DE GRUYTER Demonstratio Mathematica 2023; 56: 20220220

Research Article

Adel Abd Elaziz El-Sayed*

Pell-Lucas polynomials for numerical treatment of the nonlinear fractional-order Duffing equation

https://doi.org/10.1515/dema-2022-0220
received August 30, 2022; accepted February 28, 2023

Abstract: The nonlinear fractional-order cubic-quintic-heptic Duffing problem will be solved through a new numerical approximation technique. The suggested method is based on the Pell-Lucas polynomials' operational matrix in the fractional and integer orders. The studied problem will be transformed into a nonlinear system of algebraic equations. The numerical expansion containing unknown coefficients will be obtained numerically via applying Newton's iteration method to the claimed system. Convergence analysis and error estimates for the introduced process will be discussed. Numerical applications will be given to illustrate the applicability and accuracy of the proposed method.

Keywords: Duffing equation, Caputo fractional operator, Pell-Lucas polynomials, Fractional-order operational matrix, Tau method

MSC 2020: 26A33, 11B39, 65L05, 97N40, 34A08, 34C15, 41A25

1 Introduction

Mathematicians, physicists, and engineers have undertaken a multidisciplinary endeavor to obtain a new instrument for describing many complex problems [1–3]. One of the most important outcomes of these efforts is the fractional calculus field [4]. This branch of science enables researchers to create a flow of ideas for solving and describing many real-world problems [5,6]. Therefore, there are accurate descriptions for a lot of application problems in terms of fractional-order differential equations in a wide range of fields, such as physics [7], biology [8], mechanics [9], medical [10], astrophysics [11], engineering [12], and chemistry [13].

Other than modeling fractional-order differential equations, solutions to these models can be considered one of the important aspects as well. Several numerical techniques are used for solutions for fractional differential equations such as wavelet method [14], a domain decomposition technique [15], spectral Legendre method [16], Laplace transform approach [17], Chebyshev collocation [18], Tau procedure [19], variational iteration method and differential transformation technique [20], Homotopy analysis approach [21], operational matrix approach [22], finite difference method [23], nonstandard finite difference [24], and other techniques [25–29]. The major characteristic of using the spectral Tau method and operational matrix method is that it reduces the fractional-order problems to a system of algebraic equations. Moreover, the advantage of using operational matrices and the Tau method is their simple procedure, rapid convergence, and easy computation.

*** Corresponding author: Adel Abd Elaziz El-Sayed,** Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63514, Egypt; Department of Mathematics, College of Education, University of Technology and Applied Sciences, Al-Rustaq, Oman, e-mail: aaa18@fayoum.edu.eg, aelsayed.rus@cas.edu.om

As we know, the nonlinear differential equations appear to detail many physical phenomena located around us [21]. One of these equations is the Duffing equation, whose general form is given by

$$u'' + au' + bu + cu^3 + du^5 + eu^7 = g(t), \quad u = u(t), \ t \in [0, 1], \tag{1}$$

subject to the conditions

$$u(0) = u_0, \quad u'(0) = u_1. \tag{2}$$

Equation (1) is called cubic, if $d = e = 0$; cubic-quintic, if $e = 0$; and cubic-quintic-heptic, if all coefficients are nonzeros. Also, the nonfractional-order Duffing equation is a well-known nonlinear equation that is adopted as a strong tool to handle some significant practical phenomena in applied science [30]. This equation was used in the middle of the twentieth century to study electronics as in [31]. It is the most uncomplicated oscillator, representing catastrophic rises in amplitude and phase when the frequency of the forcing term is practiced as a gradually varying parameter. Also, the Duffing equation has a wide appearance in applications such as brain modeling [32], Duffing oscillators for passive islanding detection of inverter-based distributed generation units [33], electromagnetic pulses' nonlinear media propagation [3], radar systems and digital communication [34], nonlinear electrical circuits [35], and other applications. Moreover, there exist some trials for solving the Duffing equation numerically [30,36].

As a result of the importance and appearance of the Duffing equations in many applications, the researchers have studied the fractional form of this equation [37], but there are not many other works on this topic. Therefore, this article will introduce a numerical treatment for the general formula of the nonlinear Duffing equation (cubic-quintic-heptic equation). Consider the following formula for this equation:

$$\mathbb{D}^{\mu}u + a\mathbb{D}^{\beta}u + bu + cu^3 + du^5 + eu^7 = g(x), \quad \mu \in \,]1, 2], \quad \beta \in \,]0, 1], \ x \in [0, 1], \tag{3}$$

subject to the conditions

$$u(0) = u_0, \quad u'(0) = u_1, \tag{4}$$

where $u = u(x)$, the fractional terms $\mathbb{D}^{\mu}$ and $\mathbb{D}^{\beta}$ are described in the Caputo's definition; a, b, c, d, and e are known coefficient values; the damping controller is a; and the initial values for the problem are u_0 and u_1. Motivated and stimulated by the above-described works, we investigate a new operational matrix technique of integer and fractional order in terms of Pell-Lucas polynomials and apply these matrices to solve the problem in equation (3).

This work makes three main significant contributions: first, it introduces a new method for numerically solving a nonlinear fractional-order Duffing equation of various orders using operational matrices of fractional-order derivatives of Pell-Lucas polynomials. Second, it presents an algorithm that combines the use of the spectral and Tau methods to solve a fractional-order cubic-quintic-heptic Duffing problem. Third, pay close attention to the convergence analysis that results from the suggested Pell-Lucas expansion. Finally, it illustrates that a variety of fractional-order differential equation issues can be solved using the created operational matrix and methodology.

The organization of this work is as follows: In Section 2, briefly, some tools of the fractional calculus, in addition to definitions and mathematical formulae of Pell-Lucas polynomials, are presented. In Section 3, integer and fractional-order operational matrices in terms of Pell-Lucas polynomials are constructed. In Section 4, expression of the problem in terms of derived matrices and claimed numerical solution is given. In Section 5, a global error estimate and convergence analysis for the suggested Pell-Lucas expansion will be derived. Section 6 demonstrates the accuracy and efficiency of the proposed method by introducing some test examples. Section 7 gives the concluding remarks.

2 Preliminaries and principal formulae

Some principles of fractional calculus theory are presented in this section and will be helpful throughout the rest of the article. Additionally, a description of Pell-Lucas polynomials is provided, along with a few new formulas related to them.

2.1 Fractional and integral operators

Definition 2.1. [4] The fractional-order integral operator I^{μ} on the Lebesgue space $L_1[0,1]$ in Riemann-Liouville sense is defined as follows:

$$(I^{\mu})g(x) = \begin{cases} \dfrac{1}{\Gamma(\mu)}\displaystyle\int_0^x (x-\tau)^{\mu-1}g(\tau)\mathrm{d}\tau, & \mu > 0,\\ g(x), & \mu = 0. \end{cases} \tag{5}$$

The Riemann-Liouville integration operator meets the following criteria:

$$\begin{aligned} &(i) \quad I^{\mu}I^{\gamma} = I^{\gamma}I^{\mu},\\ &(ii) \quad I^{\mu}I^{\gamma} = I^{\mu+\gamma},\\ &(iii) \quad I^{\mu}x^{i} = \frac{\Gamma(i+1)}{\Gamma(i+\mu+1)}x^{i+\mu}, \end{aligned}$$

where $\mu, \gamma \geq 0$, and $i > -1$.

Definition 2.2. [4] The fractional-order derivative of order $\mu > 0$ in Riemann-Liouville sense is given by

$$\mathbb{D}_{*}^{\mu}g(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{k} I^{k-\mu}g(x), \quad k-1 \leq \mu < k, \ \ k \in \mathbb{N}. \tag{6}$$

Definition 2.3. [26] Consider the function $g(x)$, which has the following differential formula:

$$\mathbb{D}^{\mu}g(x) = \frac{1}{\Gamma(k-\mu)}\int_0^x (x-\tau)^{k-\mu-1}g^{(k)}(\tau)\mathrm{d}\tau, \quad \mu > 0, \ \ x > 0. \tag{7}$$

This is known as the Caputo differential operator, where $k-1 \leq \mu < k$, $k \in \mathbb{N}$.

The following relations are satisfied by the operator $\mathbb{D}$:

$$\begin{aligned} (\mathbb{D}^{\mu}I^{\mu})g(x) &= g(x),\\ (I^{\mu}\mathbb{D}^{\mu})g(x) &= g(x) - \sum_{i=0}^{m-1}\frac{g^{(i)}(0^{+})}{i!}x^{i}, \quad x > 0,\\ \mathbb{D}^{\mu}x^{r} &= \frac{\Gamma(r+1)}{\Gamma(r+1-\mu)}x^{r-\mu}, \quad r \in \mathbb{N}, \ \ r \geq \lceil \mu \rceil, \end{aligned} \tag{8}$$

where the ceiling notation is $\lceil \mu \rceil$. For a more thorough examination of the fractional-order operators for differentiation and integration, one can see [4,26].

2.2 Pell-Lucas polynomial overview

Generalized Lucas polynomials have also been widely studied and used in various areas of mathematics, such as the study of Diophantine equations, number theory, and the solution of fractional- and integer-order differential equations [38–42]. Pell-Lucas polynomials are a specific case of generalized Lucas polynomials [43]. However, the properties, applications, and uses of both types of polynomials are sometimes different depending on the properties and the application area because the properties of Pell-Lucas polynomials are relatively simple and well-known, whereas the properties of generalized Lucas polynomials are more complex [43].

Definition 2.4. The following power expression defines Pell-Lucas polynomials of degree $m \geq 1$ in the variable x.

$$\mathrm{PL}_m(x) = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m\Gamma(m-i)}{\Gamma(i+1)\Gamma(m-2i+1)} (2x)^{m-2i}, \tag{9}$$

where $\lfloor \lambda \rfloor$ is the largest integer $\leq \lambda$.

Also, Pell-Lucas polynomials, $\mathrm{PL}_m(x)$, can be produced by adopting the subsequent recurrence relation

$$\mathrm{PL}_{m+1}(x) = 2x\mathrm{PL}_m(x) + \mathrm{PL}_{m-1}(x), \quad m \geq 1, \; x \in \mathbb{R}, \tag{10}$$

with the starting functions $\mathrm{PL}_0(x) = 2$, $\mathrm{PL}_1(x) = 2x$.

Additionally, the polynomials defined by $\mathrm{PL}_m(x)$ and their Binet's formula are as follows:

$$\mathrm{PL}_m(x) = (x + \sqrt{x^2+1})^m + (x - \sqrt{x^2+1})^m. \tag{11}$$

The generating functions for $\mathrm{PL}_m(x)$ are found according to the following equation:

$$\sum_{k=0}^{\infty} \mathrm{PL}_{k+1}(x)t^k = \frac{2(x+t)}{1-2xt-t^2}. \tag{12}$$

Theorem 1. *The power function x^k can be expressed in terms of the Pell-Lucas polynomials according to the following*:

$$x^k = \frac{1}{2^k} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^i \Gamma(k+1)\xi_{k-2i}}{\Gamma(i+1)\Gamma(k-i+1)} \mathrm{PL}_{k-2i}(x), \tag{13}$$

where

$$\xi_{k-2i} = \begin{cases} \frac{1}{2}, & i = \frac{k}{2}, \\ 1, & i < \frac{k}{2}. \end{cases} \tag{14}$$

Proof. Equation (13) can be easily proved with the aid of [43]. □

Theorem 2. *The following relation can be employed to illustrate how the original functions of Pell-Lucas polynomials and their first derivative are related*:

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{PL}_m(x) = 2m \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^i \xi_{m-2i-1} \mathrm{PL}_{m-2i-1}(x), \quad m > 2. \tag{15}$$

Proof. With the help of [43], equation (15) may be demonstrated simply. □

Presently, the analytical description of Pell-Lucas polynomials that were stated in equation (9) can be rephrased as the following congruent formula:

$$\mathrm{PL}_m(x) = m \sum_{i=0}^{m} \frac{2^{i+1}\Gamma\left(\frac{m+i+2}{2}\right)\delta_{m+i}}{(m+i)\Gamma(i+1)\Gamma\left(\frac{m-i+2}{2}\right)} x^i, \; m \geq 1, \tag{16}$$

where

$$\delta_z = \begin{cases} 1, & z \text{ even}, \\ 0, & z \text{ odd}. \end{cases} \tag{17}$$

Also, equation (13) is equivalent to

$$x^m = \frac{\Gamma(m+1)}{2^m} \sum_{\substack{i=0 \\ (i+m) \text{ even}}}^{m} \frac{(-1)^{\frac{m-i}{2}} \xi i}{\Gamma\left(\frac{m+i+2}{2}\right)\Gamma\left(\frac{m-i+2}{2}\right)} \mathrm{PL}_i(x), \quad m \geq 1. \tag{18}$$

The two latter equations will be applied within some of the suggestion theorems in this article. For more details and knowledge about Pell-Lucas polynomials and their associated characteristics, see [44–46].

3 Operational matrices of derivatives for Pell-Lucas polynomials

In this section, we look into the operational matrices of Pell-Lucas polynomials for both the integer and fractional orders of derivatives.

3.1 Integer-order operational matrix of derivatives for Pell-Lucas polynomials

Consider a square Lebesgue function $u(x)$ that can be integrated on $(0, 1)$. Take into account that the Pell-Lucas polynomials can be utilized to describe the function $u(x)$ as a linear independent combination of their terms as follows:

$$u(x) = \sum_{i=0}^{\infty} c_i \mathrm{PL}_i(x). \tag{19}$$

As a result of the approximation theory, we are able to truncate all terms except for the first $(N+1)$-terms of the infinite expansion, equation (19) became as follows:

$$u(x) \approx u_N(x) = \sum_{i=0}^{N} a_i \mathrm{PL}_i(x) = \mathbf{A}^T \Omega(\boldsymbol{x}), \tag{20}$$

where

$$\mathbf{A}^T = [a_0, a_1, \ldots, a_N] \tag{21}$$

and

$$\Omega(\boldsymbol{x}) = [\mathrm{PL}_0(x), \mathrm{PL}_1(x), \ldots, \mathrm{PL}_N(x)]^T. \tag{22}$$

Let the first derivative of the vector $\Omega(\boldsymbol{x})$ be described in the matrix form expression as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}\Omega(\boldsymbol{x}) = \mathbf{W}^{(1)}\Omega(\boldsymbol{x}), \tag{23}$$

where $\mathbf{W}^{(1)} = (w_{lm}^{(1)})$ is the $(N+1) \times (N+1)$ operational matrix of integer-order derivatives. The inputs elements of $W^{(1)}$ can be achieved within Theorem 2, equation (15). These components can be represented explicitly by

$$w_{0 \leq l, m \leq N}^{(1)} = \begin{cases} 2l(-1)^{\frac{l-m+1}{2}} \xi_m, & l > m, \ (l+m) \text{ odd}, \\ 0, & l \leq m, \ (l+m) \text{ even}. \end{cases} \tag{24}$$

For example, for $N = 7$, the operational matrix of the first derivative, $\mathbf{W}^{(1)}$, is claimed by

$$\mathbf{W}^{(1)} = 2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ \frac{-1}{2} & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ \frac{1}{2} & 0 & -1 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}_{8\times 8}.$$

Hence, with the aid of Theorem 2, together with the two equations (23) and (24), we can produce an integer-order operational matrix of derivatives in the generalized description for Pell-Lucas polynomials as follows:

$$\frac{\mathrm{d}^M}{\mathrm{d}x^M}\Omega(\boldsymbol{x}) = \mathbf{W}^{(M)}\Omega(\boldsymbol{x}) = (\mathbf{W}^{(1)})^M\Omega(\boldsymbol{x}), \tag{25}$$

where M is the integer order of the derivatives $M \geq 1$.

3.2 Fractional-order operational matrix of derivatives for Pell-Lucas polynomials

Theorem 3. *Assume $\Omega(\boldsymbol{x})$ to be the Pell-Lucas polynomial vector that is defined in equation (22). For any $\mu > 0$ and for $x \in (0, L)$, one has*

$$\mathbb{D}^{\mu}\Omega(\boldsymbol{x}) = x^{-\mu}\mathbf{H}^{(\mu)}\Omega(\boldsymbol{x}), \tag{26}$$

where $\mathbf{H}^{(\mu)} = (h^{\mu}_{r,s})$ is $(N+1)\times(N+1)$-order square matrix that presents the fractional-order operational matrix of derivatives for Pell-Lucas polynomials of order μ in Caputo sense of fractional derivative and it is described explicitly as follows:

$$\mathbf{H}^{(\mu)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \zeta_\mu(\lceil\mu\rceil, 1) & \zeta_\mu(\lceil\mu\rceil, \lceil\mu\rceil) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \zeta_\mu(r, 0) & \cdots & \zeta_\mu(r, r) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \zeta_\mu(N, 0) & \zeta_\mu(N, 1) & \zeta_\mu(N, 2) & \cdots & \zeta_\mu(N, N) \end{pmatrix}. \tag{27}$$

The elements entries $(h^{\mu}_{r,s})$ of this matrix can be given through the relation

$$h^{\mu}_{r,s} = \begin{cases} \zeta_\mu(r, s), & r \geq s, r \geq \lceil\mu\rceil, \quad s = 0, 1, \ldots, r, \\ 0, & r < s, \qquad\qquad r < \lceil\mu\rceil, \end{cases} \tag{28}$$

where

$$\zeta_\mu(r, s) = r\sum_{k=\lceil\mu\rceil}^{r}\frac{(-1)^{\frac{k-s}{2}}\delta_{r+k}\delta_{s+k}\xi_s\Gamma(k+1)\Gamma\left(\frac{r+k-2}{2}\right)}{\Gamma\left(\frac{r-k+2}{2}\right)\Gamma\left(\frac{k-s+2}{2}\right)\Gamma\left(\frac{k+s+2}{2}\right)\Gamma(k-\mu+1)}. \tag{29}$$

Proof. Caputo operator $\mathbb{D}^{\mu}$ effecting on both sides of equation (16) beside the relation in equation (8) yields

$$\mathbb{D}^{\mu}\mathrm{PL}_r(x) = r\sum_{k=\lceil\mu\rceil}^{r}\frac{2^{k+1}\Gamma\left(\frac{r+k+2}{2}\right)\delta_{r+k}}{(r+k)\Gamma\left(\frac{r-k+2}{2}\right)\Gamma(k-\mu+1)}x^{k-\mu}. \tag{30}$$

We can accomplish the following if we continue with the explanation in equation (30) and carry out some extensive algebraic calculations:

$$\mathbb{D}^{\mu}\mathrm{PL}_r(x) = x^{-\mu}\sum_{s=0}^{r}\zeta_{\mu}(r,s)\mathrm{PL}_s(x), \tag{31}$$

and $\zeta_{\mu}(r,s)$ is given in (29).

Equation (31) can be alternatively rewritten as the equivalence vector formula

$$\mathbb{D}^{\mu}\mathrm{PL}_r(x) = x^{-\mu}[\zeta_{\mu}(r,0), \zeta_{\mu}(r,1), \ldots, \zeta_{\mu}(r,r), 0, 0, \ldots, 0]\Omega(\boldsymbol{x}), \quad \lceil\alpha\rceil \le i \le N. \tag{32}$$

Moreover, we can write

$$\mathbb{D}^{\alpha}\mathrm{PL}_r(x) = x^{-\mu}[0, 0, \ldots, 0], \quad 0 \le r \le \lceil\mu\rceil - 1. \tag{33}$$

The intended outcome is reached by assembling equations (32) and (33). □

4 Numerical treatment of fractional-order Duffing equation

Consider equation (3), then using equation (20) in addition to equation (25) and Theorem 3, we obtain the following matrix form:

$$\begin{aligned} &x^{-\mu}\mathbf{A}^T\mathbf{H}^{(\mu)}\Omega(\boldsymbol{x}) + a\mathbf{A}^T x^{-\beta}\mathbf{H}^{(\beta)}\Omega(\boldsymbol{x}) + b\mathbf{A}^T\Omega(\boldsymbol{x}) + c(\mathbf{A}^T\Omega(\boldsymbol{x}))^3 + d(\mathbf{A}^T\Omega(\boldsymbol{x}))^5 \\ &\quad + e(\mathbf{A}^T\Omega(\boldsymbol{x}))^7 = g(x), \;\; x \in [0,1]. \end{aligned} \tag{34}$$

The residual of equation (34) can be computed through the following formula:

$$\begin{aligned} x^{\mu+\beta}R(x) = {} & x^{\beta}\mathbf{A}^T\mathbf{H}^{(\mu)}\Omega(\boldsymbol{x}) + ax^{\mu}\mathbf{A}^T\mathbf{H}^{(\beta)}\Omega(\boldsymbol{x}) + bx^{\mu+\beta}\mathbf{A}^T\Omega(\boldsymbol{x}) \\ & + cx^{\mu+\beta}(\mathbf{A}^T\Omega(\boldsymbol{x}))^3 + dx^{\mu+\beta}(\mathbf{A}^T\Omega(\boldsymbol{x}))^5 + ex^{\mu+\beta}(\mathbf{A}^T\Omega(\boldsymbol{x}))^7 - x^{\mu+\beta}g(x). \end{aligned} \tag{35}$$

By means of Tau method implementation (see, e.g., [26]) we have

$$\int_0^1 x^{\mu+\beta}R(x)\mathrm{PL}_j(x)\mathrm{d}x = 0, \quad 0 \le j \le N-2. \tag{36}$$

Additionally, the approximate solution in matrix form (20) and the integer-order matrix of derivatives for Pell-Lucas polynomials (23) are applied on the initial conditions that are given in equation (4) to have the next description

$$\mathbf{A}^T\Omega(\mathbf{0}) = u_0, \quad \mathbf{A}^T\mathbf{W}^{(1)}\Omega(\mathbf{0}) = u_1. \tag{37}$$

Using equations (34) and (37), a system of nonlinear algebraic equations are created to represent the unknown expansion coefficients ai of $(N+1)$ dimension. The generated algebraic system can be solved using Newton's iterative technique or any other suitable technique. As a result, the main equation problem's desired approximation solution in equation (20) can be determined.

Pell-Lucas Tau algorithm for equation (3)

Step 1. Provided μ, β, and N.

Step 2. Determine $H^{(\mu)}$ and $H^{(\beta)}$.

Step 3. Evaluate $x^{\mu+\beta}R(x)$ according to equation (35).

Step 4. Calculate the results of equation (36).

Step 5. Join (Output 4, equation (37)).
Step 6. Solve numerically (Results of 5).

5 Convergence and error estimate discussion

This section will discuss the error estimate and convergence analysis of the proposed methodology. Following is an introduction to certain lemmas that are regarded essential for achieving this goal in the sequel:

Lemma 5.1. *Assume at the point $x = 0$, there exists an infinitely differentiable function $u(x)$. Then, this function can be expanded in Pell-Lucas polynomial terms as the next*

$$u(x) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(-1)^q \xi_p u^{(p+2q)}(0)}{2^{p+2q}\Gamma(q+1)\Gamma(p+q+1)} \mathrm{PL}_p(x). \tag{38}$$

Proof. In the beginning, according to Taylor series expansion, for any infinitely differential function $u(x)$, we can describe the function as follows:

$$u(x) = \sum_{k=0}^{\infty} b_k x^k, \quad b_k = \frac{u^{(k)}(0)}{\Gamma(k+1)}. \tag{39}$$

Inserting equation (18) in equation (39), we have

$$u(x) = \sum_{k=0}^{\infty} b_k \sum_{\substack{j=0 \\ (k+j)\text{ even}}}^{k} \gamma_{j,k} \mathrm{PL}_j(x), \tag{40}$$

where $\gamma_{j,k} = \frac{(-1)^{\frac{j+k}{2}} \xi_j \Gamma(k+1)}{2^k \Gamma\left(\frac{k-j+2}{2}\right)\Gamma\left(\frac{k+j+2}{2}\right)}$.

After expanding equation (40), combine the identical terms on its right-hand side. From there, the following formula can be produced:

$$u(x) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{u^{(p+2q)}(0)\gamma_{r,r+2s}}{\Gamma(p+2q+1)} \mathrm{PL}_p(x), \tag{41}$$

hence Lemma 5.1 is proved. □

Lemma 5.2. [47] *By using the well-known modified first kind Bessel function of order η which denoted by I_η, then, the next equation is valid*

$$\sum_{j=0}^{\infty} \frac{x^{\eta+2j}}{\Gamma(j+1)\Gamma(j+\upsilon+1)} = I_\eta(2x). \tag{42}$$

Lemma 5.3. [48] *I_η satisfies the following inequality*:

$$\frac{x^\eta \cosh(x)}{2^\eta \Gamma(\mu+1)} \geq |I_\eta(x)|, \quad \forall x > 0. \tag{43}$$

Lemma 5.4. *Pell-Lucas polynomials satisfy the following property*:

$$|\mathrm{PL}_n(x)| \leq 2\lambda^n, \quad \forall x \in [0, l], \quad \forall l > 0, n \in \mathbb{N}, \quad \textit{where } \lambda = l + \sqrt{l^2+1}. \tag{44}$$

Proof. We claim to prove through induction on order n. Suppose Lemma 5.4 is true for order $k-1$ and $k-2$, then the following two relations hold:

$$|\mathrm{PL}_{n-1}(x)| \le 2\lambda^{n-1}, \quad |\mathrm{PL}_{n-2}(x)| \le 2\lambda^{n-2}. \tag{45}$$

Using equation (10), we have

$$\mathrm{PL}_n(x) = 2x\mathrm{PL}_{n-1}(x) + \mathrm{PL}_{m-2}(x) \le 4l\lambda^{n-1} + 2\lambda^{n-2} = 2\lambda^{n-1}\left(2l + \frac{1}{\lambda}\right), \tag{46}$$

since

$$\frac{1}{l + \sqrt{l^2+1}} \le \sqrt{l^2+1} - l, \tag{47}$$

therefore,

$$2l + \frac{1}{\lambda} \le \lambda \tag{48}$$

From equation (48) and equation (46), Lemma 5.4 is proved. □

Theorem 4. *Let the function $u(x)$ be defined on $[0, l]$ and $|u^{(j)}(0)| \le M^j$, $j \ge 0$, where M is a positive constant. Also, $u(x)$ is expanded in Pell-Lucas polynomials (i.e., $u(x) = \sum_{i=0}^{\infty} a_i \mathrm{PL}_i(x)$). Then, the following estimation is achieved*:

$$|a_i| \le \frac{\left(\frac{M}{2}\right)^i \cosh(M)}{\Gamma(i+1)}. \tag{49}$$

Moreover, absolute convergence of the series $\sum_{i=0}^{\infty} a_i \mathrm{PL}_i(x)$ holds.

Proof. Lemma 5.1 in addition to the hypothesis of Theorem 4 enable us to write

$$|a_i| = \left| \sum_{s=0}^{\infty} \frac{(-1)^s \xi_i u^{(2s+i)}(0)}{2^{i+2s}\Gamma(s+1)\Gamma(s+i+1)} \right| \le \sum_{s=0}^{\infty} \frac{(-1)^s \xi_i M^{2s+i}}{2^{i+2s}\Gamma(s+1)\Gamma(s+i+1)}. \tag{50}$$

In virtue of Lemma 5.2, we have

$$|a_i| \le I_i(M). \tag{51}$$

Through using Lemma 5.3, we have

$$|a_i| \le \frac{\left(\frac{M}{2}\right)^i \cosh(M)}{\Gamma(i+1)}. \tag{52}$$

Hence, part one of the proof for Theorem (4) is completed.

Second, to prove $\sum_{i=0}^{\infty} a_i \mathrm{PL}_i(x)$ is convergent, the comparison idea will be applied. Beginning with the last inequality, equation (52), we obtain

$$|a_i \mathrm{PL}_i(x)| \le \left| \frac{\left(\frac{M}{2}\right)^i \cosh(M)}{\Gamma(i+1)} \mathrm{PL}_i(x) \right|. \tag{53}$$

Lemma 5.4 is applied here to gain

$$|a_i \mathrm{PL}_i(x)| \le \left| \frac{2\left(\frac{M}{2}\right)^i (l + \sqrt{l^2+1})^i \cosh(M)}{\Gamma(i+1)} \right|, \tag{54}$$

since

$$\sum_{i=0}^{\infty}\left|\frac{\left(\frac{M}{2}\right)^{i}(l+\sqrt{l^{2}+1})^{i}}{\Gamma(i+1)}\right| = e^{\frac{Ml+M\sqrt{l^{2}+1}}{2}}. \tag{55}$$

Consequently, the proof of Theorem 4 is complete.

Theorem 5. *Let* $u(x)$ *be the function that satisfies all conditions of* Theorem 4, *and assuming the global error is defined as the expansion* $E_N(x) = \sum_{i=N+1}^{\infty} a_i \mathrm{PL}_i(x)$, *then, the error estimate is described by*

$$|E_N(x)| < \frac{2e^{MQ}Q^{N+1}\cosh(M)}{\Gamma(N+2)}, \tag{56}$$

where $Q = (l + \sqrt{l^2+1})$.

Proof. Theorem 4 implies that

$$|E_N(x)| \le 2\cosh(M)\sum_{i=N+1}^{\infty}\frac{\left(\frac{Ml+M\sqrt{l^2+1}}{2}\right)^{i}}{\Gamma(i+1)}, \tag{57}$$

where $Q = (l + \sqrt{l^2+1})$. Moreover, we have

$$|E_N(x)| \le 2e^{MQ}\cosh(M)\left(1 - \frac{\Gamma(N+1, MQ)}{\Gamma(N+1)}\right), \tag{58}$$

where the symbol $\Gamma(.,.)$ indicates the incomplete gamma function. Now, applying the description formula for both gamma and gamma incomplete functions in addition to the fact $e^{-t} < 1,\ \forall t > 0$, then we have

$$\left(1 - \frac{\Gamma(N+1, MQ)}{\Gamma(N+1)}\right) < \frac{Q^{N+1}}{\Gamma(N+2)}. \tag{59}$$

Thus, the proof of Theorem 5 is completed. □

6 Numerical applications

In this section, we used the Pell-Lucas Tau operational matrix method to numerically solve the fractional-order nonlinear Duffing problem. These numerical tests are provided to demonstrate the precision, applicability, and effectiveness of the suggested method as well as to validate the theoretical findings.

Example 6.1. Consider the nonlinear cubic-quintic-heptic Duffing equation of the fractional-order as follows:

$$\mathbb{D}^{\mu}u + a\mathbb{D}^{\beta}u + bu + cu^3 + du^5 + eu^7 = g(x), \quad \mu \in]1,2], \quad \beta \in]0,1],\ x \in [0,1], \tag{60}$$

subject to the conditions

$$u(0) = 1, \quad u'(0) = 0, \tag{61}$$

where $g(x)$ is compatible according to the analytical solution of the problem, and the exact solution of equation (60) is given by $u(x) = 1 + x^3$ in the case of $\mu = 2$, $\beta = 1$.

Case one: Consider the fractional orders as integer numbers $\mu = 2, \beta = 1$, and $N = 3$. Then, take the following three different case studies of integer-orders for the problem given in Example 6.1:

$(I)\quad a = 2,\; b = 1,\; c = 8,\; d = e = 0,\quad$ (nonlinear cubic IDE).
$(II)\quad a = 2,\; b = 1,\; c = 8,\; d = 2,\; e = 0,\quad$ (nonlinear cubic-quintic IDE).
$(III)\quad a = 2,\; b = 1,\; c = 8,\; d = 2,\; e = 3,\quad$ (nonlinear cubic-quintic-heptic IDE).

Table 1 lists the numerical results that were obtained using the Pell-Lucas Tau spectral method for three distinct types of the integer-order Duffing equation (IDE). This table reports the absolute error of 6.1 case one for the three types of Duffing equations (I), (II), (III), respectively. Also, we plot Figures 1 and 2 to display the absolute error for the three case studies (I), (II), (III), respectively, for the value $N = 3$. The results in Table 1, coupled with the results acquired through Figures 1 and 2, demonstrate that the suggested strategy achieves good accuracy with a limited number of approximations in Pell-Lucas polynomial terms ($N = 3$).

Case two: Consider several cases of the fractional-orders μ and β for nonlinear cubic-quintic-heptic fractional-order Duffing equation with the parameters $a = 2$, $b = 1$, $c = 8$, $d = 4$, $e = 5$, and $N = 3$.

The numerical results using the suggested technique for Case two are plotted in Figure 3. This figure presents the numerical solution for the several choices of the fractional-order parameters. The plotted figure indicates that the numerical solutions for distinct values in the fractional-order case have behavior similar to that in the integer-order case. Also, these results supported the accuracy and applicability of our proposed technique for solving various linear and nonlinear fractional-order differential equations.

The numerical results using the suggested technique for Case two are plotted in Figure 3. This figure presents the numerical solution for the various selections of the fractional-order parameters (μ, β). Figure 3 indicates that the numerical solutions for distinct values in the fractional-orders μ and β exhibit behavior that is comparable to that in the integer-order case $(\mu, \beta) = (2, 1)$. Also, these results supported the accuracy and applicability of our proposed technique for solving various linear and nonlinear fractional-order differential equations.

Example 6.2. Consider the following nonlinear fractional-order cubic Duffing equation:

$$\mathbb{D}^{\mu}u + 4\mathbb{D}^{\beta}u - 2u + 3u^3 = g(x),\quad \mu \in \,]1, 2],\; \beta \in \,]0, 1],\; x \in [0, 1], \tag{62}$$

subject to the conditions

$$u(0) = 0.5,\quad u'(0) = -0.5, \tag{63}$$

where $g(x) = \frac{3}{8}e^{-3x} - \frac{5}{2}e^{-x}$, and the analytical solution is $u(x) = 0.5e^{-x}$ only in the case of $\mu = 2$, $\beta = 1$, otherwise, $g(x)$ is described according to the exact solution of the problem.

The approximate solutions for Example 6.2 using the introduced technique are reported in Table 2. This table lists these results for multi-choice terms for the power series approximation via Pell-Lucas polynomials $N = 3, 6, 9, 12$, respectively. From Table 2 we can conclude that the absolute error is decreased in vise versa relation of N. Also, the absolute error in these cases is presented in Figures 4 and 5. Moreover, by

Table 1: Absolute error via the proposed method for Example 6.1-case one

x	(I)	(II)	(III)
0.1	2.54144×10^{-18}	1.65877×10^{-17}	3.87176×10^{-18}
0.2	6.61306×10^{-18}	5.70251×10^{-17}	1.45989×10^{-17}
0.3	6.88578×10^{-18}	1.07323×10^{-16}	3.08491×10^{-17}
0.4	1.96947×10^{-18}	1.53493×10^{-16}	5.12900×10^{-17}
0.5	2.52818×10^{-17}	1.81547×10^{-16}	7.45896×10^{-17}
0.6	6.83802×10^{-17}	1.77494×10^{-16}	9.94154×10^{-17}
0.7	1.36594×10^{-16}	1.27348×10^{-16}	1.24435×10^{-16}
0.8	2.35252×10^{-16}	1.71176×10^{-17}	1.48317×10^{-16}
0.9	3.69683×10^{-16}	1.67184×10^{-16}	1.69728×10^{-16}

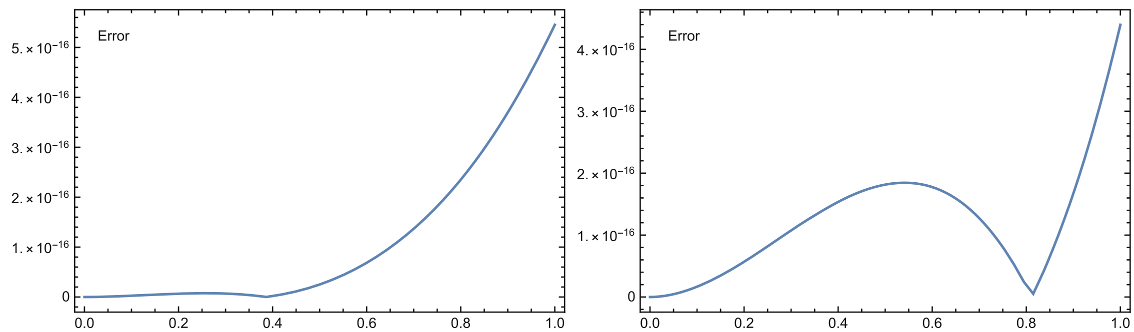

Figure 1: The absolute error of Example 6.1-case one (I), (II), respectively.

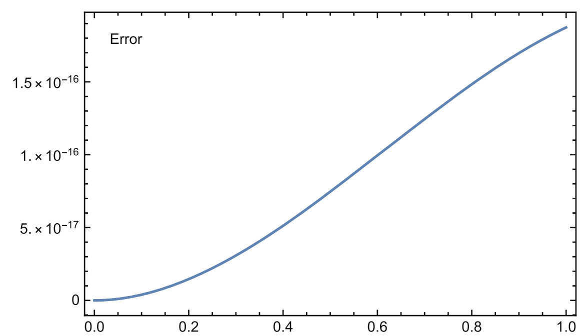

Figure 2: The absolute error for Example 6.1-case one (III).

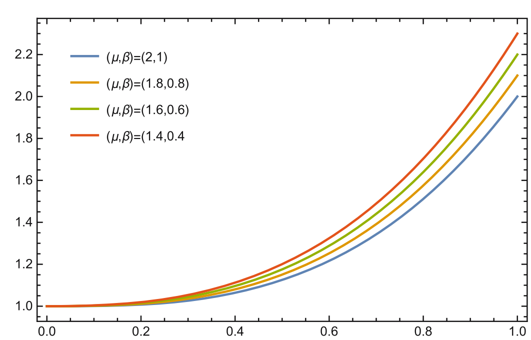

Figure 3: The absolute error for Example 6.1-case two.

Table 2: Absolute error via the proposed method for Example 6.2 in the case of $\mu = 2$, $\beta = 1$, and several values of N

x	$N = 3$	$N = 6$	$N = 9$	$N = 12$
0.1	1.0066×10^{-4}	8.5872×10^{-8}	1.2372×10^{-11}	5.5511×10^{-17}
0.2	3.1232×10^{-4}	1.9730×10^{-7}	2.4348×10^{-11}	0
0.3	5.3283×10^{-4}	2.6947×10^{-7}	3.2236×10^{-11}	5.5511×10^{-17}
0.4	7.0102×10^{-4}	3.1654×10^{-7}	3.7797×10^{-11}	1.1102×10^{-16}
0.5	7.9285×10^{-4}	3.5348×10^{-7}	4.1800×10^{-11}	0
0.6	8.1783×10^{-4}	3.8144×10^{-7}	4.4921×10^{-11}	5.5511×10^{-17}
0.7	8.1586×10^{-4}	4.0143×10^{-7}	4.7462×10^{-11}	2.7756×10^{-17}
0.8	8.5431×10^{-4}	4.2420×10^{-7}	4.9823×10^{-11}	8.3267×10^{-17}
0.9	1.0255×10^{-4}	4.4752×10^{-7}	5.1674×10^{-11}	2.7756×10^{-17}

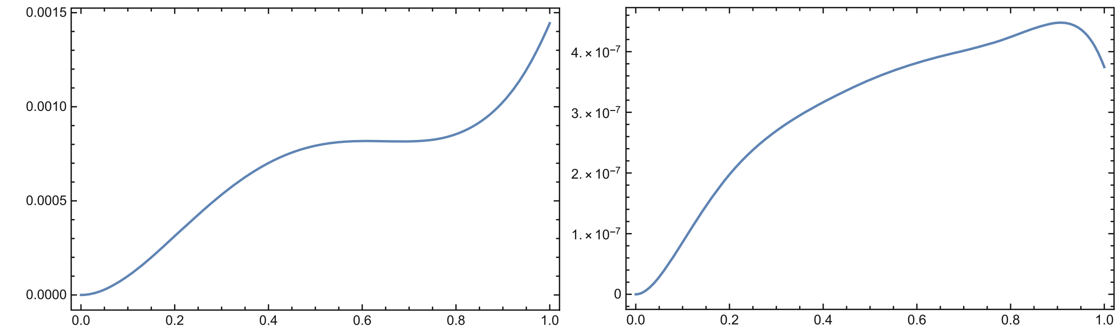

Figure 4: The absolute error for Example 6.2 at $N = 3, 6$, respectively.

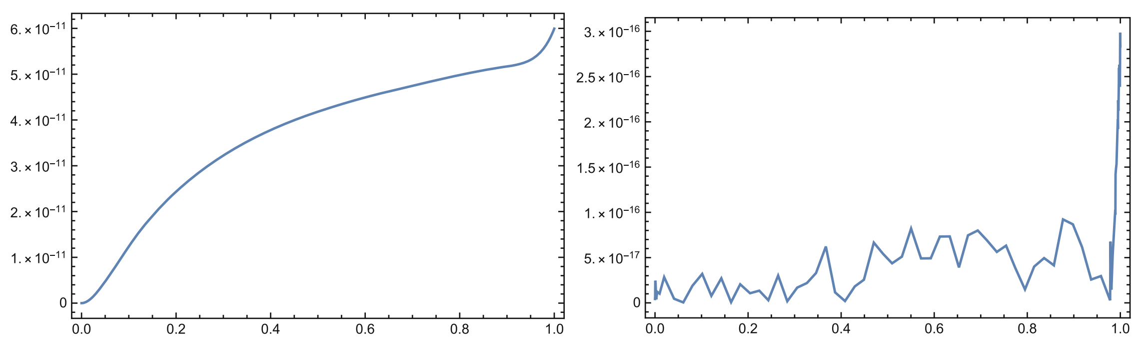

Figure 5: The absolute error of Example 6.2 in the cases of $N = 9, 12$, respectively.

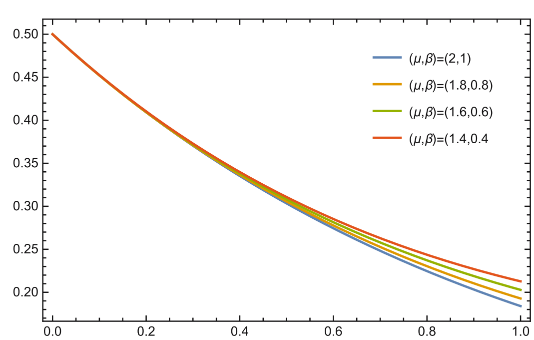

Figure 6: Comparison of analytic solution in the integer-order case and the approximate solution in fractional-order cases for Example 6.2 at $N = 3$.

Table 3: Example 6.3 absolute error results using the suggested method whenever $\mu = 2$, $\beta = 1$, and N = distinct values

x	$N = 5$	$N = 7$	$N = 9$	$N = 11$
0.1	3.9380×10^{-6}	1.9199×10^{-8}	4.0906×10^{-11}	5.9730×10^{-14}
0.2	9.2710×10^{-6}	3.9583×10^{-8}	7.7022×10^{-11}	1.0669×10^{-13}
0.3	1.1147×10^{-6}	4.5455×10^{-8}	8.6332×10^{-11}	1.1813×10^{-13}
0.4	1.0251×10^{-5}	3.9411×10^{-8}	7.3506×10^{-11}	9.9032×10^{-14}
0.5	6.9675×10^{-5}	2.5885×10^{-8}	4.6586×10^{-11}	6.1506×10^{-14}
0.6	3.0442×10^{-6}	9.3763×10^{-9}	1.5239×10^{-11}	1.8208×10^{-14}
0.7	7.4005×10^{-6}	5.6495×10^{-9}	1.3338×10^{-11}	2.0650×10^{-14}
0.8	4.0632×10^{-6}	1.7186×10^{-8}	3.4603×10^{-11}	4.9738×10^{-14}
0.9	6.0542×10^{-6}	2.5488×10^{-8}	4.8835×10^{-11}	6.7835×10^{-14}

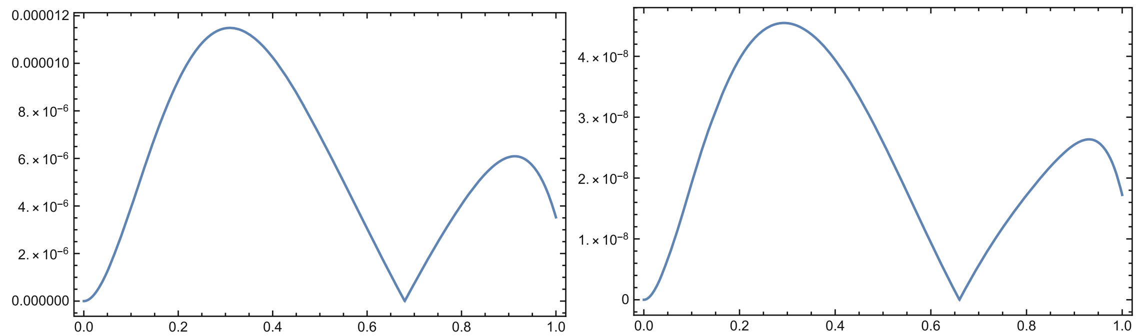

Figure 7: Example 6.3 absolute errors at the two values $N = 5$ and $N = 7$.

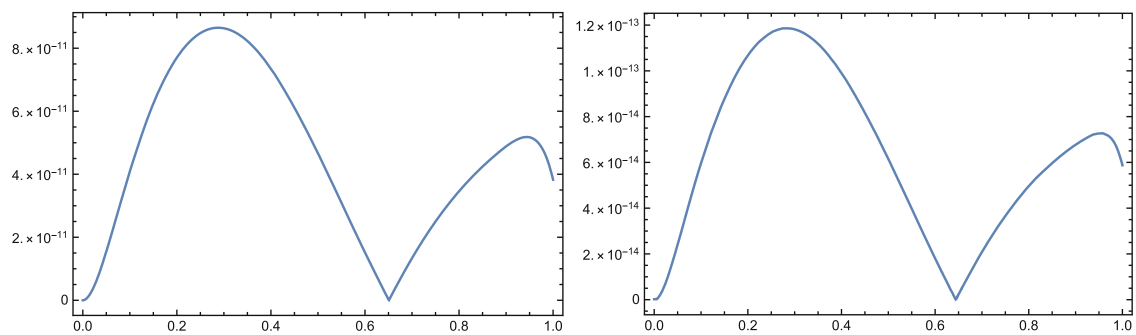

Figure 8: Example 6.3 absolute errors for the values $N = 9$ and $N = 11$.

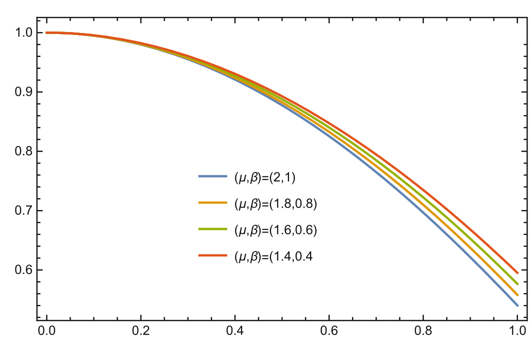

Figure 9: Analytic solution plotting in the integer-order case beside the fractional-order cases approximate solution for Example 6.3 at $N = 3$.

changing the integer-order values of μ and β into the fractional-order ones, we obtain the numerical results shown in Figure 6. All the obtained results in Table 2 and Figures 4–6 prove and support the high accuracy and efficiency of the recommended technique.

Example 6.3. Consider the following nonlinear quintic Duffing fractional-order differential equation:

$$\mathbb{D}^{\mu}u + 2\mathbb{D}^{\beta}u + u + 8u^3 + u^5 = g(x), \quad \mu \in]1, 2], \beta \in]0, 1], x \in [0, 1], \tag{64}$$

subject to the conditions

$$u(0) = 1, \quad u'(0) = 0, \tag{65}$$

where $g(x)$ is given through the exact solution of equation (64) with initial conditions equation (65), and $u(x) = \cos x$ at $\mu = 2$, $\beta = 1$.

We implement the presented method for different values of N with constant values of $\mu = 2$ and $\beta = 1$. The absolute error results for $N = 5, 7, 9, 11$, respectively, are declared in Table 3. Also, the absolute errors plotting according to these values are illustrated in Figures 7 and 8. Moreover, the numerical solution for the distinct fractional-order values of (μ, β) in addition to their integer ones $(2, 1)$ is demonstrated through Figure 9. The last illustration displays the identical style curve in both the integer-order case and the fractional-order case. The gained results in Table 3, Figures 7–9 show that the suggested methodology is capable of providing workable numerical solutions for this example and similar applications with high accuracy.

7 Conclusion

In this article, a nonlinear cubic-quintic-heptic Duffing equation of the fractional-order is solved numerically via a systematic technique. The method under investigation is based on developing new operational matrices of the integer/fractional-order derivatives of Pell-Lucas polynomials in conjunction with the use of the appropriate spectral Tau method. The fractional-order is described by the Caputo sense operator. The convergence and error estimates are examined using the new suggested technique. We solved the examples via the proposed technique with multiple possibilities for the fractional parameters μ and β. The outcomes of the numerical applications demonstrate the applicability, accuracy, and simplicity of the suggested method. Additionally, we believe that the proposed methodology can be used in a number of applications to solve various classes of linear and nonlinear fractional-order differential equations. All calculations were performed using the HP Core i7 laptop and Mathematica 11.0 with 4GB of RAM.

Acknowledgments: The author appreciates the anonymous reviewers' attentive reading of the work and their insightful remarks, which helped in improving the article to its form.

Funding information: Not applicable.

Author contributions: The study was done by the author. Also, he has agreed to take full responsibility for this manuscript's content and has given his approval for its submission.

Conflict of interest: The author states no conflict of interest.

Ethical approval: The research being done has nothing to do with using humans or animals.

Informed consent: Not applicable.

Data availability statement: This article contains all the necessary information and materials.

References

[1] B. Agheli, *Solving fractional Bratu's equations using a semi-analytical technique*, J. Math. **51** (2019), no. 9, 111–121.
[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, San Diego, 2006.
[3] A. I. Maimistov, *Propagation of an ultimately short electromagnetic pulse in a nonlinear medium described by the fifth-order Duffing model*, Opt. Spectrosc. **94** (2003), 251–257, DOI: https://doi.org/10.1134/1.1555186.
[4] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, To Methods of Their Solution and Some of Their Applications*, Academic Press, London, UK, 1999.
[5] S. Babolian, S. Javadi, and E. Moradi, *RKM for solving Bratu-type differential equations of fractional order*, Math. Meth. Appl. Sci. **39** (2015), no. 6, 1548–1557, DOI: https://doi.org/10.1002/mma.3588.

[6] A. A. El-Sayed, D. Baleanu, and P. Agarwal, *A novel Jacobi operational matrix for numerical solution of multi-term variable-order fractional differential equations*, J. Taibah Univ. Sci. **14** (2020), no. 1, 963–974, DOI: https://doi.org/10.1080/16583655.2020.1792681.

[7] P. Agarwal and A. A. El-Sayed, *Vieta-Lucas polynomials for solving a fractional-order mathematical physics model*, Adv. Differential Equations **2020** (2020), no. 626, 1–18, DOI: https://doi.org/10.1186/s13662-020-03085-y.

[8] K. M. Owolabi, *High-dimensional spatial patterns in fractional reaction-diffusion system arising in biology*, Chaos Solitons Fractals **134** (2020), 109723, DOI: https://doi.org/10.1016/j.chaos.2020.109723.

[9] L. Yuan and O. P. Agrawal, *A numerical scheme for dynamic systems containing fractional derivatives*, J. Vib. Acoust. **124** (2002), no. 2, 321–324, DOI: https://doi.org/10.1115/1.1448322.

[10] C. M. A. Pinto and A. R. M. Carvalho, *Fractional modeling of typical stages in HIV epidemics with drug-resistance*, Progr. Fract. Differ. Appl. **1** (2015), no. 2, 111–122, DOI: https://dx.doi.org/10.12785/pfda/010205.

[11] W.-X. Ma, M. M. Mousa, and M. R. Ali, *Application of a new hybrid method for solving singular fractional Lane-Emden-type equations in astrophysics*, Modern. Phys. Lett. B **34** (2020), no. 3, 2050049, DOI: https://doi.org/10.1142/S0217984920500499.

[12] P. K. Sahu and B. Mallicki, *Approximate solution of fractional order Lane-Emden type differential equation by orthonormal Bernoulli's polynomials*, Int. J. Appl. Comput. Math. **5** (2019), no. 89, 1–9, DOI: https://doi.org/10.1007/s40819-019-0677-0.

[13] S. Qureshi, A. Yusuf, A. A. Shaikh, M. Inc, and D. Baleanu, *Fractional modeling of blood ethanol concentration system with real data application*, Chaos **29** (2019), no. 1, 013143, DOI: https://doi.org/10.1063/1.5082907.

[14] M. ur Rehman, D. Baleanu, J. Alzabut, M. Ismail, and U. Saeed, *Green-Haar wavelets method for generalized fractional differential equations*, Adv. Differential Equations **2020** (2020), no. 515, 1–25, DOI: https://doi.org/10.1186/s13662-020-02974-6.

[15] P. Guo, *The Adomian decomposition method for a type of fractional differential equations*, J. Appl. Math. Phys. **7** (2019), no. 10, 2459–2466, DOI: https://doi.org/10.4236/jamp.2019.710166.

[16] A. A. El-Sayed and P. Agarwal, *Numerical solution of multiterm variable-order fractional differential equations via shifted Legendre polynomials*, Math. Meth. Appl. Sci. **41** (2019), no. 11, 3978–3991, DOI: https://doi.org/10.1002/mma.5627.

[17] E. K. Akgäl, A. Akgü l, and D. Baleanu, *Laplace transform method for economic models with constant proportional Caputo derivative*, Fractal Fract. **4** (2020), no. 3, 1–10, DOI: https://doi.org/10.3390/fractalfract4030030.

[18] N. H. Sweilam, A. M. Nagy, and A. A. El-Sayed, *Numerical approach for solving space fractional order diffusion equations using shifted Chebyshev polynomials of the fourth kind*, Turkish J. Math. **40** (2016), no. 6, 1283–1297, DOI: https://doi.org/10.3906/mat-1503-20.

[19] S. Sabermahani, Y. Ordokhani, and P. M. Lima, *A novel Lagrange operational matrix and Tau-collocation method for solving variable-order fractional differential equations*, Iran. J. Sci. Technol. Trans. Sci. **44** (202), 127–135, DOI: https://doi.org/10.1007/s40995-019-00797-z.

[20] A. Harir, S. Melliani, H. ElHarfi, and L. S. Chadli, *Variational iteration method and differential transformation method for solving the SEIR epidemic model*, Int. J. Differ. Equ. **2020** (2020), Article ID 3521936, 7 pages, DOI: https://doi.org/10.1155/2020/3521936.

[21] S. R. Saratha, M. Bagyalakshmi, and G. S. S. Krishnan, *Fractional generalised homotopy analysis method for solving nonlinear fractional differential equations*, Comp. Appl. Math. **39** (2020), no. 112, 1–32, DOI: https://doi.org/10.1007/s40314-020-1133-9.

[22] P. Agarwal, A. A. El-Sayed, and J. Tariboong, *Vieta-Fibonacci operational matrices for spectral solutions of variable-order fractional integro-differential equations*, J. Comput. Appl. Math. **382** (2021), Pages 113063, DOI: https://doi.org/10.1016/j.cam.2020.113063.

[23] E. F. Anley and Z. Zheng, *A finite difference approximation method for a space fractional convection-diffusion equation with variable coefficients*, Symmetry **12** (2020), no. 3, 1–19, DOI: https://doi.org/10.3390/sym12030485.

[24] N. H. Sweilam, A. A. El-Sayed, and S. Boulaaras, *Fractional-order advection-dispersion problem solution via the spectral collocation method and the non-standard finite difference technique*, Chaos Solitons Fractals, **144** (2021), 110736, DOI: https://doi.org/10.1016/j.chaos.2021.110736.

[25] M. Arfan, Z. A. Khan, A. Zeb, and K. Shah, *Study of numerical solution to some fractional order differential equation using Hermite polynomials*, Int. J. Appl. Comput. Math. **8** (2022), no. 60, 1–16, DOI: https://doi.org/10.1007/s40819-022-01255-y.

[26] A. A. El-Sayed, S. Boulaaras, and N. H. Sweilam, *Numerical solution of the fractional-order logistic equation via the first-kind Dickson polynomials and spectral Tau method*, Math. Meth. Appl. Sci. **46** (2021), no. 7, 1–14, DOI: https://doi.org/10.1002/mma.7345.

[27] Y. H. Youssri, W. M. Abd-Elhameed, and H. M. Ahmed, *New fractional derivative expression of the shifted third-kind Chebyshev polynomials: Application to a type of nonlinear fractional Pantograph differential equations*, J. Funct. Spaces **2022** (2022), Article ID 3966135, DOI: https://doi.org/10.1155/2022/3966135.

[28] N. A. Zabidi, Z. A. Majid, A. Kilicman, and Z. B. Ibrahim, *Numerical solution of fractional differential equations with Caputo derivative by using numerical fractional predict-correct technique*, Adv. Cont. Discr. Mod. **2022** (2022), no. 26, 1–23, DOI: https://doi.org/10.1186/s13662-022-03697-6.

[29] A. A. El-Sayed and P. Agarwal, *Spectral treatment for the fractional-order wave equation using shifted Chebyshev orthogonal polynomials*, J. Comput. Appl. Math. **424** (2023), 114933, DOI: https://doi.org/10.1016/j.cam.2022.114933.

[30] K. Tabatabaei and E. Gunerhan, *Numerical solution of Duffing equation by the differential transform method*, Appl. Math. Inf. Sci. Lett. **2** (2014), no. 1, 1–6, DOI: https://doi.org/10.12785/amisl/020101.

[31] J. Stokes, *Nonlinear Vibrations*, Intersciences, New York, 1950.

[32] E. C. Zeeman, *Duffing's equation in brain modelling*, Bull. Inst. Math. Appl. **12** (1975), 207–214.

[33] H. Vahedi, G. B. Gharehpetian, and M. Karrari, *Application of Duffing oscillators for passive islanding detection of inverter-based distributed generation units*, IEEE Trans. Power Delivery **27** (2012), no. 4, 1973–1983, DOI: https://doi.org/10.1109/TPWRD.2012.2212251.

[34] X. Y. Deng, B. Liu, and T. Long, *A new complex Duffing oscillator used in complex signal detection*, Chin. Sci. Bull. **57** (2012), 2185–2191, DOI: https://doi.org/10.1007/s11434-012-5145-8.

[35] A. H. Salas and J. E. Castillo, *Exact solutions to cubic Duffing equation for a nonlinear electrical circuit*, Visión electrónica **8** (2014), no. 1, 1–8, DOI: https://doi.org/10.14483/22484728.7861.

[36] R. Novin and Z. S. Dastjerd, *Solving Duffing equation using an improved semi-analytical method*, Commun. Adv. Comput. Sci. Appl. **2015** (2015), no. 2, 54–58, DOI: https://doi.org/10.5899/2015/cacsa-00040.

[37] P. Pirmohabbati, A. H. R. Sheikhani, H. S. Najafi, and A. A. Ziabari, *Numerical solution of full fractional Duffing equations with Cubic-Quintic-Heptic nonlinearities*, AIMS Math. **5** (2020), no. 2, 1621–1641, DOI: https://doi.org/10.3934/math.2020110.

[38] W. M. Abd-Elhameed and Y. H. Youssri, *Connection formulae between generalized Lucas polynomials and some Jacobi polynomials: Application to certain types of fourth-order BVPs*, Int. J. Appl. Comput. Math **6** (2020), no. 45, 1–19, DOI: https://doi.org/10.1007/s40819-020-0799-4.

[39] Y. H. Youssri, W. M. Abd-Elhameed, A. S. Mohamed, and S. M. Sayed, *Generalized Lucas polynomial sequence treatment of fractional pantograph differential equation*, Int. J. Appl. Comput. Math **7** (2021), no. 27, 1–16, DOI: https://doi.org/10.1007/s40819-021-00958-y.

[40] Y. H. Youssri, W. M. Abd-Elhameed, and A. G. Atta, *Spectral Galerkin treatment of linear one-dimensional telegraph type problem via the generalized Lucas polynomials*, Arab. J. Math. **11** (2022), 601–615, DOI: https://doi.org/10.1007/s40065-022-00374-0.

[41] Y. H. Youssri, S. M. Sayed, A. S. Mohamed, E. M. Aboeldahab, and W. M. Abd-Elhameed, *Modified Lucas polynomials for the numerical treatment of second-order boundary value problems*, Comput. Methods Differ. Equ. **11** (2023), no. 1, 12–31, DOI: https://doi.org/10.22034/CMDE.2022.50891.2115.

[42] Y. H. Youssri, W. M. Abd-Elhameed, and S. M. Sayed, *Generalized Lucas Tau method for the numerical treatment of the one and two-dimensional partial differential heat equation*, J. Funct. Spaces **2022** (2022), 13 pages, DOI: https://doi.org/10.1155/2022/3128586.

[43] W. M. Abd-Elhameed and Y. H. Youssri, *Generalized Lucas polynomial sequence approach for fractional differential equations*, Nonlinear Dyn. **89** (2017), 1341–1355, DOI: https://doi.org/10.1007/s11071-017-3519-9.

[44] A. F. Horadam and J. M. Mahon, *Pell and Pell-Lucas polynomials*, The Fibonacci Quart. **23** (1985), 7–20.

[45] A. Lupas, *A guide of Fibonacci and Lucas polynomials*, Octagon Math. Mag. **7** (1999), no. 1, 2–12.

[46] W. Wang and H. Wang, *Some results on convolved* (p, q)*-Fibonacci polynomials*, Integral Transforms Spec. Funct. **26** (2015), no. 5, 340–356, DOI: https://doi.org/10.1080/10652469.2015.1007502.

[47] M. Abramowitz and I. A. Stegun, Handbook of mathematical functions: with formulas, graphs, and mathematical tables, *Courier Corporation*, Series 55, 1964.

[48] Y. L. Luke, *Inequalities for generalized hypergeometric functions*, J. Approx. Theory **5** (1972), no. 1, 41–65, DOI: https://doi.org/10.1016/0021-9045(72)90028-7.