Research Article

Phakdi Charoensawan, Supreedee Dangskul*, and Pariwate Varnakovida

Common best proximity points for a pair of mappings with certain dominating property

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Abstract: This article introduces a type of dominating property, partially inherited from L. Chen's, and proves an existence and uniqueness theorem concerning common best proximity points. A certain kind of boundary value problem involving the so-called Caputo derivative can be formulated so that our result applies.

Keywords: common best proximity points, contractions, dominating property, fixed points, fractional differential equations

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1 Introduction

Problems regarding proximity points, where the closest distance between objects is of main interest, date back to Euclid or even earlier. In modern computational geometry, closest-point problems, for instance, seek estimation of the closest distance between any two points among the given n distinct points in Euclidean plane, see, e.g., [1]. One may study similar problems in a more general framework, in metric spaces, where distance is still meaningful. More precisely, given a mapping $f : A \to B$, with A, B being subsets of a metric space X, is it possible to find $x^* \in A$ such that the distance between x^* and fx^* minimizes the distance between A and B? This is known as a proximity point problem of mappings, and such a point x^* is called a *best proximity point*. For arbitrary nonempty disjoint subsets A, B of X, the answer to when a best proximity point exists merely depends on the complexity of the mapping. For example, if f is a constant mapping sending the whole A to a boundary point $b \in B$, then there exists a best proximity point.

From a fixed-point theory perspective, one may view the above-mentioned proximity point problem as a generalized existence problem of a fixed point. Some of very first articles on proximity point problems are due to Sadiq Basha and Veeramani [2,3], in which the latter imposes conditions on function-valued mappings. Furthermore, another work of Sadiq Basha [4] proves existence theorems of best proximity points for proximal contractions. As the field of fixed-point theory is rich and robust, many researchers tackle proximity point problems by various approaches producing an extensive number of publications. For instance, Karapınar and Erhan [5] and Karapınar [6] proved the existence of best proximity points for cyclic mappings; another solo work by Karapınar [7] deals with the so-called ψ-Geraghty contractions named after Geraghty [8] where the contractions are somehow controlled by a function ψ; and a recent article by

*** Corresponding author: Supreedee Dangskul,** Advanced Research Center for Computational Simulation, Chiang Mai University, Chiang Mai, 50200, Thailand; Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand, e-mail: supreedee.dangskul@cmu.ac.th
Phakdi Charoensawan: Advanced Research Center for Computational Simulation, Chiang Mai University, Chiang Mai, 50200, Thailand; Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand, e-mail: phakdi@hotmail.com
Pariwate Varnakovida: KMUTT Geospatial Engineering and Innovation Center, Faculty of Science, King Mongkut's University of Technology Thonburi, Thung Khru, Bangkok, 10140, Thailand, e-mail: pariwate.var@kmutt.ac.th

Karapınar and Khojasteh [9] proposed a way to study the existence and uniqueness of a best proximity point via a simulation function. There are many more contributions in the literature, see, e.g., [10–14], to mention but a few.

One may extend the notion of best proximity points as follows. Given two mappings $f, g : A \to B$, with A, B being subsets of a metric space (X, d), a point x^* is a *common best proximity point* if both $d(x^*, fx^*)$ and $d(x^*, gx^*)$ are exactly the distance between A and B. To the best of our knowledge, research on common best proximity points started from a work by Shahzad et al. [15]. A lone research study by Sadiq Basha [16] came out a year later dealing with some condition on subspaces A, B of X known as approximate compactness. Kumam and Mongkolkeha [17] proved common best proximity point theorems for proximity commuting mappings, improving results in [18]. Chen [19] introduced an idea of domination, where one mapping dominates the other in a particular manner, and achieved the existence and uniqueness of a common best proximity point for a pair of non-self-mappings. The reader may be referred to [20–25] for some other relevant topics. Moreover, some recent publications concerning fixed-point and common fixed point problems, which serve as special cases of common best proximity point problems, can be found in [26–30].

This article mainly aims at establishing an existence and uniqueness result of common best proximity points, Theorem 3.3, and illustrating a concrete application in fraction differential equations in Section 4. Here, our approach slightly adjusts Chen's domination of mappings in which two given mappings are made intertwined with a function α, see, e.g., Definitions 2.3 and 2.5.

This article is outlined as follows. Section 2 comprises the relevant definitions concerning common best proximity points as well as their related notions. Section 3 provides the main theorem and an example to support the result in Euclidean space. Finally, Section 4 expresses how our main result applies to guarantee that some fraction differential equations have a solution.

2 Preliminaries

Throughout Sections 2 and 3, unless otherwise stated, let (X, d) be a metric space and $f, g : A \to B$ be mappings between nonempty subsets of X. Let us adopt the following notations:

$$\begin{aligned} d(A, B) &= \inf\{d(x, y) : x \in A, y \in B\}; \\ A_0 &= \{x \in A : \ d(x, y) = d(A, B) \ \text{for some} \ y \in B\}; \\ B_0 &= \{y \in B : \ d(x, y) = d(A, B) \ \text{for some} \ x \in A\}. \end{aligned}$$

Obviously, $A_0 \neq \varnothing$ if and only if $B_0 \neq \varnothing$.

2.1 Common best proximity points

Definition 2.1. [17] An element $x^* \in A$ is said to be a *common best proximity point* of the mappings f and g if

$$d(x^*, fx^*) = d(A, B) = d(x^*, gx^*).$$

Denote by $\mathcal{CB}(f, g)$ the set of common best proximity points of f and g.

If $A \cap B \neq \varnothing$, then $d(A, B) = 0$; in this case, a common best proximity point becomes a *common fixed point*. Denote by $\mathcal{C}(f, g)$ the set of common fixed points of f and g.

2.2 Commutativity of mappings

Definition 2.2. [18] Two mappings f and g *proximally commute* if

$$d(v, fx) = d(A, B) = d(u, gx) \quad \text{implies } fu = gv,$$

for all $x, u, v \in A$.

If f and g proximally commute and $d(u, fx) = d(A, B) = d(u, gx)$ for some $u \in A$, then f and g coincide at u; such an element u is known as a *coincidence point* of f and g.

Let $\alpha : X \times X \to [0, \infty)$. Denote $A(\alpha, f, g) = \{x \in A : \alpha(fx, gx) \geq 1\}$.

Definition 2.3. A mapping f is said to be α_g*-proximal* if for any $u, v \in A$ and $x \in A(\alpha, f, g)$,

(i) $\alpha(fu, fu) \geq 1$;

(ii) $\alpha(gu, gv) \geq 1$ implies $\alpha(fu, fv) \geq 1$;

(iii) $d(u, fx) = d(A, B) = d(v, gx)$ implies $\alpha(u, v) \geq 1$.

Definition 2.4. A mapping f is said to be α_g*-proximally commutative* if f is α_g*-proximal* and f, g proximally commute.

2.3 Domination of mappings

Let us now consider a (nonempty) class of functions

$$\mathcal{B} \subseteq \{\beta : [0, \infty) \to [0, 1]; \lim_{n\to\infty} \beta(t_n) = 1 \;\Rightarrow\; \lim_{n\to\infty} t_n = 0\}.$$

Definition 2.5. A function $f : A \to B$ is said to satisfy $(\alpha_g, \mathcal{B})$*-dominating property* if for any x_1, x_2, u_1, u_2, v_1, $v_2 \in A$ with

$$d(u_1, fx_1) = d(u_2, fx_2) = d(A, B) = d(v_1, gx_1) = d(v_2, gx_2),$$

$\alpha(u_1, v_1) \geq 1$ and $\alpha(u_2, v_2) \geq 1$, there exists $\beta \in \mathcal{B}$ such that

$$d(u_1, u_2) \leq \beta(d(v_1, v_2))d(v_1, v_2). \tag{2.1}$$

In the case $\mathcal{B} = \{\beta\}$ being a singleton, we may instead say f has (α_g, β)-dominating property if (2.1) holds.

It is also worth mentioning a special case where $A = B = X$, g is the identity mapping, $\alpha \equiv 1$, and $\mathcal{B} = \{\beta\}$ with $\beta \equiv k \in [0, 1)$. In this case, a mapping satisfying Definition 2.5 is a contraction, and (2.1) becomes

$$d(fx_1, fx_2) \leq kd(x_1, x_2). \tag{2.2}$$

This means we are dealing with a general situation, for which generalized results could possibly be established.

3 Main results

Before we assert our main results, some facts need to be established.

Lemma 3.1. *Let $\{u_n\}$ be a sequence in a metric space (X, d) such that*

$$\lim_{n\to\infty} d(u_{n-1}, u_n) = 0.$$

If $\{u_n\}$ is not a Cauchy sequence, there exist subsequences $\{u_{m_k}\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ with $m_k > n_k > k$ for all $k \in \mathbb{N}$ such that

$$\lim_{k\to\infty} d\left(u_{m_k}, u_{n_k}\right) = \lim_{k\to\infty} d\left(u_{m_k+1}, u_{n_k+1}\right) = \varepsilon,$$

for some $\varepsilon > 0$.

Proof. Assume that $\{u_n\}$ is not a Cauchy sequence. Then, there exist subsequences $\{u_{m_k}\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ with $m_k > n_k > k$ for all $k \in \mathbb{N}$ such that

$$d\left(u_{m_k}, u_{n_k}\right) \geq \varepsilon, \tag{3.1}$$

for some $\varepsilon > 0$. In addition, we choose the smallest n_k satisfying (3.1) so that

$$d\left(u_{m_k}, u_{n_k-1}\right) < \varepsilon. \tag{3.2}$$

By using (3.1) and (3.2), we have that

$$\varepsilon \leq d\left(u_{m_k}, u_{n_k}\right) \leq d\left(u_{m_k}, u_{n_k-1}\right) + d\left(u_{n_k-1}, u_{n_k}\right) < \varepsilon + d\left(u_{n_k-1}, u_{n_k}\right). \tag{3.3}$$

Since $\lim_{n\to\infty} d(u_n, u_{n+1}) = 0$, taking the limit as $k \to \infty$ in (3.3) implies

$$\lim_{k\to\infty} d\left(u_{m_k}, u_{n_k}\right) = \varepsilon. \tag{3.4}$$

It now remains to show that

$$\lim_{k\to\infty} d\left(u_{m_k+1}, u_{n_k+1}\right) = \varepsilon. \tag{3.5}$$

By the triangular inequality, we obtain

$$d\left(u_{m_k}, u_{n_k}\right) \leq d\left(u_{m_k}, u_{m_k+1}\right) + d\left(u_{m_k+1}, u_{n_k+1}\right) + d\left(u_{n_k+1}, u_{n_k}\right)$$

and

$$d\left(u_{m_k+1}, u_{n_k+1}\right) \leq d\left(u_{m_k+1}, u_{m_k}\right) + d\left(u_{m_k}, u_{n_k}\right) + d\left(u_{n_k}, u_{n_k+1}\right).$$

As $k \to \infty$, we obtain

$$\lim_{k\to\infty} d\left(u_{m_k}, u_{n_k}\right) = \lim_{k\to\infty} d\left(u_{m_k+1}, u_{n_k+1}\right) = \varepsilon,$$

as required. □

Lemma 3.2. *Suppose that $f : A \to B$ with $f(A_0) \subseteq B_0$ has $(\alpha_g, \mathcal{B})$-dominating property and is α_g-proximally commutative. If $A_0 \cap C(f, g) \neq \emptyset$, then $C\mathcal{B}(f, g) \neq \emptyset$.*

Proof. Let $u \in A_0 \cap C(f, g)$. Then, we have $u \in A_0$ and $fu = gu$. Since $f(A_0) \subseteq B_0$, there exists $x^* \in A_0$ such that

$$d(x^*, fu) = d(A, B) = d(x^*, gu). \tag{3.6}$$

By the commutativity of f and g, we have

$$fx^* = gx^*.$$

Again, since $x^* \in A_0$ and $f(A_0) \subseteq B_0$, there exists $y^* \in A_0$ such that

$$d(y^*, fx^*) = d(A, B) = d(y^*, gx^*). \tag{3.7}$$

Hence, (3.6) and (3.7) become

$$d(x^*, fu) = d(y^*, fx^*) = d(A, B) = d(x^*, gu) = d(y^*, gx^*). \tag{3.8}$$

Since $\alpha(fu, gu) = \alpha(fu, fu) \geq 1$ and $\alpha(fx^*, gx^*) = \alpha(fx^*, fx^*) \geq 1$, both u and x^* belong to $A(\alpha, f, g)$. Since f is α_g-proximal, (3.8) yields

$$\alpha(x^*, x^*) \geq 1 \quad \text{and} \quad \alpha(y^*, y^*) \geq 1.$$

Next, we claim that $x^* = y^*$. Suppose that $d(x^*, y^*) > 0$. By the dominating property, we have

$$d(x^*, y^*) \leq \beta(d(x^*, y^*))d(x^*, y^*) \leq d(x^*, y^*),$$

and hence,

$$1 \leq \beta(d(x^*, y^*)) \leq 1.$$

The property of β gives $d(x^*, y^*) = 0$, which leads to a contradiction. Thus, $x^* = y^*$, and by (3.7), we obtain

$$d(x^*, fx^*) = d(A, B) = d(x^*, gx^*).$$

Therefore, $\mathcal{CB}(f, g) \neq \varnothing$. □

Our main results are now ready to be stated.

Theorem 3.3. *Let (X, d) be a complete metric space, let $f : A \to B$ with $f(A_0) \subseteq B_0$ satisfying $(\alpha_g, \mathcal{B})$-dominating property and be α_g-proximally commutative. Suppose also that the following hold:*
- (*i*) *A_0 is closed and $A_0 \cap A(\alpha, f, g) \neq \varnothing$;*
- (*ii*) *$f(A_0) \subseteq g(A_0)$;*
- (*iii*) *either*
  - (*a*) *f and g are continuous; or*
  - (*b*) *for any sequences $\{x_n\}$ and $\{u_n\}$ in A such that*

$$d(u_n, fx_n) = d(A, B) = d(u_{n-1}, gx_n),$$

if $\{u_n\}$ converges to $u \in A$ with $\alpha(u_n, u_{n-1}) \geq 1$ for all n, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d\big(u, fx_{n_k}\big) = d(A, B) = d\big(u, gx_{n_k}\big).$$

Then, $\mathcal{CB}(f, g) \neq \varnothing$. Moreover, if $\mathcal{CB}(f, g) \subseteq A(\alpha, f, g)$, then $\mathcal{CB}(f, g)$ has only one element.

The gist of the proof is to show, using Lemma 3.1, that a sequence constructed by iteration is Cauchy.

Proof. First, let $x_0 \in A_0 \cap A(\alpha, f, g)$. The assumptions (i) and (ii) inductively give rise to a sequence $\{x_n\}$ in A_0 satisfying

$$gx_{n+1} = fx_n \quad \text{and} \quad x_n \in A(\alpha, f, g), \tag{3.9}$$

and a sequence $\{u_n\}$ in A_0 satisfying

$$d(u_n, fx_n) = d(A, B), \tag{3.10}$$

for all n. Hence, (3.9) and (3.10) yield

$$d(A, B) = d(u_n, fx_n) = d(u_n, gx_{n+1}), \quad \forall n \geq 0. \tag{3.11}$$

Observe, for now, that if $u_{n_0} = u_{n_0+1}$ for some n_0, then (3.10) and (3.11) produce

$$d(A, B) = d\big(u_{n_0+1}, fx_{n_0+1}\big) = d\big(u_{n_0}, fx_{n_0}\big) = d\big(u_{n_0}, gx_{n_0+1}\big).$$

By the commutativity of f and g, we have $f(u_{n_0}) = g(u_{n_0+1}) = g(u_{n_0})$, which then fulfills all hypotheses in Lemma 3.2. Thus, $\mathcal{CB}(f, g) \neq \varnothing$.

Second, we show that

$$\lim_{n\to\infty} d(u_{n-1}, u_n) = 0,$$

provided that $u_n \neq u_{n+1}$ for all n. From (3.11), note that, for all $n \geq 1$

$$d(u_n, fx_n) = d(u_{n+1}, fx_{n+1}) = d(A, B) = d(u_{n-1}, gx_n) = d(u_n, gx_{n+1}). \tag{3.12}$$

Since f is α_g-proximal, (3.12) yields

$$\alpha(u_n, u_{n-1}) \geq 1 \quad \text{and} \quad \alpha(u_{n+1}, u_n) \geq 1, \tag{3.13}$$

for all n. By the dominating property, there exists $\beta \in \mathcal{B}$ such that

$$d(u_n, u_{n+1}) \leq \beta(d(u_{n-1}, u_n))d(u_{n-1}, u_n) \leq d(u_{n-1}, u_n), \tag{3.14}$$

for all n. It is clear that $\lim_{n\to\infty} d(u_{n-1}, u_n)$ exists. By the property of β, if $\lim_{n\to\infty} d(u_{n-1}, u_n)$ were nonzero, then $\lim_{n\to\infty} \beta(d(u_{n-1}, u_n))$ would not be 1, which contradicts (3.14) as $n \to \infty$.

Third, we claim that $\{u_n\}$ is a Cauchy sequence. Suppose, for a contradiction, that it is not the case. By Lemma 3.1, there exist subsequences $\{u_{m_k}\}$ and $\{u_{n_k}\}$ of $\{u_n\}$, with $m_k > n_k > k$ for all $k \in \mathbb{N}$ such that

$$\lim_{k\to\infty} d\left(u_{m_k}, u_{n_k}\right) = \lim_{k\to\infty} d\left(u_{m_k+1}, u_{n_k+1}\right) = \varepsilon,$$

for some $\varepsilon > 0$. Since $\{u_{m_k}\}$ and $\{u_{n_k}\}$ satisfy (3.12), we have

$$\begin{aligned} d\left(u_{n_k+1}, fx_{n_k+1}\right) &= d(A, B) = d\left(u_{n_k}, gx_{n_k+1}\right) \\ d\left(u_{m_k+1}, fx_{m_k+1}\right) &= d(A, B) = d\left(u_{m_k}, gx_{m_k+1}\right), \end{aligned} \tag{3.15}$$

for all k. It is not hard to see that the same procedure as above applies, so that we obtain

$$d\left(u_{n_k+1}, u_{m_k+1}\right) \leq \beta\left(d\left(u_{n_k}, u_{m_k}\right)\right)d\left(u_{n_k}, u_{m_k}\right) \leq d\left(u_{n_k}, u_{m_k}\right). \tag{3.16}$$

Taking $k \to \infty$ in (3.16) yields

$$\lim_{n\to\infty} \beta\left(d\left(u_{n_k}, u_{m_k}\right)\right) = 1,$$

and hence,

$$\varepsilon = \lim_{n\to\infty} d\left(u_{n_k}, u_{m_k}\right) = 0,$$

which is a contradiction.

Next, we prove the existence of a common proximity point of f and g by showing $C(f, g) \neq \varnothing$ and applying Lemma 3.2. Since A_0 is a closed subspace of X, let $\lim_{n\to\infty} u_n = u \in A_0$. If f and g are continuous, then

$$fu = \lim_{n\to\infty} fu_n = \lim_{n\to\infty} gu_{n+1} = gu,$$

which implies that $u \in C(f, g)$. If assumption (iii)(b) is satisfied, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d\left(u, fx_{n_k}\right) = d(A, B) = d\left(u, gx_{n_k}\right),$$

and hence, $fu = gu$ by commutativity; that is, $C(f, g) \neq \varnothing$.

Finally, assume $\mathcal{CB}(f, g) \subseteq A(\alpha, f, g)$. We show the uniqueness of a common best proximity point. Let $x^*, y^* \in \mathcal{CB}(f, g)$. Then,

$$d(x^*, fx^*) = d(y^*, fy^*) = d(A, B) = d(x^*, gx^*) = d(y^*, gy^*).$$

As above, since f is α_g-proximal, we have

$$\alpha(x^*, x^*) \geq 1 \quad \text{and} \quad \alpha(y^*, y^*) \geq 1.$$

By the dominating property, we again obtain

$$d(x^*, y^*) \leq \beta(d(x^*, y^*))d(x^*, y^*) \leq d(x^*, y^*).$$

Suppose for a contradiction that $x^* \neq y^*$. Then, $\beta(d(x^*, y^*)) = 1$, implying $d(x^*, y^*) = 0$. This contradicts $x^* = y^*$. □

Example 3.4. Let $X = \mathbb{R}^3$ be equipped with the standard Euclidean metric d. Also, let

$$A = \{(x, 1, 2) : 0 \leq x \leq 2\} \quad \text{and} \quad B = \{(x, -2, 6) : 0 \leq x \leq 2\}.$$

It is easy to see that $A_0 = A$, $B_0 = B$, and $d(A, B) = 5$. Define the continuous mappings $f, g : A \to B$ by

$$f(x, 1, 2) = (\ln(1 + x), -2, 6) \quad \text{and} \quad g(x, 1, 2) = (x, -2, 6),$$

for all $(x, 1, 2) \in A$, and also define $\alpha : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ by

$$\alpha((x_1, x_2, x_3), (y_1, y_2, y_3)) = \begin{cases} 1; & x_1 \leq y_1, x_2 \geq y_2, x_3 \leq y_3 \\ 0; & \text{otherwise}. \end{cases}$$

Observe that $f(A_0) \subseteq g(A_0)$.

We show that f is α_g-proximally commutative:

(1) For any $u \in A$, it is easy to see that $\alpha(fu, fu) \geq 1$.

(2) Let $u = (x, 1, 2)$ and $v = (x', 1, 2)$ be such that

$$\alpha(gu, gv) = \alpha(g(x, 1, 2), g(x', 1, 2)) = \alpha((x, -2, 6), (x', -2, 6)) \geq 1.$$

Then, we have $x \leq x'$, and hence, $(\ln(1 + x)) \leq (\ln(1 + x'))$. Thus,

$$\alpha(fu, fv) = \alpha((\ln(1 + x), -2, 6), (\ln(1 + x'), -2, 6)) \geq 1.$$

(3) Let $u = (x, 1, 2)$, $v = (x', 1, 2)$, and $z = (x'', 1, 2)$ be such that $\alpha(fz, gz) \geq 1$ satisfying

$$d(u, fz) = d(A, B) = d(v, gz).$$

It follows by school algebra that $x = \ln(1 + x'')$ and $x' = x''$. Thus, $\alpha(u, v) \geq 1$.

(4) Now it remains to show that f and g proximally commute. Let $u = (x, 1, 2)$, $v = (x', 1, 2)$, and $z = (x'', 1, 2)$ satisfy

$$d(u, fz) = d(A, B) = d(v, gz).$$

Then, $x = \ln(1 + x'')$ and $x' = x''$. Thus,

$$fv = (\ln(1 + x''), -2, 6) = gu.$$

Next, let us define $\beta : [0, \infty) \to [0, 1]$ by

$$\beta(t) = \begin{cases} 1, & t = 0, \\ \dfrac{\arctan t}{t}, & t > 0. \end{cases}$$

Note that $\left|1 - \frac{\arctan t}{t}\right|$ is close to zero only if $t \to 0$. Here, $\mathcal{B} = \{\beta\}$, and we may write β instead of $\mathcal{B}$ for convenience. We show that f has the (α_g, β)-dominating property. Let

$$z_1 = (\hat{z}_1, 1, 2), \quad z_2 = (\hat{z}_2, 1, 2),$$
$$u_1 = (\hat{u}_1, 1, 2), \quad u_2 = (\hat{u}_2, 1, 2),$$
$$v_1 = (\hat{v}_1, 1, 2), \quad v_2 = (\hat{v}_2, 1, 2),$$

satisfy

$$d(u_1, fz_1) = d(u_2, fz_2) = d(A, B) = d(v_1, gz_1) = d(v_2, gz_2).$$

Then, $\hat{u}_1 = \ln(1+\hat{z}_1), \hat{u}_2 = \ln(1+\hat{z}_2), \hat{v}_1 = \hat{z}_1, \hat{v}_2 = \hat{z}_2$, and $\hat{z}_1, \hat{z}_2 \in [0,2]$. Since $\ln(1+\hat{z}_1) \le \hat{z}_1$ and $\ln(1+\hat{z}_2) \le \hat{z}_2$, we have

$$\alpha(u_1, v_1) \ge 1 \quad \text{and} \quad \alpha(u_2, v_2) \ge 1.$$

To obtain inequality (2.1), we first assume $v_1 \ne v_2$. Then, $d(v_1, v_2) = |\hat{z}_1 - \hat{z}_2| > 0$. Hence,

$$\begin{aligned} d(u_1, u_2) &= |\hat{u}_1 - \hat{u}_2| \\ &= |\ln(1+\hat{z}_1) - \ln(1+\hat{z}_2)| \\ &\le \ln(1 + |\hat{z}_1 - \hat{z}_2|) \\ &\le \arctan(|\hat{z}_1 - \hat{z}_2|) \\ &= \left(\frac{\arctan(|\hat{z}_1 - \hat{z}_2|)}{|\hat{z}_1 - \hat{z}_2|}\right)|\hat{z}_1 - \hat{z}_2| \\ &= \beta(d(v_1, v_2))d(v_1, v_2). \end{aligned}$$

If $v_1 = v_2$, then $u_1 = u_2$; inequality (2.1) clearly holds.

Theorem 3.3 now applies, which guarantees the existence of a common best proximity point of f and g. Let $x^* \in \mathcal{CB}(f, g)$. Then,

$$d(x^*, fx^*) = d(A, B) = d(x^*, gx^*),$$

which implies that $fx^* = gx^*$. Hence, $\alpha(fx^*, gx^*) \ge 1$. That is, $\mathcal{CB}(f, g) \subseteq A(\alpha, f, g)$. Therefore, x^* is the only common best proximity point of f and g. In fact, $x^* = (0, 1, 2)$.

As a consequence of our main theorem, the following corollary includes a fixed-point theorem as a special case.

Corollary 3.5. *Suppose that all assumptions in Theorem* 3.3 *hold, and also let* $\mathcal{B} = \{\beta\}$, *where* $\beta \equiv k \in [0, 1)$. *Then,* $\mathcal{CB}(f, g) \ne \emptyset$. *Moreover, if* $\mathcal{CB}(f, g) \subseteq A(\alpha, f, g)$, *then* $\mathcal{CB}(f, g)$ *has only one element. In particular, if g is the identity on* $A = B = X$ *and* $\alpha \equiv 1$, *then we obtain Banach fixed-point theorem.*

Proof. The former part of the corollary is obvious. For the latter part, note that if $A = B = X$, then $A_0 = B_0 = X$, which implies that all the assumptions of Theorem 3.3 are met. Moreover, if g is the identity mapping, then the dominating property gives rise to a contraction satisfying (2.2). □

4 Applications to nonlinear fractional differential equations with nonlocal boundary conditions

Fractional calculus has recently become of much interest as it can provide tools for solving real-world problems, see, e.g., [31,32]. Solving fractional differential equations is generally not an easy task, and it is worth investigating if they possess a solution. There appear a number of research studies devoted to such investigation, see, e.g., [33–37]. Here, we employ a technique in fixed-point theory for the existence of a solution.

For an integer $n \ge 2$ and $n - 1 < \xi \le n$, let us consider a fractional differential equation of the form

$$({}^cD^{\xi}y)(t) = f(t, y(t)), \tag{4.1}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The so-called Caputo derivative ${}^cD^{\alpha}u$ of u of fractional order α is defined for all positive real numbers by

$${}^cD^{\alpha}u = I^{\lceil\alpha\rceil-\alpha}D^{\lceil\alpha\rceil}u,$$

where $\lceil\cdot\rceil$ is the ceiling function, and I^{ω} is the Riemann-Liouville integral operator of order $\omega > 0$ defined by

$$I^{\omega}u(t) = \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}u(s)\mathrm{d}s.$$

Recall also that Γ here denotes the gamma function. If $\xi = 0$, I^0 is the identity operator. Observe that each I^{ω} is a bounded linear operator on the set of continuous functions $C[0,1]$ with respect to supremum norm.

We are particularly concerned with finding an $(n-2)$-differentiable function $y(t)$ satisfying (4.1) together with boundary conditions

$$y(0) = y'(0) = \dots = y^{(n-2)}(0) = 0 \quad \text{and} \quad y(1) = \int_0^{\delta} y(s)\mathrm{d}s, \tag{4.2}$$

where $\delta \in [0,1]$. Equation (4.1) with conditions (4.2) may be referred to as a boundary value problem (BVP) of Caputo fractional differential equations. The general solution of (4.1) is given by

$$y(t) = a_0 + a_1 t + \dots + a_{n-1}t^{n-1} + I^{\xi}f(t, y(t)),$$

and the boundary conditions yield $a_0 = a_1 = \dots = a_{n-2} = 0$ and

$$a_{n-1} = \frac{n}{n-\delta^n}\left(\int_0^{\delta} I^{\xi}f(s, y(s))\mathrm{d}s - I^{\xi}f(1, y(1))\right).$$

Thus, the solution $y(t)$ to our BVP can be implicitly expressed as

$$y(t) = \frac{nt^{n-1}}{n-\delta^n}\left(\int_0^{\delta} I^{\xi}f(s, y(s))\mathrm{d}s - I^{\xi}f(1, y(1))\right) + I^{\xi}f(t, y(t)). \tag{4.3}$$

Let us now introduce an operator between the set of continuous mappings $C[0,1]$. Define $T : C[0,1] \to C[0,1]$ by

$$T(u)(t) = \frac{nt^{n-1}}{n-\delta^n}\left(\int_0^{\delta} I^{\xi}f(s, u(s))\mathrm{d}s - I^{\xi}f(1, u(1))\right) + I^{\xi}f(t, u(t)).$$

The dominated convergence theorem guarantees the continuity of the operator T. It also turns out that the existence of a fixed point of T gives rise to a solution to the BVP (4.1)–(4.2).

Now let us take $A = B = X = C[0,1]$ with supremum norm, which is a Banach space.

Lemma 4.1. *Suppose that there exists a function $\gamma : \mathbb{R}^2 \to \mathbb{R}$ such that*
(C1) $\gamma(Tu(t), Tu(t)) \geq 0$ *for all* $u \in C[0,1]$ *and all* $t \in [0,1]$;
(C2) $\gamma(u(t), v(t)) \geq 0$ *implies* $\gamma(Tu(t), Tu(t)) \geq 0$

for all $u, v \in C[0,1]$ *and* $t \in [0,1]$. *Then, there is a function* $\alpha : C[0,1] \times C[0,1] \to \mathbb{R}$ *such that* T *is* α_{I^0}*-proximal.*

Proof. Define

$$\alpha(u, v) = \begin{cases} 1 & \text{if } \gamma(u(t), v(t)) \geq 0 \quad \text{for all } t \in [0,1] \\ 0 & \text{otherwise}, \end{cases}$$

for $u, v \in C[0,1]$. It is easy to verify that all conditions in Definition 2.3 are met. □

Lemma 4.2. *Suppose that there exists a function $\gamma : \mathbb{R}^2 \to \mathbb{R}$ satisfying (C1) and (C2) as in Lemma 4.1, and that*

(C3) $|f(t, u(t)) - f(t, v(t))| \le K_1 \ln(1 + |u(t) - v(t)|)$ *for all* $u, v \in C[0, 1]$ *and* $t \in [0, 1]$,

where

$$K_1 \le \frac{(n - \delta^n)\Gamma(\xi + 2)}{n\delta^{\xi+1} + (\xi + 1)(2n - \delta^n)}.$$

Then, T satisfies (α_{I^0}, β)-dominating property for some $\beta : [0, \infty) \to [0, 1]$.

Proof. First of all, let us compute for $u, v \in C[0, 1]$ using (C3)

$$|Tu(t) - Tv(t)| \le \frac{nt^{n-1}K_1K_2}{(n - \delta^n)\Gamma(\xi)} \ln(1 + \|u - v\|_\infty), \tag{4.4}$$

where

$$\begin{aligned} K_2 &= \sup_{t\in[0,1]} \left(\int_0^\delta \int_0^s (s - \tau)^{\xi-1} d\tau ds + \int_0^1 (1 - s)^{\xi-1} ds + \frac{n - \delta^n}{n} \int_0^t (t - s)^{\xi-1} ds \right) \\ &= \frac{n\delta^{\xi+1} + (\xi + 1)(2n - \delta^n)}{n\xi(\xi + 1)}. \end{aligned}$$

Then, (4.4) becomes $|Tu(t) - Tv(t)| \le \ln(1 + \|u - v\|_\infty)$. Define $\beta : [0, \infty) \to [0, 1]$ by

$$\beta(t) = \begin{cases} \dfrac{\ln(1 + t)}{t}, & \text{if } t > 0 \\ 0, & \text{if } t = 0. \end{cases}$$

Observe that if t is away from zero, so is $\left|\frac{\ln(1+t)}{t} - 1\right|$; that is, β satisfies the property $\lim_{n\to\infty}\beta(t_n) = 1 \Rightarrow \lim_{n\to\infty}t_n = 0$. For any $u, v \in C[0, 1]$ with $\alpha(Tu, u) > 1$ and $\alpha(Tv, v) > 1$, we then obtain

$$\|Tu - Tv\|_\infty \le \beta(\|u - v\|_\infty)\|u - v\|_\infty.$$

It follows from Definition 2.5 that T has (α_{I^0}, β)-dominating property. □

Lemmas 4.1 and 4.2 thus give rise to a solution to our BVP.

Theorem 4.3. *Suppose that there exists a function γ satisfying (C1)–(C3) as in Lemmas 4.1 and 4.2. Additionally, assume that*

(C4) There exists $u_0 \in C[0, 1]$ such that $\gamma(Tu_0(t), u_0(t)) \ge 0$ for all $t \in [0, 1]$.

Then, $\mathcal{CB}(T, I^0) \ne \emptyset$. In other words, T has a fixed point $u^* \in C[0, 1]$, which is a solution to the BVP (4.1)–(4.2).

Proof. By (C1)–(C4), all assumptions in Theorem 3.3 are fully satisfied. □

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