

Research Article

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Investigation of hybrid fractional q -integro-difference equations supplemented with nonlocal q -integral boundary conditions

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Abstract: In this article, we introduce and study a new class of hybrid fractional q -integro-difference equations involving Riemann-Liouville q -derivatives, supplemented with nonlocal boundary conditions containing Riemann-Liouville q -integrals of different orders. The existence of a unique solution to the given problem is shown by applying Banach's fixed point theorem. We also present the existing criteria for solutions to the problem at hand by applying Krasnoselskii's fixed point theorem and Leray-Schauder's nonlinear alternative. Illustrative examples are given to demonstrate the application of the obtained results. Some new results follow as special cases of this work.

Keywords: q -difference equation, Riemann-Liouville q -derivative, q -integral, nonlocal boundary conditions, existence, fixed point

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1 Introduction

In this article, we explore the existence criteria for solutions of a new boundary value problem consisting of a nonlinear hybrid fractional q -integro-difference equation involving Riemann-Liouville q -derivatives and nonlocal q -integral boundary conditions. In precise terms, we study the following nonlocal hybrid q -fractional integral boundary value problem:

$$\nu D_q^\alpha[u(x) - f(x, u(x))] + (1 - \nu) D_q^\beta u(x) = ag(x, u(x)) + bI_q^\varepsilon h(x, u(x)), \quad 0 < x < 1, \quad (1)$$

$$u(0) = 0, \quad u(1) = \xi \int_0^\eta \frac{(\eta - qs)^{(\gamma_1-1)}}{\Gamma_q(\gamma_1)} u(s) d_qs + (1 - \xi) \int_0^\sigma \frac{(\sigma - qs)^{(\gamma_2-1)}}{\Gamma_q(\gamma_2)} u(s) d_qs, \quad \gamma_1, \gamma_2 > 0, \quad (2)$$

where $0 < q < 1$, $1 < \alpha, \beta \leq 2$, $0 < \varepsilon < 1$, $0 < \nu \leq 1$, $0 \leq \xi \leq 1$, $\alpha - \beta > 0$, $0 < \eta, \sigma < 1$, D_q^α , and D_q^β denote the Riemann-Liouville fractional q -derivatives of order α and β , respectively, I_q^ε denotes the Riemann-Liouville fractional q -integral of order ε , and $f, g, h : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, $a, b \in \mathbb{R}$.

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Let us now dwell on some recent works on fractional q -difference equations. In 2013, Zhou and Liu [1] applied Mönch's fixed point theorem together with the technique of measure of weak noncompactness to investigate the existence of solutions for the following fractional q -difference equation with boundary conditions:

$$\begin{cases} {}^c D_q^\alpha u(t) + f(t, u(t)) = 0, & 0 \leq t \leq 1, \quad 0 < q < 1, \\ u(0) = (D_q^2 u)(0) = 0, & \gamma(D_q u)(1) + \beta(D_q^2 u)(1) = 0, \end{cases}$$

where $2 < \alpha \leq 3$, $\gamma, \beta \geq 0$, and $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

In [2], the authors studied the following nonlinear boundary value problem of fractional q -integro-difference equation:

$$\begin{aligned} (\lambda D_q^\alpha + (1 - \lambda) D_q^\beta) u(t) &= a f(t, u(t)) + b I_q^\delta g(t, u(t)), \quad t \in [0, 1], \quad a, b \in \mathbb{R}^+, \\ u(0) &= 0, \quad \mu \int_0^1 \frac{(1 - qs)^{(\gamma_1-1)}}{\Gamma_q(\gamma_1)} u(s) d_qs + (1 - \mu) \int_0^1 \frac{(1 - qs)^{(\gamma_2-1)}}{\Gamma_q(\gamma_2)} u(s) d_qs = 0, \quad \gamma_1, \gamma_2 > 0, \end{aligned}$$

where $0 < q < 1$, $1 < \alpha, \beta < 2$, $0 < \delta < 1$, $0 < \lambda \leq 1$, $0 \leq \mu \leq 1$, $\alpha - \beta > 1$, and D_q^α denotes the Riemann-Liouville fractional q -derivative of order α , and $f, g: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions. For some more results on boundary value problems involving fractional q -difference operators, we refer the reader to previous studies [3–11].

Recently, in [12], a coupled system of nonlinear fractional q -integro-difference equations equipped with coupled q -integral boundary conditions was studied. In [13], the authors proved some existence results for a Langevin-type q -variant system of nonlinear fractional integro-difference equations with nonlocal boundary conditions. On the other hand, for some recent works on fractional differential equations, for example, see [14–19].

The objective of this present work is to investigate the criteria ensuring the existence and uniqueness of solutions to problems (1) and (2). We make use of the Banach fixed point theorem [20] to obtain a uniqueness result, while two existence results are established by means of Krasnoselskii's fixed point theorem [21] and Leray-Schauder's nonlinear alternative [22]. Moreover, our results generalize the ones presented in [2] in the sense that we consider a hybrid Riemann-Liouville-type fractional q -integro-difference equation subject to nonlocal Riemann-Liouville q -integral boundary conditions. The second condition in (2) can be interpreted as the value of the unknown function $u(x)$ at $x = 1$ is proportional to the sum of its two-strip contributions formulated in terms of Riemann-Liouville integral on the segments $(0, \eta)$ and $(0, \sigma)$.

We arrange the remainder of this article as follows. In Section 2, we recall some basic concepts related to this study. Section 3 contains the main results, while examples illustrating the obtained results are presented in Section 4.

2 Auxiliary material

Let us first recall some necessary concepts and definitions about q -fractional calculus and fixed point theory.

For every $a \in \mathbb{R}$, the q -number $[a]_q$ is defined by $[a]_q = \frac{1 - q^a}{1 - q}$, where $q \in (0, 1)$ is an arbitrary real number. Also, the q -shifted factorial of real number a is defined by $(a; q)_0 = 1$ and $(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j)$ for $n \in \mathbb{N} \cup \{\infty\}$. For $a, b \in \mathbb{R}$, the q -analogue of the power function $(a - b)^n$ with $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ is given by:

$$(a - b)^{(0)} = 1, \quad (a - b)^{(n)} = \prod_{j=0}^{n-1} (a - bq^j).$$

In general, if ϱ is a real number, then $(a - b)^{(\varrho)} = a^{(\varrho)} \prod_{j=0}^{\infty} \frac{a - bq^j}{a - bq^{\varrho+j}}$ and $a^{(\varrho)} = a^{\varrho}$ when $b = 0$. If $\varrho > 0$ and $0 \leq a \leq b \leq t$, then $(t - b)^{(\varrho)} \leq (t - a)^{(\varrho)}$. The q -Gamma function $\Gamma_q(\varrho)$ is defined as:

$$\Gamma_q(\varrho) = \frac{(1 - q)^{(\varrho-1)}}{(1 - q)^{\varrho-1}}, \quad \varrho \in \mathbb{R} \setminus \{0, -1, -2, \dots\},$$

which satisfies the relation $\Gamma_q(\varrho + 1) = [\varrho]_q \Gamma_q(\varrho)$ [23].

Definition 2.1. [23] Let $\varrho \geq 0$ and $u : (0, \infty) \rightarrow \mathbb{R}$ be a continuous function. The Riemann-Liouville fractional q -integral for the function u of order ϱ is defined by $(I_q^0 u)(t) = u(t)$ and

$$(I_q^{\varrho} u)(t) = \frac{1}{\Gamma_q(\varrho)} \int_0^t (t - qs)^{(\varrho-1)} u(s) d_qs = t^{\varrho} (1 - q)^{\varrho} \sum_{k=0}^{\infty} q^k \frac{(q^{\varrho}; q)_k}{(q; q)_k} u(tq^k), \quad \varrho > 0, \quad t \in (0, \infty).$$

Definition 2.2. [24] The fractional q -derivative of the Riemann-Liouville type of order $\varrho \geq 0$ is defined by $(D_q^0 u)(t) = u(t)$ and

$$(D_q^{\varrho} u)(t) = (D_q^m I_q^{m-\varrho} u)(t), \quad \varrho > 0,$$

where m is the smallest integer greater than or equal to ϱ , $I_q^{(\cdot)}$ is the Riemann-Liouville fractional q -integral of order $(\cdot)$, and D_q^m is the q -derivative of integer order m .

Alternatively, the Riemann-Liouville fractional q -derivative of order $\varrho > 0$ for a function $u : (0, \infty) \rightarrow \mathbb{R}$ is defined by [23]:

$$D_q^{\varrho} u(t) = \frac{1}{\Gamma_q(n - \varrho)} \int_0^t \frac{u(s)}{(t - qs)^{\varrho-n+1}} d_qs, \quad n - 1 < \varrho < n.$$

Recall that $I_q^{\beta} I_q^{\varrho} u(t) = I_q^{\beta+\varrho} u(t)$ for $\varrho, \beta \in \mathbb{R}^+$ [23]. Further, according to Lemma 2.8 in [13],

$$I_q^{\varrho}((x - a)^{(\sigma)}) = \frac{\Gamma_q(\sigma + 1)}{\Gamma_q(\varrho + \sigma + 1)} (x - a)^{(\varrho+\sigma)}, \quad 0 < a < x < b, \quad \varrho \in \mathbb{R}^+, \quad \sigma \in (-1, \infty).$$

In particular, for $\sigma = 0$ and $a = 0$, using q -integration by parts, we have

$$\begin{aligned} (I_q^{\varrho} 1)(x) &= \frac{1}{\Gamma_q(\varrho)} \int_0^x (x - qt)^{(\varrho-1)} d_q t \\ &= \frac{1}{\Gamma_q(\varrho)} \int_0^x \frac{D_q((x - t)^{(\varrho)})}{-[\varrho]_q} d_q t \\ &= \frac{-1}{\Gamma_q(\varrho + 1)} \int_0^x D_q((x - t)^{(\varrho)}) d_q t \\ &= \frac{1}{\Gamma_q(\varrho + 1)} x^{(\varrho)}. \end{aligned}$$

3 Main results

We begin this section with an auxiliary lemma that characterizes the structure of solutions for problems (1) and (2).

Lemma 3.1. Let $y \in C([0, 1], \mathbb{R})$ and

$$\Delta := 1 - \frac{\xi \Gamma_q(\alpha) \eta^{\alpha+\gamma_1-1}}{\Gamma_q(\alpha + \gamma_1)} - \frac{(1 - \xi) \Gamma_q(\alpha) \sigma^{\alpha+\gamma_2-1}}{\Gamma_q(\alpha + \gamma_2)} \neq 0. \quad (3)$$

The function u is a solution for the fractional q -difference boundary value problem

$$\begin{cases} \nu D_q^\alpha [u(x) - f(x, u(x))] + (1 - \nu) D_q^\beta u(x) = y(x), & 0 < x < 1, \\ u(0) = 0, \quad u(1) = \xi \int_0^\eta \frac{(\eta - qs)^{(\gamma_1-1)}}{\Gamma_q(\gamma_1)} u(s) d_qs + (1 - \xi) \int_0^\sigma \frac{(\sigma - qs)^{(\gamma_2-1)}}{\Gamma_q(\gamma_2)} u(s) d_qs, \end{cases} \quad (4)$$

if and only if u is a solution for the fractional q -integral equation

$$\begin{aligned} u(x) = & f(x, u(x)) + \frac{(\nu - 1)}{\nu \Gamma_q(\alpha - \beta)} \int_0^x (x - qs)^{(\alpha-\beta-1)} u(s) d_qs + \frac{1}{\nu \Gamma_q(\alpha)} \int_0^x (x - qs)^{(\alpha-1)} y(s) d_qs \\ & + \frac{x^{\alpha-1}}{\Delta} \left[\frac{\xi}{\Gamma_q(\gamma_1)} \int_0^\eta (\eta - qs)^{(\gamma_1-1)} f(s, u(s)) d_qs + \frac{\xi(\nu - 1)}{\nu \Gamma_q(\alpha - \beta + \gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha-\beta+\gamma_1-1)} u(s) d_qs \right. \\ & + \frac{\xi}{\nu \Gamma_q(\alpha + \gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha+\gamma_1-1)} y(s) d_qs + \frac{(1 - \xi)}{\Gamma_q(\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\gamma_2-1)} f(s, u(s)) d_qs \\ & + \frac{(1 - \xi)(\nu - 1)}{\nu \Gamma_q(\alpha - \beta + \gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha-\beta+\gamma_2-1)} u(s) d_qs + \frac{(1 - \xi)}{\nu \Gamma_q(\alpha + \gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha+\gamma_2-1)} y(s) d_qs \\ & \left. - \frac{(\nu - 1)}{\nu \Gamma_q(\alpha - \beta)} \int_0^1 (1 - qs)^{(\alpha-\beta-1)} u(s) d_qs - \frac{1}{\nu \Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha-1)} y(s) d_qs - f(1, u(1)) \right]. \end{aligned} \quad (5)$$

Proof. Let u be a solution of the q -fractional boundary value problem (4). Then, we have

$$D_q^\alpha [u(x) - f(x, u(x))] = \left(\frac{\nu - 1}{\nu} \right) D_q^\beta u(x) + \frac{1}{\nu} y(x).$$

Taking the Riemann-Liouville fractional q -integral of order α to both sides of the above equation, we obtain

$$u(x) - f(x, u(x)) = \left(\frac{\nu - 1}{\nu} \right) I_q^\alpha D_q^\beta u(x) + \frac{1}{\nu} I_q^\alpha y(x) + c_1 x^{\alpha-1} + c_2 x^{\alpha-2},$$

where c_1 and $c_2 \in \mathbb{R}$ are the arbitrary constants. Since $1 < \alpha < 2$, it follows from the first boundary condition that $c_2 = 0$. Thus,

$$u(x) = f(x, u(x)) + \left(\frac{\nu - 1}{\nu} \right) I_q^{\alpha-\beta} u(x) + \frac{1}{\nu} I_q^\alpha y(x) + c_1 x^{\alpha-1}. \quad (6)$$

On the other hand, if $\Theta \in \{\gamma_1, \gamma_2\}$, then we have

$$\begin{aligned} I_q^\Theta u(x) = & \frac{1}{\Gamma_q(\Theta)} \int_0^x (x - qs)^{(\Theta-1)} f(s, u(s)) d_qs + \frac{\nu - 1}{\nu \Gamma_q(\alpha - \beta + \Theta)} \int_0^x (x - qs)^{(\alpha-\beta+\Theta-1)} u(s) d_qs \\ & + \frac{1}{\nu \Gamma_q(\alpha + \Theta)} \int_0^x (x - qs)^{(\alpha+\Theta-1)} y(s) d_qs + c_1 \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha + \Theta)} x^{\alpha+\Theta-1}. \end{aligned}$$

Now, by using the second boundary condition and substituting the values $\Theta \in \{\gamma_1, \gamma_2\}$ into the aforementioned expression, we find that

$$\begin{aligned} c_1 = \frac{1}{\Delta} & \left[\frac{\xi}{\Gamma_q(\gamma_1)} \int_0^\eta (\eta - qs)^{(\gamma_1-1)} f(s, u(s)) d_qs + \frac{\xi(v-1)}{v\Gamma_q(\alpha-\beta+\gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha-\beta+\gamma_1-1)} u(s) d_qs \right. \\ & + \frac{\xi}{v\Gamma_q(\alpha+\gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha+\gamma_1-1)} y(s) d_qs + \frac{(1-\xi)}{\Gamma_q(\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\gamma_2-1)} f(s, u(s)) d_qs \\ & + \frac{(1-\xi)(v-1)}{v\Gamma_q(\alpha-\beta+\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha-\beta+\gamma_2-1)} u(s) d_qs + \frac{(1-\xi)}{v\Gamma_q(\alpha+\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha+\gamma_2-1)} y(s) d_qs \\ & \left. - \frac{(v-1)}{v\Gamma_q(\alpha-\beta)} \int_0^1 (1 - qs)^{(\alpha-\beta-1)} u(s) d_qs - \frac{1}{v\Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha-1)} y(s) d_qs - f(1, u(1)) \right], \end{aligned}$$

where Δ is defined in (3). Substituting the value of c_1 in (6), we obtain solution (5). Conversely, it is clear that u is a solution for the fractional q -difference equation (4) whenever u is a solution for the fractional q -integral equation (5). This completes the proof. $\square$

In relation to problems (1) and (2), we introduce an operator $\mathcal{T} : E \rightarrow E$ by

$$\begin{aligned} (\mathcal{T}u)(x) = & f(x, u(x)) + \frac{(v-1)}{v\Gamma_q(\alpha-\beta)} \int_0^x (x - qs)^{(\alpha-\beta-1)} u(s) d_qs \\ & + \frac{a}{v\Gamma_q(\alpha)} \int_0^x (x - qs)^{(\alpha-1)} g(s, u(s)) d_qs + \frac{b}{v\Gamma_q(\alpha+\varepsilon)} \int_0^x (x - qs)^{(\alpha+\varepsilon-1)} h(s, u(s)) d_qs \\ & + \frac{x^{\alpha-1}}{\Delta} \left[\frac{\xi}{\Gamma_q(\gamma_1)} \int_0^\eta (\eta - qs)^{(\gamma_1-1)} f(s, u(s)) d_qs + \frac{\xi(v-1)}{v\Gamma_q(\alpha-\beta+\gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha-\beta+\gamma_1-1)} u(s) d_qs \right. \\ & + \frac{a\xi}{v\Gamma_q(\alpha+\gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha+\gamma_1-1)} g(s, u(s)) d_qs + \frac{b\xi}{v\Gamma_q(\alpha+\varepsilon+\gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha+\varepsilon+\gamma_1-1)} h(s, u(s)) d_qs \\ & + \frac{(1-\xi)}{\Gamma_q(\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\gamma_2-1)} f(s, u(s)) d_qs + \frac{(1-\xi)(v-1)}{v\Gamma_q(\alpha-\beta+\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha-\beta+\gamma_2-1)} u(s) d_qs \\ & + \frac{a(1-\xi)}{v\Gamma_q(\alpha+\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha+\gamma_2-1)} g(s, u(s)) d_qs + \frac{b(1-\xi)}{v\Gamma_q(\alpha+\varepsilon+\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha+\varepsilon+\gamma_2-1)} h(s, u(s)) d_qs \\ & - \frac{(v-1)}{v\Gamma_q(\alpha-\beta)} \int_0^1 (1 - qs)^{(\alpha-\beta-1)} u(s) d_qs - \frac{a}{v\Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha-1)} g(s, u(s)) d_qs \\ & \left. - \frac{b}{v\Gamma_q(\alpha+\varepsilon)} \int_0^1 (1 - qs)^{(\alpha+\varepsilon-1)} h(s, u(s)) d_qs - f(1, u(1)) \right], \end{aligned} \quad (7)$$

where $u \in E$ and $x \in [0, 1]$. Here, $E = C([0, 1], \mathbb{R})$ is the Banach space of all continuous real-valued functions defined on $[0, 1]$ equipped with the norm $\|u\| = \sup_{x \in [0, 1]} |u(x)|$, $u \in E$. Here, one can note that the fixed points of the operator $\mathcal{T}$ are solutions to problems (1) and (2).

In the sequel, we set

$$\begin{aligned}
\Omega_1 &= 1 + \frac{\xi\eta^{\gamma_1}}{|\Delta|\Gamma_q(\gamma_1 + 1)} + \frac{(1 - \xi)\sigma^{\gamma_2}}{|\Delta|\Gamma_q(\gamma_2 + 1)} + \frac{1}{|\Delta|}, \\
\Omega_2 &= \frac{|a|}{\nu} \left[\frac{1}{\Gamma_q(\alpha + 1)} + \frac{\xi\eta^{\alpha+\gamma_1}}{|\Delta|\Gamma_q(\alpha + \gamma_1 + 1)} + \frac{(1 - \xi)\sigma^{\alpha+\gamma_2}}{|\Delta|\Gamma_q(\alpha + \gamma_2 + 1)} + \frac{1}{|\Delta|\Gamma_q(\alpha + 1)} \right], \\
\Omega_3 &= \frac{|b|}{\nu} \left[\frac{1}{\Gamma_q(\alpha + \varepsilon + 1)} + \frac{\xi\eta^{\alpha+\varepsilon+\gamma_1}}{|\Delta|\Gamma_q(\alpha + \varepsilon + \gamma_1 + 1)} + \frac{(1 - \xi)\sigma^{\alpha+\varepsilon+\gamma_2}}{|\Delta|\Gamma_q(\alpha + \varepsilon + \gamma_2 + 1)} + \frac{1}{|\Delta|\Gamma_q(\alpha + \varepsilon + 1)} \right], \\
\Omega_4 &= \frac{|\nu - 1|}{\nu} \left[\frac{1}{\Gamma_q(\alpha - \beta + 1)} + \frac{\xi\eta^{\alpha-\beta+\gamma_1}}{|\Delta|\Gamma_q(\alpha - \beta + \gamma_1 + 1)} + \frac{(1 - \xi)\sigma^{\alpha-\beta+\gamma_2}}{|\Delta|\Gamma_q(\alpha - \beta + \gamma_2 + 1)} + \frac{1}{|\Delta|\Gamma_q(\alpha - \beta + 1)} \right].
\end{aligned} \tag{8}$$

In our first result, we prove the uniqueness of solutions to problems (1) and (2) with the aid of the Banach contraction mapping principle [20].

Theorem 3.2. Let $f, g, h \in ([0, 1] \times \mathbb{R}, \mathbb{R})$ satisfy the following conditions:

(H1) there exists a positive constant L_1 such that, for each pair of elements $u, v \in \mathbb{R}$,

$$|f(x, u) - f(x, v)| \leq L_1|u - v|, \quad x \in [0, 1];$$

(H2) there exists a positive constant L_2 such that, for each pair of elements $u, v \in \mathbb{R}$,

$$|g(x, u) - g(x, v)| \leq L_2|u - v|, \quad x \in [0, 1];$$

(H3) there exists a positive constant L_3 such that, for each pair of elements $u, v \in \mathbb{R}$,

$$|h(x, u) - h(x, v)| \leq L_3|u - v|, \quad x \in [0, 1].$$

Then, the fractional hybrid q -difference equation (1) supplemented with q -integral nonlocal boundary conditions (2) has a unique solution on $[0, 1]$, provided that

$$\Omega := L_1\Omega_1 + L_2\Omega_2 + L_3\Omega_3 + \Omega_4 < 1, \tag{9}$$

where $\Omega_1, \Omega_2, \Omega_3$, and Ω_4 are defined by (8).

Proof. Let us verify that the operator $\mathcal{T} : E \rightarrow E$ defined by (7) satisfies the hypothesis of the Banach contraction mapping principle [20]. Setting $\sup_{x \in [0, 1]} |f(x, 0)| = K_1 < +\infty$, $\sup_{x \in [0, 1]} |g(x, 0)| = K_2 < +\infty$, and $\sup_{x \in [0, 1]} |h(x, 0)| = K_3 < +\infty$ and fixing $r \geq (K_1\Omega_1 + K_2\Omega_2 + K_3\Omega_3)/(1 - \Omega)$, we show that $\mathcal{T}B_r \subset B_r$, where $B_r = \{u \in E : \|u\| \leq r\}$. For any $u \in B_r, x \in [0, 1]$, it follows by assumptions (H_1) , (H_2) , and (H_3) that

$$\begin{aligned}
|f(x, u(x))| &\leq |f(x, u(x)) - f(x, 0)| + |f(x, 0)| \leq L_1r + K_1, \\
|g(x, u(x))| &\leq |g(x, u(x)) - g(x, 0)| + |g(x, 0)| \leq L_2r + K_2, \\
|h(x, u(x))| &\leq |h(x, u(x)) - h(x, 0)| + |h(x, 0)| \leq L_3r + K_3.
\end{aligned}$$

Then, for any $u \in B_r, x \in [0, 1]$, we have

$$\begin{aligned}
\|\mathcal{T}u\| &= \sup_{x \in [0, 1]} \left\{ |f(x, u(x))| + \frac{|\nu - 1|}{\nu\Gamma_q(\alpha - \beta)} \int_0^x (x - qs)^{(\alpha-\beta-1)} |u(s)| d_qs \right. \\
&\quad + \frac{|a|}{\nu\Gamma_q(\alpha)} \int_0^x (x - qs)^{(\alpha-1)} |g(s, u(s))| d_qs + \frac{|b|}{\nu\Gamma_q(\alpha + \varepsilon)} \int_0^x (x - qs)^{(\alpha+\varepsilon-1)} |h(s, u(s))| d_qs \\
&\quad + \frac{x^{(\alpha-1)}}{|\Delta|} \left[\frac{\xi}{\Gamma_q(\gamma_1)} \int_0^\eta (\eta - qs)^{(\gamma_1-1)} |f(s, u(s))| d_qs + \frac{\xi|\nu - 1|}{\nu\Gamma_q(\alpha - \beta + \gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha-\beta+\gamma_1-1)} |u(s)| d_qs \right. \\
&\quad \left. \left. + \frac{|a|\xi}{\nu\Gamma_q(\alpha + \gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha+\gamma_1-1)} |g(s, u(s))| d_qs + \frac{|b|\xi}{\nu\Gamma_q(\alpha + \varepsilon + \gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha+\varepsilon+\gamma_1-1)} |h(s, u(s))| d_qs \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{(1-\xi)}{\Gamma_q(\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\gamma_2-1)} |f(s, u(s))| d_qs + \frac{(1-\xi)|v-1|}{v\Gamma_q(\alpha-\beta+\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha-\beta+\gamma_2-1)} |u(s)| d_qs \\
& + \frac{|a|(1-\xi)}{v\Gamma_q(\alpha+\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha+\gamma_2-1)} |g(s, u(s))| d_qs \\
& + \frac{|b|(1-\xi)}{v\Gamma_q(\alpha+\varepsilon+\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha+\varepsilon+\gamma_2-1)} |h(s, u(s))| d_qs \\
& + \frac{|v-1|}{v\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} |u(s)| d_qs + \frac{|a|}{v\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} |g(s, u(s))| d_qs \\
& + \frac{|b|}{v\Gamma_q(\alpha+\varepsilon)} \int_0^1 (1-qs)^{(\alpha+\varepsilon-1)} |h(s, u(s))| d_qs + |f(1, u(1))| \Bigg\} \\
\leq & (L_1 r + K_1) \sup_{x \in [0,1]} \left\{ 1 + \frac{x^{(\alpha-1)}}{|\Delta|} \left[\frac{\xi}{\Gamma_q(\gamma_1)} \int_0^\eta (\eta - qs)^{(\gamma_1-1)} d_qs + \frac{(1-\xi)}{\Gamma_q(\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\gamma_2-1)} d_qs + 1 \right] \right. \\
& + (L_2 r + K_2) \sup_{x \in [0,1]} \left\{ \frac{|a|}{v\Gamma_q(\alpha)} \int_0^x (x - qs)^{(\alpha-1)} d_qs + \frac{x^{(\alpha-1)}}{|\Delta|} \left[\frac{|a|\xi}{v\Gamma_q(\alpha+\gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha+\gamma_1-1)} d_qs \right. \right. \\
& + \left. \frac{|a|(1-\xi)}{v\Gamma_q(\alpha+\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha+\gamma_2-1)} d_qs + \frac{|a|}{v\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} d_qs \right] \Bigg\} \\
& + (L_3 r + K_3) \sup_{x \in [0,1]} \left\{ \frac{|b|}{v\Gamma_q(\alpha+\varepsilon)} \int_0^x (x - qs)^{(\alpha+\varepsilon-1)} d_qs + \frac{x^{(\alpha-1)}}{|\Delta|} \left[\frac{|b|\xi}{v\Gamma_q(\alpha+\varepsilon+\gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha+\varepsilon+\gamma_1-1)} d_qs \right. \right. \\
& + \left. \frac{|b|(1-\xi)}{v\Gamma_q(\alpha+\varepsilon+\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha+\varepsilon+\gamma_2-1)} d_qs + \frac{|b|}{v\Gamma_q(\alpha+\varepsilon)} \int_0^1 (1-qs)^{(\alpha+\varepsilon-1)} d_qs \right] \Bigg\} \\
& + r \sup_{x \in [0,1]} \left\{ \frac{|v-1|}{v\Gamma_q(\alpha-\beta)} \int_0^x (x - qs)^{(\alpha-\beta-1)} d_qs + \frac{x^{(\alpha-1)}}{|\Delta|} \left[\frac{\xi|v-1|}{v\Gamma_q(\alpha-\beta+\gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha-\beta+\gamma_1-1)} d_qs \right. \right. \\
& + \left. \frac{(1-\xi)|v-1|}{v\Gamma_q(\alpha-\beta+\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha-\beta+\gamma_2-1)} d_qs + \frac{|v-1|}{v\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} d_qs \right] \Bigg\} \\
\leq & (L_1 r + K_1) \left[1 + \frac{\xi \eta^{\gamma_1}}{|\Delta| \Gamma_q(\gamma_1+1)} + \frac{(1-\xi) \sigma^{\gamma_2}}{|\Delta| \Gamma_q(\gamma_2+1)} + \frac{1}{|\Delta|} \right] \\
& + (L_2 r + K_2) \left[\frac{|a|}{v} \left(\frac{1}{\Gamma_q(\alpha+1)} + \frac{\xi \eta^{(\alpha+\gamma_1)}}{|\Delta| \Gamma_q(\alpha+\gamma_1+1)} + \frac{(1-\xi) \sigma^{(\alpha+\gamma_2)}}{|\Delta| \Gamma_q(\alpha+\gamma_2+1)} + \frac{1}{|\Delta| \Gamma_q(\alpha+1)} \right) \right] \\
& + (L_3 r + K_3) \left[\frac{|b|}{v} \left(\frac{1}{\Gamma_q(\alpha+\varepsilon+1)} + \frac{\xi \eta^{(\alpha+\varepsilon+\gamma_1)}}{|\Delta| \Gamma_q(\alpha+\varepsilon+\gamma_1+1)} + \frac{(1-\xi) \sigma^{(\alpha+\varepsilon+\gamma_2)}}{|\Delta| \Gamma_q(\alpha+\varepsilon+\gamma_2+1)} + \frac{1}{|\Delta| \Gamma_q(\alpha+\varepsilon+1)} \right) \right] \\
& + r \left[\frac{|v-1|}{v} \left(\frac{1}{\Gamma_q(\alpha-\beta+1)} + \frac{\xi \eta^{(\alpha-\beta+\gamma_1)}}{|\Delta| \Gamma_q(\alpha-\beta+\gamma_1+1)} + \frac{(1-\xi) \sigma^{(\alpha-\beta+\gamma_2)}}{|\Delta| \Gamma_q(\alpha-\beta+\gamma_2+1)} + \frac{1}{|\Delta| \Gamma_q(\alpha-\beta+1)} \right) \right],
\end{aligned}$$

which, on using (9) and definition of r , implies that $\|\mathcal{T}u\| \leq r$. Thus, $\mathcal{T}B_r \subset B_r$ as $u \in B_r$ is an arbitrary element. For any $x \in [0, 1]$ and a pair of elements $u, v \in \mathbb{R}$, we obtain

$$\begin{aligned}
\|\mathcal{T}u - \mathcal{T}v\| \leq & \sup_{x \in [0,1]} \left\{ |f(x, u(x)) - f(x, v(x))| + \frac{|v-1|}{v\Gamma_q(\alpha-\beta)} \int_0^x (x-qs)^{(\alpha-\beta-1)} |u(s) - v(s)| d_qs \right. \\
& + \frac{|a|}{v\Gamma_q(\alpha)} \int_0^x (x-qs)^{(\alpha-1)} |g(s, u(s)) - g(s, v(s))| d_qs \\
& + \frac{|b|}{v\Gamma_q(\alpha+\varepsilon)} \int_0^x (x-qs)^{(\alpha+\varepsilon-1)} |h(s, u(s)) - h(s, v(s))| d_qs \\
& + \frac{x^{\alpha-1}}{|\Delta|} \left[\frac{\xi}{\Gamma_q(\gamma_1)} \int_0^\eta (\eta-qs)^{(\gamma_1-1)} |f(s, u(s)) - f(s, v(s))| d_qs \right. \\
& + \frac{\xi|v-1|}{v\Gamma_q(\alpha-\beta+\gamma_1)} \int_0^\eta (\eta-qs)^{(\alpha-\beta+\gamma_1-1)} |u(s) - v(s)| d_qs \\
& + \frac{|a|\xi}{v\Gamma_q(\alpha+\gamma_1)} \int_0^\eta (\eta-qs)^{(\alpha+\gamma_1-1)} |g(s, u(s)) - g(s, v(s))| d_qs \\
& + \frac{|b|\xi}{v\Gamma_q(\alpha+\varepsilon+\gamma_1)} \int_0^\eta (\eta-qs)^{(\alpha+\varepsilon+\gamma_1-1)} |h(s, u(s)) - h(s, v(s))| d_qs \\
& + \frac{(1-\xi)}{\Gamma_q(\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\gamma_2-1)} |f(s, u(s)) - f(s, v(s))| d_qs \\
& + \frac{(1-\xi)|v-1|}{v\Gamma_q(\alpha-\beta+\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\alpha-\beta+\gamma_2-1)} |u(s) - v(s)| d_qs \\
& + \frac{|a|(1-\xi)}{v\Gamma_q(\alpha+\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\alpha+\gamma_2-1)} |g(s, u(s)) - g(s, v(s))| d_qs \\
& + \frac{|b|(1-\xi)}{v\Gamma_q(\alpha+\varepsilon+\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\alpha+\varepsilon+\gamma_2-1)} |h(s, u(s)) - h(s, v(s))| d_qs \\
& + \frac{|v-1|}{v\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} |u(s) - v(s)| d_qs \\
& + \frac{|a|}{v\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} |g(s, u(s)) - g(s, v(s))| d_qs \\
& + \frac{|b|}{v\Gamma_q(\alpha+\varepsilon)} \int_0^1 (1-qs)^{(\alpha+\varepsilon-1)} |h(s, u(s)) - h(s, v(s))| d_qs + |f(1, u(1)) - f(1, v(1))| \Bigg\} \\
\leq & L_1 \left[1 + \frac{\xi\eta^{\gamma_1}}{|\Delta|\Gamma_q(\gamma_1+1)} + \frac{(1-\xi)\sigma^{\gamma_2}}{|\Delta|\Gamma_q(\gamma_2+1)} + \frac{1}{|\Delta|} \right] \|u - v\| \\
& + L_2 \left[\frac{|a|}{v} \left(\frac{1}{\Gamma_q(\alpha+1)} + \frac{\xi\eta^{(\alpha+\gamma_1)}}{|\Delta|\Gamma_q(\alpha+\gamma_1+1)} + \frac{(1-\xi)\sigma^{(\alpha+\gamma_2)}}{|\Delta|\Gamma_q(\alpha+\gamma_2+1)} + \frac{1}{|\Delta|\Gamma_q(\alpha+1)} \right) \right] \|u - v\|
\end{aligned}$$

$$\begin{aligned}
& + L_3 \left[\frac{|b|}{v} \left(\frac{1}{\Gamma_q(\alpha + \varepsilon + 1)} + \frac{\xi \eta^{(\alpha + \varepsilon + \gamma_1)}}{|\Delta| \Gamma_q(\alpha + \varepsilon + \gamma_1 + 1)} + \frac{(1 - \xi) \sigma^{(\alpha + \varepsilon + \gamma_2)}}{|\Delta| \Gamma_q(\alpha + \varepsilon + \gamma_2 + 1)} + \frac{1}{|\Delta| \Gamma_q(\alpha + \varepsilon + 1)} \right) \right] \|u - v\| \\
& + \left[\frac{|v - 1|}{v} \left(\frac{1}{\Gamma_q(\alpha - \beta + 1)} + \frac{\xi \eta^{(\alpha - \beta + \gamma_1)}}{|\Delta| \Gamma_q(\alpha - \beta + \gamma_1 + 1)} + \frac{(1 - \xi) \sigma^{(\alpha - \beta + \gamma_2)}}{|\Delta| \Gamma_q(\alpha - \beta + \gamma_2 + 1)} + \frac{1}{|\Delta| \Gamma_q(\alpha - \beta + 1)} \right) \right] \|u - v\| \\
& \leq \Omega \|u - v\|,
\end{aligned}$$

which, by condition (9), implies that $\mathcal{T}$ is a contraction. In consequence, it follows by the conclusion of the Banach contraction mapping principle [20] that the operator $\mathcal{T}$ has a unique fixed point, which is indeed the unique solution of problems (1) and (2). The proof is completed. $\square$

In the next two results, we present the existence criteria for solutions to problems (1) and (2). The first result is based on Krasnoselskii's fixed point theorem [21], while the second one relies on Leray-Schauder nonlinear alternative [22].

Theorem 3.3. Assume that

(H₄) there exist ψ_1, ψ_2 , and $\psi_3 \in C([0, 1], \mathbb{R}^+)$ with

$$|f(x, u)| \leq \psi_1(x), \quad |g(x, u)| \leq \psi_2(x), \quad |h(x, u)| \leq \psi_3(x) \quad \forall (x, u) \in [0, 1] \times \mathbb{R},$$

and $\|\psi_i\| = \sup_{x \in [0, 1]} |\psi_i(x)|$, $i = 1, 2, 3$.

If $\Omega_4 < 1$, where Ω_4 is given in (8), then the fractional hybrid q -difference equation (1) with q -integral nonlocal boundary conditions (2) has at least one solution on $[0, 1]$.

Proof. Let us define $B_\rho := \{u \in E : \|u\| \leq \rho\}$ with

$$\rho \geq \frac{\|\psi_1\| \Omega_1 + \|\psi_2\| \Omega_2 + \|\psi_3\| \Omega_3}{1 - \Omega_4}, \quad \Omega_4 < 1, \quad (10)$$

where $\Omega_1, \Omega_2, \Omega_3$, and Ω_4 are given in (8). Clearly B_ρ is a closed, bounded, convex, and nonempty subset of Banach space E . Now we verify that the operator $\mathcal{T} : E \rightarrow E$ defined by (7) satisfies the hypothesis of Krasnoselskii's fixed point theorem [21]. For each $x \in [0, 1]$, we define two operators from B_ρ to E as follows:

$$\begin{aligned}
(\mathcal{T}_1 u)(x) = & \frac{(v - 1)}{v \Gamma_q(\alpha - \beta)} \int_0^x (x - qs)^{(\alpha - \beta - 1)} u(s) d_qs + \frac{x^{\alpha - 1}}{\Delta} \left[\frac{\xi(v - 1)}{v \Gamma_q(\alpha - \beta + \gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha - \beta + \gamma_1 - 1)} u(s) d_qs \right. \\
& \left. + \frac{(1 - \xi)(v - 1)}{v \Gamma_q(\alpha - \beta + \gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha - \beta + \gamma_2 - 1)} u(s) d_qs - \frac{(v - 1)}{v \Gamma_q(\alpha - \beta)} \int_0^1 (1 - qs)^{(\alpha - \beta - 1)} u(s) d_qs \right], \quad (11)
\end{aligned}$$

$$\begin{aligned}
(\mathcal{T}_2 u)(x) = & f(x, u(x)) + \frac{a}{v \Gamma_q(\alpha)} \int_0^x (x - qs)^{(\alpha - 1)} g(s, u(s)) d_qs + \frac{b}{v \Gamma_q(\alpha + \varepsilon)} \int_0^x (x - qs)^{(\alpha + \varepsilon - 1)} h(s, u(s)) d_qs \\
& + \frac{x^{\alpha - 1}}{\Delta} \left[\frac{\xi}{\Gamma_q(\gamma_1)} \int_0^\eta (\eta - qs)^{(\gamma_1 - 1)} f(s, u(s)) d_qs + \frac{a\xi}{v \Gamma_q(\alpha + \gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha + \gamma_1 - 1)} g(s, u(s)) d_qs \right. \\
& + \frac{b\xi}{v \Gamma_q(\alpha + \varepsilon + \gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha + \varepsilon + \gamma_1 - 1)} h(s, u(s)) d_qs + \frac{(1 - \xi)}{\Gamma_q(\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\gamma_2 - 1)} f(s, u(s)) d_qs \\
& + \frac{a(1 - \xi)}{v \Gamma_q(\alpha + \gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha + \gamma_2 - 1)} g(s, u(s)) d_qs + \frac{b(1 - \xi)}{v \Gamma_q(\alpha + \varepsilon + \gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha + \varepsilon + \gamma_2 - 1)} h(s, u(s)) d_qs \\
& \left. - \frac{a}{v \Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha - 1)} g(s, u(s)) d_qs - \frac{b}{v \Gamma_q(\alpha + \varepsilon)} \int_0^1 (1 - qs)^{(\alpha + \varepsilon - 1)} h(s, u(s)) d_qs - f(1, u(1)) \right]. \quad (12)
\end{aligned}$$

For any $u, v \in B_\rho$, we have

$$\begin{aligned}
|\mathcal{T}_1 u(x) + \mathcal{T}_2 v(x)| &\leq \sup_{x \in [0,1]} \left\{ |f(x, v(x))| + \frac{|v-1|}{v\Gamma_q(\alpha-\beta)} \int_0^x (x-qs)^{(\alpha-\beta-1)} |u(s)| d_qs \right. \\
&\quad + \frac{|a|}{v\Gamma_q(\alpha)} \int_0^x (x-qs)^{(\alpha-1)} |g(s, v(s))| d_qs + \frac{|b|}{v\Gamma_q(\alpha+\varepsilon)} \int_0^x (x-qs)^{(\alpha+\varepsilon-1)} |h(s, v(s))| d_qs \\
&\quad + \frac{x^{\alpha-1}}{|\Delta|} \left[\frac{\xi}{\Gamma_q(\gamma_1)} \int_0^\eta (\eta-qs)^{(\gamma_1-1)} |f(s, v(s))| d_qs + \frac{\xi|v-1|}{v\Gamma_q(\alpha-\beta+\gamma_1)} \int_0^\eta (\eta-qs)^{(\alpha-\beta+\gamma_1-1)} |u(s)| d_qs \right. \\
&\quad + \frac{|a|\xi}{v\Gamma_q(\alpha+\gamma_1)} \int_0^\eta (\eta-qs)^{(\alpha+\gamma_1-1)} |g(s, v(s))| d_qs \\
&\quad + \frac{|b|\xi}{v\Gamma_q(\alpha+\varepsilon+\gamma_1)} \int_0^\eta (\eta-qs)^{(\alpha+\varepsilon+\gamma_1-1)} |h(s, v(s))| d_qs \\
&\quad + \frac{(1-\xi)}{\Gamma_q(\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\gamma_2-1)} |f(s, v(s))| d_qs + \frac{(1-\xi)|v-1|}{v\Gamma_q(\alpha-\beta+\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\alpha-\beta+\gamma_2-1)} |u(s)| d_qs \\
&\quad + \frac{|a|(1-\xi)}{v\Gamma_q(\alpha+\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\alpha+\gamma_2-1)} |g(s, v(s))| d_qs \\
&\quad + \frac{|b|(1-\xi)}{v\Gamma_q(\alpha+\varepsilon+\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\alpha+\varepsilon+\gamma_2-1)} |h(s, v(s))| d_qs \\
&\quad + \frac{|v-1|}{v\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} |u(s)| d_qs + \frac{|a|}{v\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} |g(s, v(s))| d_qs \\
&\quad \left. + \frac{|b|}{v\Gamma_q(\alpha+\varepsilon)} \int_0^1 (1-qs)^{(\alpha+\varepsilon-1)} |h(s, v(s))| d_qs + |f(1, v(1))| \right\} \\
&\leq \|\psi_1\| \sup_{x \in [0,1]} \left\{ 1 + \frac{x^{(\alpha-1)}}{|\Delta|} \left[\frac{\xi}{\Gamma_q(\gamma_1)} \int_0^\eta (\eta-qs)^{(\gamma_1-1)} d_qs + \frac{(1-\xi)}{\Gamma_q(\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\gamma_2-1)} d_qs + 1 \right] \right\} \\
&\quad + \|\psi_2\| \sup_{x \in [0,1]} \left\{ \frac{|a|}{v\Gamma_q(\alpha)} \int_0^x (x-qs)^{(\alpha-1)} d_qs + \frac{x^{(\alpha-1)}}{|\Delta|} \left[\frac{|a|\xi}{v\Gamma_q(\alpha+\gamma_1)} \int_0^\eta (\eta-qs)^{(\alpha+\gamma_1-1)} d_qs \right. \right. \\
&\quad \left. + \frac{|a|(1-\xi)}{v\Gamma_q(\alpha+\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\alpha+\gamma_2-1)} d_qs + \frac{|a|}{v\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} d_qs \right] \Big\} \\
&\quad + \|\psi_3\| \sup_{x \in [0,1]} \left\{ \frac{|b|}{v\Gamma_q(\alpha+\varepsilon)} \int_0^x (x-qs)^{(\alpha+\varepsilon-1)} d_qs + \frac{x^{(\alpha-1)}}{|\Delta|} \left[\frac{|b|\xi}{v\Gamma_q(\alpha+\varepsilon+\gamma_1)} \int_0^\eta (\eta-qs)^{(\alpha+\varepsilon+\gamma_1-1)} d_qs \right. \right. \\
&\quad \left. + \frac{|b|(1-\xi)}{v\Gamma_q(\alpha+\varepsilon+\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\alpha+\varepsilon+\gamma_2-1)} d_qs + \frac{|b|}{v\Gamma_q(\alpha+\varepsilon)} \int_0^1 (1-qs)^{(\alpha+\varepsilon-1)} d_qs \right] \Big\}
\end{aligned}$$

$$\begin{aligned}
& + \rho \sup_{x \in [0,1]} \left\{ \frac{|\nu - 1|}{v\Gamma_q(\alpha - \beta)} \int_0^x (x - qs)^{(\alpha - \beta - 1)} d_qs + \frac{x^{(\alpha - 1)}}{|\Delta|} \left[\frac{\xi|\nu - 1|}{v\Gamma_q(\alpha - \beta + \gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha - \beta + \gamma_1 - 1)} d_qs \right. \right. \\
& \left. \left. + \frac{(1 - \xi)|\nu - 1|}{v\Gamma_q(\alpha - \beta + \gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha - \beta + \gamma_2 - 1)} d_qs + \frac{|\nu - 1|}{v\Gamma_q(\alpha - \beta)} \int_0^1 (1 - qs)^{(\alpha - \beta - 1)} d_qs \right] \right\} \\
& \leq \|\psi_1\| \left[1 + \frac{\xi\eta^{\gamma_1}}{|\Delta|\Gamma_q(\gamma_1 + 1)} + \frac{(1 - \xi)\sigma^{\gamma_2}}{|\Delta|\Gamma_q(\gamma_2 + 1)} + \frac{1}{|\Delta|} \right] \\
& + \|\psi_2\| \times \left[\frac{|a|}{v} \left(\frac{1}{\Gamma_q(\alpha + 1)} + \frac{\xi\eta^{(\alpha + \gamma_1)}}{|\Delta|\Gamma_q(\alpha + \gamma_1 + 1)} + \frac{(1 - \xi)\sigma^{(\alpha + \gamma_2)}}{|\Delta|\Gamma_q(\alpha + \gamma_2 + 1)} + \frac{1}{|\Delta|\Gamma_q(\alpha + 1)} \right) \right] \\
& + \|\psi_3\| \left[\frac{|b|}{v} \left(\frac{1}{\Gamma_q(\alpha + \varepsilon + 1)} + \frac{\xi\eta^{(\alpha + \varepsilon + \gamma_1)}}{|\Delta|\Gamma_q(\alpha + \varepsilon + \gamma_1 + 1)} + \frac{(1 - \xi)\sigma^{(\alpha + \varepsilon + \gamma_2)}}{|\Delta|\Gamma_q(\alpha + \varepsilon + \gamma_2 + 1)} + \frac{1}{|\Delta|\Gamma_q(\alpha + \varepsilon + 1)} \right) \right] \\
& + \rho \left[\frac{|\nu - 1|}{v} \left(\frac{1}{\Gamma_q(\alpha - \beta + 1)} + \frac{\xi\eta^{(\alpha - \beta + \gamma_1)}}{|\Delta|\Gamma_q(\alpha - \beta + \gamma_1 + 1)} + \frac{(1 - \xi)\sigma^{(\alpha - \beta + \gamma_2)}}{|\Delta|\Gamma_q(\alpha - \beta + \gamma_2 + 1)} + \frac{1}{|\Delta|\Gamma_q(\alpha - \beta + 1)} \right) \right] \\
& \leq \|\psi_1\|\Omega_1 + \|\psi_2\|\Omega_2 + \|\psi_3\|\Omega_3 + \rho\Omega_4,
\end{aligned}$$

which implies that $\|\mathcal{T}_1 u + \mathcal{T}_2 v\| \leq \rho$ by condition (10) and so $\mathcal{T}_1 u + \mathcal{T}_2 v \in B_\rho$ for all $u, v \in B_\rho$. From the continuity of f and g , it follows that the operator $\mathcal{T}_2$ is continuous on B_ρ .

In the next step, we show that the operator $\mathcal{T}_2$ is compact. Let us first show that $\mathcal{T}_2$ is uniformly bounded. For each $u \in B_\rho$ and $x \in [0, 1]$, we have

$$\begin{aligned}
\|\mathcal{T}_2 u\| & \leq \sup_{x \in [0,1]} \left\{ |f(x, u(x))| + \frac{|a|}{v\Gamma_q(\alpha)} \int_0^x (x - qs)^{(\alpha - 1)} |g(s, u(s))| d_qs + \frac{|b|}{v\Gamma_q(\alpha + \varepsilon)} \int_0^x (x - qs)^{(\alpha + \varepsilon - 1)} |h(s, u(s))| d_qs \right. \\
& + \frac{x^{\alpha - 1}}{|\Delta|} \left[\frac{\xi}{\Gamma_q(\gamma_1)} \int_0^\eta (\eta - qs)^{(\gamma_1 - 1)} |f(s, u(s))| d_qs + \frac{|a|\xi}{v\Gamma_q(\alpha + \gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha + \gamma_1 - 1)} |g(s, u(s))| d_qs \right. \\
& + \frac{|b|\xi}{v\Gamma_q(\alpha + \varepsilon + \gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha + \varepsilon + \gamma_1 - 1)} |h(s, u(s))| d_qs + \frac{(1 - \xi)}{\Gamma_q(\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\gamma_2 - 1)} |f(s, u(s))| d_qs \\
& + \frac{|a|(1 - \xi)}{v\Gamma_q(\alpha + \gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha + \gamma_2 - 1)} |g(s, u(s))| d_qs + \frac{|b|(1 - \xi)}{v\Gamma_q(\alpha + \varepsilon + \gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha + \varepsilon + \gamma_2 - 1)} |h(s, u(s))| d_qs \\
& \left. \left. + \frac{|a|}{v\Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha - 1)} |g(s, u(s))| d_qs + \frac{|b|}{v\Gamma_q(\alpha + \varepsilon)} \int_0^1 (1 - qs)^{(\alpha + \varepsilon - 1)} |h(s, u(s))| d_qs + |f(1, u(1))| \right] \right\} \\
& \leq \|\psi_1\| \left[1 + \frac{\xi\eta^{\gamma_1}}{|\Delta|\Gamma_q(\gamma_1 + 1)} + \frac{(1 - \xi)\sigma^{\gamma_2}}{|\Delta|\Gamma_q(\gamma_2 + 1)} + \frac{1}{|\Delta|} \right] + \|\psi_2\| \\
& \times \left[\frac{|a|}{v} \left(\frac{1}{\Gamma_q(\alpha + 1)} + \frac{\xi\eta^{(\alpha + \gamma_1)}}{|\Delta|\Gamma_q(\alpha + \gamma_1 + 1)} + \frac{(1 - \xi)\sigma^{(\alpha + \gamma_2)}}{|\Delta|\Gamma_q(\alpha + \gamma_2 + 1)} + \frac{1}{|\Delta|\Gamma_q(\alpha + 1)} \right) \right] \\
& + \|\psi_3\| \left[\frac{|b|}{v} \left(\frac{1}{\Gamma_q(\alpha + \varepsilon + 1)} + \frac{\xi\eta^{(\alpha + \varepsilon + \gamma_1)}}{|\Delta|\Gamma_q(\alpha + \varepsilon + \gamma_1 + 1)} + \frac{(1 - \xi)\sigma^{(\alpha + \varepsilon + \gamma_2)}}{|\Delta|\Gamma_q(\alpha + \varepsilon + \gamma_2 + 1)} + \frac{1}{|\Delta|\Gamma_q(\alpha + \varepsilon + 1)} \right) \right] \\
& = \|\psi_1\|\Omega_1 + \|\psi_2\|\Omega_2 + \|\psi_3\|\Omega_3.
\end{aligned}$$

In order to establish the equicontinuity of the operator $\mathcal{T}_2$, we assume that $x_1, x_2 \in [0, 1]$ such that $x_2 > x_1$. We will show that $\mathcal{T}_2$ maps bounded sets into equicontinuous sets. For each $u \in B_\rho$, we have

$$\begin{aligned}
|\mathcal{T}_2 u(x_2) - \mathcal{T}_2 u(x_1)| &\leq \sup_{x \in [0,1]} \left\{ |f(x_2, u(x_2)) - f(x_1, u(x_1))| + \frac{|a|}{v\Gamma_q(\alpha)} \int_0^{x_1} [(x_2 - qs)^{(\alpha-1)} - (x_1 - qs)^{(\alpha-1)}] |g(s, u(s))| d_qs \right. \\
&\quad + \frac{|b|}{v\Gamma_q(\alpha + \varepsilon)} \int_0^{x_1} [(x_2 - qs)^{(\alpha+\varepsilon-1)} - (x_1 - qs)^{(\alpha+\varepsilon-1)}] |h(s, u(s))| d_qs \\
&\quad + \frac{|a|}{v\Gamma_q(\alpha)} \int_{x_1}^{x_2} (x_2 - qs)^{(\alpha-1)} |g(s, u(s))| d_qs + \frac{|b|}{v\Gamma_q(\alpha + \varepsilon)} \int_{x_1}^{x_2} (x_2 - qs)^{(\alpha+\varepsilon-1)} |h(s, u(s))| d_qs \\
&\quad + \frac{x_2^{(\alpha-1)} - x_1^{(\alpha-1)}}{|\Delta|} \left[\frac{\xi}{\Gamma_q(\gamma_1)} \int_0^\eta (\eta - qs)^{(\gamma_1-1)} |f(s, u(s))| d_qs \right. \\
&\quad + \frac{|a|\xi}{v\Gamma_q(\alpha + \gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha+\gamma_1-1)} |g(s, u(s))| d_qs \\
&\quad + \frac{|b|\xi}{v\Gamma_q(\alpha + \varepsilon + \gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha+\varepsilon+\gamma_1-1)} |h(s, u(s))| d_qs \\
&\quad + \frac{(1-\xi)}{\Gamma_q(\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\gamma_2-1)} |f(s, u(s))| d_qs \\
&\quad + \frac{|a|(1-\xi)}{v\Gamma_q(\alpha + \gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha+\gamma_2-1)} |g(s, u(s))| d_qs \\
&\quad + \frac{|b|(1-\xi)}{v\Gamma_q(\alpha + \varepsilon + \gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha+\varepsilon+\gamma_2-1)} |h(s, u(s))| d_qs \\
&\quad \left. + \frac{|a|}{v\Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha-1)} |g(s, u(s))| d_qs + \frac{|b|}{v\Gamma_q(\alpha + \varepsilon)} \int_0^1 (1 - qs)^{(\alpha+\varepsilon-1)} |h(s, u(s))| d_qs \right\} \\
&\leq |f(x_2, u(x_2)) - f(x_1, u(x_1))| + \frac{|a|\|\psi_2\|}{v\Gamma_q(\alpha)} \int_0^{x_1} [(x_2 - qs)^{(\alpha-1)} - (x_1 - qs)^{(\alpha-1)}] d_qs \\
&\quad + \frac{|b|\|\psi_3\|}{v\Gamma_q(\alpha + \varepsilon)} \int_0^{x_1} [(x_2 - qs)^{(\alpha+\varepsilon-1)} - (x_1 - qs)^{(\alpha+\varepsilon-1)}] d_qs \\
&\quad + \frac{|a|\|\psi_2\|}{v\Gamma_q(\alpha)} \int_{x_1}^{x_2} (x_2 - qs)^{(\alpha-1)} d_qs + \frac{|b|\|\psi_3\|}{v\Gamma_q(\alpha + \varepsilon)} \int_{x_1}^{x_2} (x_2 - qs)^{(\alpha+\varepsilon-1)} d_qs \\
&\quad + \frac{(x_2^{(\alpha-1)} - x_1^{(\alpha-1)})}{|\Delta|} \left[\frac{\|\psi_1\|\xi\eta^{\gamma_1}}{\Gamma_q(\gamma_1 + 1)} + \frac{|a|\xi\|\psi_2\|\eta^{(\alpha+\gamma_1)}}{v\Gamma_q(\alpha + \gamma_1 + 1)} + \frac{|b|\xi\|\psi_3\|\eta^{(\alpha+\varepsilon+\gamma_1)}}{v\Gamma_q(\alpha + \varepsilon + \gamma_1 + 1)} \right. \\
&\quad + \frac{\|\psi_1\|(1-\xi)\sigma^{\gamma_2}}{\Gamma_q(\gamma_2 + 1)} + \frac{|a|\|\psi_2\|(1-\xi)\sigma^{(\alpha+\gamma_2)}}{v\Gamma_q(\alpha + \gamma_2 + 1)} + \frac{|b|\|\psi_3\|(1-\xi)\sigma^{(\alpha+\varepsilon+\gamma_2)}}{v\Gamma_q(\alpha + \varepsilon + \gamma_2 + 1)} \\
&\quad \left. + \frac{|a|\|\psi_2\|}{v\Gamma_q(\alpha)} + \frac{|b|\|\psi_3\|}{v\Gamma_q(\alpha + \varepsilon)} \right].
\end{aligned}$$

Observe that the right-hand side of the above inequality is independent of $u \in B_\rho$ and tends to zero as $x_1 \rightarrow x_2$. This shows that $\mathcal{T}_2$ is equicontinuous. Therefore, the operator $\mathcal{T}_2$ is relatively compact on B_ρ , and hence, the Arzelà-Ascoli theorem implies that $\mathcal{T}_2$ is completely continuous and so $\mathcal{T}_2$ is the compact operator on B_ρ .

Finally, we prove that the operator $\mathcal{T}_1$ is a contraction. For any $u, v \in B_\rho$ and $x \in [0, 1]$, one can write

$$\begin{aligned} \|\mathcal{T}_1 u - \mathcal{T}_1 v\| \leq & \sup_{x \in [0,1]} \left\{ \frac{|v-1|}{v\Gamma_q(\alpha-\beta)} \int_0^x (x-qs)^{(\alpha-\beta-1)} |u(s) - v(s)| d_qs \right. \\ & + \frac{x^{\alpha-1}}{|\Delta|} \left[\frac{\xi|v-1|}{v\Gamma_q(\alpha-\beta+\gamma_1)} \int_0^\eta (\eta-qs)^{(\alpha-\beta+\gamma_1-1)} |u(s) - v(s)| d_qs \right. \\ & + \frac{(1-\xi)|v-1|}{v\Gamma_q(\alpha-\beta+\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\alpha-\beta+\gamma_2-1)} |u(s) - v(s)| d_qs \\ & \left. \left. + \frac{|v-1|}{v\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} |u(s) - v(s)| d_qs \right] \right\} \\ & \leq \Omega_4 \|u - v\|. \end{aligned}$$

Since $\Omega_4 < 1$, so $\mathcal{T}_1$ is a contraction. Thus, all the assumptions of Krasnoselskii's fixed point theorem [21] are satisfied. Therefore, the fractional hybrid q -difference equation (1) with q -integral nonlocal boundary conditions (2) has at least one solution on $[0, 1]$ and the proof is completed. $\square$

Theorem 3.4. Assume that:

- (H5) there exist continuous nondecreasing functions $\chi_1, \chi_2, \chi_3 : [0, \infty) \rightarrow (0, \infty)$ and functions $\phi_1, \phi_2, \phi_3 \in C([0, 1], \mathbb{R}^+)$ such that $|f(x, u)| \leq \phi_1(x)\chi_1(|u|)$, $|g(x, u)| \leq \phi_2(x)\chi_2(|u|)$, and $|h(x, u)| \leq \phi_3(x)\chi_3(|u|)$ for each $(x, u) \in [0, 1] \times \mathbb{R}$;
- (H6) there exists a constant $G > 0$ such that

$$\frac{(1 - \Omega_4)G}{\|\phi_1\|_{\chi_1}(G)\Omega_1 + \|\phi_2\|_{\chi_2}(G)\Omega_2 + \|\phi_3\|_{\chi_3}(G)\Omega_3} > 1, \quad \Omega_4 < 1,$$

where $\Omega_1, \Omega_2, \Omega_3$, and Ω_4 are defined by (8).

Then fractional hybrid q -difference equation (1) with q -integral nonlocal boundary conditions (2) has at least one solution on $[0, 1]$.

Proof. We verify the hypothesis of Leray-Schauder's nonlinear alternative [22] in several steps. Let us first show that the operator $\mathcal{T}$, defined by (7), maps bounded sets (balls) into bounded sets in E . For a positive number ω , let $B_\omega = \{u \in E : \|u\| \leq \omega\}$ be a bounded ball in E . Then, for $x \in [0, 1]$, we have

$$\begin{aligned} \|\mathcal{T}u\| \leq & \sup_{x \in [0,1]} \left\{ |f(x, u(x))| + \frac{|v-1|}{v\Gamma_q(\alpha-\beta)} \int_0^x (x-qs)^{(\alpha-\beta-1)} |u(s)| d_qs \right. \\ & + \frac{|a|}{v\Gamma_q(\alpha)} \int_0^x (x-qs)^{(\alpha-1)} |g(s, u(s))| d_qs + \frac{|b|}{v\Gamma_q(\alpha+\varepsilon)} \int_0^x (x-qs)^{(\alpha+\varepsilon-1)} |h(s, u(s))| d_qs \\ & + \frac{x^{\alpha-1}}{|\Delta|} \left[\frac{\xi}{\Gamma_q(\gamma_1)} \int_0^\eta (\eta-qs)^{(\gamma_1-1)} |f(s, u(s))| d_qs + \frac{\xi|v-1|}{v\Gamma_q(\alpha-\beta+\gamma_1)} \int_0^\eta (\eta-qs)^{(\alpha-\beta+\gamma_1-1)} |u(s)| d_qs \right. \\ & + \frac{|a|\xi}{v\Gamma_q(\alpha+\gamma_1)} \int_0^\eta (\eta-qs)^{(\alpha+\gamma_1-1)} |g(s, u(s))| d_qs + \frac{|b|\xi}{v\Gamma_q(\alpha+\varepsilon+\gamma_1)} \int_0^\eta (\eta-qs)^{(\alpha+\varepsilon+\gamma_1-1)} |h(s, u(s))| d_qs \\ & \left. + \frac{(1-\xi)}{\Gamma_q(\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\gamma_2-1)} |f(s, u(s))| d_qs + \frac{(1-\xi)|v-1|}{v\Gamma_q(\alpha-\beta+\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\alpha-\beta+\gamma_2-1)} |u(s)| d_qs \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{|a|(1-\xi)}{v\Gamma_q(\alpha+\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\alpha+\gamma_2-1)} |g(s, u(s))| d_qs + \frac{|b|(1-\xi)}{v\Gamma_q(\alpha+\varepsilon+\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\alpha+\varepsilon+\gamma_2-1)} |h(s, u(s))| d_qs \\
& + \frac{|v-1|}{v\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} |u(s)| d_qs + \frac{|a|}{v\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} |g(s, u(s))| d_qs \\
& + \frac{|b|}{v\Gamma_q(\alpha+\varepsilon)} \int_0^1 (1-qs)^{(\alpha+\varepsilon-1)} |h(s, u(s))| d_qs + |f(1, u(1))| \Bigg] \Bigg\} \\
& \leq \|\phi_1\|_{\chi_1}(\omega) \sup_{x \in [0,1]} \left\{ 1 + \frac{x^{(\alpha-1)}}{|\Delta|} \left[\frac{\xi}{\Gamma_q(\gamma_1)} \int_0^\eta (\eta-qs)^{(\gamma_1-1)} d_qs + \frac{(1-\xi)}{\Gamma_q(\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\gamma_2-1)} d_qs + 1 \right] \right. \\
& + \|\phi_2\|_{\chi_2}(\omega) \sup_{x \in [0,1]} \left\{ \frac{|a|}{v\Gamma_q(\alpha)} \int_0^x (x-qs)^{(\alpha-1)} d_qs + \frac{x^{(\alpha-1)}}{|\Delta|} \left[\frac{|a|\xi}{v\Gamma_q(\alpha+\gamma_1)} \int_0^\eta (\eta-qs)^{(\alpha+\gamma_1-1)} d_qs \right. \right. \\
& + \frac{|a|(1-\xi)}{v\Gamma_q(\alpha+\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\alpha+\gamma_2-1)} d_qs + \left. \left. \frac{|a|}{v\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} d_qs \right] \right\} \\
& + \|\phi_3\|_{\chi_3}(\omega) \sup_{x \in [0,1]} \left\{ \frac{|b|}{v\Gamma_q(\alpha+\varepsilon)} \int_0^x (x-qs)^{(\alpha+\varepsilon-1)} d_qs \right. \\
& + \frac{x^{(\alpha-1)}}{|\Delta|} \left[\frac{|b|\xi}{v\Gamma_q(\alpha+\varepsilon+\gamma_1)} \int_0^\eta (\eta-qs)^{(\alpha+\varepsilon+\gamma_1-1)} d_qs \right. \\
& + \left. \left. \frac{|b|(1-\xi)}{v\Gamma_q(\alpha+\varepsilon+\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\alpha+\varepsilon+\gamma_2-1)} d_qs + \frac{|b|}{v\Gamma_q(\alpha+\varepsilon)} \int_0^1 (1-qs)^{(\alpha+\varepsilon-1)} d_qs \right] \right\} \\
& + \omega \sup_{x \in [0,1]} \left\{ \frac{|v-1|}{v\Gamma_q(\alpha-\beta)} \int_0^x (x-qs)^{(\alpha-\beta-1)} d_qs + \frac{x^{(\alpha-1)}}{|\Delta|} \left[\frac{\xi|v-1|}{v\Gamma_q(\alpha-\beta+\gamma_1)} \int_0^\eta (\eta-qs)^{(\alpha-\beta+\gamma_1-1)} d_qs \right. \right. \\
& + \left. \left. \frac{(1-\xi)|v-1|}{v\Gamma_q(\alpha-\beta+\gamma_2)} \int_0^\sigma (\sigma-qs)^{(\alpha-\beta+\gamma_2-1)} d_qs + \frac{|v-1|}{v\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} d_qs \right] \right\} \\
& \leq \|\phi_1\|_{\chi_1}(\omega)\Omega_1 + \|\phi_2\|_{\chi_2}(\omega)\Omega_2 + \|\phi_3\|_{\chi_3}(\omega)\Omega_3 + \omega\Omega_4.
\end{aligned}$$

Second, we show that $\mathcal{T}$ maps bounded sets into equicontinuous sets of E . Let $x_1, x_2 \in [0, 1]$ with $x_1 < x_2$ and $u \in B_\omega$. Then, we have

$$\begin{aligned}
|\mathcal{T}u(x_2) - \mathcal{T}u(x_1)| & \leq \sup_{x \in [0,1]} \{|f(x_2, u(x_2)) - f(x_1, u(x_1))|\} \\
& + \frac{|v-1|}{v\Gamma_q(\alpha-\beta)} \int_0^{x_1} [(x_2-qs)^{(\alpha-\beta-1)} - (x_1-qs)^{(\alpha-\beta-1)}] |u(s)| d_qs \\
& + \frac{|v-1|}{v\Gamma_q(\alpha-\beta)} \int_{x_1}^{x_2} (x_2-qs)^{(\alpha-\beta-1)} |u(s)| d_qs \\
& + \frac{|a|}{v\Gamma_q(\alpha)} \int_0^{x_1} [(x_2-qs)^{(\alpha-1)} - (x_1-qs)^{(\alpha-1)}] |g(s, u(s))| d_qs
\end{aligned}$$

$$\begin{aligned}
& + \frac{|b|}{v\Gamma_q(\alpha + \varepsilon)} \int_0^{x_1} [(x_2 - qs)^{(\alpha+\varepsilon-1)} - (x_1 - qs)^{(\alpha+\varepsilon-1)}] |h(s, u(s))| d_qs \\
& + \frac{|a|}{v\Gamma_q(\alpha)} \int_{x_1}^{x_2} (x_2 - qs)^{(\alpha-1)} |g(s, u(s))| d_qs \\
& + \frac{|b|}{v\Gamma_q(\alpha + \varepsilon)} \int_{x_1}^{x_2} (x_2 - qs)^{(\alpha+\varepsilon-1)} |h(s, u(s))| d_qs \\
& + \frac{x_2^{(\alpha-1)} - x_1^{(\alpha-1)}}{|\Delta|} \left[\frac{\xi}{\Gamma_q(\gamma_1)} \int_0^\eta (\eta - qs)^{(\gamma_1-1)} |f(s, u(s))| d_qs \right. \\
& + \frac{\xi|v-1|}{v\Gamma_q(\alpha - \beta + \gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha-\beta+\gamma_1-1)} |u(s)| d_qs \\
& + \frac{|a|\xi}{v\Gamma_q(\alpha + \gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha+\gamma_1-1)} |g(s, u(s))| d_qs \\
& + \frac{|b|\xi}{v\Gamma_q(\alpha + \varepsilon + \gamma_1)} \int_0^\eta (\eta - qs)^{(\alpha+\varepsilon+\gamma_1-1)} |h(s, u(s))| d_qs \\
& + \frac{(1-\xi)}{\Gamma_q(\gamma_2)} \int_0^\sigma (\sigma - qs)^{(\gamma_2-1)} |f(s, u(s))| d_qs \\
& + \frac{(1-\xi)|v-1|}{v\Gamma_q(\alpha - \beta + \gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha-\beta+\gamma_2-1)} |u(s)| d_qs \\
& + \frac{|a|(1-\xi)}{v\Gamma_q(\alpha + \gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha+\gamma_2-1)} |g(s, u(s))| d_qs \\
& + \frac{|b|(1-\xi)}{v\Gamma_q(\alpha + \varepsilon + \gamma_2)} \int_0^\sigma (\sigma - qs)^{(\alpha+\varepsilon+\gamma_2-1)} |h(s, u(s))| d_qs \\
& + \frac{|v-1|}{v\Gamma_q(\alpha - \beta)} \int_0^1 (1 - qs)^{(\alpha-\beta-1)} |u(s)| d_qs + \frac{|a|}{v\Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha-1)} |g(s, u(s))| d_qs \\
& \left. + \frac{|b|}{v\Gamma_q(\alpha + \varepsilon)} \int_0^1 (1 - qs)^{(\alpha+\varepsilon-1)} |h(s, u(s))| d_qs \right] \Bigg\} \\
& \leq |f(x_2, u(x_2)) - f(x_1, u(x_1))| + \frac{\omega|v-1|}{v\Gamma_q(\alpha - \beta)} \int_0^{x_1} [(x_2 - qs)^{(\alpha-\beta-1)} - (x_1 - qs)^{(\alpha-\beta-1)}] d_qs \\
& + \frac{\omega|v-1|}{v\Gamma_q(\alpha - \beta)} \int_{x_1}^{x_2} (x_2 - qs)^{(\alpha-\beta-1)} d_qs + \frac{|a|\|\phi_2\|\chi_2(\omega)}{v\Gamma_q(\alpha)} \int_0^{x_1} [(x_2 - qs)^{(\alpha-1)} - (x_1 - qs)^{(\alpha-1)}] d_qs \\
& + \frac{|b|\|\phi_3\|\chi_3(\omega)}{v\Gamma_q(\alpha + \varepsilon)} \int_0^{x_1} [(x_2 - qs)^{(\alpha+\varepsilon-1)} - (x_1 - qs)^{(\alpha+\varepsilon-1)}] d_qs \\
& + \frac{|a|\|\phi_2\|\chi_2(\omega)}{v\Gamma_q(\alpha)} \int_{x_1}^{x_2} (x_2 - qs)^{(\alpha-1)} d_qs + \frac{|b|\|\phi_3\|\chi_3(\omega)}{v\Gamma_q(\alpha + \varepsilon)} \int_{x_1}^{x_2} (x_2 - qs)^{(\alpha+\varepsilon-1)} d_qs
\end{aligned}$$

$$\begin{aligned}
& + \frac{(x_2^{(\alpha-1)} - x_1^{(\alpha-1)})}{|\Delta|} \left[\frac{\|\phi_1\| \chi_1(\omega) \xi \eta^{\gamma_1}}{\Gamma_q(\gamma_1 + 1)} + \frac{\xi \omega |v - 1| \eta^{\alpha - \beta + \gamma_1}}{v \Gamma_q(\alpha - \beta + \gamma_1 + 1)} + \frac{|a| \xi \|\phi_2\| \chi_2(\omega) \eta^{(\alpha + \gamma_1)}}{v \Gamma_q(\alpha + \gamma_1 + 1)} \right. \\
& + \frac{|b| \xi \|\phi_3\| \chi_3(\omega) \eta^{(\alpha + \varepsilon + \gamma_1)}}{v \Gamma_q(\alpha + \varepsilon + \gamma_1 + 1)} + \frac{\|\phi_1\| \chi_1(\omega) (1 - \xi) \sigma^{\gamma_2}}{\Gamma_q(\gamma_2 + 1)} + \frac{\omega (1 - \xi) |v - 1| \sigma^{(\alpha - \beta + \gamma_2)}}{v \Gamma_q(\alpha - \beta + \gamma_2 + 1)} \\
& + \frac{|a| \|\phi_2\| \chi_2(\omega) (1 - \xi) \sigma^{(\alpha + \gamma_2)}}{v \Gamma_q(\alpha + \gamma_2 + 1)} + \frac{|b| \|\phi_3\| \chi_3(\omega) (1 - \xi) \sigma^{(\alpha + \varepsilon + \gamma_2)}}{v \Gamma_q(\alpha + \varepsilon + \gamma_2 + 1)} \\
& \left. + \frac{\omega |v - 1|}{v \Gamma_q(\alpha - \beta + 1)} + \frac{|a| \|\phi_2\| \chi_2(\omega)}{v \Gamma_q(\alpha)} + \frac{|b| \|\phi_3\| \chi_3(\omega)}{v \Gamma_q(\alpha + \varepsilon)} \right].
\end{aligned}$$

Obviously, the right-hand side of the aforementioned inequality tends to zero independently of $u \in B_\omega$ as $x_2 \rightarrow x_1$. Therefore, it follows by the Arzelà-Ascoli theorem that $\mathcal{T} : E \rightarrow E$ is completely continuous.

In order to complete the hypothesis of the Leray-Schauder nonlinear alternative [22], it will be shown that the set of all solutions to the equation $u = \theta \mathcal{T}u$ is bounded for $\theta \in [0, 1]$. For that, let u be a solution of $u = \theta \mathcal{T}u$ for $\theta \in [0, 1]$. Then, for $x \in [0, 1]$, we apply the strategy used in the first step to obtain

$$\|u\| \leq \Omega_1 \|\phi_1\| \chi_1(\|u\|) + \Omega_2 \|\phi_2\| \chi_2(\|u\|) + \Omega_3 \|\phi_3\| \chi_3(\|u\|) + \Omega_4 \|u\|.$$

Consequently, we have

$$\frac{(1 - \Omega_4) \|u\|}{\|\phi_1\| \chi_1(\|u\|) \Omega_1 + \|\phi_2\| \chi_2(\|u\|) \Omega_2 + \|\phi_3\| \chi_3(\|u\|) \Omega_3} \leq 1.$$

By the condition (H_6) , we can find a positive number G such that $\|u\| \neq G$. Introduce a set

$$U = \{u \in E : \|u\| < G\}, \quad (13)$$

and observe that the operator $\mathcal{T} : \bar{U} \rightarrow E$ is continuous and completely continuous ($\bar{U}$ is closure of U). With this choice of U , we cannot find $u \in \partial U$ (∂U is boundary of U) satisfying $u = \theta \mathcal{T}u$. Therefore, it follows by the nonlinear alternative of Leray-Schauder type [22] that the operator $\mathcal{T}$ has a fixed point in $\bar{U}$. Thus, there exists a solution of problems (1) and (2) on $[0, 1]$. The proof is complete. $\square$

4 Examples

Example 4.1. (Illustration of Theorem 3.2)

Let us consider the fractional hybrid q -difference equation

$$0.99 D_{0.5}^{1.5} [u(x) - f(x, u(x))] + (1 - 0.99) D_{0.5}^{1.01} u(x) = 0.25 g(x, u(x)) + 0.25 I_{0.5}^{1/5} h(x, u(x)), \quad (14)$$

with q -integral nonlocal boundary conditions

$$u(0) = 0, \quad u(1) = \int_0^{0.25} \frac{(0.25 - qs)^{(0.1-1)}}{\Gamma_q(0.1)} u(s) d_qs + (1 - 0.1) \int_0^{0.35} \frac{(0.35 - qs)^{(0.1-1)}}{\Gamma_q(0.1)} u(s) d_qs, \quad (15)$$

where $\alpha = 1.5$, $q = 0.5$, $\beta = 1.01$, $\eta = 0.25$, $\sigma = 0.35$, $\varepsilon = 1/5$, $v = 0.99$, $\xi = 0.1$, $\gamma_1 = \gamma_2 = 0.1$, $a = b = 0.25$, $x \in [0, 1]$, and

$$f(x, u(x)) = \frac{0.07 |\cos(\pi x)| |u(x)|}{3 + |u(x)|}, \quad g(x, u(x)) = \frac{90 |u(x)|}{1,000(1 + |u(x)|)}, \quad h(x, u(x)) = \frac{8 |u(x)|}{100(1 + |u(x)|)}.$$

Then, $L_1 = 7/100$, $L_2 = 9/100$, $L_3 = 8/100$ as

$$|f(x, u(x)) - f(x, v(x))| \leq \frac{7}{100} (|u(x) - v(x)|), \quad |g(x, u(x)) - g(x, v(x))| \leq \frac{9}{100} (|u(x) - v(x)|),$$

$$|h(x, u(x)) - h(x, v(x))| \leq \frac{8}{100}(|u(x) - v(x)|).$$

With the given data, it is found that $\Delta \approx 0.480477$, $\Omega_1 \approx 5.01705$, $\Omega_2 \approx 0.729128$, $\Omega_3 \approx 0.654832$, $\Omega_4 \approx 0.0458087$, and $\Omega \approx 0.51501 < 1$. Clearly, the assumptions of Theorem 3.2 hold, and hence, problems (14) and (15) have a unique solution on $[0, 1]$ by the conclusion of Theorem 3.2.

Example 4.2. (Illustration of Theorem 3.3)

Consider the fractional hybrid q -difference equation

$$0.99D_{0.5}^{1.5}[u(x) - f(x, u(x))] + (1 - 0.99)D_{0.5}^{1.01}u(x) = 0.25g(x, u(x)) + 0.25I_{0.5}^{1/5}h(x, u(x)), \quad (16)$$

subject to q -integral nonlocal boundary conditions

$$u(0) = 0, \quad u(1) = \int_0^{0.25} \frac{(0.25 - qs)^{(0.1-1)}}{\Gamma_q(0.1)} u(s) d_qs + (1 - 0.1) \int_0^{0.35} \frac{(0.35 - qs)^{(0.1-1)}}{\Gamma_q(0.1)} u(s) d_qs. \quad (17)$$

Here, $\alpha = 1.5$, $q = 0.5$, $\beta = 1.01$, $\eta = 0.25$, $\sigma = 0.35$, $\varepsilon = 1/5$, $\nu = 0.99$, $\xi = 0.1$, $\gamma_1 = \gamma_2 = 0.1$, $a = b = 0.25$, $x \in [0, 1]$, and

$$f(x, u) = \frac{1}{9} \left(\frac{x^2 |u(x)|}{1 + |u(x)|} + 4 \right), \quad g(x, u) = \frac{1}{2} \left(\frac{|\sin u| x}{1 + |\sin u|} + \frac{1}{4} \right), \quad h(x, u) = \frac{1}{5} \left(\frac{x |u(x)|}{1 + |u(x)|} + 7 \right).$$

On the other hand, there exists a continuous function $\psi_1(x) = (x + 3)5$, $\psi_2(x) = (4x + 1)8$, and $\psi_3(x) = (x + 7)5$ on $[0, 1]$. Also, we have $\|\psi_1\| = \sup_{x \in [0, 1]} \psi_1(x) \approx 0.055556$, $\|\psi_2\| = \sup_{x \in [0, 1]} \psi_2(x) \approx 0.3125$, and $\|\psi_3\| = \sup_{x \in [0, 1]} \psi_3(x) \approx 1.6$. Using the given values, we have that $\Omega_4 \approx 0.0458087 < 1$. Clearly, all the assumptions of Theorem 3.3 are satisfied. Therefore, the conclusion of Theorem 3.3 applies, and hence, the fractional q -integro-difference equation (16) with q -integral nonlocal boundary conditions (17) has at least one solution on $[0, 1]$.

Example 4.3. (Illustration of Theorem 3.4) Consider the fractional hybrid q -difference equation

$$0.99D_{0.5}^{1.5}[u(x) - f(x, u(x))] + (1 - 0.99)D_{0.5}^{1.01}u(x) = 0.25g(x, u(x)) + 0.25I_{0.5}^{1/5}h(x, u(x)), \quad (18)$$

subject to q -integral nonlocal boundary conditions

$$u(0) = 0, \quad u(1) = \int_0^{0.25} \frac{(0.25 - qs)^{(0.1-1)}}{\Gamma_q(0.1)} u(s) d_qs + (1 - 0.1) \int_0^{0.35} \frac{(0.35 - qs)^{(0.1-1)}}{\Gamma_q(0.1)} u(s) d_qs. \quad (19)$$

Here, $\alpha = 1.5$, $q = 0.5$, $\beta = 1.01$, $\eta = 0.25$, $\sigma = 0.35$, $\varepsilon = 1/5$, $\nu = 0.99$, $\xi = 0.1$, $\gamma_1 = \gamma_2 = 0.1$, $a = b = 0.25$, $x \in [0, 1]$, and

$$f(x, u) = \frac{3 \cos(u)}{(3 + x)^2}, \quad g(x, u) = \frac{(x + 1)u^2}{u^2 + 1}, \quad h(x, u) = \frac{4 \sin(u)}{(x + 4)^2}.$$

With the given data, we obtain $\Omega_1 \approx 5.01705$, $\Omega_2 \approx 0.729128$, $\Omega_3 \approx 0.654832$, $\Omega_4 \approx 0.0458087$, and

$$|f(x, u)| \leq \frac{3 \cos(|u|)}{(3 + x)^2} = \phi_1(x) \chi_1(|u|), \quad |g(x, u)| \leq \frac{(x + 1)|u|^2}{|u|^2 + 1} = \phi_2(x) \chi_2(|u|), \\ |h(x, u)| \leq \frac{4 \sin(|u|)}{(x + 4)^2} = \phi_3(x) \chi_3(|u|).$$

Clearly, $\phi_1(x) = \frac{3}{(3 + x)^2}$, $\chi_1(|u|) = \cos(|u|)$, $\phi_2(x) = x + 1$, $\chi_2(|u|) = \frac{|u|^2}{|u|^2 + 1}$, $\phi_3(x) = \frac{4}{(x + 4)^2}$, and $\chi_3(|u|) = \sin(|u|)$. Moreover, the condition (H_6) implies that there exists $G > 2.28404$. Thus, the hypotheses of Theorem 3.4 are satisfied. Therefore, the conclusion of Theorem 3.4 implies that the hybrid q -difference equation (18) with nonlocal q -integral boundary conditions (19) has at least one solution on $[0, 1]$.

5 Conclusion

We have developed the existence theory for hybrid fractional q -integro-difference equations involving Riemann-Liouville q -derivatives of different orders, complemented with nonlocal Riemann-Liouville q -integral boundary conditions. Our results are new and contribute significantly to the known literature on nonlocal hybrid q -fractional integral boundary value problems. In particular, our work generalizes the results proved in [2]. Several special cases also follow from the results of this article, for instance, see the following examples.

- (i) For $\xi = 1$, we deduce the results for the nonlinear hybrid Riemann-Liouville fractional q -integro-difference equation (1) subject to the nonlocal boundary conditions of the form:

$$u(0) = 0, \quad u(1) = \int_0^{\eta} \frac{(\eta - qs)^{(\gamma_1-1)}}{\Gamma_q(\gamma_1)} u(s) d_qs, \quad \gamma_1 > 0.$$

- (ii) Our results correspond to the following problem for $\nu = \xi = 1$, $a = 1$, and $b = 0$:

$$D_q^\alpha[u(x) - f(x, u(x))] = g(x, u(x)), \quad 0 < x < 1,$$

$$u(0) = 0, \quad u(1) = \int_0^{\eta} \frac{(\eta - qs)^{(\gamma_1-1)}}{\Gamma_q(\gamma_1)} u(s) d_qs, \quad \gamma_1 > 0.$$

- (iii) We obtain the results for the following problem if we fix $\nu = 1/2$, $a = 0$, $b = 1/2$ in the present results:

$$D_q^\alpha[u(x) - f(x, u(x))] + D_q^\beta u(x) = I_q^\varepsilon h(x, u(x)), \quad 0 < x < 1,$$

$$u(0) = 0, \quad u(1) = \xi \int_0^{\eta} \frac{(\eta - qs)^{(\gamma_1-1)}}{\Gamma_q(\gamma_1)} u(s) d_qs + (1 - \xi) \int_0^{\sigma} \frac{(\sigma - qs)^{(\gamma_2-1)}}{\Gamma_q(\gamma_2)} u(s) d_qs, \quad \gamma_1, \gamma_2 > 0.$$

The aforementioned results are indeed new.

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