Research Article

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On local fractional integral inequalities via generalized $(\tilde{h}_1, \tilde{h}_2)$-preinvexity involving local fractional integral operators with Mittag-Leffler kernel

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Abstract: Local fractional integral inequalities of Hermite-Hadamard type involving local fractional integral operators with Mittag-Leffler kernel have been previously studied for generalized convexities and preinvexities. In this article, we analyze Hermite-Hadamard-type local fractional integral inequalities via generalized $(\tilde{h}_1, \tilde{h}_2)$-preinvex function comprising local fractional integral operators and Mittag-Leffler kernel. In addition, two examples are discussed to ensure that the derived consequences are correct. As an application, we construct an inequality to establish central moments of a random variable.

Keywords: generalized $(\tilde{h}_1, \tilde{h}_2)$-preinvex functions, local fractional integrals, generalized Hermite-Hadamard inequality, fractal sets, Mittag-Leffler kernel

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1 Introduction

In the area of mathematical inequalities, convex functions are connected with a variety of assumptions. The most well-known inequality with its extensive geometrical structure is the Hermite-Hadamard inequality, presented by Jacques Hadamard in 1881 (see [1]). It is well understood that a function Q is a convex function on an interval J if and only if it meets the following inequality:

$$Q\left(\frac{s+r}{2}\right) \leq \frac{1}{r-s}\int_{s}^{r} Q(y)\mathrm{d}y \leq \frac{Q(s)+Q(r)}{2}, \tag{1}$$

for $s, r \in J$ and $s < r$. This classical inequality has been studied broadly in terms of extensions, generalizations, and improvements via various generalizations of convex function (we refer to [2–8] and references therein for more details).

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One of the famous extended convex functions is *preinvex function*. Throughout this work, we assume that $\mathcal{D} \subseteq \mathbb{R}^n$ is a non-empty set, $Q : \mathcal{D} \to \mathbb{R}^n$ a continuous mapping, and $\eta(.,.) : \mathcal{D} \times \mathcal{D} \to \mathbb{R}^n$ be a bi-function. Recall that a set $\mathcal{D} \subseteq \mathbb{R}^n$ is said to be *invex* with respect to the bi-function $\eta(.,.) : \mathcal{D} \times \mathcal{D} \to \mathbb{R}^n$ if

$$s + \varsigma\eta(r, s) \in \mathcal{D},$$

where $s, r \in \mathcal{D}$, and $0 \le \varsigma \le 1$. If $\eta(r, s) = s - r$ holds, the invex set $\mathcal{D}$ becomes convex (see [9, p. 1]).

Definition 1. [10, p. 30] Given an invex set $\mathcal{D}$ with respect to $\eta(.,.)$. Then, $Q : \mathcal{D} \to \mathbb{R}^n$ is the preinvex function on $\mathcal{D}$ if

$$Q(s + \varsigma\eta(r, s)) \le (1 - \varsigma)Q(s) + \varsigma Q(r), \tag{2}$$

where $s, r \in \mathcal{D}$ and $0 \le \varsigma \le 1$. When $-Q$ is preinvex, Q becomes preconcave.

Definition 2. [11, p. 2] Given two non-negative functions $\tilde{h}_1, \tilde{h}_2 : (0, 1) \subseteq J \to \mathbb{R}$. A function Q on the invex set $\mathcal{D}$ is said to be $(\tilde{h}_1, \tilde{h}_2)$-preinvex, if

$$Q(s + \varsigma\eta(r, s)) \le \tilde{h}_1(1 - \varsigma)\tilde{h}_2(\varsigma)Q(s) + \tilde{h}_1(\varsigma)\tilde{h}_2(1 - \varsigma)Q(r), \tag{3}$$

where $s, r \in \mathcal{D}$ and $0 \le \varsigma \le 1$.

Note that the Jensen-type $(\tilde{h}_1, \tilde{h}_2)$-preinvex function is obtained if $\varsigma = \frac{1}{2}$ in (3), that is,

$$Q\left(\frac{2s + \eta(r, s)}{2}\right) \le \tilde{h}_1\left(\frac{1}{2}\right)\tilde{h}_2\left(\frac{1}{2}\right)[Q(s) + Q(r)].$$

Yang presented the theory of local fractional calculus in [12,13]. This theory has valid applications in communication engineering, control theory, physics, and random walk process (see [14–17]). In the context of ordinary and partial differential equations, local fractional calculus can be used to study the dynamics of systems that exhibit non-differentiable behavior at certain points in space as well as time (see [18–25]). On the other hand, different studies were conducted to extend the concept of convex functions on fractal sets to explore Hermite-Hadamard inequalities (we refer [26–33] for more details). Some important generalizations of convex functions in fractal sense include the results presented by Sun [34–40]. Ahmad et al. [41] studied functional inequalities for convex functions emphasizing novel integral operators of fractional order with exponential kernel. Sun in [42] defined two fractal integral operators with Mittag-Leffler kernel to study local fractional inequalities of Hermite-Hadamard-type via generalized h-convex functions involving these operators.

Definition 3. [42, p. 4989] Let $Q \in L(s, r)$. The fractional integrals $\boldsymbol{I}_s^{\xi}(g)$ and $\boldsymbol{I}_r^{\xi}(g)$ of order $\xi \in (0, 1)$ are given as:

$$\boldsymbol{I}_s^{\xi}(x) = \frac{1}{\xi^{\xi}\Gamma(1 + \xi)}\int_s^g E_{\xi}\left(-\frac{1 - \xi}{\xi}(g - k)\right)Q(k)\mathrm{d}k, \quad g > s, \tag{4}$$

and

$$\boldsymbol{I}_r^{\xi}(x) = \frac{1}{\xi^{\xi}\Gamma(1 + \xi)}\int_g^r E_{\xi}\left(-\frac{1 - \xi}{\xi}(k - g)\right)Q(k)\mathrm{d}k, \quad g < r. \tag{5}$$

Theorem 1. [42, p. 4989] *Let the function $Q : [s, r] \to \mathbb{R}^{\xi}$ be a positive function with $0 \le s < r$ and $Q(y) \in I_y^{(\xi)}[s, r]$. If Q is a generalized h-convex function on $[s, r]$, then*

$$\frac{[1^{\xi}-E_{\xi}(-\rho)^{\xi}]}{\rho^{\xi}h^{\xi}\left(\frac{1}{2}\right)}Q\left(\frac{s+r}{2}\right)\leq\left(\frac{\xi}{(r-s)}\right)^{\xi}\left[\boldsymbol{I}_{s^{+}}^{(\xi)}Q(r)+\boldsymbol{I}_{r^{-}}^{(\xi)}Q(s)\right] \tag{6}$$
$$\leq[Q(s)+Q(r)]_0I_1^{(\xi)}E_{\xi}(-\rho\varsigma)^{\xi}[h^{\xi}(\varsigma)+h^{\xi}(1-\varsigma)],$$

where $\rho=\frac{1-\xi}{\xi}(r-s)$.

Sun in [43, Lemma 5] proved the following identity for generalized preinvex functions.

Lemma 1. *Let* $\mathcal{D}\subseteq\mathbb{R}$ *be an open invex subset with respect to the bi-function* $\eta:\mathcal{D}\times\mathcal{D}\to\mathbb{R}$, *where* $s,r\in\mathcal{D}$, $s<s+\eta(r,s)$. *Assume that* $Q:\mathbb{I}\to\mathbb{R}^{\xi}$ *is a local fractional differentiable function with* $\eta(r,s)\geq 0$, $s,r\in\mathcal{D}$. *If* $Q(y)^{\xi}\in I_y^{(\xi)}[s,s+\eta(r,s)]$, *then the following equality holds*:

$$\frac{Q(s)+Q(s+\eta(r,s))}{2^{\xi}}-\frac{(1-\xi)^{\xi}}{2^{\xi}[1^{\xi}-E_{\xi}(-\rho)^{\xi}]}[\boldsymbol{I}_{s^{+}}^{(\xi)}Q(s+\eta(r,s))+\boldsymbol{I}_{(s+\eta(r,s))^{-}}^{(\xi)}Q(s)]$$
$$=\frac{\eta^{\xi}(r,s)}{2^{\xi}[1^{\xi}-E_{\xi}(-\rho)^{\xi}]}\left[\frac{1}{\Gamma(1+\xi)}\int_0^1E_{\xi}(-\rho\varsigma)^{\xi}Q^{\xi}(s+(1-\varsigma)\eta(r,s))(\mathrm{d}\varsigma)^{\xi}\right.$$
$$\left.-\frac{1}{\Gamma(1+\xi)}\int_0^1E_{\xi}(-\rho\varsigma)^{\xi}Q^{\xi}(s+\varsigma\eta(r,s))(\mathrm{d}\varsigma)^{\xi}\right].$$

2 Preliminaries

We recall some properties for addition and multiplication operations on $\mathbb{R}^{\xi}$ for $0<\xi\leq 1$. If $d^{\xi},e^{\xi},f^{\xi}\in\mathbb{R}^{\xi}$, then

- $d^{\xi}+e^{\xi}\in\mathbb{R}^{\xi}$, $d^{\xi}e^{\xi}\in\mathbb{R}^{\xi}$,
- $d^{\xi}+e^{\xi}=d^{\xi}+e^{\xi}=(d+e)^{\xi}=(e+d)^{\xi}$,
- $d^{\xi}+(e^{\xi}+f^{\xi})=(d+e)^{\xi}+f^{\xi}$,
- $d^{\xi}e^{\xi}=e^{\xi}d^{\xi}=(de)^{\xi}=(ed)^{\xi}$,
- $d^{\xi}(e^{\xi}f^{\xi})=(d^{\xi}e^{\xi})f^{\xi}$,
- $d^{\xi}(e^{\xi}+f^{\xi})=d^{\xi}e^{\xi}+d^{\xi}f^{\xi}$,
- $d^{\xi}+0^{\xi}=0^{\xi}+d^{\xi}=d^{\xi}$, and $d^{\xi}1^{\xi}=1^{\xi}d^{\xi}=d^{\xi}$,
- If $d^{\xi}<e^{\xi}$, then $d^{\xi}+f^{\xi}<e^{\xi}+f^{\xi}$,
- If $0^{\xi}<d^{\xi}$, $0^{\xi}<e^{\xi}$, then $0^{\xi}<d^{\xi}e^{\xi}$.

We now review the concepts of local fractional continuity, derivative, and integral on $\mathbb{R}^{\xi}$ (see [12, chapter 2] and [13, chapter 1] for more extensive study).

Definition 4. A non-differentiable mapping $Q:\mathbb{R}\to\mathbb{R}^{\xi}$, $y\to Q(y)$ is said to be local fractional continuous at y_0; if for any $\varepsilon>0$, there exists $\delta>0$ such that

$$|Q(y)-Q(y_0)|<\varepsilon^{\xi}$$

holds whenever $|y-y_0|<\delta$, with $\varepsilon,\delta\in\mathbb{R}$. If $Q(y)$ is the local fractional continuous function on (b,c), we denote $Q(y)\in C_{\xi}(b,c)$.

Definition 5. The local fractional derivative of $Q(y)$ of order ξ at $y=y_0$ is defined as follows:

$$Q^{(\xi)}(y_0)=\left.\frac{\mathrm{d}^{\xi}Q(y)}{\mathrm{d}\varsigma^{\xi}}\right|_{y=y_0}=\lim_{y\to y_0}\frac{\Gamma(1+\xi)(Q(y)-Q(y_0))}{(y-y_0)^{\xi}}.$$

In this case, $D_{\xi}(b,c)$ is called ξ-local derivative set. If there exists $Q^{((K+1)\xi)}(y)=\overbrace{D_y^{\xi}\ldots D_y^{\xi}}^{(n+1)\text{times}}Q(y)$ for any $y\in\mathbb{I}\subseteq\mathbb{R}$, we denote $Q\in D_{(n+1)\xi}(I)$, and $n=0,1,2,\ldots$.

Definition 6. Let $Q(y) \in C_\xi[b,c]$. The local fractional integral of $Q(y)$ can be defined by:

$$ {}_bI_c^\xi Q(y) = \frac{1}{\Gamma(1+\xi)}\int_b^c Q(\varsigma)(\mathrm{d}\varsigma)^\xi = \frac{1}{\Gamma(1+\xi)}\lim_{\Delta\varsigma\to 0}\sum_{j=0}^{N-1} Q(\varsigma_j)(\Delta\varsigma_j)^\xi, $$

where $b = \varsigma_0 < \varsigma_1 < \ldots < \varsigma_{N-1} < \varsigma_N = c$, $[\varsigma_j, \varsigma_{j+1}]$ is a partition of $[b,c]$, $\Delta\varsigma_j = \Delta\varsigma_{j+1} - \Delta\varsigma_j$, $\Delta\varsigma = \max\{\varsigma_0, \varsigma_1 \ldots \varsigma_{N-1}\}$.

Note that ${}_bI_c^\xi Q(y) = 0$ if $b = c$, and ${}_cI_b^\xi Q(y) = -{}_bI_c^\xi Q(y)$ if $b < c$. We denote $Q(y) \in I_y^\xi[b,c]$ if there exists ${}_bI_y^\xi Q(y)$ for any $y \in [b,c]$.

Mittag-Leffler function of fractal order ξ $(0 < \xi \le 1)$ on Yang's fractal sets can be defined by:

$$ E_\xi(y^\xi) = \sum_{k=0}^{\infty} \frac{y^{k\xi}}{\Gamma(1+k\xi)} \quad y \in \mathbb{R}. \tag{7} $$

Formulas for local fractional calculus of Mittag-Leffler function are presented as (see [12, chapter 2] and [13, chapter 1]).

Lemma 2. *The local fractional derivative and local fractional integration of Mittag-Leffler function can be given as*:

$$ \frac{\mathrm{d}^\xi E_\xi(ky^\xi)}{\mathrm{d}y^\xi} = kE_\xi(ky^\xi), \quad k \text{ is a constant}, \tag{8} $$

$$ {}_bI_c^\xi E_\xi(y^\xi) = E_\xi(c^\xi) - E_\xi(b^\xi). \tag{9} $$

Lemma 3.

(1) *Let* $g(y) = f^{(\xi)}(y) \in C_\xi[b,c]$. *Then,*

$$ {}_bI_c^\xi g(y) = f(c) - f(b). $$

(2) *Let* $g(y), f(y) \in D_\xi[b,c]$ *and* $g^{(\xi)}(y), f^{(\xi)}(y) \in C_\xi[b,c]$. *Then,*

$$ {}_bI_c^\xi g(y) f^{(\xi)}(y) = g(y)f(y)|_b^c - {}_bI_c^\xi g^{(\xi)}(y) f(y). $$

Lemma 4. *Let* $Q(y) \in C_\xi[b,c]$. *The local fractional derivative and integral of an elementary function* $Q(y) = y^{s\xi}$ *is given by*:

$$ \frac{\mathrm{d}^\xi y^{s\xi}}{\mathrm{d}u^\xi} = \frac{\Gamma(1+s\xi)}{\Gamma(1+(s-1)\xi)} y^{(s-1)\xi}, $$

$$ \frac{1}{\Gamma(\xi+1)}\int_b^c y^{s\xi}(du)^\xi = \frac{\Gamma(1+s\xi)}{\Gamma(1+(s+1)\xi)}(c^{(s+1)\xi} - b^{(s+1)\xi}), \quad s > 0. $$

Lemma 5. *Let* $Q(y) = 1$. *Then, by property of mean value theorem for local fractional integrals, we have*

$$ {}_bI_c^\xi 1^\xi = \frac{(c-b)^\xi}{\Gamma(1+\xi)}. $$

Lemma 6. (Generalized Hölder's inequality) *Let* $p, q > 1$ *with* $p^{-1} + q^{-1} = 1$. *Let* $g(y), f(y) \in C_\xi[b,c]$. *Then,*

$$ \frac{1}{\Gamma(\xi+1)}\int_b^c |g(y)f(y)|(\mathrm{d}y)^\xi \le \left(\frac{1}{\Gamma(\xi+1)}\int_b^c |g(y)|^p(\mathrm{d}y)^\xi\right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\xi+1)}\int_b^c |f(y)|^q(\mathrm{d}y)^\xi\right)^{\frac{1}{q}}. \tag{10} $$

We recall the generalized beta function, which will be used throughout the work:

$$B_\xi(y,x)=\frac{1}{\Gamma(1+\xi)}\int_0^1\varsigma^{(y-1)\xi}(1-\varsigma)^{(x-1)\xi}(\mathrm{d}\varsigma)^\xi,\quad y>0,x>0. \tag{11}$$

3 Main results

We begin by introducing the definition of generalized $(\tilde{h}_1,\tilde{h}_2)$-preinvex with respect to a bi-function η.

Definition 7. Suppose that $\tilde{h}_1,\tilde{h}_2:(0,1)\subseteq\mathbb{J}\to\mathbb{R}$ are two non-negative functions with $\tilde{h}_1^{\xi}\not\equiv 0^\xi$, $\tilde{h}_2^{\xi}\not\equiv 0^\xi$, and $(0,1)\subseteq\mathbb{J}$ is any interval in $\mathbb{R}$. Let $\mathcal{D}$ be an invex set with respect to η. A function $Q:\mathcal{D}\to\mathbb{R}^\xi(0<\xi\le 1)$ is said to be *generalized* $(\tilde{h}_1,\tilde{h}_2)$*-preinvex* with respect to η; if for each $s,r\in\mathcal{D}$ and $\varsigma\in[0,1]$, we have

$$Q(r+\varsigma\eta(s,r))\le\tilde{h}_1^{\xi}(1-\varsigma)\tilde{h}_2^{\xi}(\varsigma)Q(s)+\tilde{h}_1^{\xi}(\varsigma)\tilde{h}_2^{\xi}(1-\varsigma)Q(r). \tag{12}$$

Moreover, the function Q is *generalized* $(\tilde{h}_1,\tilde{h}_2)$*-preconcave* with respect to η if it satisfies the aforementioned reverse inequality.

Remark 1. If we assume $\xi=1$ in (12), then a *generalized* $(\tilde{h}_1,\tilde{h}_2)$*-preinvex* function becomes a classical $(\tilde{h}_1,\tilde{h}_2)$-preinvex function. If $\tilde{h}_1^{\xi}(\varsigma)=\varsigma^{s\xi}$ and $\tilde{h}_2^{\xi}(v)=v^{s\xi}$with $s\in[0,1]$, then (12) becomes *generalized s-preinvex* function. Moreover, if $\tilde{h}_1^{\xi}(1-\varsigma)\tilde{h}_2^{\xi}(\varsigma)+\tilde{h}_1^{\xi}(\varsigma)\tilde{h}_2^{\xi}(1-\varsigma)=1$, then (12) becomes *generalized Toader-like preinvex* function (see [44, p. 79]).

For what follows, we set $\rho=\frac{1-\xi}{\eta(r,s)}$.

Theorem 2. *Let $\mathcal{D}\subseteq\mathbb{R}$ be an open invex subset with respect to the bi-function $\eta:\mathcal{D}\times\mathcal{D}\to\mathbb{R}$, where $s,r\in\mathcal{D}$, and $s<s+\eta(r,s)$. Let $Q:\mathcal{D}\to\mathbb{R}^\xi$ be a positive function with $\eta(r,s)\ge 0$ and $Q(y)\in I_y^{(\xi)}[s,s+\eta(r,s)]$. If Q is generalized $(\tilde{h}_1,\tilde{h}_2)$-preinvex function on $[s,s+\eta(r,s)]$, then the following inequalities hold:*

$$\begin{aligned}\frac{[1^\xi-E_\xi(-\rho)^\xi]}{\rho^\xi\tilde{h}_1^{\xi}\left(\frac{1}{2}\right)\tilde{h}_2^{\xi}\left(\frac{1}{2}\right)}Q\left(\frac{2s+\eta(r,s)}{2}\right)&\le\left(\frac{\xi}{\eta(s,r)}\right)^\xi\left[\boldsymbol{I}_{s^+}^{(\xi)}Q(s+\eta(r,s))+\boldsymbol{I}_{(s+\eta(r,s))^-}^{(\xi)}Q(s)\right]\\&\le[Q(s)+Q(r)]{}_0I_1^{(\xi)}E_\xi(-\rho\varsigma)^\xi[\tilde{h}_1^{\xi}(1-\varsigma)\tilde{h}_2^{\xi}(\varsigma)+\tilde{h}_1^{\xi}(\varsigma)\tilde{h}_2^{\xi}(1-\varsigma)].\end{aligned} \tag{13}$$

Proof. Since Q is a *generalized* $(\tilde{h}_1,\tilde{h}_2)$*-preinvex* function, then

$$Q\left(\frac{2w+\eta(x,w)}{2}\right)\le\tilde{h}_1\left(\frac{1}{2}\right)\tilde{h}_2\left(\frac{1}{2}\right)[Q(w)+Q(x)].$$

Let $w=s+\varsigma\eta(r,s)$, and $x=s+(1-\varsigma)\eta(r,s)$. Then,

$$\frac{1^\xi}{\tilde{h}_1^{\xi}\left(\frac{1}{2}\right)\tilde{h}_2^{\xi}\left(\frac{1}{2}\right)}Q\left(\frac{2s+\eta(r,s)}{2}\right)\le Q(s+\varsigma\eta(r,s))+Q(s+(1-\varsigma)\eta(r,s)). \tag{14}$$

By multiplying both sides of the inequality in (14) by $E_\xi(-\rho\varsigma)^\xi$, then using local fractional integration corresponding to ς over $[0,1]$ of the resulting inequality, we obtain

$$\begin{aligned}&\frac{1^\xi}{\tilde{h}_1^{\xi}\left(\frac{1}{2}\right)\tilde{h}_2^{\xi}\left(\frac{1}{2}\right)}Q\left(\frac{2s+\eta(r,s)}{2}\right)\frac{1}{\Gamma(1+\xi)}\int_0^1E_\xi(-\rho\varsigma)^\xi(\mathrm{d}\varsigma)^\xi\\&\le\frac{1}{\Gamma(1+\xi)}\int_0^1E_\xi(-\rho\varsigma)^\xi Q(s+\varsigma\eta(r,s))(\mathrm{d}\varsigma)^\xi+\frac{1}{\Gamma(1+\xi)}\int_0^1E_\xi(-\rho\varsigma)^\xi Q(s+(1-\varsigma)\eta(r,s))(\mathrm{d}\varsigma)^\xi.\end{aligned} \tag{15}$$

By setting $s+\varsigma\eta(r,s)=u$, and $s+(1-\varsigma)\eta(r,s)=v$, we have

$$
\begin{aligned}
&\frac{1^{\xi}}{\rho^{\xi}\tilde{h}_1^{\xi}\left(\frac{1}{2}\right)\tilde{h}_2^{\xi}\left(\frac{1}{2}\right)}Q\left(\frac{2s+\eta(r,s)}{2}\right)\\
&\le \left(\frac{1}{\eta(r,s)}\right)^{\xi}\left[\frac{1}{\Gamma(1+\xi)}\int_s^{s+\eta(r,s)} E_{\xi}\left(-\rho\left(\frac{s+\eta(r,s)-u}{\eta(r,s)}\right)\right)^{\xi} Q(u)(\mathrm{d}u)^{\xi}\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\xi)}\int_s^{s+\eta(r,s)} E_{\xi}\left(-\rho\left(\frac{v-s}{\eta(r,s)}\right)\right)^{\xi} Q(v)(\mathrm{d}v)^{\xi}\right]\\
&\le \left(\frac{1}{\eta(s,s)}\right)^{\xi}\left[\frac{1}{\Gamma(1+\xi)}\int_s^{s+\eta(r,s)} E_{\xi}\left(-\rho\left(-\frac{1-\xi}{\xi}\right)\right)^{\xi}(s+\eta(r,s)-u)^{\xi} Q(u)(\mathrm{d}u)^{\xi}\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\xi)}\int_s^{s+\eta(r,s)} E_{\xi}\left(-\rho\left(-\frac{1-\xi}{\xi}\right)\right)^{\xi}(v-s)^{\xi} Q(v)(\mathrm{d}v)^{\xi}\right]\\
&= \left(\frac{\xi}{\eta(r,s)}\right)^{\xi}[\boldsymbol{I}_{s^+}^{(\xi)}Q(s+\eta(r,s))+\boldsymbol{I}_{(s+\eta(r,s))^-}^{(\xi)}Q(s)].
\end{aligned}
\tag{16}
$$

Hence, we established the right side of the inequality in (13). Now to prove the left side of the inequality, note that since Q is *generalized* $(\tilde{h}_1, \tilde{h}_2)$*-preinvex* functions on $[s, s+\eta(r,s)]$, we have

$$Q(s+\varsigma\eta(r,s)) \le \tilde{h}_1^{\xi}(1-\varsigma)\tilde{h}_2^{\xi}(\varsigma)Q(r)+\tilde{h}_1^{\xi}(\varsigma)\tilde{h}_2^{\xi}(1-\varsigma)Q(s),$$

and

$$Q(s+(1-\varsigma)\eta(r,s)) \le \tilde{h}_1^{\xi}(1-\varsigma)\tilde{h}_2^{\xi}(\varsigma)Q(r)+\tilde{h}_1^{\xi}(\varsigma)\tilde{h}_2^{\xi}(1-\varsigma)Q(s).$$

Furthermore, we have

$$Q(s+\varsigma\eta(r,s))+Q(s+(1-\varsigma)\eta(r,s)) \le [Q(s)+Q(r)][\tilde{h}_1^{\xi}(\varsigma)\tilde{h}_2^{\xi}(1-\varsigma)+\tilde{h}_1^{\xi}(1-\varsigma)\tilde{h}_2^{\xi}(\varsigma)]. \tag{17}$$

By multiplying both sides of the inequality in (17) by $E_{\xi}(-\rho\varsigma)^{\xi}$, then using local fractional integration corresponding to ς over $[0,1]$ of the resulting inequality, we obtain

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\xi)}\int_0^1 E_{\xi}(-\rho\varsigma)^{\xi}Q(s+\varsigma\eta(r,s))(\mathrm{d}\varsigma)^{\xi}+\frac{1}{\Gamma(1+\xi)}\int_0^1 E_{\xi}(-\rho\varsigma)^{\xi}Q(s+(1-\varsigma)\eta(r,s))(\mathrm{d}\varsigma)^{\xi}\\
&\le [Q(s)+Q(r)]\frac{1}{\Gamma(1+\xi)}\int_0^1 E_{\xi}(-\rho\varsigma)^{\xi}[\tilde{h}_1^{\xi}(1-\varsigma)\tilde{h}_2^{\xi}(\varsigma)]+\tilde{h}_1^{\xi}(\varsigma)\tilde{h}_2^{\xi}(1-\varsigma)](\mathrm{d}\varsigma)^{\xi}.
\end{aligned}
$$

From (15) and (16), we obtain

$$
\begin{aligned}
&\left(\frac{\xi}{\eta(s,r)}\right)^{\xi}\left[\boldsymbol{I}_{s^+}^{(\xi)}Q(s+\eta(r,s))+\boldsymbol{I}_{(s+\eta(r,s))^-}^{(\xi)}Q(s)\right]\\
&\le [Q(s)+Q(r)]\frac{1}{\Gamma(1+\xi)}\int_0^1 E_{\xi}(-\rho\varsigma)^{\xi}[\tilde{h}_1^{\xi}(1-\varsigma)\tilde{h}_2^{\xi}(\varsigma)]+\tilde{h}_1^{\xi}(\varsigma)\tilde{h}_2^{\xi}(1-\varsigma)](\mathrm{d}\varsigma)^{\xi}\\
&\le [Q(s)+Q(r)]{}_0I_1^{(\xi)}E_{\xi}(-\rho\varsigma)^{\xi}[\tilde{h}_1^{\xi}(1-\varsigma)\tilde{h}_2^{\xi}(\varsigma)+\tilde{h}_1^{\xi}(\varsigma)\tilde{h}_2^{\xi}(1-\varsigma)].
\end{aligned}
\tag{18}
$$

Hence, the left side of (18) holds. □

Corollary 1. *If* $\tilde{h}_1^{\xi}(\varsigma)=\frac{1}{2^{\xi}}$ *and* $\tilde{h}_2^{\xi}(\varsigma)=\frac{1}{2^{\xi}}$ *in* (18), *then we obtain*

$$Q\left(\frac{2s+\eta(r,s)}{2}\right) \le \frac{2^{\xi}(1-\xi)^{\xi}}{[1^{\xi}-E_{\xi}(-\rho)^{\xi}]}[\boldsymbol{I}_{s^+}^{(\xi)}Q(s+\eta(r,s)) + \boldsymbol{I}_{(s+\eta(r,s))^-}^{(\xi)}Q(s)] \le [Q(s)+Q(r)].$$

Corollary 2. *Using the fact that*

$$\lim_{\xi\to 1}\frac{1^{\xi}-E_{\xi}(-\rho)^{\xi}}{\rho^{\xi}} = 1, \tag{19}$$

we obtain inequality for classical $(\tilde{h}_1, \tilde{h}_2)$*-preinvex function*

$$\frac{1}{2\tilde{h}_1\left(\frac{1}{2}\right)\tilde{h}_2\left(\frac{1}{2}\right)}Q\left(\frac{2s+\eta(r,s)}{2}\right) \le \left(\frac{1}{\eta(r,s)}\right)\int_{s}^{s+\eta(r,s)} Q(y)\mathrm{d}y$$
$$\le Q(s)\int_0^1 \tilde{h}_1(1-\varsigma)\tilde{h}_2(\varsigma)\mathrm{d}\varsigma + Q(r)\int_0^1 \tilde{h}_1(\varsigma)\tilde{h}_2(1-\varsigma)\mathrm{d}\varsigma.$$

Theorem 3. *Let* $\mathcal{D} \subseteq \mathbb{R}$ *be an open invex subset with respect to the bi-function* $\eta : \mathcal{D}\times\mathcal{D} \to \mathbb{R}$, *where* $s, r \in \mathcal{D}$, $s < s+\eta(r,s)$. *Assume that the function* $Q : \mathcal{D} \to \mathbb{R}^{\xi}$ *is a local fractional differentiable function with* $\eta(r,s) \ge 0$ *and* $Q(y)^{\xi} \in I_y^{(\xi)}[s, s+\eta(r,s)]$. *If* $|Q^{(\xi)}|^q$ *is the generalized* $(\tilde{h}_1, \tilde{h}_2)$*-preinvex function on* $[s, s+\eta(r,s)]$ *with* $p^{-1}+q^{-1}=1$, $p, q > 1$, *then the following inequality holds*:

$$\left|\frac{Q(s)+Q(s+\eta(r,s))}{2^{\xi}} - \frac{(1-\xi)^{\xi}}{2^{\xi}[1^{\xi}-E_{\xi}(-\rho)^{\xi}]}\left[\boldsymbol{I}_{s^+}^{(\xi)}Q(s+\eta(r,s)) + \boldsymbol{I}_{(s+\eta(r,s))^-}^{(\xi)}Q(s)\right]\right|$$
$$\le \frac{\eta^{\xi}(r,s)}{[1^{\xi}-E_{\xi}(-\rho)^{\xi}]}\left(\frac{1^{\xi}-E_{\xi}(-p\rho)^{\xi}}{(-p\rho)^{\xi}}\right)^{\frac{1}{p}} \tag{20}$$
$$\times\left([|Q^{(\xi)}(s)|^q + |Q^{(\xi)}(r)|^q]\frac{1}{\Gamma(1+\xi)}\int_0^1[\tilde{h}_1^{\ \xi}(1-\varsigma)\tilde{h}_2^{\ \xi}(\varsigma) + \tilde{h}_1^{\ \xi}(\varsigma)\tilde{h}_2^{\ \xi}(1-\varsigma)](\mathrm{d}\varsigma)^{\xi}\right)^{\frac{1}{q}}.$$

Proof. Utilizing Lemma 1 and the generalized Hölder's inequality (10), we have

$$\left|\frac{Q(s)+Q(s+\eta(r,s))}{2^{\xi}} - \frac{(1-\xi)^{\xi}}{2^{\xi}[1^{\xi}-E_{\xi}(-\rho)^{\xi}]}\left[\boldsymbol{I}_{s^+}^{(\xi)}Q(s+\eta(r,s)) + \boldsymbol{I}_{(s+\eta(r,s))^-}^{(\xi)}Q(s)\right]\right|$$
$$\le \frac{\eta^{\xi}(r,s)}{2^{\xi}[1^{\xi}-E_{\xi}(-\rho)^{\xi}]}\left[\frac{1}{\Gamma(1+\xi)}\int_0^1 E_{\xi}(-\rho\varsigma)^{\xi}|Q^{\xi}(s+(1-\varsigma)\eta(r,s))|(\mathrm{d}\varsigma)^{\xi}\right.$$
$$\left.+\frac{1}{\Gamma(1+\xi)}\int_0^1 E_{\xi}(-\rho\varsigma)^{\xi}|Q^{\xi}(s+\varsigma\eta(r,s))|(\mathrm{d}\varsigma)^{\xi}\right]$$
$$\le \frac{\eta^{\xi}(r,s)}{2^{\xi}[1^{\xi}-E_{\xi}(-\rho)^{\xi}]}\left[\left(\frac{1}{\Gamma(1+\xi)}\int_0^1 E_{\xi}(-p\rho\varsigma)^{\xi}(\mathrm{d}\varsigma)^{\xi}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\xi)}\int_0^1|Q^{\xi}(s+(1-\varsigma)\eta(r,s))|^q(\mathrm{d}\varsigma)^{\xi}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{\Gamma(1+\xi)}\int_0^1 E_{\xi}(-p\rho\varsigma)^{\xi}(\mathrm{d}\varsigma)^{\xi}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\xi)}\int_0^1|Q^{\xi}(s+\varsigma\eta(r,s))|^q(\mathrm{d}\varsigma)^{\xi}\right)^{\frac{1}{q}}\right].$$

On the other hand, since $|Q^{(\xi)}|^q$ is the *generalized* $(\tilde{h}_1, \tilde{h}_2)$*-preinvex* function on $[s, s+\eta(r,s)]$, then

$$\frac{1}{\Gamma(1+\xi)}\int_0^1 |Q^{\xi}(s+\varsigma\eta(r,s))|^q (d\varsigma)^{\xi} \le \frac{1}{\Gamma(1+\xi)}\int_0^1 \tilde{h}_1^{\xi}(1-\varsigma)\tilde{h}_2^{\xi}(\varsigma)|Q(s)|^q + \frac{1}{\Gamma(1+\xi)}\int_0^1 \tilde{h}_1^{\xi}(\varsigma)\tilde{h}_2^{\xi}(1-\varsigma)|Q(r)|^q$$

$$\frac{1}{\Gamma(1+\xi)}\int_0^1 |Q^{\xi}(s+\varsigma\eta(r,s))|^q (d\varsigma)^{\xi} \le \frac{1}{\Gamma(1+\xi)}\int_0^1 \tilde{h}_1^{\xi}(\varsigma)\tilde{h}_2^{\xi}(1-\varsigma)|Q(s)|^q + \frac{1}{\Gamma(1+\xi)}\int_0^1 \tilde{h}_1^{\xi}(1-\varsigma)\tilde{h}_2^{\xi}(\varsigma)|Q(r)|^q. \tag{21}$$

and

$$\frac{1}{\Gamma(1+\xi)}\int_0^1 E_{\xi}(-p\rho\varsigma)^{\xi}(d\varsigma)^{\xi} = \frac{1^{\xi} - E_{\xi}(-p\rho)^{\xi}}{(-p\rho)^{\xi}}. \tag{22}$$

By substituting (21) to (22) in (7), we obtain the desired result. □

Corollary 3. *Choosing* $\tilde{h}_1(\varsigma) = \varsigma^{\xi}$ *and* $\tilde{h}_2(\varsigma) = \varsigma^{\xi}$ *in Theorem* 3, *we obtain*

$$\left|\frac{Q(s)+Q(s+\eta(r,s))}{2^{\xi}} - \frac{(1-\xi)^{\xi}}{2^{\xi}[1^{\xi} - E_{\xi}(-\rho)^{\xi}]}\left[\boldsymbol{I}_{s^+}^{(\xi)}Q(s+\eta(r,s)) + \boldsymbol{I}_{(s+\eta(r,s))^-}^{(\xi)}Q(s)\right]\right|$$
$$\le \frac{2^{\xi}\eta^{\xi}(r,s)}{[1^{\xi} - E_{\xi}(-\rho)^{\xi}]}\left(\frac{1^{\xi} - E_{\xi}(-p\rho)^{\xi}}{(-p\rho)^{\xi}}\right)([|Q^{(\xi)}(s)|^q + |Q^{(\xi)}(r)|^q]\beta_{\xi}(2,2))^{\frac{1}{q}}.$$

Lemma 7. *Let* $\mathcal{D} \subseteq \mathbb{R}$ *be an open invex subset with respect to the bi-function* $\eta : \mathcal{D}\times\mathcal{D} \to \mathbb{R}$, *where* $s, r \in \mathcal{D}$, $s < s+\eta(r,s)$. *Suppose that* $Q : \mathcal{D} \to \mathbb{R}^{\xi}$ *is a local fractional differentiable function with* $\eta(r,s) \ge 0$ *and* $Q(y)^{\xi} \in I_y^{(\xi)}[s, s+\eta(r,s)]$. *If* Q *is the generalized* $(\tilde{h}_1, \tilde{h}_2)$*-preinvex function on* $[s, s+\eta(r,s)]$, *then*

$$\frac{Q(s)+Q(s+\eta(r,s))}{2^{\xi}} - \frac{(1-\xi)^{\xi}}{2^{\xi}[1^{\xi} - E_{\xi}(-\rho)^{\xi}]}\left[\boldsymbol{I}_{s^+}^{(\xi)}Q(s+\eta(r,s)) + \boldsymbol{I}_{(s+\eta(r,s))^-}^{(\xi)}Q(s)\right]$$
$$= \frac{\eta(r,s)^{\xi}}{2^{\xi}[1^{\xi} - E_{\xi}(-\rho)^{\xi}]}\left[\frac{1}{\Gamma(1+\xi)}\int_0^1 E_{\xi}(-\rho\varsigma)^{\xi}Q^{\xi}(s+\varsigma\eta(r,s))(d\varsigma)^{\xi} - \frac{1}{\Gamma(1+\xi)}\int_0^1 E_{\xi}(-\rho(1-\varsigma))^{\xi}Q^{\xi}(s+\varsigma\eta(r,s))(d\varsigma)^{\xi}\right]. \tag{23}$$

Proof. By local fractional integration by parts, and letting $u = s + \varsigma\eta(r,s)$, we have

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\xi)}\int_0^1 E_\xi(-\rho\varsigma)^\xi Q^\xi(s+\varsigma\eta(r,s))(\mathrm{d}\varsigma)^\xi\\
&=\left(\frac{1}{\eta(r,s)}\right)^\xi\left[E_\xi(-\rho\varsigma)^\xi Q(s+\varsigma\eta(r,s))(\mathrm{d}\varsigma)|_0^1-\frac{1}{\Gamma(1+\xi)}\int_0^1 Q^\xi(s+\varsigma\eta(r,s))(E_\xi(-\rho\varsigma)^\xi)^\xi(\mathrm{d}\varsigma)^\xi\right]\\
&=\left(\frac{1}{\eta(r,s)}\right)^\xi\left[E_\xi(-\rho\varsigma)^\xi Q(s+\varsigma\eta(r,s))(\mathrm{d}\varsigma)|_0^1-\frac{1}{\Gamma(1+\xi)}\int_0^1 Q^\xi(s+\varsigma\eta(r,s))(E_\xi(-\rho\varsigma)^\xi)^\xi(\mathrm{d}\varsigma)^\xi\right]\\
&=\left(\frac{1}{\eta(r,s)}\right)^\xi\left[E_\xi(-\rho)^\xi Q(s+\eta(r,s))-Q(s)-\frac{(-\rho)^\xi}{\Gamma(1+\xi)}\int_0^1 Q^\xi(s+\varsigma\eta(r,s))E_\xi(-\rho\varsigma)^\xi(\mathrm{d}\varsigma)^\xi\right]\\
&=\frac{E_\xi(-\rho)^\xi Q(s+\eta(r,s))-Q(s)}{\eta(r,s)^\xi}\\
&\quad-\rho^\xi\xi^\xi\left(\frac{1}{\eta(r,s)}\right)^{2\xi}\frac{1}{\xi\Gamma(1+\xi)}\int_s^{s+\eta(r,s)} E_\xi\left(-\rho\left(\frac{s+\eta(r,s)-u}{\eta(r,s)}\right)\right)^\xi Q(u)(\mathrm{d}u)^\xi\\
&=\frac{E_\xi(-\rho)^\xi Q(s+\eta(r,s))-Q(s)}{\eta(r,s)^\xi}-\left(\frac{1-\xi}{\eta(r,s)}\right)^\xi \boldsymbol{I}_{s^+}^{(\xi)}Q(s+\eta(r,s)).
\end{aligned}
\tag{24}
$$

Similarly, we obtain

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\xi)}\int_0^1 E_\xi(-\rho(1-\varsigma))^\xi Q^\xi(s+\varsigma\eta(r,s))(\mathrm{d}\varsigma)^\xi\\
&=\frac{E_\xi(-\rho)^\xi Q(s)-Q(s+\eta(r,s))}{\eta^\xi(r,s)}-\left(\frac{1-\xi}{\eta(r,s)}\right)^\xi \boldsymbol{I}_{(s+\eta(r,s))^-}^{(\xi)}Q(s).
\end{aligned}
\tag{25}
$$

Subtracting (25) from (24), we obtain

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\xi)}\int_0^1 E_\xi(-\rho\varsigma)^\xi Q^\xi(s+\varsigma\eta(r,s))(\mathrm{d}\varsigma)^\xi-\frac{1}{\Gamma(1+\xi)}\int_0^1 E_\xi(-\rho(1-\varsigma))^\xi Q^\xi(s+\varsigma\eta(r,s))(\mathrm{d}\varsigma)^\xi\\
&=\frac{[1^\xi-E_\xi(-\rho)^\xi][Q(s)+Q(s+\eta(r,s))]}{\eta^\xi(r,s)}-\left(\frac{1-\xi}{\eta(r,s)}\right)^\xi\left[\boldsymbol{I}_{s^+}^{(\xi)}Q(s+\eta(r,s))+\boldsymbol{I}_{(s+\eta(r,s))^-}^{(\xi)}Q(s)\right],
\end{aligned}
\tag{26}
$$

which completes the proof. □

Theorem 4. *Let $\mathcal{D}\subseteq\mathbb{R}$ be an open invex subset with respect to the bi-function $\eta:\mathcal{D}\times\mathcal{D}\to\mathbb{R}$, where $s,r\in\mathcal{D}$, $s<s+\eta(r,s)$. Assume that $Q:\mathcal{D}\to\mathbb{R}^\xi$ is a local fractional differentiable function with $\eta(r,s)\geq 0$ and $Q^\xi(y)\in I_y^{(\xi)}[s,s+\eta(r,s)]$. If $|Q^{(\xi)}|$ is a the generalized $(\tilde h_1,\tilde h_2)$-preinvex function on $[s,s+\eta(r,s)]$, then the following inequality holds*:

$$
\begin{aligned}
&\left|\frac{Q(s)+Q(r)}{2^\xi}-\frac{(1-\xi)^\xi}{2^\xi[1^\xi-E_\xi(-\rho)^\xi]}(\boldsymbol{I}_{s^+}^{(\xi)}Q(s+\eta(r,s))+\boldsymbol{I}_{(s+\eta(r,s))^-}^{(\xi)}Q(s))\right|\\
&\leq\frac{\eta^\xi(r,s)(|Q^\xi(s)|+|Q^\xi(r)|)}{2^\xi[1^\xi-E_\xi(-\rho)^\xi]}\frac{1}{\Gamma(1+\xi)}\int_0^{\frac12}(E_\xi(-\rho\varsigma)^\xi-E_\xi(-\rho(1-\varsigma))^\xi)[\tilde h_1{}^\xi(1-\varsigma)\tilde h_2{}^\xi(\varsigma)\\
&\quad+\tilde h_1{}^\xi(\varsigma)\tilde h_2{}^\xi(1-\varsigma)](\mathrm{d}\varsigma)^\xi.
\end{aligned}
\tag{27}
$$

Proof. Since $|Q^{(\xi)}|$ is a *generalized* $(\tilde{h}_1, \tilde{h}_2)$*-preinvex* function on $[s, s+\eta(r,s)]$ and $(\tilde{h}_1, \tilde{h}_2)$ is a non-negative function, then by utilizing Lemma 7, we can establish

$$\begin{aligned}
&\left|\frac{Q(s)+Q(r)}{2^{\xi}} - \frac{(1-\xi)^{\xi}}{2^{\xi}[1^{\xi}-E_{\xi}(-\rho)^{\xi}]}\left(\boldsymbol{I}_{s^{+}}^{(\xi)}Q(s+\eta(r,s)) + \boldsymbol{I}_{(s+\eta(r,s))^{-}}^{(\xi)}Q(s)\right)\right| \\
&\leq \frac{\eta^{\xi}(r,s)}{2^{\xi}[1^{\xi}-E_{\xi}(-\rho)^{\xi}]}\frac{1}{\Gamma(1+\xi)}\int_{0}^{1}|E_{\xi}(-\rho\varsigma)^{\xi}-E_{\xi}(-\rho(1-\varsigma))^{\xi}||Q^{(\xi)}(s+\varsigma\eta(r,s))|(\mathrm{d}\varsigma)^{\xi} \\
&\leq \frac{\eta^{\xi}(r,s)}{2^{\xi}[1^{\xi}-E_{\xi}(-\rho)^{\xi}]}\left[\frac{1}{\Gamma(1+\xi)}\int_{0}^{\frac{1}{2}}(E_{\xi}(-\rho\varsigma)^{\xi}-E_{\xi}(-\rho(1-\varsigma))^{\xi})|\tilde{h}_1^{\xi}(1-\varsigma)\tilde{h}_2^{\xi}(\varsigma)|Q^{(\xi)}(r)|\right. \\
&\quad +\tilde{h}_1^{\xi}(\varsigma)\tilde{h}_2^{\xi}(1-\varsigma)|Q^{(\xi)}(s)||(\mathrm{d}\varsigma)^{\xi} + \frac{1}{\Gamma(1+\xi)}\int_{\frac{1}{2}}^{1}(E_{\xi}(-\rho(1-\varsigma))^{\xi} \\
&\quad \left. -E_{\xi}(-\rho\varsigma)^{\xi})|\tilde{h}_1^{\xi}(1-\varsigma)\tilde{h}_2^{\xi}(\varsigma)|Q^{(\xi)}(r)|+\tilde{h}_1^{\xi}(\varsigma)\tilde{h}_2^{\xi}(1-\varsigma)|Q^{(\xi)}(s)||(\mathrm{d}\varsigma)^{\xi}\right] \\
&= \frac{\eta^{\xi}(r,s)}{2^{\xi}[1^{\xi}-E_{\xi}(-\rho)^{\xi}]}\left[\frac{1}{\Gamma(1+\xi)}\int_{0}^{\frac{1}{2}}(E_{\xi}(-\rho\varsigma)^{\xi}-E_{\xi}(-\rho(1-\varsigma))^{\xi})|\tilde{h}_1^{\xi}(1-\varsigma)\tilde{h}_2^{\xi}(\varsigma)|Q^{(\xi)}(r)|\right. \\
&\quad +\tilde{h}_1^{\xi}(\varsigma)\tilde{h}_2^{\xi}(1-\varsigma)|Q^{(\xi)}(s)|\;|(\mathrm{d}\varsigma)^{\xi} + \frac{1}{\Gamma(1+\xi)}\int_{0}^{\frac{1}{2}}(E_{\xi}(-\rho\varsigma)^{\xi}-E_{\xi}(-\rho(1-\varsigma))^{\xi}) \\
&\quad \left.\times|\tilde{h}_1^{\xi}(1-\varsigma)\tilde{h}_2^{\xi}(\varsigma)|Q^{(\xi)}(r)|+\tilde{h}_1^{\xi}(\varsigma)\tilde{h}_2^{\xi}(1-\varsigma)|Q^{(\xi)}(s)||(\mathrm{d}\varsigma)^{\xi}\right] \\
&= \frac{\eta^{\xi}(r,s)}{2^{\xi}[1^{\xi}-E_{\xi}(-\rho)^{\xi}]}\frac{1}{\Gamma(1+\xi)}\int_{0}^{\frac{1}{2}}(E_{\xi}(-\rho\varsigma)^{\xi}-E_{\xi}(-\rho(1-\varsigma))^{\xi})\times(\tilde{h}_1^{\xi}(1-\varsigma)\tilde{h}_2^{\xi}(\varsigma)+\tilde{h}_1^{\xi}(\varsigma)\tilde{h}_2^{\xi}(1-\varsigma))(|Q^{(\xi)}(r)| \\
&\quad +|Q^{(\xi)}(s)|)(\mathrm{d}\varsigma)^{\xi} \\
&= \frac{\eta^{\xi}(r,s)(|Q^{(\xi)}(r)|+|Q^{(\xi)}(s)|)}{2^{\xi}[1^{\xi}-E_{\xi}(-\rho)^{\xi}]}\times\frac{1}{\Gamma(1+\xi)}\int_{0}^{\frac{1}{2}}(E_{\xi}(-\rho\varsigma)^{\xi}-E_{\xi}(-\rho(1-\varsigma))^{\xi})(\tilde{h}_1^{\xi}(1-\varsigma)\tilde{h}_2^{\xi}(\varsigma) \\
&\quad +\tilde{h}_1^{\xi}(\varsigma)\tilde{h}_2^{\xi}(1-\varsigma))(\mathrm{d}\varsigma)^{\xi},
\end{aligned}$$

which completes the proof. □

Corollary 4. (Dragomir-Agarwal-type inequality) *Suppose that* $\tilde{h}_1^{\xi}(\varsigma)=\frac{1}{2^{\xi}}$ *and* $\tilde{h}_2^{\xi}(\varsigma)=\frac{1}{2^{\xi}}$ *in* (27), *then we obtain*

$$\begin{aligned}
&\left|\frac{Q(s)+Q(r)}{2^{\xi}} - \frac{(1-\xi)^{\xi}}{2^{\xi}[1^{\xi}-E_{\xi}(-\rho)^{\xi}]}\left(\boldsymbol{I}_{s^{+}}^{(\xi)}Q(s+\eta(r,s)) + \boldsymbol{I}_{(s+\eta(r,s))^{-}}^{(\xi)}Q(s)\right)\right| \\
&\leq \frac{1}{2^{\xi}}\left[\frac{\eta^{\xi}(r,s)|(Q^{(\xi)}(s)|+|Q^{(\xi)}(r)|)}{2^{\xi}\rho^{\xi}}\tanh_{\xi}\left(\frac{\rho}{4}\right)^{\xi}\right].
\end{aligned} \tag{28}$$

Proof. If $\tilde{h}_1^{\xi}(\varsigma)=\frac{1}{2^{\xi}}$ and $\tilde{h}_2^{\xi}(\varsigma)=\frac{1}{2^{\xi}}$ in (27), then

$$\begin{aligned}
&\frac{1}{\Gamma(1+\xi)}\int\limits_0^{\frac{1}{2}}(E_\xi(-\rho\varsigma)^\xi - E_\xi(-\rho(1-\varsigma))^\xi)(\tilde{h}_1{}^\xi(1-\varsigma)\tilde{h}_2{}^\xi(\varsigma) + \tilde{h}_1{}^\xi(\varsigma)\tilde{h}_2{}^\xi(1-\varsigma))(\mathrm{d}\varsigma)^\xi \\
&= \frac{1}{2^\xi\Gamma(1+\xi)}\int\limits_0^{\frac{1}{2}}(E_\xi(-\rho\varsigma)^\xi - E_\xi(-\rho(1-\varsigma))^\xi)(\mathrm{d}\varsigma)^\xi \\
&= \frac{[1^\xi - 2^\xi E_\xi(\frac{-\rho}{2})^\xi + E_\xi(-\rho)^\xi]}{2^\xi\rho^\xi} \\
&= \frac{\left(1^\xi - E_\xi(\frac{-\rho}{2})^\xi\right)^2}{2^\xi\rho^\xi}.
\end{aligned} \tag{29}$$

Substituting (29) into (28), the intended result yields after simplification. □

Corollary 5. *For $\xi \to 1$, we obtain*

$$\lim_{\xi\to 1}\frac{(1-\xi)^\xi}{2^\xi[1^\xi - E_\xi(-\rho)^\xi]} = \frac{1}{2\eta(r,s)}, \tag{30}$$

and

$$\lim_{\xi\to 1}\frac{(E_\xi(-\rho\varsigma)^\xi - E_\xi(-\rho(1-\varsigma))^\xi)}{2^\xi[1^\xi - E_\xi(-\rho)^\xi]} = \frac{1-2\varsigma}{2}. \tag{31}$$

Letting $\xi \to 1$ in Theorem 4, we have inequality for classical generalized $(\tilde{h}_1, \tilde{h}_2)$-preinvex function,

$$\begin{aligned}
&\left|\frac{Q(s)+Q(r)}{2} - \frac{1}{\eta(r,s)}\int\limits_s^{s+\eta(r,s)} Q(y)\mathrm{d}y\right| \\
&\le \frac{\eta(r,s)(|Q'(s)| + |Q'(r)|)}{2}\int\limits_0^{\frac{1}{2}}(1-2\varsigma)[\tilde{h}_1(1-\varsigma)\tilde{h}_2(\varsigma) + \tilde{h}_1(\varsigma)\tilde{h}_2(1-\varsigma)](\mathrm{d}\varsigma).
\end{aligned}$$

4 Examples

Example 1. For $0 \le s < s + \eta(r,s)$, the following inequality holds:

$$\frac{(2s+\eta(r,s))^3}{4} \le \frac{4s^3 + 6s^2\eta(s,s) + 4s\eta^2(s,s) + \eta^3(s,s)}{4} \le \frac{s^3 + r^3}{6}. \tag{32}$$

Proof. Let $Q(y) = y^3$, $y \in (0,\infty)$. Then, $Q(y)$ is a classical *generalized* $(\tilde{h}_1, \tilde{h}_2)$*-preinvex* function for $\tilde{h}_1(\varsigma) = \varsigma$ and $\tilde{h}_2(\varsigma) = \varsigma$; then, using the fact given in Corollary 2, we obtain

$$2\left(\frac{2s+\eta(r,s)}{2}\right)^3 \le \frac{1}{\eta(s,s)}\int\limits_s^{s+\eta(s,s)} y^3\mathrm{d}y \le s^3\int\limits_0^1\varsigma(1-\varsigma)(\mathrm{d}\varsigma) + r^3\int\limits_0^1\varsigma(1-\varsigma)(\mathrm{d}\varsigma).$$

After simplifying the aforementioned inequality, we obtain (32). □

Example 2. For $0 \le s < s + \eta(r,s)$, the following inequality holds:

$$\left|\frac{s^3+r^3}{2} - \frac{4s^3 + 6s^2\eta(s,s) + 4s\eta^2(s,s) + \eta^3(s,s)}{4}\right| \le \frac{3\eta(r,s)(s^2+r^2)}{40}. \tag{33}$$

Proof. Letting $Q(y) = y^3$, $y \in (0, \infty)$. Since $Q(y)$ is a classical *generalized* $(\tilde{h}_1, \tilde{h}_2)$*-preinvex* function for $\tilde{h}_1(\varsigma) = (\varsigma + 1)^2$ and $\tilde{h}_2(\varsigma) = (\varsigma + 1)^2$. Hence, using the fact given in Corollary 5, we obtain (33). □

5 Application to random variables

Assume that X is a continuous random variable and $Q : \mathcal{D} \to \mathbb{R}^{\xi}$ is a generalized probability density function. Let $\mathcal{D} \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$, where $s, r \in \mathcal{D}$, $s < s + \eta(r, s)$. The generalized τ^{th} central moment about an arbitrary point $\zeta \in \mathbb{R}$ of X, $\tau \geq 0$ is given by:

$$M_{\xi}^{\tau}(\zeta) = \frac{1}{\Gamma(1+\xi)} \int_{s}^{r} (y - \zeta)^{\tau\xi} g(y) d(y)^{\xi}, \quad \tau = 1, 2, 3, \ldots. \tag{34}$$

Moreover, we have

$$(M_{\xi}^{\tau}(\zeta))^{\xi} = -\frac{\Gamma(1+\tau\xi)\Gamma(1+\xi)}{\Gamma(1+(\tau-1)\xi)} \frac{1}{\Gamma(1+\xi)} \int_{s}^{r} (y - \zeta)^{(\tau-1)\xi} g(y) d(y)^{\xi}$$

and

$$(M_{\xi}^{\tau}(\zeta))^{\xi} = -\frac{\Gamma(1+\tau\xi)\Gamma(1+\xi)}{\Gamma(1+(\tau-1)\xi)} M_{\xi}^{\tau-1}(\zeta)^{\xi}.$$

Proposition 1. *Let $\mathcal{D} \subseteq \mathbb{R}$ be an open invex subset with respect to the bi-function $\eta : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$, where $s, r \in \mathcal{D}$, $s < s + \eta(r, s)$. Assume the hypothesis of Corollary 3. Let $Q(\xi) = M_{\xi}^{\tau}(\zeta)$. If $|(M_{\xi}^{\tau}(\zeta))^{\xi}|^q$ is a generalized $(\tilde{h}_1, \tilde{h}_2)$-preinvex function on $[s, s + \eta(r, s)]$, then for $p^{-1} + q^{-1} = 1, p, q > 1$, the following inequality involving generalized moment holds,*

$$\left| \frac{M_{\xi}^{\tau}(s) + M_{\xi}^{\tau}(s + \eta(r, s))}{2^{\xi}} - \frac{(1-\xi)^{\xi}}{2^{\xi}[1^{\xi} - E_{\xi}(-\rho)^{\xi}]} \left[\boldsymbol{I}_{s^{+}}^{(\xi)} M_{\xi}^{\tau}(s + \eta(r, s)) + \boldsymbol{I}_{(s+\eta(r,s))^{-}}^{(\xi)} M_{\xi}^{\tau}(s) \right] \right|$$
$$\leq \frac{2^{\xi}\eta^{\xi}(r, s)}{[1^{\xi} - E_{\xi}(-\rho)^{\xi}]} \frac{\Gamma(1+\tau\xi)\Gamma(1+\xi)}{\Gamma(1+(\tau-1)\xi)} \left(\frac{1^{\xi} - E_{\xi}(-p\rho)^{\xi}}{(-p\rho)^{\xi}} \right)^{\frac{1}{p}} ([|M_{\xi}^{\tau(\xi)}(s)|^q + |M_{\xi}^{\tau(\xi)}(r)|^q] \beta_{\xi}(2, 2))^{\frac{1}{q}}.$$

6 Conclusion

Utilizing the notion of integral operators of fractional order with Mittag-Leffler kernel on Yang's fractal sets defined by [42], this work intended to acquire some new identities of generalized local fractional integral Hermite-Hadamard-type inequalities via generalized $(\tilde{h}_1, \tilde{h}_2)$-preinvex functions. Our results give definite estimations for the variation between the left part and middle part together with the middle part and right part in concerned inequality. Some novel generalized special cases manifested the imposing execution of local fractional integration. The derived results have been illustrated by two examples to check the accuracy of the obtained results. As special application, we constructed an integral inequality corresponding to central moments of continuous random variables.

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