Research Article

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Approximation of the image of the L_p ball under Hilbert-Schmidt integral operator

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Abstract: In this article, an approximation of the image of the closed ball of the space L_p $(p > 1)$ centered at the origin with radius r under Hilbert-Schmidt integral operator $F(\cdot) : L_p \to L_q$, $\frac{1}{p} + \frac{1}{q} = 1$ is considered. An error evaluation for the given approximation is obtained.

Keywords: Hilbert-Schmidt integral operator, image of a set, input–output system, approximation, error evaluation

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1 Introduction

Integral operators arise in various problems of theory and applications and are one of the important tools to investigate different types of problems in mathematics. For example, integral operators are used in integral equations of the Fredholm, Volterra, Urysohn-Hammerstein and other types and play a crucial role in the definition of solution concepts for different types of initial and boundary value problems of differential equations (see, e.g., [1–3] and references therein). It is necessary to underline that the theory of linear integral equations is considered one of the origins of contemporary functional analysis [4–6]. In particular, the integral operators are used to describe the behavior of some input–output systems (see, e.g., [7–9]).

In this article an approximation of the image of the closed ball of the space L_p $(p > 1)$ centered at the origin under the Hilbert-Schmidt integral operator is considered. The presented approximation method allows for every $\varepsilon > 0$ to construct a finite ε-net on the image of the closed ball, which consists of the images of a finite number of piecewise-constant functions. An approximation of the image of a given closed ball can be used in infinite-dimensional optimization problems for predetermining the desirable inputs for the input–output system described by Hilbert-Schmidt integral operator. Note that the input functions with integral constraints are usually applied when the input resources of the system are exhausted by consumption, such as energy, fuel, and finance (see, e.g., [10–12] and references therein). An error evaluation of the Hausdorff distance between the image of the closed ball and its approximation, which consists of a finite number of functions, is given.

The article is organized as follows. In Section 2, the conditions and auxiliary propositions that are used in the following arguments are formulated. In Section 3, the image of the integral operator is approximated by the set, consisting of a finite number of functions. An error estimation depending on the approximation parameters is given (Theorem 1).

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2 Preliminaries

Consider the Hilbert-Schmidt integral operator

$$F(x(\cdot))|(\xi) = \int_{\Omega} K(\xi, s)x(s)\mathrm{d}s \quad \text{for almost all } \xi \in E, \tag{1}$$

where $x(s) \in \mathbb{R}^n$, $K(\xi, s)$ is an $m \times n$ dimensional matrix function, $(\xi, s) \in E \times \Omega$, $E \subset \mathbb{R}^b$ and $\Omega \subset \mathbb{R}^k$ are compact sets.

For given $p > 1$ and $r > 0$, we denote

$$B_p(r) = \{x(\cdot) \in L_p(\Omega; \mathbb{R}^n) : \|x(\cdot)\|_p \le r\},$$

where $L_p(\Omega; \mathbb{R}^n)$ is the space of Lebesgue measurable functions $x(\cdot) : \Omega \to \mathbb{R}^n$ such that $\|x(\cdot)\|_p < +\infty$, $\|x(\cdot)\|_p = \left(\int_{\Omega} \|x(s)\|^p \mathrm{d}s\right)^{\frac{1}{p}}$, and $\|\cdot\|$ denotes the Euclidean norm.

It is assumed that the matrix function $K(\cdot,\cdot) : E \times \Omega \to \mathbb{R}^{m\times n}$ is Lebesgue measurable and

$$\int_E \int_{\Omega} \|K(\xi, s)\|^q \mathrm{d}s\mathrm{d}\xi < +\infty,$$

where $\frac{1}{q} + \frac{1}{p} = 1$. Denote

$$\mathcal{F}_p(r) = \{F(x(\cdot))|(\cdot) : x(\cdot) \in B_p(r)\}. \tag{2}$$

It is obvious that the set $\mathcal{F}_p(r)$ is the image of the set $B_p(r)$ under Hilbert-Schmidt integral operator (1). Since operator $F(\cdot)$ is linear and compact one, then the set $\mathcal{F}_p(r)$ is a convex and compact subset of the space $L_q(E; \mathbb{R}^m)$.

Since the set of continuous functions $\Phi(\cdot,\cdot) : E \times \Omega \to \mathbb{R}^{m\times n}$ is dense in the space $L_q(E \times \Omega; \mathbb{R}^{m\times n})$ (see, e.g., [13], p. 318), then, for every $\lambda > 0$, there exists a continuous function $K_\lambda(\cdot,\cdot) : E \times \Omega \to \mathbb{R}^{m\times n}$ such that

$$\left(\int_E \int_{\Omega} \|K(\xi, s) - K_\lambda(\xi, s)\|^q \mathrm{d}s\mathrm{d}\xi\right)^{\frac{1}{q}} \le \frac{\lambda}{2r}. \tag{3}$$

Denote

$$M(\lambda) = \max\{\|K_\lambda(\xi, s)\| : (\xi, s) \in E \times \Omega\}, \tag{4}$$

$$\omega_\lambda(\Delta) = \max\{\|K_\lambda(\xi, s_2) - K_\lambda(\xi, s_1)\| : (\xi, s_2) \in E \times \Omega, \quad (\xi, s_1) \in E \times \Omega, \quad \|s_2 - s_1\| \le \Delta\}, \tag{5}$$

where $\Delta > 0$ is a given number. The compactness of the sets $E \subset \mathbb{R}^b$ and $\Omega \subset \mathbb{R}^k$ and continuity of the function $K_\lambda(\cdot,\cdot) : E \times \Omega \to \mathbb{R}^{m\times n}$ imply that, for each fixed $\lambda > 0$, we have $\omega_\lambda(\Delta) \to 0$ as $\Delta \to 0^+$ and $\omega_\lambda(\Delta_1) \le \omega_\lambda(\Delta_2)$ if $\Delta_1 < \Delta_2$.

Let us define a finite Δ-partition of a given set $D \subset \mathbb{R}^{n_0}$, which will be used in following arguments.

Definition 1. Let $\Delta > 0$ and $D \subset \mathbb{R}^{n_0}$. A finite family of sets $\Lambda = \{D_1, D_2, \dots, D_l\}$ is called a finite Δ-partition of the set D, if

(1) $D_i \subset D$ and D_i is Lebesgue measurable for every $i = 1, 2, \dots, l$;
(2) $D_i \bigcap D_j = \varnothing$ for every $i \neq j$, where $i = 1, 2, \dots, l$ and $j = 1, 2, \dots, l$;
(3) $D = \bigcup_{i=1}^{l} D_i$;
(4) diam $(D_i) \le \Delta$ for every $i = 1, 2, \dots, l$, where diam $(D_i) = \sup\{\|x - y\| : x \in D_i, y \in D_i\}$.

Proposition 1. *Let $D \subset \mathbb{R}^{n_0}$ be a compact set. Then, for each $\Delta > 0$, it has a finite Δ-partition.*

3 Approximation

Let $\gamma > 0$ and $\sigma > 0$ be given numbers, $\Lambda = \{\Omega_1, \Omega_2, \dots, \Omega_N\}$ be a finite Δ-partition of the compact set $\Omega \subset \mathbb{R}^k$, $\Lambda_* = \{0 = r_0, r_1, \dots, r_a = \gamma\}$ be a uniform partition of the closed interval $[0, \gamma]$, $\delta = r_{j+1} - r_j$, $j = 0, 1, \dots, a-1$, be a diameter of the partition Λ_*, $P = \{x \in \mathbb{R}^n : \|x\| = 1\}$, and $P_\sigma = \{e_1, e_2, \dots, e_c\}$ be a finite σ-net on P. Denote

$$B_p^{\gamma,\Delta,\delta,\sigma}(r) = \Big\{x(\cdot) : \Omega \to \mathbb{R}^n : x(s) = r_{j_i} e_{l_i} \quad \text{for every } s \in \Omega_i, \quad \text{where } r_{j_i} \in \Lambda_*, \quad e_{l_i} \in P_\sigma, \quad i = 1, 2, \dots, N, \quad \sum_{i=1}^{N} \mu(\Omega_i) r_{j_i}^p \le r^p \Big\}, \tag{6}$$

$$\mathcal{F}_p^{\gamma,\Delta,\delta,\sigma}(r) = \{F(x(\cdot))|(\cdot) : x(\cdot) \in B_p^{\gamma,\Delta,\delta,\sigma}(r)\}, \tag{7}$$

where $\mu(\cdot)$ means the Lebesgue measure of a set. It is obvious that the set $\mathcal{F}_p^{\gamma,\Delta,\delta,\sigma}(r)$ consists of a finite number of functions. We set

$$c_* = 2r^p[\mu(E)]^{\frac{1}{q}}, \tag{8}$$

$$\psi_\lambda(\Delta) = 2r[\mu(\Omega)\cdot \mu(E)]^{\frac{1}{q}} \omega_\lambda(\Delta), \tag{9}$$

$$\varphi_\lambda(\delta) = M(\lambda)\mu(\Omega)\cdot[\mu(E)]^{\frac{1}{q}} \delta, \tag{10}$$

$$\alpha_\lambda(\gamma, \sigma) = M(\lambda)\mu(\Omega)\cdot[\mu(E)]^{\frac{1}{q}} \gamma\sigma, \tag{11}$$

where $M(\lambda)$ is defined by (4).

The Hausdorff distance between the sets $U \subset L_q(E; \mathbb{R}^m)$ and $V \subset L_q(E; \mathbb{R}^m)$ is denoted by $h_q(U, V)$.

Theorem 1. *For every $\lambda > 0$, $\gamma > 0$, finite Δ-partition of the compact set $\Omega \subset \mathbb{R}^k$, uniform δ-partition of the closed interval $[0, \gamma]$, and $\sigma > 0$, the inequality*

$$h_q(\mathcal{F}_p(r), \mathcal{F}_p^{\gamma,\Delta,\delta,\sigma}(r)) \le \lambda + \frac{c_* M(\lambda)}{\gamma^{p-1}} + \psi_\lambda(\Delta) + \varphi_\lambda(\delta) + \alpha_\lambda(\gamma, \sigma)$$

is satisfied, where the sets $\mathcal{F}_p(r)$ and $\mathcal{F}_p^{\gamma,\Delta,\delta,\sigma}(r)$ are defined by (2) and (7), respectively.

Proof. Denote

$$F_\lambda(x(\cdot))|(\xi) = \int_\Omega K_\lambda(\xi, s)x(s)\mathrm{d}s \quad \text{for every } \xi \in E \tag{12}$$

and

$$\mathcal{F}_p^\lambda(r) = \{F_\lambda(x(\cdot))|(\cdot) : x(\cdot) \in B_p(r)\},$$

where $K_\lambda(\cdot,\cdot)$ is defined in (3). The set $\mathcal{F}_p^\lambda(r)$ is the image of the closed ball $B_p(r)$ under the Hilbert-Schmidt integral operator (12), and the compactness of the operator $F_\lambda(\cdot)$ implies that the set $\mathcal{F}_p^\lambda(r)$ is a compact subset of the space $C(E; \mathbb{R}^m)$, where $C(E; \mathbb{R}^m)$ is the space of continuous functions $y(\cdot) : E \to \mathbb{R}^m$ with norm $\|y(\cdot)\|_C = \max\{\|y(\xi)\| : \xi \in E\}$.

Applying (3) and Hölder's inequality, it is not difficult to show that

$$h_q(\mathcal{F}_p(r), \mathcal{F}_p^\lambda(r)) \le \frac{\lambda}{2}. \tag{13}$$

Now we set

$$B_p^\gamma(r) = \{x(\cdot) \in B_p(r) : \|x(s)\| \le \gamma \quad \text{for almost all } s \in \Omega\},$$

and let

$$\mathcal{F}_p^{\lambda,\gamma}(r) = \{F_\lambda(x(\cdot))|(\cdot) : x(\cdot) \in B_p^{\gamma}(r)\}.$$

Let $y(\cdot) \in \mathcal{F}_p^{\lambda}(r)$ be an arbitrarily chosen function, which is the image of $x(\cdot) \in B_p(r)$ under operator (12). Define the function $x_*(\cdot) : \Omega \to \mathbb{R}^n$, setting

$$x_*(s) = \begin{cases} x(s) & \text{if } \|x(s)\| \le \gamma, \\ \dfrac{x(s)}{\|x(s)\|}\gamma & \text{if } \|x(s)\| > \gamma, \end{cases}$$

where $s \in \Omega$. It is not difficult to verify that $x_*(\cdot) \in B_p^{\gamma}(r)$. Let $y_*(\cdot) \in \mathcal{F}_p^{\lambda,\gamma}(r)$ be the image of $x_*(\cdot) \in B_p^{\gamma}(r)$ under operator (12). Denote $W = \{s \in \Omega : \|x(s)\| > \gamma\}$. From inclusion $x(\cdot) \in B_p(r)$ and Tchebyshev's inequality (see, [14], p. 82), it follows that

$$\mu(W) \le \frac{r^p}{\gamma^p}. \tag{14}$$

Thus, from (8), (14), and Hölder's inequality, we obtain that

$$\|y(\xi) - y_*(\xi)\| \le \int\limits_W \|K_\lambda(\xi, s)\| \|x(s) - x_*(s)\| \mathrm{d}s \le 2rM(\lambda)[\mu(W)]^{\frac{1}{q}} \le \frac{2r^p M(\lambda)}{\gamma^{p-1}}$$

for every $\xi \in E$, and consequently,

$$\|y(\cdot) - y_*(\cdot)\|_q \le \frac{2r^p M(\lambda)}{\gamma^{p-1}}[\mu(E)]^{\frac{1}{q}} = \frac{c_* M(\lambda)}{\gamma^{p-1}},$$

where $M(\lambda)$ is defined by (4). Since $y(\cdot) \in \mathcal{F}_p^{\lambda}(r)$ is arbitrarily chosen, we obtain from the last inequality that

$$\mathcal{F}_p^{\lambda}(r) \subset \mathcal{F}_p^{\lambda,\gamma}(r) + \frac{c_* M(\lambda)}{\gamma^{p-1}} B_q(1), \tag{15}$$

where

$$B_q(1) = \{y(\cdot) \in L_q(E; \mathbb{R}^m) : \|y(\cdot)\|_q \le 1\}. \tag{16}$$

The inclusion $\mathcal{F}_p^{\lambda,\gamma}(r) \subset \mathcal{F}_p^{\lambda}(r)$ and (15) yield that

$$h_q(\mathcal{F}_p^{\lambda}(r), \mathcal{F}_p^{\lambda,\gamma}(r)) \le \frac{c_* M(\lambda)}{\gamma^{p-1}}. \tag{17}$$

For given $\Delta > 0$ and finite Δ-partition $\Lambda = \{\Omega_1, \Omega_2, \ldots, \Omega_N\}$ of the compact set $\Omega \subset \mathbb{R}^k$, we denote

$$B_p^{\gamma,\Delta}(r) = \{x(\cdot) \in B_p^{\gamma}(r) : x(s) = x_i \quad \text{for every } s \in \Omega_i, \quad i = 1, 2, \ldots, N\} \tag{18}$$

and

$$\mathcal{F}_p^{\lambda,\gamma,\Delta}(r) = \{F_\lambda(x(\cdot))|(\cdot) : x(\cdot) \in B_p^{\gamma,\Delta}(r)\}.$$

Choose an arbitrary $z(\cdot) \in \mathcal{F}_p^{\lambda,\gamma}(r)$, which is the image of $v(\cdot) \in B_p^{\gamma}(r)$ under operator (12). Define the function $v_*(\cdot) : \Omega \to \mathbb{R}^n$, setting

$$v_*(s) = \frac{1}{\mu(\Omega_i)} \int\limits_{\Omega_i} v(\tau)\mathrm{d}\tau, \quad s \in \Omega_i, \quad i = 1, 2, \ldots, N. \tag{19}$$

Taking into consideration the inclusion $v(\cdot) \in B_p^{\gamma}(r)$ and equality (19), we obtain that $\|v_*(s)\| \le \gamma$ for every $s \in \Omega$ and

$$\int\limits_{\Omega_i} \|v_*(s)\|^p \mathrm{d}s \le \int\limits_{\Omega_i} \|v(s)\|^p \mathrm{d}s$$

for every $i = 1, 2, \ldots, N$. Since $\|v(\cdot)\|_p \le r$, then it follows from last inequality that $\|v_*(\cdot)\|_p \le \|v(\cdot)\|_p \le r$. Thus, from (18) and (19), we obtain that $v_*(\cdot) \in B_p^{\gamma,\Delta}(r)$.

Let $z_*(\cdot) \in \mathcal{F}_p^{\lambda,\gamma,\Delta}(r)$ be the image of $v_*(\cdot)$ under operator (12). We have

$$\|z_*(\xi) - z(\xi)\| = \left\| \sum_{i=1}^{N} \int_{\Omega_i} K_\lambda(\xi, s)[v_*(s) - v(s)]\mathrm{d}s \right\| \tag{20}$$

for every $\xi \in E$. Equation (19) yields that

$$\int_{\Omega_i} v_*(s)\mathrm{d}s = \int_{\Omega_i} v(s)\mathrm{d}s \tag{21}$$

for every $i = 1, 2, \ldots, N$. Let $\xi \in E$ and $i = 1, 2, \ldots, N$ be fixed. Now let us choose an arbitrary $s_i \in \Omega_i$. From (21), it follows that

$$\left\| \int_{\Omega_i} K_\lambda(\xi, s)[v_*(s) - v(s)]\mathrm{d}s \right\| \le \int_{\Omega_i} \|K_\lambda(\xi, s) - K_\lambda(\xi, s_i)\| \|v_*(s) - v(s)\|\mathrm{d}s. \tag{22}$$

Since $\Lambda = \{\Omega_1, \Omega_2, \ldots, \Omega_N\}$ is a finite Δ-partition of the compact Ω, $s_i \in \Omega_i$, then from Definition 1 we obtain that $\|s - s_i\| \le \Delta$ for every $s \in \Omega_i$. Finally, by virtue of (5), we have that

$$\|K_\lambda(\xi, s) - K_\lambda(\xi, s_i)\| \le \omega_\lambda(\Delta) \tag{23}$$

for every $s \in \Omega_i$. Thus, from (22) and (23), it follows that

$$\left\| \int_{\Omega_i} K_\lambda(\xi, s)[v_*(s) - v(s)]\mathrm{d}s \right\| \le \omega_\lambda(\Delta) \int_{\Omega_i} \|v_*(s) - v(s)\|\mathrm{d}s. \tag{24}$$

Since $v_*(\cdot) \in B_p(r)$ and $v(\cdot) \in B_p(r)$, then (24) yields that

$$\begin{aligned} \left\| \sum_{i=1}^{N} \int_{\Omega_i} K_\lambda(\xi, s)[v_*(s) - v(s)]\mathrm{d}s \right\| &\le \omega_\lambda(\Delta) \sum_{i=1}^{N} \int_{\Omega_i} \|v_*(s) - v(s)\|\mathrm{d}s = \omega_\lambda(\Delta) \int_{\Omega} \|v_*(s) - v(s)\|\mathrm{d}s \\ &\le 2r\omega_\lambda(\Delta)[\mu(\Omega)]^{\frac{1}{q}}. \end{aligned} \tag{25}$$

Equations (9), (20), and (25) imply that

$$\|z_*(\xi) - z(\xi)\| \le 2r\omega_\lambda(\Delta)[\mu(\Omega)]^{\frac{1}{q}}$$

for every $\xi \in E$ and, consequently,

$$\|z_*(\cdot) - z(\cdot)\|_q \le 2r\omega_\lambda(\Delta)[\mu(\Omega) \cdot \mu(E)]^{\frac{1}{q}} = \psi_\lambda(\Delta).$$

Since $z(\cdot) \in \mathcal{F}_p^{\lambda,\gamma}(r)$ is arbitrarily chosen, the last inequality yields

$$\mathcal{F}_p^{\lambda,\gamma}(r) \subset \mathcal{F}_p^{\lambda,\gamma,\Delta}(r) + \psi_\lambda(\Delta)B_q(1), \tag{26}$$

where $B_q(1)$ is defined by (16).

From inclusion $\mathcal{F}_p^{\lambda,\gamma,\Delta}(r) \subset \mathcal{F}_p^{\lambda,\gamma}(r)$ and (26), we obtain

$$h_q(\mathcal{F}_p^{\lambda,\gamma}(r), \mathcal{F}_p^{\lambda,\gamma,\Delta}(r)) \le \psi_\lambda(\Delta). \tag{27}$$

For given $\Delta > 0$, $\delta > 0$, finite Δ-partition $\Lambda = \{\Omega_1, \Omega_2, \ldots, \Omega_N\}$ of the compact set $\Omega \subset \mathbb{R}^k$, and uniform δ-partition $\Lambda_* = \{0 = r_0, r_1, \ldots, r_a = \gamma\}$ of the closed interval $[0, \gamma]$, we set

$$B_p^{\gamma,\Delta,\delta}(r) = \{x(\cdot) \in B_p^{\gamma,\Delta}(r) : x(s) = x_i \quad \text{for every } s \in \Omega_i, \|x_i\| \in \Lambda_* \text{ for every } i = 1, 2, \ldots, N\},$$

$$\mathcal{F}_p^{\lambda,\gamma,\Delta,\delta}(r) = \{F_\lambda(x(\cdot))|(\cdot) : x(\cdot) \in B_p^{\gamma,\Delta,\delta}(r)\}.$$

Let $y_0(\cdot) \in \mathcal{F}_p^{\lambda,\gamma,\Delta}(r)$ be an arbitrarily chosen function, which is the image of $x_0(\cdot) \in B_p^{\gamma,\Delta}(r)$ under operator (12). From inclusion $x_0(\cdot) \in B_p^{\gamma,\Delta}(r)$, it follows that

$$x_0(s) = x_i, \quad s \in \Omega_i, \ i = 1, 2, \ldots, N, \tag{28}$$

where

$$\sum_{i=1}^{N} \mu(\Omega_i)\|x_i\|^p \le r^p, \quad \|x_i\| \le \gamma \quad \text{for every } i = 1, 2, \ldots, N. \tag{29}$$

If $\|x_i\| < \gamma$, then there exists $r_{j_i} \in \Lambda_*$ such that

$$\|x_i\| \in [r_{j_i}, r_{j_i+1}). \tag{30}$$

Define new function $v_0(\cdot) : \Omega \to \mathbb{R}^n$, setting

$$v_0(s) = \begin{cases} \dfrac{x_i}{\|x_i\|} r_{j_i} & \text{if } 0 < \|x_i\| < \gamma, \\ x_i & \text{if } \|x_i\| = 0 \quad \text{or} \quad \|x_i\| = \gamma, \end{cases} \tag{31}$$

where $s \in \Omega_i$, $i = 1, 2, \ldots, N$, and $r_{j_i} \in \Lambda_*$ is defined by (30). It is not difficult to observe that $\|v_0(s)\| \le \|x_0(s)\|$ for every $s \in \Omega$. Moreover, from equations (28)–(31) it follows that $v_0(\cdot) \in B_p^{\gamma,\Delta,\delta}(r)$ and

$$\|x_0(s) - v_0(s)\| \le \delta \tag{32}$$

for every $s \in \Omega$. Now, let $z_0(\cdot) \in \mathcal{F}_p^{\lambda,\gamma,\Delta,\delta}(r)$ be the image of $v_0(\cdot) \in B_p^{\gamma,\Delta,\delta}(r)$ under operator (12). Thus, (4) and (32) imply that

$$\|z_0(\xi) - y_0(\xi)\| \le M(\lambda)\mu(\Omega)\delta$$

for every $\xi \in E$. From the last inequality and (10), we conclude that

$$\|z_0(\cdot) - y_0(\cdot)\|_q \le M(\lambda)\mu(\Omega)\cdot[\mu(E)]^{\frac{1}{q}}\delta = \varphi_\lambda(\delta). \tag{33}$$

Since $y_0(\cdot) \in \mathcal{F}_p^{\lambda,\gamma,\Delta}(r)$ is arbitrarily chosen and $z_0(\cdot) \in \mathcal{F}_p^{\lambda,\gamma,\Delta,\delta}(r)$, the inequality (33) yields that

$$\mathcal{F}_p^{\lambda,\gamma,\Delta}(r) \subset \mathcal{F}_p^{\lambda,\gamma,\Delta,\delta}(r) + \varphi_\lambda(\delta)B_q(1). \tag{34}$$

The inclusion $\mathcal{F}_p^{\lambda,\gamma,\Delta,\delta}(r) \subset \mathcal{F}_p^{\lambda,\gamma,\Delta}(r)$ and (34) imply that

$$h_q(\mathcal{F}_p^{\lambda,\gamma,\Delta}(r), \mathcal{F}_p^{\lambda,\gamma,\Delta,\delta}(r)) \le \varphi_\lambda(\delta). \tag{35}$$

Let us set

$$\mathcal{F}_p^{\lambda,\gamma,\Delta,\delta,\sigma}(r) = \{F_\lambda(x(\cdot))|(\cdot) : x(\cdot) \in B_p^{\gamma,\Delta,\delta,\sigma}(r)\},$$

where the set $B_p^{\gamma,\Delta,\delta,\sigma}(r)$ is defined by (6).

Choose an arbitrary $w(\cdot) \in \mathcal{F}_p^{\lambda,\gamma,\Delta,\delta}(r)$, which is the image of $u(\cdot) \in B_p^{\gamma,\Delta,\delta}(r)$ under operator (12). From inclusion $u(\cdot) \in B_p^{\gamma,\Delta,\delta}(r)$, it follows that there exist $r_{j_i} \in \Lambda_*$, $g_i \in P$ $(i = 1, 2, \ldots, N)$ such that

$$u(s) = r_{j_i} g_i, \quad s \in \Omega_i, \ i = 1, 2, \ldots, N, \tag{36}$$

where

$$\sum_{i=1}^{N} \mu(\Omega_i) r_{j_i}^p \le r^p. \tag{37}$$

Since $g_i \in P$ for every $i = 1, 2, \ldots, N$, P_σ is a finite σ-net on P, then, for each $g_i \in P$, there exists $e_{l_i} \in P_\sigma$ such that

$$\|g_i - e_{l_i}\| \le \sigma. \tag{38}$$

Define function $u_*(\cdot) : \Omega \to \mathbb{R}^n$, setting

$$u_*(s) = r_{j_i} e_{l_i}, \quad s \in \Omega_i, \ i = 1, 2, \ldots, N. \tag{39}$$

From equations (36)–(39) it follows that $u_*(\cdot) \in B_p^{\gamma,\Delta,\delta,\sigma}(r)$ and

$$\|u_*(s) - u(s)\| \le r_{j_i}\|g_i - e_{l_i}\| \le \gamma\sigma \tag{40}$$

for every $s \in \Omega$. Now let $w_*(\cdot) \in \mathcal{F}_p^{\lambda,\gamma,\Delta,\delta,\sigma}(r)$ be the image of $u_*(\cdot) \in B_p^{\gamma,\Delta,\delta,\sigma}(r)$ under operator (12). Then, (40) yields

$$\|w_*(\xi) - w(\xi)\| \le M(\lambda)\mu(\Omega)\gamma\sigma$$

for every $\xi \in E$ and hence

$$\|w_*(\cdot) - w_0(\cdot)\|_q \le M(\lambda)\mu(\Omega)\cdot[\mu(E)]^{\frac{1}{q}}\gamma\sigma = \alpha_\lambda(\gamma, \sigma), \tag{41}$$

where $M(\lambda)$ is defined by (4) and $\alpha_\lambda(\gamma, \sigma)$ is defined by (11). Thus, for arbitrary chosen $w(\cdot) \in \mathcal{F}_p^{\lambda,\gamma,\Delta,\delta}(r)$, there exists $w_*(\cdot) \in \mathcal{F}_p^{\lambda,\gamma,\Delta,\delta,\sigma}(r)$ such that the inequality (41) is satisfied. This means that

$$\mathcal{F}_p^{\lambda,\gamma,\Delta,\delta}(r) \subset \mathcal{F}_p^{\lambda,\gamma,\Delta,\delta,\sigma}(r) + \alpha_\lambda(\gamma, \sigma)B_q(1).$$

The last inclusion and inclusion $\mathcal{F}_p^{\lambda,\gamma,\Delta,\delta,\sigma}(r) \subset \mathcal{F}_p^{\lambda,\gamma,\Delta,\delta}(r)$ imply that

$$h_q(\mathcal{F}_p^{\lambda,\gamma,\Delta,\delta,\sigma}(r), \mathcal{F}_p^{\lambda,\gamma,\Delta,\delta}(r)) \le \alpha_\lambda(\gamma, \sigma). \tag{42}$$

Analogous to (13), it is not difficult to show that

$$h_q(\mathcal{F}_p^{\lambda,\gamma,\Delta,\delta,\sigma}(r), \mathcal{F}_p^{\gamma,\Delta,\delta,\sigma}(r)) \le \frac{\lambda}{2}, \tag{43}$$

where the set $\mathcal{F}_p^{\gamma,\Delta,\delta,\sigma}(r)$ is defined by (7).

Finally, the proof of the theorem follows from the inequalities (13), (17), (27), (35), (42), and (43). □

From Theorem 1, it follows the validity of the following corollary.

Corollary 1. *For every* $\varepsilon > 0$, *there exist* $\lambda(\varepsilon) > 0$, $\gamma_*(\varepsilon) = \gamma(\varepsilon, \lambda(\varepsilon)) > 0$, $\Delta_*(\varepsilon) = \Delta(\varepsilon, \lambda(\varepsilon)) > 0$, $\delta_*(\varepsilon) = \delta(\varepsilon, \lambda(\varepsilon)) > 0$, *and* $\sigma_*(\varepsilon) = \sigma(\varepsilon, \lambda(\varepsilon), \gamma_*(\varepsilon)) > 0$ *such that, for every finite* Δ*-partition of the compact set* $\Omega \subset \mathbb{R}^k$, *uniform* δ*-partition of the closed interval* $[0, \gamma_*(\varepsilon)]$, *and* $\sigma > 0$, *the inequality*

$$h_q(\mathcal{F}_p(r), \mathcal{F}_p^{\gamma_*(\varepsilon),\Delta,\delta,\sigma}(r)) \le \varepsilon$$

is satisfied for each $\Delta \in (0, \Delta_*(\varepsilon)]$, $\delta \in (0, \delta_*(\varepsilon)]$, *and* $\sigma \in (0, \sigma_*(\varepsilon)]$.

Proof. Let us choose $\lambda(\varepsilon) = \frac{\varepsilon}{5}$, and fix it, and let

$$\gamma_*(\varepsilon) = \gamma(\varepsilon, \lambda(\varepsilon)) = \left(\frac{5c_* M(\lambda(\varepsilon))}{\varepsilon}\right)^{\frac{1}{p-1}}.$$

By virtue of (5) and (9), we have that $\psi_\lambda(\Delta) \to 0$ as $\Delta \to 0^+$. Define $\Delta_*(\varepsilon) = \Delta(\varepsilon, \lambda(\varepsilon)) > 0$ such that the inequality

$$\psi_{\lambda(\varepsilon)}(\Delta) \le \frac{\varepsilon}{5}$$

is satisfied for every $\Delta \in (0, \Delta_*(\varepsilon)]$.

Furthermore, we denote

$$\delta_*(\varepsilon) = \delta(\varepsilon, \lambda(\varepsilon)) = \frac{\varepsilon}{5M(\lambda(\varepsilon))\mu(\Omega)\cdot[\mu(E)]^{\frac{1}{q}}},$$

$$\sigma_*(\varepsilon) = \sigma(\varepsilon, \lambda(\varepsilon), \gamma_*(\varepsilon)) = \frac{\varepsilon}{5M(\lambda(\varepsilon))\mu(\Omega)\cdot[\mu(E)]^{\frac{1}{q}}\gamma_*(\varepsilon)}.$$

Now the proof of the corollary follows from Theorem 1. □

4 Conclusion

In this article, an approximation of the image of the closed ball $B_p(r) = \{x(\cdot) \in L_p(\Omega; \mathbb{R}^m) : \|x(\cdot)\|_p \leq r\}$ under the Hilbert-Schmidt integral operator is presented. The closed ball $B_p(r) \subset L_p(\Omega; \mathbb{R}^m)$ is replaced by the set, which consists of a finite number of piecewise constant functions, and it is proved that the images of these functions form an approximation of the image of the ball $B_p(r)$. An error evaluation for the Hausdorff distance between the image of the closed ball and its approximation, which consists of a finite number of functions, is given. The obtained result can be used to construct a set of outputs of the input–output system with integrally constrained inputs, which will allow the design of system outputs with prescribed properties.

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