Research Article

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The structure of fuzzy fractals generated by an orbital fuzzy iterated function system

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Abstract: In this article, we present a structure result concerning fuzzy fractals generated by an orbital fuzzy iterated function system $((X, d), (f_i)_{i\in I}, (\rho_i)_{i\in I})$. Our result involves the following two main ingredients: (a) the fuzzy fractal associated with the canonical iterated fuzzy function system $((I^{\mathbb{N}}, d_\Lambda), (\tau_i)_{i\in I}, (\rho_i)_{i\in I})$, where d_Λ is Baire's metric on the code space $I^{\mathbb{N}}$ and $\tau_i : I^{\mathbb{N}} \to I^{\mathbb{N}}$ is given by $\tau_i((\omega_1, \omega_2, \ldots)) := (i, \omega_1, \omega_2, \ldots)$ for every $(\omega_1, \omega_2, \ldots) \in I^{\mathbb{N}}$ and every $i \in I$; (b) the canonical projections of certain iterated function systems associated with the fuzzy fractal under consideration.

Keywords: orbital fuzzy iterated function system, fuzzy Hutchinson-Barnsley operator, fuzzy fractal, canonical projection

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1 Introduction

1.1 Generalities concerning iterated function systems, iterated fuzzy set systems and orbital fuzzy iterated function systems

Iterated function systems (which were initiated by Hutchinson in [1]) provide a standard framework for self-similarity. Some nice surveys on this theory are presented in [2–5]. Because of their attribute to squeeze large amount of data into a few number of parameters, they have nice applications in image processing via the so-called inverse problem, which asks to find an iterated function system whose attractor approximates a target set with a prescribed precision. In the case of image with gray levels the aforementioned theory involves a class of measures generated by adding a probability system to the iterated function system. An alternative approach, based on considering images as functions rather that measures, was introduced by Cabrelli and Molter [6] and Cabrelli et al. [7]. More precisely, they combined the idea of representation of a image as a fuzzy set with the Hutchinson's theory and introduced the concept of iterated fuzzy set system (IFZS). In this way, they succeeded in generating images with gray levels as attractors of IFZSs, i.e., as the fixed point of corresponding fuzzy operators. In addition, they proved that one can find an IFZS whose attractor approximates a target set with a imposed precision. For some other results along this line of research, see [8–10].

The articles [11–13] deal with the concept of orbital iterated function system, which is an iterated function system consisting of continuous functions satisfying Banach's orbital condition. It is a real

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generalization of the idea of the iterated function system since its associated fractal operator is weak Picard, but it is not necessarily Picard. The fuzzification idea applied to the framework of orbital iterated function systems leads to the study of the orbital fuzzy iterated function system in [14], whose corresponding fuzzy operator is weak Picard. Its fixed points are called fuzzy fractals. In [15], we presented a structure result concerning fuzzy fractals associated with an orbital fuzzy iterated function system. We proved that such an object is perfectly determined by the action of the initial term of the Picard iteration sequence on the closure of the orbits of certain elements.

1.2 The results of this article

Given a fuzzy iterated function system $\mathcal{S}_Z = ((X, d), (f_i)_{i\in I}, (\rho_i)_{i\in I})$, let us denote by $\mathbf{u}_{\mathcal{S}}$ its fuzzy fractal and by $\mathbf{u}_\Lambda$ the fuzzy fractal of the canonical iterated fuzzy function system $((\Lambda(I), d_\Lambda), (\tau_i)_{i\in I}, (\rho_i)_{i\in I})$, where d_Λ is Baire's metric on the code space $\Lambda(I)$ and $\tau_i : \Lambda(I) \to \Lambda(I)$ is given by $\tau_i(\omega) = i\omega$ for every $\omega \in \Lambda(I)$ and every $i \in I$.

First, we prove (Theorem 3.4) that $\mathbf{u}_S = \pi(\mathbf{u}_\Lambda)$, where π is the canonical projection associated with the iterated function system $((X, d), (f_i)_{i\in I})$. In addition (Theorem 3.2), we prove that $\mathbf{u}_\Lambda$ is perfectly determined by the admissible system of gray level maps $(\rho_i)_{i\in I}$.

Finally (Theorem 3.5), on the basis of the main result from [15] and the aforementioned results, we are able to provide a structure result concerning the fuzzy fractals generated by an orbital fuzzy iterated function system $\mathcal{S}_Z$. It involves $\mathbf{u}_\Lambda$ and the canonical projections of certain iterated function systems associated with the fuzzy fractal under consideration.

2 Preliminaries

2.1 Some basic notations

For a function $f : X \to X$ and $n \in \mathbb{N} \overset{\text{def}}{=} \{1, 2, \ldots\}$, by $f^{[n]}$, we designate the composition of f by itself n times.

Given a metric space (X, d), a function $f : X \to X$ is called weak Picard operator if there exists $\lim_{n\to\infty} f^{[n]}(x)$, and it is a fixed point of f for every $x \in X$. If a weak Picard operator has a unique fixed point, then it is called Picard operator.

Given a metric space (X, d), by h we designate the Hausdorff-Pompeiu metric on X, i.e. the function $h : P_{b,cl}(X) \times P_{b,cl}(X) \to [0, \infty)$, described by

$$h(K_1, K_2) = \max\left\{\sup_{x\in K_1} d(x, K_2), \sup_{x\in K_2} d(x, K_1)\right\},$$

for every $K_1, K_2 \in P_{b,cl}(X) = \{A \subseteq X | A \neq \varnothing \text{ and } A \text{ is bounded and closed}\}$.

For a family of functions $(f_i)_{i\in I}$, where $f_i : X \to \mathbb{R}$, we shall use the following notation:

$$\sup_{i\in I} f_i \overset{\text{not}}{=} \bigvee_{i\in I} f_i.$$

Lemma 2.1. *Given a family $(a_{ij})_{i\in I, j\in J}$ of real numbers, where I is finite and J is infinite, we have*

$$\max_{i\in I} \max_{j\in J} a_{ij} = \max_{j\in J} \max_{i\in I} a_{ij},$$

provided that

$$\sup_{j\in J} a_{ij} = \max_{j\in J} a_{ij},$$

for every $i \in I$.

Proof. For $j \in J$, we shall consider $i_j \in I$ such that

$$\max_{i\in I} a_{ij} = a_{i_j j}.$$

On the one hand, we have

$$\max_{i\in I}\max_{j\in J} a_{ij} \leq \max_{j\in J}\max_{i\in I} a_{ij}. \tag{1}$$

Indeed,

$$a_{ij} \leq a_{i_j j},$$

for all $i \in I$ and $j \in J$. Hence,

$$\sup_{j\in J} a_{ij} \leq \sup_{j\in J} a_{i_j j},$$

so

$$\max_{j\in J} a_{ij} \leq \max_{j\in J} a_{i_j j} = \max_{j\in J}\max_{i\in I} a_{ij},$$

for all $i \in I$ and the justification of (1) is completed.

On the other hand, we have

$$\max_{j\in J}\max_{i\in I} a_{ij} \leq \max_{i\in I}\max_{j\in J} a_{ij}. \tag{2}$$

Indeed,

$$a_{ij} \leq \sup_{j\in J} a_{ij} = \max_{j\in J} a_{ij},$$

for all $i \in I$ and $j \in J$. Therefore,

$$\max_{i\in I} a_{ij} \leq \max_{i\in I}\max_{j\in J} a_{ij},$$

i.e.,

$$a_{i_j j} \leq \max_{i\in I}\max_{j\in J} a_{ij},$$

for every $j \in J$. Hence,

$$\sup_{j\in J} a_{i_j j} = \max_{j\in J} a_{i_j j} = \max_{j\in J}\max_{i\in I} a_{ij} \leq \max_{i\in I}\max_{j\in J} a_{ij}$$

and the justification of (2) is finished.

By taking into account (1) and (2), the proof is completed. □

2.2 Fuzzy sets

Let us consider the sets X and Y.

The elements of

$$\{u : X \to [0,1]\} \overset{\text{not}}{=} \mathcal{F}_X$$

are called fuzzy subsets of X.

If there exists $x \in X$ such that $u(x) = 1$, then we say that $u \in \mathcal{F}_X$ is normal.

A nonzero function $\rho : [0,1] \to [0,1]$ is called a gray level map.

For a gray level map ρ and $u \in \mathcal{F}_X$, one can consider

$$\rho \circ u \overset{\text{not}}{=} \rho(u) \in \mathcal{F}_X.$$

Given $u \in \mathcal{F}_X$ and $\alpha \in (0,1]$, we shall use the following notations:

$$\{x \in X | u(x) \geq \alpha\} \overset{\text{not}}{=} [u]^\alpha$$

and

$$\{x \in X | u(x) > 0\} \overset{\text{not}}{=} [u]^*.$$

Given $u \in \mathcal{F}_X$ and $f : X \to Y$, one can consider $f(u) \in \mathcal{F}_Y$, which is described in the following way:

$$f(u)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(\{y\})} u(x), & \text{if } f^{-1}(\{y\}) \neq \emptyset \\ 0, & \text{if } f^{-1}(\{y\}) = \emptyset, \end{cases}$$

for every $y \in Y$.

Given a metric space (X, d) and $u \in \mathcal{F}_X$, we shall use the following notations:

$$\overline{[u]^*} \overset{\text{not}}{=} \operatorname{supp} u$$

$$\{u \in \mathcal{F}_X | u \text{ is normal and } \operatorname{supp} u \text{ is compact}\} \overset{\text{not}}{=} \mathcal{F}_X^{**}$$

and

$$\{u \in \mathcal{F}_X^{**} | u \text{ is upper semicontinuous}\} \overset{\text{not}}{=} \mathcal{F}_X^*.$$

Given a metric space (X, d), the function $d_\infty : \mathcal{F}_X^* \times \mathcal{F}_X^* \to [0, \infty)$, given by

$$d_\infty(u, v) \overset{\text{def}}{=} \sup_{\alpha \in [0,1]} h([u]^\alpha, [v]^\alpha) \overset{\text{Lemma 2.5 from [14]}}{=} \sup_{\alpha \in (0,1]} h([u]^\alpha, [v]^\alpha),$$

for every $u, v \in \mathcal{F}_X^*$, is a distance on $\mathcal{F}_X^*$. See [16] for more details.

Lemma 2.2. *Let us consider a continuous function $f : X \to X$ and an increasing function $\rho : [0,1] \to [0,1]$ such that $\rho(0) = 0$. Then*

$$\rho(f(u)) = f(\rho(u)),$$

for every $u \in \mathcal{F}_X^*$.

Proof. If for $x \in X$ we have $f^{-1}(\{x\}) = \emptyset$, then

$$f(\rho(u))(x) = 0$$

and

$$f(u)(x) = 0,$$

so

$$\rho(f(u))(x) = \rho(f(u)(x)) = \rho(0) = 0.$$

Therefore,

$$f(\rho(u))(x) = \rho(f(u))(x), \tag{3}$$

for every $x \in X$ such that $f^{-1}(\{x\}) = \emptyset$.

If for $x \in X$ we have $f^{-1}(\{x\}) \neq \emptyset$, then

$$\rho(f(u))(x) = \rho(f(u)(x)) = \rho\Big(\sup_{y \in f^{-1}(\{x\})} u(y)\Big) = \rho\Big(\sup_{y \in f^{-1}(\{x\}) \cap \operatorname{supp} u} u(y)\Big).$$

Since u is upper semicontinuous and $f^{-1}(\{x\}) \cap \operatorname{supp} u$ is compact, we obtain

$$\rho(f(u))(x) = \rho\Big(\max_{y \in f^{-1}(\{x\}) \cap \operatorname{supp} u} u(y)\Big) \overset{\rho \text{ increasing}}{=} \max_{y \in f^{-1}(\{x\}) \cap \operatorname{supp} u} \rho(u(y)) = f(\rho(u))(x), \tag{4}$$

for every $x \in X$ such that $f^{-1}(\{x\}) \neq \emptyset$.

In view of (3) and (4), the proof is completed. □

2.3 The code space

Let us consider a nonempty set I and $n \in \mathbb{N}$.

The set $I^{\mathbb{N}}$ will be denoted by $\Lambda(I)$. A standard element ω of $\Lambda(I)$ takes the form $\omega = \omega_1\omega_2 \dots \omega_n\omega_{n+1} \dots$.

The set $I^{\{1,2,\dots,n\}}$ will be denoted by $\Lambda_n(I)$. A standard element ω of $\Lambda_n(I)$ takes the form $\omega = \omega_1\omega_2 \dots \omega_n$.

Given $m \in \mathbb{N}$ and $\omega = \omega_1\omega_2 \dots \omega_n\omega_{n+1} \dots \in \Lambda(I)$, the word $\omega_1\omega_2 \dots \omega_m \in \Lambda_m(I)$ will be denoted by $[\omega]_m$.

One can easily check that $d_\Lambda : \Lambda(I) \times \Lambda(I) \to [0, \infty)$, given by

$$d_\Lambda(\omega, \theta) = \begin{cases} 0, & \text{if } \omega = \theta \\ \dfrac{1}{2^{\min\{k \in \mathbb{N} \mid \omega_k \neq \theta_k\}}}, & \text{if } \omega \neq \theta \end{cases},$$

for every $\omega = \omega_1\omega_2\omega_3 \dots \omega_n\omega_{n+1} \dots$, $\theta = \theta_1\theta_2\theta_3 \dots \theta_n\theta_{n+1} \dots \in \Lambda(I)$ is a metric on $\Lambda(I)$.

Note that the metric space $(\Lambda(I), d_\Lambda)$ is compact provided that I is finite.

For every $i \in I$, the function $\tau_i : \Lambda(I) \to \Lambda(I)$ given by

$$\tau_i(\omega) = i\omega_1\omega_2 \dots \omega_n\omega_{n+1} \dots,$$

for every $\omega = \omega_1\omega_2 \dots \omega_n\omega_{n+1} \dots \in \Lambda(I)$, is a contraction since

$$d_\Lambda(\tau_i(\omega), \tau_i(\theta)) = d_\Lambda(i\omega, i\theta) \leq \frac{1}{2} d_\Lambda(\omega, \theta),$$

for every $\omega, \theta \in \Lambda(I)$.

2.4 Iterated function systems, iterated fuzzy function systems, orbital iterated function systems, and orbital fuzzy iterated function systems

By an iterated function system, we mean a pair $((X, d), (f_i)_{i \in I}) \overset{\text{not}}{=} \mathcal{S}$, where:

- (X, d) is a complete metric space;
- $(f_i)_{i \in I}$ is a finite family of contractions $f_i : X \to X$.

The fractal operator associated with $\mathcal{S}$ is the function $F_{\mathcal{S}} : P_{cp}(X) \to P_{cp}(X)$, given by

$$F_{\mathcal{S}}(K) = \bigcup_{i \in I} f_i(K),$$

for all $K \in P_{cp}(X) = \{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is compact}\}$.

Let us recall that (see [1]):

(α) $F_{\mathcal{S}}$ is a contraction with respect to h. As $(P_{cp}(X), h)$ is a complete metric space, $F_{\mathcal{S}}$ is a Picard operator. Its fixed point is denoted by $A_{\mathcal{S}}$ and is called the attractor of $\mathcal{S}$.

(β) For each $\omega \in \Lambda(I)$, the set $\cap_{n \in \mathbb{N}} f_{[\omega]_n}(A_{\mathcal{S}})$ is a singleton and its element, denoted by a_ω, belongs to $A_{\mathcal{S}}$. The function $\pi : \Lambda(I) \to A_{\mathcal{S}}$, defined by

$$\pi(\omega) = a_\omega,$$

for each $\omega \in \Lambda(I)$, has the following properties:

- It is a continuous surjection;
- $$\pi \circ \tau_i = f_i \circ \pi,$$

  for each $i \in I$.

It is called the canonical projection associated with $\mathcal{S}$.

By an iterated fuzzy function system, we mean a triple $((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I}) \overset{\text{not}}{=} \mathcal{S}_Z$, where:

- $((X,d),(f_i)_{i\in I})$ is an iterated function system
- $(\rho_i)_{i\in I}$ is an admissible system of gray level maps, i.e., $\rho_i(0)=0$, ρ_i is nondecreasing and right continuous for every $i\in I$ and there exists $j\in I$ such that $\rho_j(1)=1$.

The fuzzy Hutchinson-Barnsley operator associated with $\mathcal{S}_Z$ is the function $Z:\mathcal{F}_X^*\to\mathcal{F}_X^*$, given by

$$Z(u)=\bigvee_{i\in I}\rho_i(f_i(u)),$$

for all $u\in\mathcal{F}_X^*$.

Let us recall (see [6,7]) that Z is a Banach contraction with respect to d_∞. As $(\mathcal{F}_X^*,d_\infty)$ is a complete metric space, Z is a Picard operator. Its unique fixed point is denoted by $\mathbf{u}_{\mathcal{S}}$ and is called the fuzzy fractal generated by $\mathcal{S}_Z$.

The canonical iterated fuzzy function system

$$((\Lambda(I),d_\Lambda),(\tau_i)_{i\in I},(\rho_i)_{i\in I}) \overset{\text{not}}{=} \mathcal{S}_\Lambda$$

will play a central role in this article.

The fuzzy Hutchinson-Barnsley operator associated with $\mathcal{S}_\Lambda$ will be denoted by Z_Λ and $\mathbf{u}_{\mathcal{S}_\Lambda}$ will be denoted by $\mathbf{u}_\Lambda$. Hence,

$$Z_\Lambda(\mathbf{u}_\Lambda)=\mathbf{u}_\Lambda$$

and

$$\lim_{n\to\infty} d_\infty(Z_\Lambda^{[n]}(u),\mathbf{u}_\Lambda)=0,$$

for every $u\in\mathcal{F}_{\Lambda(I)}^*$.

By an orbital iterated function system, we mean a pair $((X,d),(f_i)_{i\in I}) \overset{\text{not}}{=} \mathcal{S}$, where:

- (X,d) is a complete metric space;
- $(f_i)_{i\in I}$ is a finite family of continuous functions $f_i:X\to X$ having the property that there exists $C\in[0,1)$ such that

$$d(f_i(y),f_i(z))\le Cd(y,z),$$

for every $i\in I$, $x\in X$ and $y,z\in O(x)=\{x\}\cup\cup_{\omega\in\Lambda(I),n\in\mathbb{N}} f_{[\omega]_n}(x)$.

In contrast with the case of iterated function systems, the fractal operator associated with an orbital iterated function system is a weak Picard operator. Its fixed points are called the attractors of the system.

By an orbital fuzzy iterated function system, we mean a triple $((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I}) \overset{\text{not}}{=} \mathcal{S}_Z$, where:

- $((X,d),(f_i)_{i\in I})$ is an orbital iterated function system.
- $(\rho_i)_{i\in I}$ is an admissible system of gray level maps.

Let us mention that if:

- Z designates the fuzzy Hutchinson-Barnsley operator associated with $\mathcal{S}_Z$
- $$\{u\in\mathcal{F}_X^{**}|\text{ for each } x\in[u]^* \text{ there exist } w_x,y_x\in X \text{ such that } x,y_x\in O(w_x) \text{ and } u(y_x)=1\} \overset{\text{not}}{=} \mathcal{F}_{\mathcal{S}}^{**}$$
- $$\{u\in\mathcal{F}_{\mathcal{S}}^{**}|u \text{ is upper semicontinuous}\} \overset{\text{not}}{=} \mathcal{F}_{\mathcal{S}}^*,$$

then $\mathbf{Z}:\mathcal{F}_{\mathcal{S}}^*\to\mathcal{F}_{\mathcal{S}}^*$, given by

$$\mathbf{Z}(u) = Z(u),$$

for every $u \in \mathcal{F}^*_{\mathcal{S}}$, is weak Picard and its fixed points are called fuzzy fractals generated by $\mathcal{S}_Z$.

For $u \in \mathcal{F}^*_{\mathcal{S}}$ and $x \in [u]^*$ (hence, there exist $w_x, y_x \in X$ such that $x, y_x \in O(w_x)$ and $u(y_x) = 1$), we shall use the following notations:

$$\lim_{n\to\infty} Z^{[n]}(u) \overset{\text{not}}{=} \mathbf{u}_u \in \mathcal{F}^*_{\mathcal{S}},$$

and

$$\lim_{n\to\infty} Z^{[n]}(u^x) \overset{\text{not}}{=} \mathbf{u}_x \in \mathcal{F}^*_{\mathcal{S}}$$

where $u^x \in \mathcal{F}^*_{\mathcal{S}}$ is described by

$$u^x(y) = \begin{cases} u(y), & \text{if } y \in \overline{O(w_x)} \\ 0, & \text{otherwise} \end{cases}.$$

Remark 2.3. (see Lemma 3.2 from [14]). Let $\mathcal{S}_Z = ((X, d), (f_i)_{i\in I}, (\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system, $f : X \to X$ a continuous function and $(u_j)_{j\in J}$ a family of elements from $\mathcal{F}^{**}_X$ having the property that there exists $K \in P_{cp}(X)$ such that $\operatorname{supp} u_j \subseteq K$ for all $j \in J$.

Then

$$f(\bigvee_{j\in J} u_j) = \bigvee_{j\in J} f(u_j).$$

Lemma 2.4. *Let $\mathcal{S}_Z = ((X, d), (f_i)_{i\in I}, (\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system. Then*

$$Z^{[n]}(u) = \bigvee_{\omega\in\Lambda_n(I)} \rho_\omega(f_\omega(u)),$$

for every $n \in \mathbb{N}$ and every $u \in \mathcal{F}^*_X$.

Proof. Let us consider a fixed, but arbitrarily chosen $u \in \mathcal{F}^*_X$.

We are going to use the mathematical induction method to prove the thesis of our lemma.

For $n = 1$, the thesis is valid in view of the definition of Z.

Now supposing that it is valid for $n \in \mathbb{N}$, we have

$$\begin{aligned} Z^{[n+1]}(u) = Z^{[n]}(Z(u)) &= \bigvee_{\omega\in\Lambda_n(I)} \rho_\omega(f_\omega(Z(u))) \\ &= \bigvee_{\omega\in\Lambda_n(I)} \rho_\omega(f_\omega(\bigvee_{i\in I} \rho_i(f_i(u)))) \overset{\text{Remark 2.3}}{=} \\ &= \bigvee_{\omega\in\Lambda_n(I)} \rho_\omega(\bigvee_{i\in I} f_\omega(\rho_i(f_i(u))))^{\rho_\omega \text{ increasing and } I \text{ finite}} \overset{}{=} \\ &= \bigvee_{\omega\in\Lambda_n(I)} \bigvee_{i\in I} \rho_\omega(f_\omega(\rho_i(f_i(u)))) \overset{\text{Lemma 2.2}}{=} \bigvee_{\omega\in\Lambda_n(I)} \bigvee_{i\in I} \rho_\omega(\rho_i(f_\omega(f_i(u)))) \\ &= \bigvee_{\omega\in\Lambda_n(I)} \bigvee_{i\in I} \rho_{\omega i}(f_{\omega i}(u)) = \bigvee_{\omega\in\Lambda_{n+1}(I)} \rho_\omega(f_\omega(u)), \end{aligned}$$

i.e., the statement is valid for $n + 1$. □

Theorem 2.5. (see Theorem 3.9 from [15]). *Let $\mathcal{S}_Z = ((X, d), (f_i)_{i\in I}, (\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system and $u \in \mathcal{F}^*_{\mathcal{S}}$. Then*

$$\mathbf{u}_u = \bigvee_{x\in[u]^*} \mathbf{u}_x = \bigvee_{x\in[u]^1} \mathbf{u}_x = \max_{x\in[u]^*} \mathbf{u}_x = \max_{x\in[u]^1} \mathbf{u}_x.$$

3 The main results

3.1 A characterization of the fuzzy fractal $\mathbf{u}_\Lambda$

For a set I, let us consider the canonical iterated fuzzy function system $\mathcal{S}_\Lambda = ((\Lambda(I), d_\Lambda), (\tau_i)_{i\in I}, (\rho_i)_{i\in I})$.

We shall denote by **1** the element of $\mathcal{F}^*_{\mathcal{S}_\Lambda}$ given by

$$\mathbf{1}(\omega) = 1,$$

for every $\omega \in \Lambda(I)$.

Proposition 3.1. *In the aforementioned framework, we have*

$$Z_\Lambda^{[n]}(\mathbf{1})(\omega) = \rho_{[\omega]_n}(1),$$

for every $\omega \in \Lambda(I)$ and every $n \in \mathbb{N}$.

Proof. For $\omega \in \Lambda(I)$, $n \in \mathbb{N}$, and $\theta \in \Lambda_n(I)$, we have

$$\tau_\theta(\mathbf{1})(\omega) = \begin{cases} \sup\limits_{\alpha\in\tau_\theta^{-1}(\{\omega\})} \mathbf{1}(\alpha), & \text{if } \tau_\theta^{-1}(\{\omega\}) \neq \varnothing \\ 0, & \text{if } \tau_\theta^{-1}(\{\omega\}) = \varnothing \end{cases} = \begin{cases} 1, & \text{if } [\omega]_n = \theta \\ 0, & \text{if } [\omega]_n \neq \theta \end{cases},$$

so we obatin

$$\rho_\theta(\tau_\theta(\mathbf{1}))(\omega) = \begin{cases} \rho_\theta(1), & \text{if } [\omega]_n = \theta \\ \rho_\theta(0) = 0, & \text{if } [\omega]_n \neq \theta \end{cases}.$$

Consequently, we conclude that

$$Z_\Lambda^{[n]}(\mathbf{1})(\omega) \overset{\text{Lemma 2.4}}{=} \bigvee_{\theta\in\Lambda_n(I)} \rho_\theta(\tau_\theta(\mathbf{1}))(\omega) = \rho_{[\omega]_n}(1).$$ □

Theorem 3.2. *In the aforementioned framework, we have*

$$\mathbf{u}_\Lambda = \lim_{n\to\infty} u_n,$$

where $u_n \in \mathcal{F}^*_{\mathcal{S}_\Lambda}$ is given by

$$u_n(\omega) = \rho_{[\omega]_n}(1),$$

for every $n \in \mathbb{N}$ and every $\omega \in \Lambda(I)$.

Proof. We have

$$\lim_{n\to\infty} d_\infty(Z_\Lambda^{[n]}(\mathbf{1}), \mathbf{u}_\Lambda) = 0,$$

so, in view of Proposition 3.1, we obtain

$$\lim_{n\to\infty} d_\infty(u_n, \mathbf{u}_\Lambda) = 0.$$ □

Remark 3.3. The aforementioned result shows that $\mathbf{u}_\Lambda$ is perfectly determined by the admissible system of gray level maps $(\rho_i)_{i\in I}$.

3.2 A structure result concerning fuzzy fractal of a fuzzy iterated function system

Theorem 3.4. *In the aforementioned framework, for every fuzzy iterated function system* $\mathcal{S}_Z = ((X, d), (f_i)_{i\in I}, (\rho_i)_{i\in I})$, *we have*

$$\mathbf{u}_S = \pi(\mathbf{u}_\Lambda),$$

where π *is the canonical projection associated with the iterated function system* $((X, d), (f_i)_{i\in I})$.

Proof. It suffices to prove that

$$Z(\pi(\mathbf{u}_\Lambda)) = \pi(\mathbf{u}_\Lambda).$$

We have

$$\begin{aligned} Z(\pi(\mathbf{u}_\Lambda))(x) &= \max_{i\in I} \rho_i(f_i(\pi(\mathbf{u}_\Lambda))(x)) = \max_{i\in I} \rho_i((f_i \circ \pi)(\mathbf{u}_\Lambda))(x) \\ &= \max_{i\in I} \rho_i\left(\sup_{\omega\in\Lambda(I) \text{such that} (f_i\circ\pi)(\omega)=x} \mathbf{u}_\Lambda(\omega)\right) \\ &= \max_{i\in I} \rho_i\left(\sup_{\omega\in\Lambda(I) \text{such that} (\pi\circ\tau_i)(\omega)=x} \mathbf{u}_\Lambda(\omega)\right) \\ &= \max_{i\in I} \rho_i\left(\sup_{\theta\in\Lambda(I) \text{such that} \pi(\theta)=x} \ \sup_{\omega\in\Lambda(I) \text{ such that } \tau_i(\omega)=\theta} \mathbf{u}_\Lambda(\omega)\right), \end{aligned}$$

for every $x \in X$.

As $\mathbf{u}_\Lambda$ is upper semicontinuous, τ_i is continuous and $\Lambda(I)$ is compact, we infer that

$$\begin{aligned} Z(\pi(\mathbf{u}_\Lambda))(x) &= \max_{i\in I} \rho_i\left(\max_{\theta\in\Lambda(I) \text{ such that } \pi(\theta)=x} \tau_i(\mathbf{u}_\Lambda)(\theta)\right) \overset{\rho_i \text{ is nondecreasing}}{=} \\ &= \max_{i\in I} \max_{\theta\in\Lambda(I) \text{ such that } \pi(\theta)=x} \rho_i(\tau_i(\mathbf{u}_\Lambda)(\theta)) \overset{\text{Lemma 2.1}}{=} \\ &= \max_{\theta\in\Lambda(I) \text{such that} \pi(\theta)=x} \max_{i\in I} \rho_i(\tau_i(\mathbf{u}_\Lambda)(\theta)) = \sup_{\theta\in\Lambda(I) \text{ such that } \pi(\theta)=x} Z_\Lambda(\mathbf{u}_\Lambda)(\theta) \\ &= \sup_{\theta\in\Lambda(I) \text{such that} \pi(\theta)=x} \mathbf{u}_\Lambda(\theta) = \pi(\mathbf{u}_\Lambda)(x), \end{aligned}$$

for every $x \in X$. □

3.3 A structure result concerning the attractors of an orbital fuzzy iterated function system

Let $\mathcal{S}_Z = ((X, d), (f_i)_{i\in I}, (\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system and $u \in \mathcal{F}^*_{\mathcal{S}}$.

Then for every $x \in [u]^*$, there exist $w_x, y_x \in X$ such that $x, y_x \in \mathcal{O}(w_x)$ and $u(y_x) = 1$.

We consider the fuzzy iterated function system

$$((\overline{\mathcal{O}(w_x)}, d), (\tilde{f}_i)_{i\in I}, (\rho_i)_{i\in I}) \overset{\text{not}}{=} S_{w_x},$$

where $\tilde{f}_i : \overline{\mathcal{O}(w_x)} \to \overline{\mathcal{O}(w_x)}$ is given by

$$\tilde{f}_i(y) = f_i(y),$$

for every $y \in \overline{\mathcal{O}(w_x)}$ and we denote by π_x its canonical projection and by Z_{w_x} its fuzzy Hutchinson-Barnsley operator.

Let us also denote by $\pi_x(\widetilde{\mathbf{u}}_\Lambda)$ the function given by

$$\pi_x(\widetilde{\mathbf{u}}_\Lambda)(y) = \begin{cases} \pi_x(\mathbf{u}_\Lambda)(y), & \text{if } y \in \overline{O(w_x)} \\ 0, & \text{if } y \in X\backslash\overline{O(w_x)} \end{cases}.$$

Theorem 3.5. *In the aforementioned framework, we have*

$$\mathbf{u}_u = \bigvee_{x\in[u]^*} \pi_x(\widetilde{\mathbf{u}}_\Lambda) = \bigvee_{x\in[u]^1} \pi_x(\widetilde{\mathbf{u}}_\Lambda) = \max_{x\in[u]^*} \pi_x(\widetilde{\mathbf{u}}_\Lambda) = \max_{x\in[u]^1} \pi_x(\widetilde{\mathbf{u}}_\Lambda).$$

Proof. Let us note that

$$\mathbf{u}_x = \pi_x(\widetilde{\mathbf{u}}_\Lambda), \tag{5}$$

for every $x \in [u]^*$.

Indeed, we have

$$d_\infty(\mathbf{u}_x, \pi_x(\widetilde{\mathbf{u}}_\Lambda)) \le d_\infty(\mathbf{u}_x, Z^{[n]}(u^x)) + d_\infty(Z^{[n]}(u^x), \pi_x(\widetilde{\mathbf{u}}_\Lambda)) = d_\infty(\mathbf{u}_x, Z^{[n]}(u^x)) + d_\infty\left(Z^{[n]}_{w_x}\left(u_{|\overline{O(w_x)}}\right), \pi_x(\mathbf{u}_\Lambda)\right), \tag{6}$$

for every $n \in \mathbb{N}$ and every $x \in [u]^*$. By passing to limit as n goes to ∞ in (6), we conclude that $\mathbf{u}_x = \pi_x(\widetilde{\mathbf{u}}_\Lambda)$ for every $x \in [u]^*$.

As, in view of Theorem 2.5, we have

$$\mathbf{u}_u = \bigvee_{x\in[u]^*} \mathbf{u}_x = \bigvee_{x\in[u]^1} \mathbf{u}_x = \max_{x\in[u]^*} \mathbf{u}_x = \max_{x\in[u]^1} \mathbf{u}_x,$$

and via (5), we obtain

$$\mathbf{u}_u = \bigvee_{x\in[u]^*} \pi_x(\widetilde{\mathbf{u}}_\Lambda) = \bigvee_{x\in[u]^1} \pi_x(\widetilde{\mathbf{u}}_\Lambda) = \max_{x\in[u]^*} \pi_x(\widetilde{\mathbf{u}}_\Lambda) = \max_{x\in[u]^1} \pi_x(\widetilde{\mathbf{u}}_\Lambda).$$ □

4 Conclusion

In this article, we presented a structure result concerning fuzzy fractals generated by an orbital fuzzy iterated function system $((X, d), (f_i)_{i\in I}, (\rho_i)_{i\in I})$.

In Theorem 3.4, we proved that $\mathbf{u}_S = \pi(\mathbf{u}_\Lambda)$, where π is the canonical projection associated with the iterated function system $((X, d), (f_i)_{i\in I})$. Moreover, in Theorem 3.2, we proved that $\mathbf{u}_\Lambda$ is perfectly determined by the admissible system of gray level maps $(\rho_i)_{i\in I}$. Finally, in Theorem 3.5, we provided a structure result concerning the fuzzy fractals generated by an orbital fuzzy iterated function system $\mathcal{S}_Z$.

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