcal{K}}(V)| \tag{3}$$

$$|\overline{S}_\rho^{\mathcal{K}}(V)| \leq |\overline{S}_\rho(V)|. \tag{4}$$

It comes from (3) and (4) that $\frac{|\underline{S}_\rho(V)|}{|\overline{S}_\rho(V)|} \leq \frac{|\underline{S}_\rho^{\mathcal{K}}(V)|}{|\overline{S}_\rho^{\mathcal{K}}(V)|}$, as required. □

Example 3.10. Let $(\mathcal{U}, \mathcal{R})$ be an AS, where $\mathcal{U} = \{\delta_1, \delta_2, \delta_3\}$ be a universe set, and $\mathcal{R} = \{(\delta_1, \delta_1), (\delta_1, \delta_3), (\delta_2, \delta_3)\}$ be a binary relation on $\mathcal{U}$. We compute the systems of N_ρ-neighborhoods and S_ρ-neighborhoods in Tables 1 and 2, respectively.

Table 1: N_ρ-neighborhoods

Subset	N_r	N_l	N_i	N_u	$N_{\langle r\rangle}$	$N_{\langle l\rangle}$	$N_{\langle i\rangle}$	$N_{\langle u\rangle}$
δ_1	$\{\delta_1, \delta_3\}$	$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1, \delta_3\}$	$\{\delta_1, \delta_3\}$	$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1, \delta_3\}$
δ_2	$\{\delta_3\}$	ϕ	ϕ	$\{\delta_3\}$	ϕ	$\{\delta_1, \delta_2\}$	ϕ	$\{\delta_1, \delta_2\}$
δ_3	ϕ	$\{\delta_1, \delta_2\}$	ϕ	$\{\delta_1, \delta_2\}$	$\{\delta_3\}$	ϕ	ϕ	$\{\delta_3\}$

Table 2: S_ρ-neighborhoods

Subset	S_r	S_l	S_i	S_u	$S_{\langle r\rangle}$	$S_{\langle l\rangle}$	$S_{\langle i\rangle}$	$S_{\langle u\rangle}$
δ_1	$\{\delta_1\}$	$\{\delta_1, \delta_3\}$	$\{\delta_1\}$	$\{\delta_1, \delta_3\}$	$\{\delta_1\}$	$\{\delta_1, \delta_2\}$	$\{\delta_1\}$	$\{\delta_1, \delta_2\}$
δ_2	$\{\delta_1, \delta_2\}$	$\mathcal{U}$	$\{\delta_1, \delta_2\}$	$\mathcal{U}$	$\mathcal{U}$	$\{\delta_2\}$	$\{\delta_2\}$	$\mathcal{U}$
δ_3	$\mathcal{U}$	$\{\delta_3\}$	$\{\delta_3\}$	$\mathcal{U}$	$\{\delta_1, \delta_3\}$	$\mathcal{U}$	$\{\delta_1, \delta_3\}$	$\mathcal{U}$

First, we generate eight topologies from the system of S_ρ-neighborhoods.

$$\begin{cases} \Omega_{Sr} = \{\phi, \{\delta_1\}, \{\delta_1, \delta_2\}, \mathcal{U}\}; \\ \Omega_{Sl} = \{\phi, \{\delta_3\}, \{\delta_1, \delta_3\}, \mathcal{U}\}; \\ \Omega_{Si} = \{\phi, \{\delta_1\}, \{\delta_3\}, \{\delta_1, \delta_3\}, \{\delta_1, \delta_2\}, \mathcal{U}\}; \\ \Omega_{Su} = \{\phi, \mathcal{U}\}; \\ \Omega_{S\langle r\rangle} = \{\phi, \{\delta_1\}, \{\delta_1, \delta_3\}, \mathcal{U}\}; \\ \Omega_{S\langle l\rangle} = \{\phi, \{\delta_2\}, \{\delta_1, \delta_2\}, \mathcal{U}\}; \\ \Omega_{S\langle i\rangle} = \{\phi, \{\delta_1\}, \{\delta_2\}, \{\delta_1, \delta_2\}, \{\delta_1, \delta_3\}, \mathcal{U}\}; \\ \Omega_{S\langle u\rangle} = \{\phi, \mathcal{U}\}. \end{cases} \tag{5}$$

In Tables 3–5, we compute the approximation operators $\underline{S}_\rho$ and $\overline{S}_\rho$ and accuracy measures $\lambda_{S\rho}$ of each subset of $\mathcal{U}$ produced by the approach given in Definition 3.5.

Table 3: $\underline{S}_\rho$-approximations

Subset	$\underline{S}_r$	$\underline{S}_l$	$\underline{S}_i$	$\underline{S}_u$	$\underline{S}_{\langle r\rangle}$	$\underline{S}_{\langle l\rangle}$	$\underline{S}_{\langle i\rangle}$	$\underline{S}_{\langle u\rangle}$
$\{\delta_1\}$	$\{\delta_1\}$	ϕ	$\{\delta_1\}$	ϕ	$\{\delta_1\}$	ϕ	$\{\delta_1\}$	ϕ
$\{\delta_2\}$	ϕ	ϕ	ϕ	ϕ	ϕ	$\{\delta_2\}$	$\{\delta_2\}$	ϕ
$\{\delta_3\}$	ϕ	$\{\delta_3\}$	$\{\delta_3\}$	ϕ	ϕ	ϕ	ϕ	ϕ
$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	ϕ	$\{\delta_1, \delta_2\}$	ϕ	$\{\delta_1\}$	$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	ϕ
$\{\delta_1, \delta_3\}$	$\{\delta_1\}$	$\{\delta_1, \delta_3\}$	$\{\delta_1, \delta_3\}$	ϕ	$\{\delta_1, \delta_3\}$	ϕ	$\{\delta_1, \delta_3\}$	ϕ
$\{\delta_2, \delta_3\}$	ϕ	$\{\delta_3\}$	$\{\delta_3\}$	ϕ	ϕ	$\{\delta_2\}$	$\{\delta_2\}$	ϕ

Table 4: $\overline{S}_\rho$-approximations

Subset	$\overline{S}_r$	$\overline{S}_l$	$\overline{S}_i$	$\overline{S}_u$	$\overline{S}_{\langle r\rangle}$	$\overline{S}_{\langle l\rangle}$	$\overline{S}_{\langle i\rangle}$	$\overline{S}_{\langle u\rangle}$
$\{\delta_1\}$	$\mathcal{U}$	$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	$\mathcal{U}$	$\mathcal{U}$	$\{\delta_1, \delta_3\}$	$\{\delta_1, \delta_3\}$	$\mathcal{U}$
$\{\delta_2\}$	$\{\delta_2, \delta_3\}$	$\{\delta_2\}$	$\{\delta_2\}$	$\mathcal{U}$	$\{\delta_2\}$	$\mathcal{U}$	$\{\delta_2\}$	$\mathcal{U}$
$\{\delta_3\}$	$\{\delta_3\}$	$\mathcal{U}$	$\{\delta_3\}$	$\mathcal{U}$	$\{\delta_2, \delta_3\}$	$\{\delta_3\}$	$\{\delta_3\}$	$\mathcal{U}$
$\{\delta_1, \delta_2\}$	$\mathcal{U}$	$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$
$\{\delta_1, \delta_3\}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\{\delta_1, \delta_3\}$	$\{\delta_1, \delta_3\}$	$\mathcal{U}$
$\{\delta_2, \delta_3\}$	$\{\delta_2, \delta_3\}$	$\mathcal{U}$	$\{\delta_2, \delta_3\}$	$\mathcal{U}$	$\{\delta_2, \delta_3\}$	$\mathcal{U}$	$\{\delta_2, \delta_3\}$	$\mathcal{U}$

Table 5: $\lambda_{S\rho}$-accuracy measures

Subset	λ_{Sr}	λ_{Sl}	λ_{Si}	λ_{Su}	$\lambda_{S\langle r\rangle}$	$\lambda_{S\langle l\rangle}$	$\lambda_{S\langle i\rangle}$	$\lambda_{S\langle u\rangle}$
$\{\delta_1\}$	$\frac{1}{3}$	0	$\frac{1}{2}$	0	$\frac{1}{3}$	0	$\frac{1}{2}$	0
$\{\delta_2\}$	0	0	0	0	0	$\frac{1}{3}$	1	0
$\{\delta_3\}$	0	$\frac{1}{3}$	1	0	0	0	0	0
$\{\delta_1, \delta_2\}$	$\frac{2}{3}$	0	1	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	0
$\{\delta_1, \delta_3\}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	0	$\frac{2}{3}$	0	1	0
$\{\delta_2, \delta_3\}$	0	$\frac{1}{3}$	$\frac{1}{2}$	0	0	$\frac{1}{3}$	$\frac{1}{2}$	0

Second, we consider $\mathcal{K} = \{\phi, \{\delta_1\}\}$ and generate eight topologies from the system of S_ρ-neighborhoods and ideal $\mathcal{K}$.

$$\begin{cases} \Omega_{Sr}^{\mathcal{K}} = \{\phi, \{\delta_1\}, \{\delta_2\}, \{\delta_1, \delta_2\}, \{\delta_2, \delta_3\}, \mathcal{U}\}; \\ \Omega_{Sl}^{\mathcal{K}} = \{\phi, \{\delta_3\}, \{\delta_1, \delta_3\}, \{\delta_2, \delta_3\}, \mathcal{U}\}; \\ \Omega_{Si}^{\mathcal{K}} = P(\mathcal{U}); \\ \Omega_{Su}^{\mathcal{K}} = \{\phi, \{\delta_2, \delta_3\}, \mathcal{U}\}; \\ \Omega_{S\langle r\rangle}^{\mathcal{K}} = \{\phi, \{\delta_1\}, \{\delta_3\}, \{\delta_1, \delta_3\}, \{\delta_2, \delta_3\}, \mathcal{U}\}; \\ \Omega_{S\langle l\rangle}^{\mathcal{K}} = \{\phi, \{\delta_2\}, \{\delta_1, \delta_2\}, \{\delta_2, \delta_3\}, \mathcal{U}\}; \\ \Omega_{S\langle i\rangle}^{\mathcal{K}} = P(\mathcal{U}); \\ \Omega_{S\langle u\rangle}^{\mathcal{K}} = \{\phi, \{\delta_2, \delta_3\}, \mathcal{U}\}. \end{cases} \tag{6}$$

In Tables 6–8, we compute the approximation operators $\underline{S}_\rho^{\mathcal{K}}$ and $\overline{S}_\rho^{\mathcal{K}}$ and accuracy measures $\lambda_{S\rho}^{\mathcal{K}}$ of each subset of $\mathcal{U}$ produced by the current approach given Definition 3.5.

Table 6: $\underline{S}_\rho^{\mathcal{K}}$-approximations

Subset	$\underline{S}_r^{\mathcal{K}}$	$\underline{S}_l^{\mathcal{K}}$	$\underline{S}_i^{\mathcal{K}}$	$\underline{S}_u^{\mathcal{K}}$	$\underline{S}_{\langle r\rangle}^{\mathcal{K}}$	$\underline{S}_{\langle l\rangle}^{\mathcal{K}}$	$\underline{S}_{\langle i\rangle}^{\mathcal{K}}$	$\underline{S}_{\langle u\rangle}^{\mathcal{K}}$
$\{\delta_1\}$	$\{\delta_1\}$	ϕ	$\{\delta_1\}$	ϕ	$\{\delta_1\}$	ϕ	$\{\delta_1\}$	ϕ
$\{\delta_2\}$	$\{\boldsymbol{\delta_2}\}$	ϕ	$\{\boldsymbol{\delta_2}\}$	ϕ	ϕ	$\{\delta_2\}$	$\{\delta_2\}$	ϕ
$\{\delta_3\}$	ϕ	$\{\delta_3\}$	$\{\delta_3\}$	ϕ	$\{\boldsymbol{\delta_3}\}$	ϕ	$\{\boldsymbol{\delta_3}\}$	ϕ
$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	ϕ	$\{\delta_1, \delta_2\}$	ϕ	$\{\delta_1\}$	$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	ϕ
$\{\delta_1, \delta_3\}$	$\{\delta_1\}$	$\{\delta_1, \delta_3\}$	$\{\delta_1, \delta_3\}$	ϕ	$\{\delta_1, \delta_3\}$	ϕ	$\{\delta_1, \delta_3\}$	ϕ
$\{\boldsymbol{\delta_2}, \boldsymbol{\delta_3}\}$	$\{\boldsymbol{\delta_2}, \boldsymbol{\delta_3}\}$	$\{\boldsymbol{\delta_2}, \boldsymbol{\delta_3}\}$	$\{\boldsymbol{\delta_2}, \boldsymbol{\delta_3}\}$	$\{\boldsymbol{\delta_2}, \boldsymbol{\delta_3}\}$	$\{\boldsymbol{\delta_2}, \boldsymbol{\delta_3}\}$	$\{\boldsymbol{\delta_2}, \boldsymbol{\delta_3}\}$	$\{\boldsymbol{\delta_2}, \boldsymbol{\delta_3}\}$	$\{\boldsymbol{\delta_2}, \boldsymbol{\delta_3}\}$

Table 7: $\overline{S}_\rho^{\mathcal{K}}$-approximations

Subset	$\overline{S}_r^{\mathcal{K}}$	$\overline{S}_l^{\mathcal{K}}$	$\overline{S}_i^{\mathcal{K}}$	$\overline{S}_u^{\mathcal{K}}$	$\overline{S}_{\langle r\rangle}^{\mathcal{K}}$	$\overline{S}_{\langle l\rangle}^{\mathcal{K}}$	$\overline{S}_{\langle i\rangle}^{\mathcal{K}}$	$\overline{S}_{\langle u\rangle}^{\mathcal{K}}$
$\{\boldsymbol{\delta_1}\}$	$\{\boldsymbol{\delta_1}\}$	$\{\boldsymbol{\delta_1}\}$	$\{\boldsymbol{\delta_1}\}$	$\{\boldsymbol{\delta_1}\}$	$\{\boldsymbol{\delta_1}\}$	$\{\boldsymbol{\delta_1}\}$	$\{\boldsymbol{\delta_1}\}$	$\{\boldsymbol{\delta_1}\}$
$\{\delta_2\}$	$\{\delta_2, \delta_3\}$	$\{\delta_2\}$	$\{\delta_2\}$	$\mathcal{U}$	$\{\delta_2\}$	$\mathcal{U}$	$\{\delta_2\}$	$\mathcal{U}$
$\{\delta_3\}$	$\{\delta_3\}$	$\mathcal{U}$	$\{\delta_3\}$	$\mathcal{U}$	$\{\delta_2, \delta_3\}$	$\{\delta_3\}$	$\{\delta_3\}$	$\mathcal{U}$
$\{\delta_1, \delta_2\}$	$\mathcal{U}$	$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	$\mathcal{U}$	$\{\boldsymbol{\delta_1}, \boldsymbol{\delta_2}\}$	$\mathcal{U}$	$\{\boldsymbol{\delta_1}, \boldsymbol{\delta_2}\}$	$\mathcal{U}$
$\{\delta_1, \delta_3\}$	$\{\boldsymbol{\delta_1}, \boldsymbol{\delta_3}\}$	$\mathcal{U}$	$\{\boldsymbol{\delta_1}, \boldsymbol{\delta_3}\}$	$\mathcal{U}$	$\mathcal{U}$	$\{\delta_1, \delta_3\}$	$\{\delta_1, \delta_3\}$	$\mathcal{U}$
$\{\delta_2, \delta_3\}$	$\{\delta_2, \delta_3\}$	$\mathcal{U}$	$\{\delta_2, \delta_3\}$	$\mathcal{U}$	$\{\delta_2, \delta_3\}$	$\mathcal{U}$	$\{\delta_2, \delta_3\}$	$\mathcal{U}$

Table 8: $\lambda^{\mathcal{K}}_{S\rho}$-accuracy measures

Subset	$\lambda^{\mathcal{K}}_{Sr}$	$\lambda^{\mathcal{K}}_{Sl}$	$\lambda^{\mathcal{K}}_{Si}$	$\lambda^{\mathcal{K}}_{Su}$	$\lambda^{\mathcal{K}}_{S\langle r\rangle}$	$\lambda^{\mathcal{K}}_{S\langle l\rangle}$	$\lambda^{\mathcal{K}}_{S\langle i\rangle}$	$\lambda^{\mathcal{K}}_{S\langle u\rangle}$
$\{\delta_1\}$	**1**	0	**1**	0	**1**	0	**1**	0
$\{\delta_2\}$	$\frac{1}{2}$	0	**1**	0	0	$\frac{1}{3}$	1	0
$\{\delta_3\}$	0	$\frac{1}{3}$	1	0	$\frac{1}{2}$	0	**1**	0
$\{\delta_1, \delta_2\}$	$\frac{2}{3}$	0	1	0	$\frac{1}{2}$	$\frac{2}{3}$	**1**	0
$\{\delta_1, \delta_3\}$	$\frac{1}{2}$	$\frac{2}{3}$	**1**	0	$\frac{2}{3}$	0	1	0
$\{\delta_2, \delta_3\}$	**1**	**$\frac{2}{3}$**	**1**	**$\frac{2}{3}$**	**1**	**$\frac{2}{3}$**	**1**	**$\frac{2}{3}$**

It can be seen from Tables 3–8 that the current approach maximizes the lower approximation and minimizes the upper approximation; hence, heightening the value of accuracy. This gives an advantage for our approach in terms of improving the approximation operators and increasing the accuracy measure of a subset compared to the approach studied in [40] for all cases of ρ under any binary relation. The cells given in bold in Tables 6–8 validate this fact.

Remark 3.11. It was proved in Proposition 6 of [40] that the topological approaches and their counterparts given in [16] are identical for $\rho \in \{r, l, i\langle r\rangle, \langle l\rangle, \langle i\rangle\}$. So, according to the previous results and example presented herein, we infer that the current approach is also better than the approach given in [16] for $\rho \in \{r, l, i\langle r\rangle, \langle l\rangle, \langle i\rangle\}$.

Remark 3.12. The current approach and the approach introduced in [37] are independent of each other. This matter can be confirmed by the computations given in Tables 6–8, and computing their counterparts induced by the system of containment neighborhoods.

In the next two results, we point out that the current approach is better than the approach discussed in [41] under a reflexive relation.

Proposition 3.13. *Let V be a subset of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ such that $\mathcal{R}$ is reflexive. Then, $\underline{E}^{\mathcal{K}}_{\rho}(V) \subseteq \underline{S}^{\mathcal{K}}_{\rho}(V)$ and $\overline{S}^{\mathcal{K}}_{\rho}(V) \subseteq \overline{E}^{\mathcal{K}}_{\rho}(V)$ for each ρ.*

Proof. Let $v \in \underline{E}^{\mathcal{K}}_{\rho}(V)$. Then, there is an open subset $W \subseteq V$ in $\Omega^{\mathcal{K}}_{E\rho}$ containing v. This implies that $E_\rho(w) - W \in \mathcal{K}$ for each $w \in W$. By reflexivity of $\mathcal{R}$, it follows from (i) of Lemma 2.10 that $S_\rho(w) \subseteq E_\rho(w)$ for each ρ. This means that W is also an open subset in $\Omega^{\mathcal{K}}_{S\rho}$. Thus, $v \in \underline{S}^{\mathcal{K}}_{\rho}(V)$. Hence, we obtain the desired result. Following a similar technique, we prove that $\overline{S}^{\mathcal{K}}_{\rho}(V) \subseteq \overline{E}^{\mathcal{K}}_{\rho}(V)$. □

Corollary 3.14. *Let V be a subset of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ such that $\mathcal{R}$ is reflexive. Then, $\lambda^{\mathcal{K}}_{E\rho} \leq \lambda^{\mathcal{K}}_{S\rho}$ for each ρ.*

Following similar arguments, one can prove the next two results, which show the advantages of the current approach compared to the approaches displayed in [35,41] to improve approximation operators and increase the value of accuracy.

Proposition 3.15. *Let V be a subset of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ such that $\mathcal{R}$ is similarity. Then, $\underline{E}^{\mathcal{K}}_{\rho}(V) \subseteq \underline{N}^{\mathcal{K}}_{\rho}(V) \subseteq \underline{S}^{\mathcal{K}}_{\rho}(V)$ and $\overline{S}^{\mathcal{K}}_{\rho}(V) \subseteq \overline{N}^{\mathcal{K}}_{\rho}(V) \subseteq \overline{E}^{\mathcal{K}}_{\rho}(V)$ for each $\rho \in \{r, l, i, u\}$.*

Proof. Let $\rho \in \{r, l, i, u\}$. It follows from (ii) of Lemma 2.10 that $S_\rho(v) \subseteq N_\rho(v) \subseteq E_\rho(v)$. Since $\mathcal{K}$ is closed under subset relation, we obtain $E_\rho(v) - V \in \mathcal{K} \Rightarrow N_\rho(v) - V \in \mathcal{K} \Rightarrow S_\rho(v) - V \in \mathcal{K}$. Therefore, $\Omega^{\mathcal{K}}_{E\rho} \subseteq \Omega^{\mathcal{K}}_{N\rho} \subseteq \Omega^{\mathcal{K}}_{S\rho}$. Hence, the proof is complete. □

Corollary 3.16. *Let V be a subset of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ such that $\mathcal{R}$ is similarity. Then, $\lambda^{\mathcal{K}}_{E\rho} \leq \lambda^{\mathcal{K}}_{N\rho} \leq \lambda^{\mathcal{K}}_{S\rho}$ for each $\rho \in \{r, l, i, u\}$.*

Definition 3.17. A subset V of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ is called $S^{\mathcal{K}}_{\rho}$-exact if $\underline{S}^{\mathcal{K}}_{\rho}(V) = \overline{S}^{\mathcal{K}}_{\rho}(V) = V$. Otherwise, it is called an $S^{\mathcal{K}}_{\rho}$-rough set.

Proposition 3.18. *A subset V of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ is $S^{\mathcal{K}}_{\rho}$-exact iff $\mathcal{B}^{\mathcal{K}}_{\rho}(V) = \phi$.*

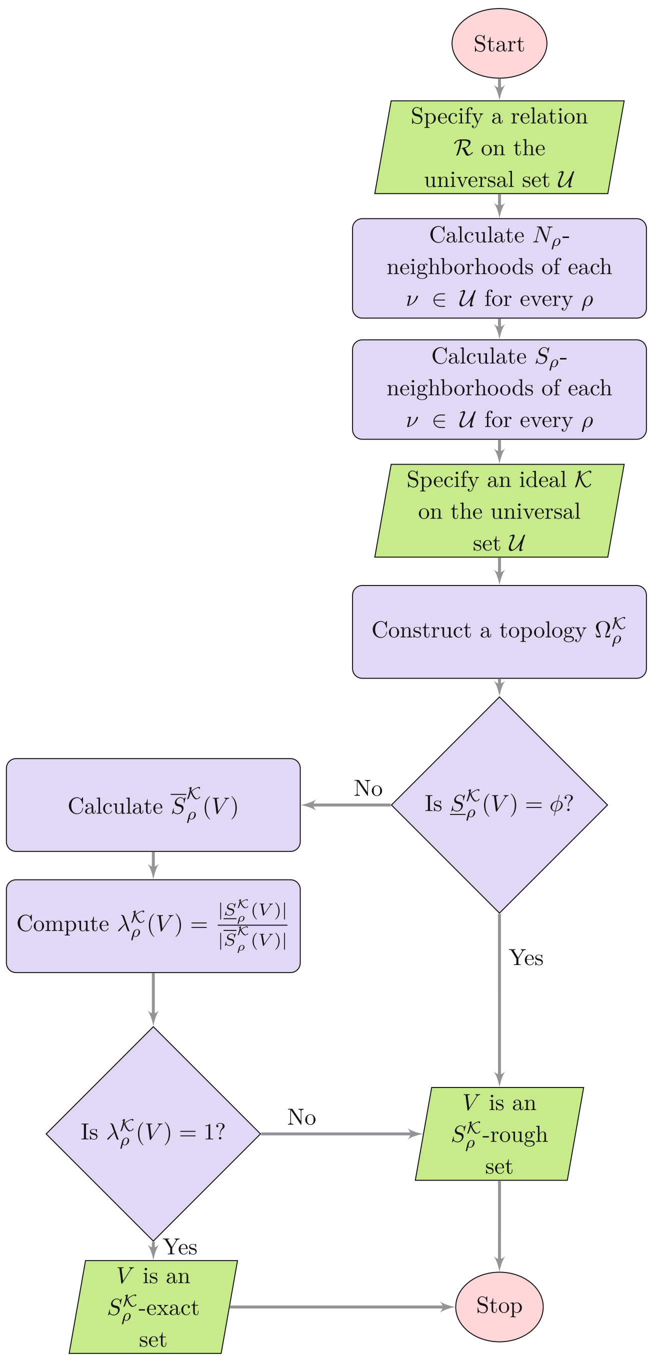

Figure 1: Flow chart of determining $S^{\mathcal{K}}_{\rho}$-exact and $S^{\mathcal{K}}_{\rho}$-rough sets.

Proof. Let V be an $S_\rho^{\mathcal{K}}$-exact set. Then, $\mathcal{B}_\rho^{\mathcal{K}}(V) = \overline{S}_\rho^{\mathcal{K}}(V) \backslash \underline{S}_\rho^{\mathcal{K}}(V) = \overline{S}_\rho^{\mathcal{K}}(V) \backslash \overline{S}_\rho^{\mathcal{K}}(V) = \phi$. Conversely, let $\mathcal{B}_\rho^{\mathcal{K}}(V) = \phi$. Then, $\overline{S}_\rho^{\mathcal{K}}(V) \backslash \underline{S}_\rho^{\mathcal{K}}(V) = \phi$. So, $\overline{S}_\rho^{\mathcal{K}}(V) \subseteq \underline{S}_\rho^{\mathcal{K}}(V)$. But $\underline{S}_\rho^{\mathcal{K}}(V) \subseteq \overline{S}_\rho^{\mathcal{K}}(V)$. Hence, $\underline{S}_\rho^{\mathcal{K}}(V) = \overline{S}_\rho^{\mathcal{K}}(V)$, which means that V is $S_\rho^{\mathcal{K}}$-exact. □

In the end of this section, we provide Algorithm 1 and Figure 1 to illustrate how it can be determined whether a subset of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ is $S_\rho^{\mathcal{K}}$-exact or $S_\rho^{\mathcal{K}}$-rough.

Algorithm 1: The algorithm of determining $S_\rho^{\mathcal{K}}$-exact and $S_\rho^{\mathcal{K}}$-rough sets in an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$.

Output: An IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$.
Output: Classification a set in an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ into two categories: $S_\rho^{\mathcal{K}}$-exact or $S_\rho^{\mathcal{K}}$-rough.

Specify a relation $\mathcal{R}$ over the universal set $\mathcal{U}$;
for *each* ρ **do**
| Calculate N_ρ -neighborhoods of all $v \in \mathcal{U}$;
| Calculate S_ρ-neighborhoods of all $v \in \mathcal{U}$
end
Specify an ideal $\mathcal{K}$ over the universal set $\mathcal{U}$;
for *each* ρ **do**
| Build a topology $\Omega_\rho^{\mathcal{K}}$ using Theorem 3.1
end
for *each nonempty subset* V *of* $\mathcal{U}$
| Calculate its lower approximation $\underline{S}_\rho^{\mathcal{K}}(V)$;
| **if** $\underline{S}_\rho^{\mathcal{K}}(V) = \phi$ **then**
| | **return** V is an $S_\rho^{\mathcal{K}}$ -rough set
| **else**
| | Calculate its upper approximation $\overline{S}_\rho^{\mathcal{K}}(V)$;
| | Compute $\lambda_\rho^{\mathcal{K}}(V) = \frac{|\underline{S}_\rho^{\mathcal{K}}(V)|}{|\overline{S}_\rho^{\mathcal{K}}(V)|}$;
| | **if** $\lambda_\rho^{\mathcal{K}}(V) = 1$ **then**
| | | **return** V is an $S_\rho^{\mathcal{K}}$ -exact set
| | **else**
| | | **return** V is an $S_\rho^{\mathcal{K}}$ -rough set
| | **end**
| **end**
end

4 ASs generated by subset neighborhoods and ideal

In this section, we give another method to produce ASs from subset neighborhoods and ideals in a direct way. We reveal the relationships between them and discuss their essential features. Also, we mention the characterizations of Pawlak approximation operators that are missing via the current approach. Compared to the approach given in the previous section, we demonstrate that the current approach improves the approximation operators for each ρ and produces higher accuracy for the cases u and $\langle u\rangle$. On the other hand, an open question is put for the relationship between the other cases of accuracy measures. Moreover, the current approach are better than approaches displayed in [16] under any arbitrary relation, [41] under a similarity relation and [34] under a reflexive relation for each ρ.

Definition 4.1. Let $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ be an IAS. The lower approximation $\underline{H}^{\mathcal{K}}_{S\rho}$ and upper approximation $\overline{H}^{\mathcal{K}}_{S\rho}$ of a subset V of $\mathcal{U}$ induced from $\mathcal{R}$ and $\mathcal{K}$ with respect to S_ρ-neighborhood are, respectively, defined as follows:

$$\underline{H}^{\mathcal{K}}_{S\rho}(V) = \{v \in \mathcal{U} : S_\rho(v) - V \in \mathcal{K}\}$$
$$\overline{H}^{\mathcal{K}}_{S\rho}(V) = \{v \in \mathcal{U} : S_\rho(v) \cap V \notin \mathcal{K}\}.$$

The accuracy measure induced from the above approximation operators is given by

$$\mathcal{H}^{\mathcal{K}}_{S\rho}(V) = \frac{|\underline{H}^{\mathcal{K}}_{\rho}(V) \cap V|}{|\overline{H}^{\mathcal{K}}_{\rho}(V) \cup V|}$$

The following results present the main properties and characterizations of the approximation operators $\underline{H}^{\mathcal{K}}_{S\rho}$ and $\overline{H}^{\mathcal{K}}_{S\rho}$.

Theorem 4.2. *Let $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ be an IAS and let V and W be subsets of $\mathcal{U}$. Then,*

(1) $\underline{H}^{\mathcal{K}}_{S\rho}(\mathcal{U}) = \mathcal{U}$ *and* $\overline{H}^{\mathcal{K}}_{S\rho}(\phi) = \phi$.

(2) *If* $V \subseteq W$, *then* $\underline{H}^{\mathcal{K}}_{S\rho}(V) \subseteq \underline{H}^{\mathcal{K}}_{S\rho}(W)$ *and* $\overline{H}^{\mathcal{K}}_{S\rho}(V) \subseteq \overline{H}^{\mathcal{K}}_{S\rho}(W)$.

(3) $\underline{H}^{\mathcal{K}}_{S\rho}(V \cap W) = \underline{H}^{\mathcal{K}}_{S\rho}(V) \cap \underline{H}^{\mathcal{K}}_{S\rho}(W)$ *and* $\overline{H}^{\mathcal{K}}_{S\rho}(V \cup W) = \overline{H}^{\mathcal{K}}_{S\rho}(V) \cup \overline{H}^{\mathcal{K}}_{S\rho}(W)$ *for each* $\rho \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$.

(4) $\underline{H}^{\mathcal{K}}_{S\rho}(V^c) = [\overline{H}^{\mathcal{K}}_{S\rho}(V)]^c$ *and* $\overline{H}^{\mathcal{K}}_{S\rho}(V^c) = [\underline{H}^{\mathcal{K}}_{S\rho}(V)]^c$.

Proof. We suffice by proving the theorem in case of lower approximation $\underline{H}^{\mathcal{K}}_{S\rho}$, and the results of upper approximation $\overline{H}^{\mathcal{K}}_{S\rho}$ can be proved in the same way.

(1) Since $S_\rho(v) - \mathcal{U} = \phi \in \mathcal{K}$, we obtain $v \in \underline{H}^{\mathcal{K}}_{S\rho}(\mathcal{U})$ for each $v \in \mathcal{U}$. So, $\underline{H}^{\mathcal{K}}_{S\rho}(\mathcal{U}) = \mathcal{U}$.

(2) Let $V \subseteq W$ and $v \in \underline{H}^{\mathcal{K}}_{S\rho}(V)$. Then, $S_\rho(v) - V \in \mathcal{K}$. Since $S_\rho(v) - W \subseteq S_\rho(v) - V$, it follows from condition (ii) of $\mathcal{K}$ that $S_\rho(v) - W \in \mathcal{K}$ as well. Hence, $v \in \underline{H}^{\mathcal{K}}_{S\rho}(W)$, as required.

(3) From (2) above, we obtain $\underline{H}^{\mathcal{K}}_{S\rho}(V \cap W) \subseteq \underline{H}^{\mathcal{K}}_{S\rho}(V) \cap \underline{H}^{\mathcal{K}}_{S\rho}(W)$. Conversely, let $v \in \underline{H}^{\mathcal{K}}_{S\rho}(V) \cap \underline{H}^{\mathcal{K}}_{S\rho}(W)$. Then, $v \in \underline{H}^{\mathcal{K}}_{S\rho}(V)$ and $v \in \underline{H}^{\mathcal{K}}_{S\rho}(N)$. This means that $S_\rho(v) - V \in \mathcal{K}$ and $S_\rho(v) - W \in \mathcal{K}$. It follows from the condition of $\mathcal{K}$ that $[S_\rho(v) - V] \cup [S_\rho(v) - W] = S_\rho(v) - (V \cap W) \in \mathcal{K}$. Thus, $v \in \underline{H}^{\mathcal{K}}_{S\rho}(V \cap W)$. Hence, we obtain the required equality.

(4) $v \in \underline{H}^{\mathcal{K}}_{S\rho}(V^c) \Leftrightarrow S_\rho(v) - V^c \in \mathcal{K} \Leftrightarrow S_\rho(v) \cap V \in \mathcal{K} \Leftrightarrow v \notin \overline{H}^{\mathcal{K}}_{S\rho}(V) \Leftrightarrow v \in (\overline{H}^{\mathcal{K}}_{S\rho}(V))^c$. □

Corollary 4.3. *Let $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ be an IAS and let V and W be subsets of $\mathcal{U}$. Then,* $\underline{H}^{\mathcal{K}}_{S\rho}(V) \cup \underline{H}^{\mathcal{K}}_{S\rho}(W) \subseteq \underline{H}^{\mathcal{K}}_{S\rho}(V \cup W)$ *and* $\overline{H}^{\mathcal{K}}_{S\rho}(V \cap W) \subseteq \overline{H}^{\mathcal{K}}_{S\rho}(V) \cap \overline{H}^{\mathcal{K}}_{S\rho}(W)$.

Proof. Obviously, $V \subseteq V \cup W$ and $W \subseteq V \cup W$, so it follows from (2) of Theorem 4.2 that $\underline{H}^{\mathcal{K}}_{S\rho}(V) \subseteq \underline{H}^{\mathcal{K}}_{S\rho}(V \cup W)$ and $\underline{H}^{\mathcal{K}}_{S\rho}(W) \subseteq \underline{H}^{\mathcal{K}}_{S\rho}(V \cup W)$. Hence, $\underline{H}^{\mathcal{K}}_{S\rho}(V) \cup \underline{H}^{\mathcal{K}}_{S\rho}(W) \subseteq \underline{H}^{\mathcal{K}}_{S\rho}(V \cup W)$. Similarly, it can be proved that $\overline{H}^{\mathcal{K}}_{S\rho}(V \cap W) \subseteq \overline{H}^{\mathcal{K}}_{S\rho}(V) \cap \overline{H}^{\mathcal{K}}_{S\rho}(W)$. □

The next properties of approximation operators in Pawlak model are missing in the current model.

(1) $\underline{H}^{\mathcal{K}}_{S\rho}(V) \subseteq V \subseteq \overline{H}^{\mathcal{K}}_{S\rho}(V)$.

(2) $\underline{H}^{\mathcal{K}}_{S\rho}(\phi) = \phi$ and $\overline{H}^{\mathcal{K}}_{S\rho}(\mathcal{U}) = \mathcal{U}$.

(3) $\underline{H}^{\mathcal{K}}_{S\rho}[\underline{H}^{\mathcal{K}}_{S\rho}(V)] = \underline{H}^{\mathcal{K}}_{S\rho}(V)$ and $\overline{H}^{\mathcal{K}}_{S\rho}[\overline{H}^{\mathcal{K}}_{S\rho}(V)] = \overline{H}^{\mathcal{K}}_{S\rho}(V)$.

(4) $\underline{H}^{\mathcal{K}}_{S\rho}(V \cap W) = \underline{H}^{\mathcal{K}}_{S\rho}(V) \cap \underline{H}^{\mathcal{K}}_{S\rho}(W)$ and $\overline{H}^{\mathcal{K}}_{S\rho}(V \cup W) = \overline{H}^{\mathcal{K}}_{S\rho}(V) \cup \overline{H}^{\mathcal{K}}_{S\rho}(W)$ for $\rho \in \{u, \langle u\rangle\}$.

Theorem 4.4. *Let $(\mathcal{U}, \mathcal{R}, \mathcal{J})$ and $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ be IASs and let V be a subset of $\mathcal{U}$. Then,*

(1) *If $V \in \mathcal{J}$, then $\underline{H}^{\mathcal{J}}_{S\rho}(V^c) = \mathcal{U}$ and $\overline{H}^{\mathcal{J}}_{S\rho}(V) = \phi$.*
(2) *If $\mathcal{J} \subseteq \mathcal{K}$, then $\underline{H}^{\mathcal{J}}_{S\rho}(V) \subseteq \underline{H}^{\mathcal{K}}_{S\rho}(V)$ and $\overline{H}^{\mathcal{K}}_{S\rho}(V) \subseteq \overline{H}^{\mathcal{J}}_{S\rho}(V)$.*
(3) *If $\mathcal{K} = P(\mathcal{U})$, then $\underline{H}^{\mathcal{K}}_{S\rho}(V) = \mathcal{U}$ and $\overline{H}^{\mathcal{K}}_{S\rho}(V) = \phi$.*
(4) *$\underline{H}^{\mathcal{J}\cap\mathcal{K}}_{S\rho}(V) = \underline{H}^{\mathcal{J}}_{S\rho}(V) \cap \underline{H}^{\mathcal{K}}_{S\rho}(V)$ and $\overline{H}^{\mathcal{J}\cap\mathcal{K}}_{S\rho}(V) = \overline{H}^{\mathcal{J}}_{S\rho}(V) \cup \overline{H}^{\mathcal{K}}_{S\rho}(V)$.*
(5) *$\underline{H}^{\mathcal{J}\vee\mathcal{K}}_{S\rho}(V) = \underline{H}^{\mathcal{J}}_{S\rho}(V) \cup \underline{H}^{\mathcal{K}}_{S\rho}(V)$ and $\overline{H}^{\mathcal{J}\vee\mathcal{K}}_{S\rho}(V) = \overline{H}^{\mathcal{J}}_{S\rho}(V) \cup \overline{H}^{\mathcal{K}}_{S\rho}(V)$.*

Proof. We suffice by proving the theorem in case of upper approximation $\overline{H}^{\mathcal{K}}_{S\rho}$, and the results of lower approximation $\underline{H}^{\mathcal{K}}_{S\rho}$ can be proved in the same way.

(1) Since $V \in \mathcal{J}$, we find $S_\rho(v) \cap V \in \mathcal{J}$ for each $v \in \mathcal{U}$. Hence, $v \notin \overline{H}^{\mathcal{J}}_{S\rho}(V)$ for each $v \in \mathcal{U}$; i.e., $\overline{H}^{\mathcal{J}}_{S\rho}(V) = \phi$.
(2) Let $v \in \overline{H}^{\mathcal{K}}_{S\rho}(V)$. Then, $S_\rho(v) \cap V \notin \mathcal{K}$. By assumption $\mathcal{J} \subseteq \mathcal{K}$, we obtain $S_\rho(v) \cap V \notin \mathcal{J}$. Hence, $\overline{H}^{\mathcal{K}}_{S\rho}(V) \subseteq \overline{H}^{\mathcal{J}}_{S\rho}(V)$.
(3) It is obvious.
(4)

$$\begin{aligned}\overline{H}^{\mathcal{J}\cap\mathcal{K}}_{S\rho}(V) &= \{v \in \mathcal{U} : S_\rho(v) \cap V \notin \mathcal{J} \cap \mathcal{K}\}.\\ &= \{v \in \mathcal{U} : S_\rho(v) \cap V \notin \mathcal{J}\} \text{ or } \{v \in \mathcal{U} : S_\rho(v) \cap V \notin \mathcal{K}\}.\\ &= \{v \in \mathcal{U} : S_\rho(v) \cap V \notin \mathcal{J}\} \cup \{v \in \mathcal{U} : S_\rho(v) \cap V \notin \mathcal{K}\}.\\ &= \overline{H}^{\mathcal{J}}_{S\rho}(V) \cup \overline{H}^{\mathcal{K}}_{S\rho}(V).\end{aligned}$$

(5)

$$\begin{aligned}\overline{H}^{\mathcal{J}\vee\mathcal{K}}_{S\rho}(V) &= \{v \in \mathcal{U} : S_\rho(v) \cap V \notin \mathcal{J} \vee \mathcal{K}\}.\\ &= \{v \in \mathcal{U} : S_\rho(v) \cap V \notin \mathcal{J} \cup \mathcal{K}\}.\\ &= \{v \in \mathcal{U} : S_\rho(v) \cap V \notin \mathcal{J}\} \text{ and } \{v \in \mathcal{U} : S_\rho(v) \cap V \notin \mathcal{K}\}.\\ &= \{v \in \mathcal{U} : S_\rho(v) \cap V \notin \mathcal{J}\} \cap \{v \in \mathcal{U} : S_\rho(v) \cap V \notin \mathcal{K}\}.\\ &= \overline{H}^{\mathcal{J}}_{S\rho}(V) \cap \overline{H}^{\mathcal{K}}_{S\rho}(V).\end{aligned}$$

□

In the next two results, we elucidate that the current ASs produce the best approximation operators and accuracy measures in cases of i and $\langle i\rangle$.

Proposition 4.5. *Let $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ be an IAS and $V \subseteq \mathcal{U}$. Then,*

(1) $\underline{H}^{\mathcal{K}}_{Su}(V) \subseteq \underline{H}^{\mathcal{K}}_{Sr}(V) \subseteq \underline{H}^{\mathcal{K}}_{Si}(V)$.
(2) $\underline{H}^{\mathcal{K}}_{Su}(V) \subseteq \underline{H}^{\mathcal{K}}_{Sl}(V) \subseteq \underline{H}^{\mathcal{K}}_{Si}(V)$.
(3) $\underline{H}^{\mathcal{K}}_{S\langle u\rangle}(V) \subseteq \underline{H}^{\mathcal{K}}_{S\langle r\rangle}(V) \subseteq \underline{H}^{\mathcal{K}}_{S\langle i\rangle}(V)$.
(4) $\underline{H}^{\mathcal{K}}_{S\langle u\rangle}(V) \subseteq \underline{H}^{\mathcal{K}}_{S\langle l\rangle}(V) \subseteq \underline{H}^{\mathcal{K}}_{S\langle i\rangle}(V)$.
(5) $\overline{H}^{\mathcal{K}}_{Si}(V) \subseteq \overline{H}^{\mathcal{K}}_{Sr}(V) \subseteq \overline{H}^{\mathcal{K}}_{Su}(V)$.
(6) $\overline{H}^{\mathcal{K}}_{Si}(V) \subseteq \overline{H}^{\mathcal{K}}_{Sl}(V) \subseteq \overline{H}^{\mathcal{K}}_{Su}(V)$.
(7) $\overline{H}^{\mathcal{K}}_{S\langle i\rangle}(V) \subseteq \overline{H}^{\mathcal{K}}_{S\langle r\rangle}(V) \subseteq \overline{H}^{\mathcal{K}}_{S\langle u\rangle}(V)$.
(8) $\overline{H}^{\mathcal{K}}_{S\langle i\rangle}(V) \subseteq \overline{H}^{\mathcal{K}}_{S\langle l\rangle}(V) \subseteq \overline{H}^{\mathcal{K}}_{S\langle u\rangle}(V)$.

Proof. We suffice by proving (i), and the other cases can be proved following a similar technique. Let $v \in \underline{H}^{\mathcal{K}}_{Su}(V)$. Then, $S_u(v) - V \in \mathcal{K}$. Since $S_r(v) \subseteq S_u(v)$, it follows from the condition of $\mathcal{K}$ that $S_r(v) - V \in \mathcal{K}$. Therefore, $v \in \underline{H}^{\mathcal{K}}_{Sr}(V)$. Thus, $\underline{H}^{\mathcal{K}}_{Su}(V) \subseteq \underline{H}^{\mathcal{K}}_{Sr}(V)$. In a similar way, we obtain $\underline{H}^{\mathcal{K}}_{Sr}(V) \subseteq \underline{H}^{\mathcal{K}}_{Si}(V)$. □

Corollary 4.6. *Let V be a subset of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$. Then,*

(1) $\mathcal{H}^{\mathcal{K}}_{Su}(V) \le \mathcal{H}^{\mathcal{K}}_{Sr}(V) \le \mathcal{H}^{\mathcal{K}}_{Si}(V)$.

(2) $\mathcal{H}^{\mathcal{K}}_{Su}(V) \le \mathcal{H}^{\mathcal{K}}_{Sl}(V) \le \mathcal{H}^{\mathcal{K}}_{Si}(V)$.

(3) $\mathcal{H}^{\mathcal{K}}_{S\langle u\rangle}(V) \le \mathcal{H}^{\mathcal{K}}_{S\langle r\rangle}(V) \le \mathcal{H}^{\mathcal{K}}_{S\langle i\rangle}(V)$.

(4) $\mathcal{H}^{\mathcal{K}}_{S\langle u\rangle}(V) \le \mathcal{H}^{\mathcal{K}}_{S\langle l\rangle}(V) \le \mathcal{H}^{\mathcal{K}}_{S\langle i\rangle}(V)$.

Proof. We suffice by proving (i), and the other cases can be proved following a similar technique. To do this, note that $\underline{H}^{\mathcal{K}}_{Su}(V) \subseteq \underline{H}^{\mathcal{K}}_{Sr}(V) \subseteq \underline{H}^{\mathcal{K}}_{Si}(V)$. This automatically leads to the next equality.

$$|\underline{H}^{\mathcal{K}}_{Su}(V)| \le |\underline{H}^{\mathcal{K}}_{Sr}(V)| \le |\underline{H}^{\mathcal{K}}_{Si}(V)| \tag{7}$$

In addition, note that $\overline{H}^{\mathcal{K}}_{Si}(V) \subseteq \overline{H}^{\mathcal{K}}_{Sr}(V) \subseteq \overline{H}^{\mathcal{K}}_{Su}(V)$, which automatically leads to the next equality.

$$\frac{1}{|\overline{H}^{\mathcal{K}}_{Su}(V) \cup V|} \le \frac{1}{|\overline{H}^{\mathcal{K}}_{Sr}(V) \cup V|} \le \frac{1}{|\overline{H}^{\mathcal{K}}_{Si}(V) \cup V|} \tag{8}$$

By (7) and (8), we obtain

$$\frac{|\underline{H}^{\mathcal{K}}_{Su}(V) \cap V|}{|\overline{H}^{\mathcal{K}}_{Su}(V) \cup V|} \le \frac{|\underline{H}^{\mathcal{K}}_{Sr}(V) \cap V|}{|\overline{H}^{\mathcal{K}}_{Sr}(V) \cup V|} \le \frac{|\underline{H}^{\mathcal{K}}_{Si}(V) \cap V|}{|\overline{H}^{\mathcal{K}}_{Si}(V) \cup V|},$$

which ends the proof. □

The next results elaborates that the approach presented in this section is better than the approach given in the previous section in terms of approximation operators.

Theorem 4.7. *Let X be a subset of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$. Then, $\underline{S}^{\mathcal{K}}_{\rho}(X) \subseteq \underline{H}^{\mathcal{K}}_{S\rho}(X)$ and $\overline{H}^{\mathcal{K}}_{S\rho}(X) \subseteq \overline{S}^{\mathcal{K}}_{\rho}(X)$ for eby the computations givenach ρ.*

Proof. First, let $v \in \underline{S}^{\mathcal{K}}_{\rho}(X)$. Then $v \in \mathrm{int}^{\mathcal{K}}_{S\rho}(X)$. Therefore there exists an open subset V such that $v \in V \subseteq X$. According to the definition of $\Omega^{\mathcal{K}}_{S\rho}$ in Theorem 3.1, we obtain $S_{\rho}(v) - V \in \mathcal{K}$. Thus, $v \in \underline{H}^{\mathcal{K}}_{S\rho}(X)$. Hence, $\underline{S}^{\mathcal{K}}_{\rho}(X) \subseteq \underline{H}^{\mathcal{K}}_{S\rho}(X)$. Second, let $v \notin \overline{S}^{\mathcal{K}}_{\rho}(X)$. Then, $v \notin \mathrm{cl}^{\mathcal{K}}_{S\rho}(X)$. So there exists a closed set W containing X such that $v \notin W$. Now, $W^c \cap X = \phi$ where W^c is an open set containing v. This implies that $S_r(v) - W^c \in \mathcal{K}$, so $S_{\rho}(v) \cap W \in \mathcal{K}$. Thus, $v \notin \overline{H}^{\mathcal{K}}_{S\rho}(X)$. Hence, $\overline{H}^{\mathcal{K}}_{S\rho}(X) \subseteq \overline{S}^{\mathcal{K}}_{\rho}(X)$. □

To show that the converse of Theorem 4.7 fails, we consider an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ given in Example 3.10 and compute the $\underline{H}^{\mathcal{K}}_{S\rho}$-lower approximations and $\overline{H}^{\mathcal{K}}_{S\rho}$-upper approximations of every subset in Tables 9 and 10, respectively.

Table 9: $\underline{H}_{S\rho}$-lower approximations

Subset	$\underline{H}_{Sr}$	$\underline{H}_{Sl}$	$\underline{H}_{Si}$	$\underline{H}_{Su}$	$\underline{H}_{S\langle r\rangle}$	$\underline{H}_{S\langle l\rangle}$	$\underline{H}_{S\langle i\rangle}$	$\underline{H}_{S\langle u\rangle}$
$\{\delta_1\}$	$\{\delta_1\}$	ϕ	$\{\delta_1\}$	ϕ	$\{\delta_1\}$	ϕ	$\{\delta_1\}$	ϕ
$\{\delta_2\}$	**$\{\delta_1, \delta_2\}$**	ϕ	**$\{\delta_1, \delta_2\}$**	ϕ	**$\{\delta_1\}$**	**$\{\delta_1, \delta_2\}$**	**$\{\delta_1, \delta_2\}$**	**$\{\delta_1\}$**
$\{\delta_3\}$	**$\{\delta_1\}$**	**$\{\delta_1, \delta_3\}$**	**$\{\delta_1, \delta_3\}$**	**$\{\delta_1\}$**	**$\{\delta_1, \delta_3\}$**	ϕ	**$\{\delta_1, \delta_3\}$**	ϕ
$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	ϕ	$\{\delta_1, \delta_2\}$	ϕ	$\{\delta_1\}$	$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	**$\{\delta_1\}$**
$\{\delta_1, \delta_3\}$	$\{\delta_1\}$	$\{\delta_1, \delta_3\}$	$\{\delta_1, \delta_3\}$	**$\{\delta_1\}$**	$\{\delta_1, \delta_3\}$	ϕ	$\{\delta_1, \delta_3\}$	ϕ
$\{\delta_2, \delta_3\}$	**U**	**U**	**U**	**U**	**U**	**U**	**U**	**U**

Table 10: $\overline{H}_{S\rho}$-upper approximations

Subset	$\overline{H}_{Sr}$	$\overline{H}_{Sl}$	$\overline{H}_{Si}$	$\overline{H}_{Su}$	$\overline{H}_{S\langle r\rangle}$	$\overline{H}_{S\langle l\rangle}$	$\overline{H}_{S\langle i\rangle}$	$\overline{H}_{S\langle u\rangle}$
$\{\delta_1\}$	$\boldsymbol{\phi}$	$\boldsymbol{\phi}$	$\boldsymbol{\phi}$	$\boldsymbol{\phi}$	$\boldsymbol{\phi}$	$\boldsymbol{\phi}$	$\boldsymbol{\phi}$	$\boldsymbol{\phi}$
$\{\delta_2\}$	$\{\delta_2, \delta_3\}$	$\{\delta_2\}$	$\{\delta_2\}$	$\{\boldsymbol{\delta_2}, \boldsymbol{\delta_3}\}$	$\{\delta_2\}$	$\mathcal{U}$	$\{\delta_2\}$	$\mathcal{U}$
$\{\delta_3\}$	$\{\delta_3\}$	$\mathcal{U}$	$\{\delta_3\}$	$\mathcal{U}$	$\{\delta_2, \delta_3\}$	$\{\delta_3\}$	$\{\delta_3\}$	$\{\boldsymbol{\delta_2}, \boldsymbol{\delta_3}\}$
$\{\delta_1, \delta_2\}$	$\{\boldsymbol{\delta_2}, \boldsymbol{\delta_3}\}$	$\{\boldsymbol{\delta_2}\}$	$\{\boldsymbol{\delta_2}\}$	$\{\boldsymbol{\delta_2}, \boldsymbol{\delta_3}\}$	$\{\boldsymbol{\delta_2}\}$	$\mathcal{U}$	$\{\boldsymbol{\delta_2}\}$	$\mathcal{U}$
$\{\delta_1, \delta_3\}$	$\{\boldsymbol{\delta_3}\}$	$\mathcal{U}$	$\{\boldsymbol{\delta_3}\}$	$\mathcal{U}$	$\{\boldsymbol{\delta_2}, \boldsymbol{\delta_3}\}$	$\{\boldsymbol{\delta_3}\}$	$\{\boldsymbol{\delta_3}\}$	$\{\boldsymbol{\delta_2}, \boldsymbol{\delta_3}\}$
$\{\delta_2, \delta_3\}$	$\{\delta_2, \delta_3\}$	$\mathcal{U}$	$\{\delta_2, \delta_3\}$	$\mathcal{U}$	$\{\delta_2, \delta_3\}$	$\mathcal{U}$	$\{\delta_2, \delta_3\}$	$\mathcal{U}$

It can be seen from the bold cells given in Tables 9 and 10 and their counterparts given in Tables 6 and 7 that the performance of the current approach to increase the lower approximation and decrease the upper approximation is better than the approach given in the previous section for each ρ.

Corollary 4.8. *Let X be a subset of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$. Then, $\lambda^{\mathcal{K}}_{S\rho}(X) \leq \mathcal{H}^{\mathcal{K}}_{S\rho}(X)$ for each ρ.*

Proof. Since $\underline{S}^{\mathcal{K}}_{\rho}(X) \subseteq X$ and $X \subseteq \overline{S}^{\mathcal{K}}_{\rho}(X)$, it follows from Theorem 4.7 that $\underline{S}^{\mathcal{K}}_{\rho}(X) \subseteq \underline{H}^{\mathcal{K}}_{S\rho}(X) \cap X$ and $\overline{H}^{\mathcal{K}}_{S\rho}(X) \cup X \subseteq \overline{S}^{\mathcal{K}}_{\rho}(X)$ for each ρ. This automatically leads to that $\frac{|\underline{S}^{\mathcal{K}}_{\rho}(X)|}{|\overline{S}^{\mathcal{K}}_{\rho}(X)|} \leq \frac{|\underline{H}^{\mathcal{K}}_{S\rho}(X) \cap X|}{|\overline{H}^{\mathcal{K}}_{S\rho}(X) \cup X|}$. This ends the proof that $\lambda^{\mathcal{K}}_{S\rho}(X) \leq \mathcal{H}^{\mathcal{K}}_{S\rho}(X)$ for each ρ. □

Now, the direct question put itself is: Is the converse of Corollary 4.8 hold true? In fact, we partially answer this question by showing that the converse generally fails for the two cases u and $\langle u\rangle$. To confirm this matter, take two subsets $\{\delta_1, \delta_3\}$ and $\{\delta_1, \delta_2\}$ of $\mathcal{U}$. It is clear that $\mathcal{H}^{\mathcal{K}}_{Su}(\{\delta_1, \delta_3\}) = \mathcal{H}^{\mathcal{K}}_{S\langle u\rangle}(\{\delta_1, \delta_2\}) = \frac{1}{3}$, whereas $\lambda^{\mathcal{K}}_{Su}(\{\delta_1, \delta_3\}) = \lambda^{\mathcal{K}}_{S\langle u\rangle}(\{\delta_1, \delta_2\}) = 0$. The other cases are still an open question.

Question 4.9. Is the converse of Corollary 4.8 hold true for $\rho \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$?

Remark 4.10. The current approach and the approach introduced in [37] are independent of each other. This matter can be confirmed by the computations given in Tables 9 and 10 and computing their counterparts induced by the system of containment neighborhoods.

To confirm the good performance of the current approach, we show that the current approach improves the ASs more than their counterparts given in [41] under a reflexive relation and [34] under a similarity relation.

Proposition 4.11. *Let V be a subset of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ such that $\mathcal{R}$ is reflexive. Then,*
(i) $\underline{H}^{\mathcal{K}}_{E\rho}(V) \subseteq \underline{H}^{\mathcal{K}}_{S\rho}(V)$ for each ρ.
(ii) $\overline{H}^{\mathcal{K}}_{S\rho}(V) \subseteq \overline{H}^{\mathcal{K}}_{E\rho}(V)$ for each ρ.

Proof. (i): Let $v \in \underline{H}^{\mathcal{K}}_{E\rho}(V)$. Then $E_\rho(v) - V \in \mathcal{K}$. By reflexivity of $\mathcal{R}$, we obtain $S_\rho(v) \subseteq E_\rho(v)$. Therefore, $S_\rho(v) - V \in \mathcal{K}$ as well. Thus, $v \in \underline{H}^{\mathcal{K}}_{S\rho}(V)$. Hence, we obtain the desired result.

One can prove (ii) similarly. □

Corollary 4.12. *Let V be a subset of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ such that $\mathcal{R}$ is reflexive. Then, $\mathcal{H}^{\mathcal{K}}_{E\rho} \leq \mathcal{H}^{\mathcal{K}}_{S\rho}$ for each ρ.*

Following similar arguments given in Proposition 4.11, one can prove the following result.

Proposition 4.13. *Let V be a subset of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ such that $\mathcal{R}$ is similarity. Then,*
(i) $\underline{H}_{E\rho}(V) \subseteq \underline{H}_{N\rho}(V) \subseteq \underline{H}^{\mathcal{K}}_{S\rho}(V)$ *for each* ρ.
(ii) $\overline{H}_{S\rho}(V) \subseteq \overline{H}_{N\rho}(V) \subseteq \overline{H}^{\mathcal{K}}_{E\rho}(V)$ *for each* ρ.

Corollary 4.14. *Let V be a subset of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ such that $\mathcal{R}$ is similarity. Then $\mathcal{H}^{\mathcal{K}}_{E\rho} \leq \mathcal{H}^{\mathcal{K}}_{N\rho} \leq \mathcal{H}^{\mathcal{K}}_{S\rho}$ for each ρ.*

Definition 4.15. A subset V of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ is called $H^{\mathcal{K}}_{S\rho}$-exact if $\underline{H}^{\mathcal{K}}_{S\rho}(V) = \overline{H}^{\mathcal{K}}_{S\rho}(V) = V$. Otherwise, it is called an $H^{\mathcal{K}}_{S\rho}$-rough set.

Proposition 4.16. *If V is $H^{\mathcal{K}}_{S\rho}$-exact, then $\mathcal{B}^{\mathcal{K}}_{H\rho}(V) = \phi$.*

Proof. Straightforward. □

In the end of this section, we provide Algorithm 2 to show how we can determine whether a subset of an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ is $H^{\mathcal{K}}_{S\rho}$-exact or $H^{\mathcal{K}}_{S\rho}$-rough.

Algorithm 2: The algorithm of determining $H^{\mathcal{K}}_{S\rho}$-exact and $H^{\mathcal{K}}_{S\rho}$-rough sets in an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$.

Input: An IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$.
Output: Classification a set in an IAS $(\mathcal{U}, \mathcal{R}, \mathcal{K})$ into two categories: $H^{\mathcal{K}}_{S\rho}$-exact or $H^{\mathcal{K}}_{S\rho}$-rough.
Specify a relation $\mathcal{R}$ over the universal set $\mathcal{U}$;
for *each* ρ **do**
  | Calculate N_ρ-neighborhoods of all $v \in \mathcal{U}$;
  | Calculate S_ρ-neighborhoods of all $v \in \mathcal{U}$
end
Specify an ideal $\mathcal{K}$ over the universal set $\mathcal{U}$;
for *each nonempty subset V of $\mathcal{U}$* **do**
  | Calculate its lower approximation $\underline{H}^{\mathcal{K}}_{S\rho}(V)$;
  | Calculate its upper approximation $\overline{H}^{\mathcal{K}}_{S\rho}(V)$;
  | **if** $\underline{H}^{\mathcal{K}}_{S\rho}(V) = \overline{H}^{\mathcal{K}}_{S\rho}(V) = V$ **then**
  | | **return** V is an $H^{\mathcal{K}}_{S\rho}$-exact set
  | **else**
  | | V is an $H^{\mathcal{K}}_{S\rho}$-rough set
  | **end**
end

---

5 Medical example: Dengue fever

In this section, we examine the performance of the methods given herein and some previous ones via the information system of dengue fever. This disease, according to World Health Organization [45], is a viral infection transmitted to humans through the bite of infected mosquitoes; it is a leading cause of serious illness and death in more than 120 countries around the world, mainly, in Latin American and Asian countries. There is no specific treatment for dengue; however, early detection of disease progression associated with severe dengue and access to proper medical care reduce fatality rates of severe dengue to below 1%. Now, we exploited the proposed approaches to analyze the data of some patients given in Table 11. To illustrate the data of this table, the symptoms of dengue fever (attributes) [46] are presented in the columns as follows: A_1 is a temperature, A_2 is muscle and joint pains, A_3 is a headache with vomiting, and A_4 is a characteristic skin rash. The last column D is the decision of disease with two values "infected" or "uninfected." The set of patients under consideration $\mathcal{U} = \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ is put in rows. The attribute A_1 takes the values very high (vh), high (h), and normal (n), whereas all the other attributes take two values: "Yes" and "No" which, respectively, denoted the possession of a symptom or not by patients.

To be able to handle the variables descriptions of attributes of Table 11, we compute their quantity values that demonstrate the similarities degrees between the patients' symptoms in Table 12. It is well known that the similarity degree $s(\delta_i, \delta_j)$ between two patients δ_i, δ_j is given as follows:

$$s(\delta_i, \delta_j) = \frac{\sum_{k=1}^{n}(A_k(\delta_i) = A_k(\delta_j))}{n}, \tag{9}$$

where n denotes the number of conditions attributes.

Now, we ask the experts in charge of the system to propose a relation that connects between the patients according to their symptoms. Let us consider that they provide the following relation: $\delta_i \mathcal{R} \delta_p \Leftrightarrow 0.5 \leq s(\delta_i, \delta_j) \leq 1$, where $s(\delta_i, \delta_j)$ is calculated by equation (9). It is should be noted that the suggested relations $\leq$ and the values 0.5, 1 can be changed depending on the estimation of experts of system. To construct the neighborhood systems applied to establish the ASs, we, first, note that $\mathcal{R}$ is a symmetry relation, so there will exist two types of N_ρ-neighborhoods and S_ρ-neighborhoods. In addition, it is a reflexive relation. On the other hand, $\mathcal{R}$ is not a transitive relation because $(\delta_2, \delta_1) \notin \mathcal{R}$ in spite of $(\delta_2, \delta_3) \in \mathcal{R}$ and $(\delta_3, \delta_1) \in \mathcal{R}$.

Sake for the brevity, the upcoming computations will do in the case of r (Table 13).

Now, we consider the ideal is $\mathcal{I} = \{\phi, \{\delta_5\}\}$.

For a set of patients with infection from dengue fever $A = \{\delta_1, \delta_3, \delta_4\}$ and a set of patients without infection from dengue fever $B = \{\delta_2, \delta_5\}$, we calculate their lower and upper approximations, boundary regions, and the accuracy measures induced from approaches displayed in [34,41] and our approach given in the previous section.

(i) For patients with infection from dengue fever, $A = \{\delta_1, \delta_3, \delta_4\}$
 - Hosny et al.'s approach [41]: The lower and upper approximations are $\underline{H}_{E\rho}^{\mathcal{K}}(A) = \phi$ and $\overline{H}_{E\rho}^{\mathcal{K}}(A) = \mathcal{U}$, respectively. Therefore, the boundary region is $B_{E\rho}^{\mathcal{K}}(A) = \mathcal{U}$ and the accuracy measure is $\mathcal{H}_{E\rho}^{\mathcal{K}}(A) = 0$.
 - Kandil et al.'s approach [34]: The lower and upper approximations are $\underline{H}_{N\rho}^{\mathcal{K}}(A) = \{\delta_1, \delta_5\}$ and $\overline{H}_{N\rho}^{\mathcal{K}}(A) = \mathcal{U}$, respectively. Therefore, the boundary region is $B_{N\rho}^{\mathcal{K}}(A) = \{\delta_2, \delta_3, \delta_4\}$ and the accuracy measure is $\mathcal{H}_{N\rho}^{\mathcal{K}}(A) = \frac{1}{5}$.
 - Our approach: The lower and upper approximations are $\underline{H}_{S\rho}^{\mathcal{K}}(A) = \{\delta_1, \delta_3, \delta_4, \delta_5\}$ and $\overline{H}_{S\rho}^{\mathcal{K}}(A) = \{\delta_1, \delta_3, \delta_4\}$, respectively. Therefore, the boundary region is $B_{S\rho}^{\mathcal{K}}(A) = \phi$ and the accuracy measure is $\mathcal{H}_{S\rho}^{\mathcal{K}}(A) = 1$.

Table 11: information system of dengue fever

U	A_1	A_2	A_3	A_4	Dengue fever
δ_1	h	Yes	Yes	Yes	Infected
δ_2	h	No	No	No	Uninfected
δ_3	h	Yes	No	No	Infected
δ_4	vh	No	Yes	No	Infected
δ_5	n	Yes	Yes	No	Uninfected

Table 12: The similarity degrees among symptoms of five patients

	δ_1	δ_2	δ_3	δ_4	δ_5
δ_1	1	0.25	0.5	0.25	0.5
δ_2	0.25	1	0.75	0.5	0.25
δ_3	0.5	0.75	1	0.25	0.5
δ_4	0.25	0.5	0.25	1	0.5
δ_5	0.5	0.25	0.5	0.5	1

Table 13: N_r-neighborhoods, E_r-neighborhoods, and S_r-neighborhoods for all patients

	N_r	E_r	S_r
δ_1	$\{\delta_1, \delta_3, \delta_5\}$	$\mathcal{U}$	$\{\delta_1, \delta_3, \delta_5\}$
δ_2	$\{\delta_2, \delta_3, \delta_4\}$	$\mathcal{U}$	$\{\delta_2\}$
δ_3	$\{\delta_1, \delta_2, \delta_3, \delta_5\}$	$\mathcal{U}$	$\{\delta_3\}$
δ_4	$\{\delta_2, \delta_4, \delta_5\}$	$\mathcal{U}$	$\{\delta_4\}$
δ_5	$\{\delta_1, \delta_3, \delta_4, \delta_5\}$	$\mathcal{U}$	$\{\delta_5\}$

(ii) For patients without infection from dengue fever, $A = \{\delta_2, \delta_5\}$

- Hosny et al.'s approach [41]: The lower and upper approximations are $\underline{H}^{\mathcal{K}}_{E\rho}(A) = \phi$ and $\overline{H}^{\mathcal{K}}_{E\rho}(A) = \mathcal{U}$, respectively. Therefore, the boundary region is $B^{\mathcal{K}}_{E\rho}(A) = \mathcal{U}$ and the accuracy measure is $\mathcal{H}^{\mathcal{K}}_{E\rho}(A) = 0$.
- Kandil et al.'s approach [34]: The lower and upper approximations are $\underline{H}^{\mathcal{K}}_{N\rho}(A) = \phi$ and $\overline{H}^{\mathcal{K}}_{N\rho}(A) = \{\delta_2, \delta_3, \delta_4\}$, respectively. Therefore, the boundary region is $B^{\mathcal{K}}_{N\rho}(A) = \{\delta_2, \delta_3, \delta_4\}$ and the accuracy measure is $\mathcal{H}^{\mathcal{K}}_{N\rho}(A) = 0$.
- Our approach: The lower and upper approximations are $\underline{H}^{\mathcal{K}}_{S\rho}(A) = \{\delta_2, \delta_5\}$ and $\overline{H}^{\mathcal{K}}_{S\rho}(A) = \{\delta_2, \delta_3, \delta_4\}$, respectively. Therefore, the boundary region is $B^{\mathcal{K}}_{S\rho}(A) = \{\delta_3, \delta_4\}$ and the accuracy measure is $\mathcal{H}^{\mathcal{K}}_{S\rho}(A) = \frac{1}{2}$.

From the above computations, it is obtained that the boundary regions of a subset of patients without infection from dengue fever and a subset of patients with infection from dengue fever inspired by the approach given in [34,41] are $\mathcal{U}$ and $\{\delta_2, \delta_3, \delta_4\}$, respectively. In this case, we are unable to decide whether these individuals are infected from dengue fever or not, which enlarges the area of uncertainty/vagueness and affects the precision of made decision. Whereas the boundary regions of these two subsets inspired by our approach are the empty set and $\{\delta_3, \delta_4\}$, which means that for a subset without infection, we completely cancel the uncertainty in the data and minimize the vagueness in the data for a subset with infection. This automatically leads to increasing the accuracy measure and enhancing the confidence of the made decision.

6 Conclusion

Rough set was introduced to deal with intelligent systems characterized by insufficient and incomplete information. It has been proposed several ways to develop and extend this theory; one of them follows from the abstract ideas of "neighborhoods and ideals."

Through this work, we have applied the concepts of subset neighborhoods and ideals to introduce two versions of ASs. The first version was inspired by topologies, whereas the second one was directly generated by subset neighborhoods and ideals. The main properties and features of these ASs have been investigated, and the relationships between them have been illustrated with the help of illustrative examples. Some comparisons between the current approaches and previous ones introduced in [16,17,35,40,41] have been conducted under different types of binary relations aiming to show the advantages of the current approaches to maximize the accuracy measure of a subset by increasing lower approximation and decreasing upper approximation.

To confirm that there is a need for investigation considering the various sorts of neighborhoods systems so that these findings may contribute to remove uncertainty of real data, we examine and analyze the performance of the current methods and some foregoing ones via the information system of dengue fever. The obtained results concluded that the approaches proposed herein were more general and accurate.

In upcoming studies, we will adopt another type of neighborhoods to obtain rid of vagueness in data. Also, we will research the current models depending on a finite family of arbitrary relations instead of one relation. Moreover, we discuss the presented approaches in the content of soft rough set and fuzzy rough set.

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