Research Article

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Regularity criteria via horizontal component of velocity for the Boussinesq equations in anisotropic Lorentz spaces

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Abstract: In this article, we study the regularity criteria of the weak solutions to the Boussinesq equations involving the horizontal component of velocity or the horizontal derivatives of the two components of velocity in anisotropic Lorentz spaces. This result reveals that the velocity field plays a dominant role in regularity theory of the Boussinesq equations.

Keywords: Boussinesq equations, regularity criterion, weak solutions, anisotropic Lorentz space

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1 Introduction and main result

The 3D Boussinesq system for the incompressible fluid flows in $\mathbb{R}^3$:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla \pi = \theta e_3, \\ \partial_t \theta - \Delta \theta + (u \cdot \nabla)\theta = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x), \end{cases} \tag{1.1}$$

where $u = u(x, t)$ is the velocity vector field, $\theta = \theta(x, t)$ is the scalar temperature, in which case the forcing term θe_3 in the momentum equation (1.1) describes the action of the buoyancy force on fluid motion, $\pi = \pi(x, t)$ is the scalar pressure, while u_0 and θ_0 are the given initial velocity and initial temperature with $\nabla \cdot u_0 = 0$ in the sense of distributions, and $e_3 = (0, 0, 1)^T$ denotes the vertical unit vector. For simplicity, the kinematic viscosity and thermal conductivity are normalized.

The Boussinesq equations play an important role in atmospheric sciences (see, for example, [1,2]), as well as a model in many geophysical applications, and have received significant attention in the mathematical fluid dynamics community because of their close connection to the multi-dimensional incompressible flows (see [3,4]). When θ_0 is identically zero (or some constant), equations (1.1) reduce to the well-known Navier-Stokes equations, which are extremely important equations to describe incompressible fluids. These equations have attracted great interests, among many analysts, and there have been many important developments (see, for example, Lions [5], Temam [6] for survey of important developments).

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Let us review some previous works about the viscous Boussinesq equations. For the two-dimensional case, the well-posedness problem is well understood. For more results in this direction, see [7–11] and references therein. For the three-dimensional case, however, the global existence of strong or smooth solutions in 3D is still an open problem. One can refer to [12–18] and references therein for recent developments along this line. These motivate us to find some possible blow-up criterions of regular solutions to (1.1), especially of strong solutions. The first blow-up criterion in the Lebesgue space was proved by Ishimura and Morimoto [19]:

$$\nabla u \in L^1(0, T; L^\infty(\mathbb{R}^3)).$$

Later, Qiu et al. [20] refined the following blow-up criterion in the largest critical Besov spaces by applying Bony paraproduct decomposition to both the momentum equations and the diffusive equation:

$$u \in L^p(0, T; B^r_{q,\infty}(\mathbb{R}^3)),$$

where $\frac{2}{p} + \frac{3}{q} = 1 + r,\ \frac{3}{1+s} < q \le \infty,\ -1 < r \le 1,$ and $(q, r) \neq (\infty, 1)$. Subsequently, considerable works are devoted to the Boussinesq equations (we refer the readers to the interesting works [21,22] and references cited therein).

The mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^3)$, with $\vec{p} \in (0, \infty]^3$, as a natural generalization of the classical Lebesgue space $L^p(\mathbb{R}^3)$ via replacing the constant exponent p by an exponent vector $\vec{p}$, was investigated by Benedek and Panzone [23]. Motivated by the aforementioned work of Benedek and Panzone [23] on the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^3)$, Fernandez [24] first introduced the anisotropic Lorentz spaces. Indeed, these function spaces with mixed norms have attracted considerable attention, and there has been made great progress since Benedek-Panzone's work. We refer to the readers to consult the recent preprint [25] and references therein. Since the Lorentz spaces with mixed norms have finer structures than the corresponding classical function spaces, they naturally arise in the studies on the solutions of partial differential equations used to model physical processes involving functions in n dimension space variable x and one-dimensional time variable t (see, for instance, [25]).

As a continuation of the previous work [26], in this article, we focus on the Cauchy problem of the three-dimensional incompressible Boussinesq equations to give a further observation on the global regularity of the solution for System (1.1) via two velocity components in anisotropic Lorentz spaces, which are more general than the classical Lorentz spaces $L^{p,q}$ (see, e.g., [23,27–30]). The method presented here may be applicable to similar situations involving other partial differential equations. This work can be modeled to apply for Environmental Health and Safety by using the relationship with in fluid dynamic in atmosphere sciences by some sort of reduction of the Navier-Stokes equation as well as the related equations of viscosity.

Before stating the precise result, let us recall the weak formulation of (1.1).

Definition 1.1. (Weak solution) Given $\theta_0 \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, a pair of functions $\{u(t), \theta(t)\}$ defined for $t \ge 0$ is called a weak solution of the initial value Problem (1.1) if the following statements are valid:

(1)

$$(u, \theta) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)), \theta \in L^\infty(0, T; L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)), \tag{1.2}$$

for all $T > 0$.

(2) $(1.1)_{1,2,3}$ are satisfied in the sense of distributions;

(3) the energy inequality

$$\|u(\cdot,t)\|^2_{L^2} + 2\int_0^t \|\nabla u(\cdot,\tau)\|^2_{L^2} \mathrm{d}\tau \le \|u_0\|^2_{L^2} + 2\int_0^t \int_{\mathbb{R}^3} \theta u_3 \mathrm{d}x \mathrm{d}\tau,$$

$$\|\theta(\cdot,t)\|^2_{L^2} + 2\int_0^t \|\nabla \theta(\cdot,\tau)\|^2_{L^2} \mathrm{d}\tau \le \|\theta_0\|^2_{L^2},$$

for all $0 \le t \le T$.

A solution of (1.1) that is in

$$(u,\theta)\in L^\infty(0,T;\,H^1(\mathbb{R}^3))\cap L^2(0,T;\,H^2(\mathbb{R}^3))$$

is a strong solution. It is known that strong solutions are actually smooth [31] (see also [32], p. 364).

Remark 1.1. A weak solution of (1.1) is regular on time interval I if the Sobolev norm $\|(u,\theta)\|_{H^s}$ is continuous for $s>\frac{1}{2}$ on I. One can apply a standard bootstrap argument to show if a solution is regular, then u and θ are smooth.

Remark 1.2. System (1.1) has scaling property that if (u,θ,π) is a solution of System (1.1), then for any $\lambda>0$, the functions

$$u_\lambda(x,t)=\lambda u(\lambda x,\lambda^2 t),\quad \theta_\lambda(x,t)=\lambda^3\theta(\lambda x,\lambda^2 t),\quad \pi_\lambda(x,t)=\lambda^2\pi(\lambda x,\lambda^2 t)$$

are also solutions of (1.1), with the corresponding initial data

$$u_\lambda(x,0)=\lambda u_0(\lambda x),\quad \theta_\lambda(x,0)=\lambda^3\theta_0(\lambda x).$$

This motivates the study of (1.1) in function spaces that are left invariant by the aforementioned scaling.

$\nabla_h=(\partial_1,\partial_2)$ denotes the horizontal gradient operator, and $\tilde{u}=(u_1,u_2)$ is the horizontal components of the velocity field u. Now, our main results can be stated in the following.

Theorem 1.2. *Denote $\tilde{r}=(r_1,r_2,r_3)>2$. Let $(u_0,\theta_0)\in H^1(\mathbb{R}^3)$ with $\nabla\cdot u_0=0$. Assume that (u,θ) is a weak solution of (1.1) on some interval $[0,T)$ with $0<T<\infty$. If the tangential components of the velocity $\tilde{u}$ satisfy one of the following two conditions:*

(1)

$$\tilde{u}\in L^{\frac{2}{1-(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3})}}(0,T;\,L^{\tilde{r},\infty}(\mathbb{R}^3))\quad \textit{with}\ \sum_{i=1}^{3}\frac{1}{r_i}<1. \tag{1.3}$$

(2) *Assume that $\sum_{i=1}^3\frac{1}{r_i}=1$. Then, there exists $\delta>0$ such that if*

$$\tilde{u}\in L^\infty([0,T];\,L^{\tilde{r},\infty}(\mathbb{R}^3))\quad \textit{and}\quad \|\tilde{u}\|_{L^\infty([0,T];\,L^{\tilde{r},\infty}(\mathbb{R}^3))}\le\delta, \tag{1.4}$$

then the solution $(u,\theta)\in C^\infty((0,T)\times\mathbb{R}^3)$.

The second result deals with regularity criterion expressed by the horizontal derivatives of horizontal components of the velocity under the anisotropic Lorentz space framework.

Theorem 1.3. *Denote $\tilde{r}=(r_1,r_2,r_3)>1$. Let $(u_0,\theta_0)\in H^1(\mathbb{R}^3)$ with $\nabla\cdot u_0=0$. Assume that (u,θ) is a weak solution of (1.1) on some interval $[0,T)$ with $0<T<\infty$. If*

$$\nabla_h\tilde{u}\in L^{\frac{2}{2-(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3})}}(0,T;\,L^{\tilde{r},\infty}(\mathbb{R}^3))\ \textit{with}\ \sum_{i=1}^{3}\frac{1}{r_i}<2, \tag{1.5}$$

then the solution $(u,\theta)\in C^\infty((0,T)\times\mathbb{R}^3)$.

Remark 1.3. Our results improve almost all known regularity criteria involving Lorentz spaces or two components.

Remark 1.4. Since the blow-up criteria in Theorems 1.2 and 1.3 are only on the velocity field u and there is no any additional condition on the temperature field θ, on the one hand, the velocity vector field plays a very important role than the temperature field in the regularity criterion for the Boussinesq equations. On the other hand, thanks to the fact that Boussinesq equations (1.1) with $\theta=0$ reduces the Navier-Stokes equations, Theorems 1.2 and 1.3 cover the previous results [33] on Navier-Stokes equations in the largest anisotropic Lorentz spaces.

We recall some tools from the theories of the anisotropic Lebesgue and Lorentz spaces (for details, see [30]). Throughout this article, we denote C a universal constant depending only on prescribed quantities and possibly varying from line to line. We endow the usual Lebesgue space $L^p(\mathbb{R}^3)$ with the norm $\|\cdot\|_{L^p}$. We denote by $\partial_i = \frac{\partial}{\partial x_i}$ the partial derivative in the x_i-direction. Recall that the anisotropic Lebesgue space consists of all measurable real-valued functions $h = h(x_1, x_2, x_3)$ with the finite norm:

$$\|h\|_{L^{\vec{q}}(\mathbb{R}^3)} = \|h\|_{L^{q_1}_{x_1}L^{q_2}_{x_2}L^{q_3}_{x_3}(\mathbb{R}^3)} = \| \| \|h\|_{L^{q_1}_{x_1}(\mathbb{R})} \|_{L^{q_2}_{x_2}(\mathbb{R})} \|_{L^{q_3}_{x_3}(\mathbb{R})} < \infty.$$

Let us recall the notation of anisotropic Lorentz space, which is a natural generalization of classical Lorentz space.

Definition 1.4. Let multi-indexes $\vec{p} = (p_1, p_2, p_3)$ and $\vec{q} = (q_1, q_2, q_3)$ with $0 < p_i < \infty$, $0 < q_i \le \infty$ for all $i \in \{1, 2, 3\}$, the anisotropic Lorentz space $L^{\vec{p},\vec{q}}(\mathbb{R}^3)$ is a set of measurable functions $h(x_1, x_2, x_3)$ on $\mathbb{R}^3$ such that

$$\|h\|_{L^{\vec{p},\vec{q}}(\mathbb{R}^3)} = \|h\|_{L^{p_1,q_1}_{x_1}L^{p_2,q_2}_{x_2}L^{p_3,q_3}_{x_3}(\mathbb{R}^3)} = \| \| \|h\|_{L^{p_1,q_1}_{x_1}(\mathbb{R})} \|_{L^{p_2,q_2}_{x_2}(\mathbb{R})} \|_{L^{p_3,q_3}_{x_3}(\mathbb{R})} < \infty.$$

For $p_1 = p_2 = p_3 = p$ and $q_1 = q_2 = q_3 = q$, the spaces $L^{\vec{p},\vec{q}}(\mathbb{R}^3)$ coincide with the usual Lorentz space $L^{p,q}(\mathbb{R}^3)$. Note that the anisotropic Lorentz spaces $L^{\vec{p},\vec{q}}(\mathbb{R}^3)$ introduced in [4] are another widely used generalization of isotropic Lebesgue and Lorentz spaces, a natural question arises whether the velocity u in scaling-invariant anisotropic Lorentz spaces also means the regularity of weak solutions.

Lemma 1.5. *Let $\vec{p} = (p_1, p_2, p_3)$ and $\vec{q} = (q_1, q_2, q_3)$ with $1 \le p_i < \infty$, $0 < q_i \le \infty$ for all $i \in \{1, 2, 3\}$.*
(1) *For any $0 < \kappa < \infty$, if $|h|^\kappa \in L^{\vec{p},\vec{q}}(\mathbb{R}^3)$, then $h \in L^{\kappa\vec{p},\kappa\vec{q}}(\mathbb{R}^3)$ and*

$$\| |h|^\kappa \|_{L^{\vec{p},\vec{q}}(\mathbb{R}^3)} = \|h\|^\kappa_{L^{\kappa\vec{p},\kappa\vec{q}}(\mathbb{R}^3)},$$

where $\kappa\vec{p} = (\kappa p_1, \kappa p_2, \kappa p_3)$ and $\kappa\vec{q} = (\kappa q_1, \kappa q_2, \kappa q_3)$.
(2) *If $f, g \in L^{\vec{p},\vec{q}}(\mathbb{R}^3)$, then $f + g \in L^{\vec{p},\vec{q}}(\mathbb{R}^3)$ and*

$$\|f + g\|_{L^{\vec{p},\vec{q}}(\mathbb{R}^3)} \le C(\|f\|_{L^{\vec{p},\vec{q}}(\mathbb{R}^3)} + \|g\|_{L^{\vec{p},\vec{q}}(\mathbb{R}^3)}),$$

where $C = C(\vec{p}, \vec{q})$ is a positive constant.

In the following, we recall the following embedding results for the anisotropic Lorentz spaces with mixed-norm.

Lemma 1.6. *Let $\vec{p} = (p_1, p_2, p_3)$, $\vec{q} = (q_1, q_2, q_3)$, and $\vec{r} = (r_1, r_2, r_3)$ be the vector-valued indices. If $1 < p_i < \infty$, $1 \le q_i \le r_i \le \infty$ for all $i \in \{1, 2, 3\}$, then we have the following sequence of continuous embeddings:*

$$L^{\vec{p},\vec{q}}(\mathbb{R}^3) \hookrightarrow L^{\vec{p},\vec{r}}(\mathbb{R}^3) \hookrightarrow L^{\vec{p},\infty}(\mathbb{R}^3),$$

with

$$\|f\|_{L^{\vec{p},\infty}(\mathbb{R}^3)} \le C\|f\|_{L^{\vec{p},\vec{r}}(\mathbb{R}^3)} \le C\|f\|_{L^{\vec{p},\vec{q}}(\mathbb{R}^3)},$$

where $C = C(\vec{p}, \vec{q}, \vec{r})$ is a positive constant.

Similar to the usual Lebesgue and Sobolev spaces, there is also the Hö lder inequality for the anisotropic Lorentz spaces with mixed-norm.

Lemma 1.7. *Let $1 < p_i, s_i, \alpha_i < \infty$ and $1 \le q_i, r_i, \beta_i \le \infty$ for all $i \in \{1, 2, 3\}$. Then, if for each $i = 1, 2, 3$, it holds the following relations:*

$$\frac{1}{p_i}=\frac{1}{s_i}+\frac{1}{\alpha_i}\quad \text{and}\quad \frac{1}{q_i}=\frac{1}{r_i}+\frac{1}{\beta_i},$$

then for any $f\in L^{\tilde{\alpha},\tilde{\beta}}(\mathbb{R}^3)$ *and* $g\in L^{\tilde{s},\tilde{r}}(\mathbb{R}^3)$, *we have* $fg\in L^{\tilde{p},\tilde{q}}(\mathbb{R}^3)$ *with the estimate*

$$\|fg\|_{L^{\tilde{p},\tilde{q}}(\mathbb{R}^3)}\le \|f\|_{L^{\tilde{\alpha},\tilde{\beta}}(\mathbb{R}^3)}\|g\|_{L^{\tilde{s},\tilde{r}}(\mathbb{R}^3)}.$$

In order to prove the main result, we need to recall the following version of the three-dimensional Sobolev inequality in anisotropic Lebesgue spaces in the whole space $\mathbb{R}^3$, which is proved in [34]. In fact, since $L^{\tilde{q},2}\hookrightarrow L^{\tilde{q},\tilde{q}}$ for $2\le q_i\le\infty$ for all $i\in\{1,2,3\}$, we have

Lemma 1.8. *Let us assume that* $2\le q_1,q_2,q_3<\infty$ *and*

$$\frac{1}{2}\le\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}\le\frac{3}{2}.$$

Then, for $f\in C_0^\infty(\mathbb{R}^3)$, *we have the following estimate*:

$$\begin{aligned}\|f\|_{L^{\tilde{q},2}}&\le C\|\partial_1 f\|_{L^2}^{\frac{q_1-2}{2q_1}}\|\partial_2 f\|_{L^2}^{\frac{q_2-2}{2q_2}}\|\partial_3 f\|_{L^2}^{\frac{q_3-2}{2q_3}}\|f\|_{L^2}^{(\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3})-\frac{1}{2}}\\&\le C\|f\|_{L^2}^{(\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3})-\frac{1}{2}}\|\nabla f\|_{L^2}^{\frac{3}{2}-(\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3})}.\end{aligned}\tag{1.6}$$

and C *is a constant independent of* f.

Here, we make use of the following Gronwall-type inequality, which is a variant of the standard Gronwall's inequality as presented in [35, Appendix B.2.j].

Lemma 1.9. *Let* $\eta(\cdot)$ *be a nonnegative, absolutely continuous function on* $[0,T]$, *which satisfies for a.e.* t *the inequality*

$$\eta'(t)+\psi(t)\le\varphi(t)\eta(t),$$

where $\varphi(t)$ *and* $\psi(t)$ *are nonnegative, summable functions on* $[0,T]$. *Then,*

$$\eta(t)+\int_0^t\psi(\tau)\mathrm{d}\tau\le\eta(0)\exp\left(\int_0^t\varphi(\tau)\mathrm{d}\tau\right).$$

The proof follows the same idea as that presented in [35] and is omitted.

2 Proof of Theorem 1.2

We introduce the main ideas of the proof of Theorems 1.2 and 1.3. It is well known that there exists a unique local strong solution to 3D Boussinesq equations (see [36]). For $(u_0,\theta_0)\in H^1(\mathbb{R}^3)$ with $\nabla\cdot u_0=0$, the weak solution is the same as the strong solution in short interval $(0,T)$. If we can find *a priori* uniform H^1-bound in $(0,T)$ for the strong solution with the regularity condition of our main Theorem 1.2, then the solution can be continuously extended to the time $t=T$ argued by standard continuation process (see, e.g., [37]). Thus, the main Theorem 1.2 is reduced to establish the uniform H^1-bound for such strong solution.

We are ready to present the proof of Theorem 1.2.

Proof. Let $T>0$ be any given fixed time. From the second equation of (1.1), one may easily show that for any $s\in[2,\infty]$,

$$\|\theta(\cdot,t)\|_{L^s}\le\|\theta_0\|_{L^s},\quad t\in[0,T],\tag{2.1}$$

where we have used the divergence-free condition $\nabla\cdot u = 0$, so

$$\theta \in L^{\infty}(0, T; L^{s}(\mathbb{R}^3)). \tag{2.2}$$

Next, taking the L^2-inner product of the first equation and the second equation in (1.1) with $(-\Delta u)$ and $(-\Delta\theta)$, respectively, and integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla\theta(\cdot,t)\|_{L^2}^2) + \|\Delta u(\cdot,t)\|_{L^2}^2 + \|\Delta\theta(\cdot,t)\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\mathrm{d}x + \int_{\mathbb{R}^3}(u\cdot\nabla)\theta\cdot\Delta\theta\mathrm{d}x - \int_{\mathbb{R}^3}\theta e_3\cdot\Delta u\mathrm{d}x \\ &= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \end{aligned} \tag{2.3}$$

where

$$\mathcal{I}_1 = \int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\mathrm{d}x, \quad \mathcal{I}_2 = \int_{\mathbb{R}^3}(u\cdot\nabla)\theta\cdot\Delta\theta\mathrm{d}x \quad \text{and} \quad \mathcal{I}_3 = -\int_{\mathbb{R}^3}\theta e_3\cdot\Delta u\mathrm{d}x.$$

For $\mathcal{I}_3$, it follows from the Cauchy-Schwarz inequality that

$$\mathcal{I}_3 = \sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k\theta\partial_k u_3\mathrm{d}x \leq \|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2} \leq \frac{1}{2}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).$$

By the Sobolev inequality $\|f\|_{L^6} \leq C\|\nabla f\|_{L^2}$, the Hölder inequality, and Young inequality, we obtain

$$\begin{aligned} \mathcal{I}_2 &\leq \|\Delta\theta\|_{L^2}\|u\|_{L^6}\|\nabla\theta\|_{L^3} \\ &\leq C\|\Delta\theta\|_{L^2}\|\nabla u\|_{L^2}\|\nabla\theta_0\|_{L^2}^{\frac{1}{2}}\|\Delta\theta\|_{L^2}^{\frac{1}{2}} \\ &\leq C\|\nabla u\|_{L^2}^4\|\nabla\theta_0\|_{L^2}^2 + \frac{1}{2}\|\Delta\theta\|_{L^2}^2, \end{aligned}$$

where we have used the following interpolation inequality with (2.1):

$$\|\nabla\theta\|_{L^3} \leq C\|\theta\|_{L^6}^{\frac{1}{2}}\|\Delta\theta\|_{L^2}^{\frac{1}{2}} \leq C\|\theta_0\|_{L^6}^{\frac{1}{2}}\|\Delta\theta\|_{L^2}^{\frac{1}{2}} \leq C\|\nabla\theta_0\|_{L^2}^{\frac{1}{2}}\|\Delta\theta\|_{L^2}^{\frac{1}{2}},$$

due to the interpolation inequality and the energy inequality in Definition 1.1.

Now, we work on the first term on the right-hand side $\mathcal{I}_1$. Using integration by parts, we obtain

$$\begin{aligned} \mathcal{I}_1 &= -\int_{\mathbb{R}^3}\nabla(u\cdot\nabla)u\cdot\nabla u\mathrm{d}x = -\sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u\cdot\nabla u\cdot\partial_k u\mathrm{d}x \\ &= -\sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k\tilde{u}\cdot\nabla_h u\partial_k u\mathrm{d}x - \sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_3\partial_3 u\partial_k u\mathrm{d}x \\ &= \sum_{k=1}^{3}\int_{\mathbb{R}^3}\tilde{u}\cdot\nabla_h\partial_k u\partial_k u\mathrm{d}x + \sum_{k=1}^{3}\int_{\mathbb{R}^3}\tilde{u}\cdot\nabla_h u\partial_k\partial_k u\mathrm{d}x - \sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_3\partial_3 u\partial_k u\mathrm{d}x \\ &= \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13}. \end{aligned}$$

Case 1. Let

$$\tilde{u} \in L^{\frac{2}{1-(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3})}}(0, T; L^{\vec{r},\infty}(\mathbb{R}^3)) \quad \text{with} \ \sum_{i=1}^{3}\frac{1}{r_i} < 1. \tag{2.4}$$

Applying for the Hölder inequality in anisotropic Lorentz spaces and Lemma 1.8, we estimate the first integral $\mathcal{I}_{11}$ as:

$$
\begin{aligned}
|\mathcal{I}_{11}| &\le C\int_{\mathbb{R}^3}|\tilde{u}||\nabla u||\Delta u|\mathrm{d}x \\
&\le C\|\tilde{u}\|_{L^{\vec{r},\infty}}\|\nabla u\|_{L^{\left(\frac{2r_1}{r_1-2},\frac{2r_2}{r_2-2},\frac{2r_3}{r_3-2}\right),2}}\|\Delta u\|_{L^2} \\
&\le C\|\tilde{u}\|_{L^{\vec{r},\infty}}\|\nabla u\|_{L^2}^{\left(\frac{r_1-2}{2r_1}+\frac{r_2-2}{2r_2}+\frac{r_3-2}{2r_3}\right)-\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{3}{2}-\left(\frac{r_1-2}{2r_1}+\frac{r_2-2}{2r_2}+\frac{r_3-2}{2r_3}\right)}\|\Delta u\|_{L^2} \\
&= C\|\tilde{u}\|_{L^{\vec{r},\infty}}\|\nabla u\|_{L^2}^{1-\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)}\|\Delta u\|_{L^2}^{1+\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)} \\
&\le \frac{1}{6}\|\Delta u\|_{L^2}^2 + C\|\tilde{u}\|_{L^{\vec{r},\infty}}^{\frac{2}{1-\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)}}\|\nabla u\|_{L^2}^2.
\end{aligned}
$$

Similarly, the second integral $\mathcal{I}_{12}$ can be estimated as:

$$
|\mathcal{I}_{12}| \le \frac{1}{6}\|\Delta u\|_{L^2}^2 + C\|\tilde{u}\|_{L^{\vec{r},\infty}}^{\frac{2}{1-\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)}}\|\nabla u\|_{L^2}^2.
$$

Using integration by parts twice and divergence-free condition, we infer that

$$
\begin{aligned}
\mathcal{I}_{13} &= -\sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_3\partial_3\tilde{u}\partial_k\tilde{u}\mathrm{d}x - \sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_3\partial_3 u_3\partial_k u_3\mathrm{d}x \\
&= \sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_3\partial_k u_3\tilde{u}\partial_k\tilde{u}\mathrm{d}x + \sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_3\tilde{u}\partial_3\partial_k\tilde{u}\mathrm{d}x + \sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_3(\partial_1 u_1 + \partial_2 u_2)\partial_k u_3\mathrm{d}x \\
&= \sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_3\partial_k u_3\tilde{u}\partial_k\tilde{u}\mathrm{d}x + \sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_3\tilde{u}\partial_3\partial_k\tilde{u}\mathrm{d}x - 2\sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_1\partial_k u_3 u_1\partial_k u_3\mathrm{d}x - 2\sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_2\partial_k u_3 u_2\partial_k u_3\mathrm{d}x.
\end{aligned}
$$

By a slight modification of the proof of $\mathcal{I}_{11}$, we find that

$$
|\mathcal{I}_{13}| \le \frac{1}{6}\|\Delta u\|_{L^2}^2 + C\|\tilde{u}\|_{L^{\vec{r},\infty}}^{\frac{2}{1-\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)}}\|\nabla u\|_{L^2}^2.
$$

Substituting $\mathcal{I}_{11}$, $\mathcal{I}_{12}$, and $\mathcal{I}_{13}$ into $\mathcal{I}_1$, we obtain

$$
\mathcal{I}_1 \le \frac{1}{2}\|\Delta u\|_{L^2}^2 + C\|\tilde{u}\|_{L^{\vec{r},\infty}}^{\frac{2}{1-\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)}}\|\nabla u\|_{L^2}^2.
$$

Inserting the aforementioned estimates into (2.3), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla\theta(\cdot,t)\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2 \\
&\le C(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + C\|\nabla u\|_{L^2}^4\|\nabla\theta_0\|_{L^2}^2 + C\|\tilde{u}\|_{L^{\vec{r},\infty}}^{\frac{2}{1-\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)}}\|\nabla u\|_{L^2}^2 \\
&\le C(1 + \|\tilde{u}\|_{L^{\vec{r},\infty}}^{\frac{2}{1-\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)}} + \|\nabla u\|_{L^2}^2\|\nabla\theta_0\|_{L^2}^2)(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).
\end{aligned}
$$

Hence, using Gronwall's inequality and (1.2), (2.2) together with (2.4), we obtain

$$
(u,\theta) \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)).
$$

Thus, (u,θ) can be extended smoothly beyond $T > 0$.

Case 2. Next, we consider the case of

$$\tilde{u} \in L^\infty([0,T]; L^{\vec{r},\infty}(\mathbb{R}^3)) \quad \text{with} \quad \|\tilde{u}\|_{L^\infty([0,T];\, L^{\vec{r},\infty}(\mathbb{R}^3))} \le \delta \quad \text{and} \quad \sum_{i=1}^{3} \frac{1}{r_i} = 1. \tag{2.5}$$

Using Hölder's the inequality in anisotropic Lorentz spaces and Sobolev imbedding theory, we have

$$\begin{aligned}
|\mathcal{I}_{11}| &\le C \int_{\mathbb{R}^3} |\tilde{u}||\nabla u||\Delta u| \mathrm{d}x \\
&\le \frac{1}{6}\|\Delta u\|_{L^2}^2 + C\||\tilde{u}||\nabla u|\|_{L^2}^2 \\
&\le \frac{1}{6}\|\Delta u\|_{L^2}^2 + C\|\tilde{u}\|_{L^{\vec{r},\infty}}^2 \|\nabla u\|^2_{L^{\left(\frac{2r_1}{r_1-2}, \frac{2r_2}{r_2-2}, \frac{2r_3}{r_3-2}\right),2}} \\
&\le \frac{1}{6}\|\Delta u\|_{L^2}^2 + C\|\tilde{u}\|_{L^{\vec{r},\infty}}^2 \|\nabla u\|_{L^2}^{2-2(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3})} \|\Delta u\|_{L^2}^{2\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)} \\
&\le \frac{1}{6}\|\Delta u\|_{L^2}^2 + C\|\tilde{u}\|_{L^{\vec{r},\infty}}^2 \|\Delta u\|_{L^2}^2.
\end{aligned}$$

Similarly, we have

$$|\mathcal{I}_{12}| \le \frac{1}{6}\|\Delta u\|_{L^2}^2 + C\|\tilde{u}\|_{L^{\vec{r},\infty}}^2 \|\Delta u\|_{L^2}^2.$$

The third integral $\mathcal{I}_{13}$ can be treated in a similar way. Then, exactly following a similar argument as before, we may show our result in this case. Here, we do not repeat the details here.

Inserting the aforementioned estimates into (2.3), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla\theta(\cdot,t)\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2 \\
&\quad \le \|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 \|\nabla\theta_0\|_{L^2}^2 + C\|\tilde{u}\|_{L^{\vec{r},\infty}}^2 \|\Delta u\|_{L^2}^2.
\end{aligned}$$

Taking $\delta > 0$ sufficiently small such that

$$\|\tilde{u}\|_{L^\infty([0,T];\, L^{\vec{r},\infty}(\mathbb{R}^3))} \le \delta,$$

and absorbing the term, $C\delta^2\|\Delta u\|_{L^2}^2$ on the left hand, we derive the following estimate:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla\theta(\cdot,t)\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2 \\
&\quad \le \|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 \|\nabla\theta_0\|_{L^2}^2 \\
&\quad \le C(1 + \|\nabla u\|_{L^2}^2 \|\nabla\theta_0\|_{L^2}^2)(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).
\end{aligned}$$

Gronwall inequality together with (2.5) then implies that

$$(u,\theta) \in L^\infty(0,T; H^1(\mathbb{R}^3)) \cap L^2(0,T; H^2(\mathbb{R}^3)).$$

Then, $(u,\theta) \in C^\infty((0,T)\times\mathbb{R}^3)$. This completes the proof of Theorem 1.2. □

3 Proof of Theorem 1.3

In this section, we will complete the proof of Theorem 1.3. As for proving Theorem 1.2, we first assume that (u,θ) is a weak solution and need to derive only some *a priori* strong estimates of ∇u and $\nabla\theta$, which are uniform in $t \in [0,T]$.

Proof. Let

$$\nabla_h \tilde{u} \in L^s(0,T;\, L^{\tilde{r},\infty}(\mathbb{R}^3)) \quad \text{with } \frac{2}{s} + \sum_{i=1}^{3} \frac{1}{r_i} = 2. \tag{3.1}$$

We estimate $\mathcal{I}_2$ and $\mathcal{I}_3$ like in the previous theorem. Now, we split $\mathcal{I}_1$ as follows:

$$\begin{aligned}\mathcal{I}_1 = &-\sum_{i,j,k=1}^{2} \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j \mathrm{d}x - \sum_{i,j=1}^{2} \int_{\mathbb{R}^3} \partial_3 u_i \partial_i u_j \partial_3 u_j \mathrm{d}x - \sum_{i,k=1}^{2} \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_3 \partial_k u_3 \mathrm{d}x - \sum_{i=1}^{2} \int_{\mathbb{R}^3} \partial_3 u_i \partial_i u_3 \partial_3 u_3 \mathrm{d}x \\ &- \sum_{j,k=1}^{2} \int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_j \partial_k u_j \mathrm{d}x - \sum_{j=1}^{2} \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3 u_j \partial_3 u_j \mathrm{d}x - \sum_{k=1}^{2} \int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_3 \partial_k u_3 \mathrm{d}x \\ = &\sum_{m=1}^{7} J_{1m}.\end{aligned}$$

Taking advantage of the definition of $\nabla_h \tilde{u}$, we have

$$\left| \sum_{m=1}^{4} J_{1m} \right| \le C \int_{\mathbb{R}^3} |\nabla_h \tilde{u}| |\nabla u|^2 \mathrm{d}x.$$

Since $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$, it readily follows that

$$\left| \sum_{m=5}^{7} J_{1m} \right| \le C \int_{\mathbb{R}^3} |-\partial_1 u_1 - \partial_2 u_2| |\nabla u|^2 \mathrm{d}x \le C \int_{\mathbb{R}^3} |\nabla_h \tilde{u}| |\nabla u|^2 \mathrm{d}x.$$

Thus, we obtain

$$\mathcal{I}_1 = \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u \mathrm{d}x \le C \int_{\mathbb{R}^3} |\nabla_h \tilde{u}| |\nabla u|^2 \mathrm{d}x.$$

Using Hölder's inequality in anisotropic Lorentz spaces and Lemma 1.8, $\mathcal{I}_1$ can be estimated as:

$$\begin{aligned}|\mathcal{I}_1| &\le C \|\nabla_h \tilde{u}\|_{L^{\tilde{r},\infty}} \|\nabla u\|^2_{L^{\left(\frac{2r_1}{r_1-1}, \frac{2r_2}{r_2-1}, \frac{2r_3}{r_3-1}\right),2}} \\ &\le C \|\nabla_h \tilde{u}\|_{L^{\tilde{r},\infty}} \|\nabla u\|_{L^2}^{2-\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)} \|\Delta u\|_{L^2}^{\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)} \\ &\le \frac{1}{8} \|\Delta u\|_{L^2}^2 + C \|\nabla_h \tilde{u}\|_{L^{\tilde{r},\infty}}^{\frac{2}{2-\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)}} \|\nabla u\|_{L^2}^2.\end{aligned}$$

By combining the estimates for $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$, we deduce that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla \theta(\cdot,t)\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta \theta\|_{L^2}^2 \\ &\le (\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) + C \|\nabla u\|_{L^2}^4 \|\nabla \theta_0\|_{L^2}^2 + C \|\nabla_h \tilde{u}\|_{L^{\tilde{r},\infty}}^{\frac{2}{2-\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)}} \|\nabla u\|_{L^2}^2 \\ &\le C\Big(1 + \|\nabla_h \tilde{u}\|_{L^{\tilde{r},\infty}}^{\frac{2}{2-\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)}} + \|\nabla u\|_{L^2}^2 \|\nabla \theta_0\|_{L^2}^2\Big)(\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2).\end{aligned}$$

Gronwall inequality together with (2.4) then implies that

$$(u,\theta) \in L^\infty(0,T;\, H^1(\mathbb{R}^3)) \cap L^2(0,T;\, H^2(\mathbb{R}^3)).$$

Then, $(u,\theta) \in C^\infty((0,T) \times \mathbb{R}^3)$. This completes the proof of Theorem 1.3. □

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