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Research Article

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Graded weakly 1-absorbing primary ideals

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Abstract: Let G be a group and R be a G-graded commutative ring with nonzero unity 1. In this article, we introduce the concept of graded weakly 1-absorbing primary ideals which is a generalization of graded 1-absorbing primary ideal. A proper graded ideal P of R is said to be a graded weakly 1-absorbing primary ideal of R if whenever nonunit elements $x, y, z \in h(R)$ such that $0 \neq xyz \in P$, then $xy \in P$ or $z^n \in P$, for some $n \in \mathbb{N}$. Several properties of graded weakly 1-absorbing primary ideals are investigated.

Keywords: graded prime ideal, graded weakly prime ideal, graded primary ideal, graded weakly primary ideal, graded 2-absorbing primary ideal, graded weakly 2-absorbing primary ideal, graded 2-absorbing ideal, graded weakly 2-absorbing ideal, graded n-absorbing ideal

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1 Introduction

Since graded prime and graded primary ideals have essential roles in graded commutative ring theory, many authors have studied generalizations of graded prime and graded primary ideals. Atani introduced in [1] the concept of graded weakly prime ideals. A proper graded ideal P of R is said to be a graded weakly prime ideal of R if whenever $x, y \in h(R)$ such that $0 \neq xy \in P$, then $x \in P$ or $y \in P$. Also, Atani introduced the notion of graded weakly primary ideals which is a generalization of graded primary ideals in [2]. A proper graded ideal P of R is said to be a graded weakly primary ideal of R if whenever $x, y \in h(R)$ such that $0 \neq xy \in P$, then $x \in P$ or $y \in \text{Grad}(P)$. For distinct generalizations of graded prime ideals and graded weakly prime ideals, the frameworks of graded 2-absorbing and graded weakly 2-absorbing ideals were defined. According to [3], a proper graded ideal P of R is said to be a graded 2-absorbing (graded weakly 2-absorbing) ideal of R if whenever $a, b, c \in h(R)$ such that $abc \in P$ ($0 \neq abc \in P$), then either $ab \in P$ or $ac \in P$ or $bc \in P$. As a generalization of graded 2-absorbing and graded weakly 2-absorbing ideals, graded 2-absorbing primary and graded weakly 2-absorbing primary ideals were defined in [4]. A proper graded ideal P of R is said to be a graded 2-absorbing primary (graded weakly 2-absorbing primary) ideal of R if whenever $a, b, c \in h(R)$ with $abc \in P$, then $ab \in P$ or $ac \in \text{Grad}(P)$ or $bc \in \text{Grad}(P)$. In a recent study in [5], we call a proper graded ideal P of R a graded 1-absorbing primary ideal of R if whenever nonunit elements $a, b, c \in h(R)$ such that $abc \in P$, then $ab \in P$ or $c \in \text{Grad}(P)$.

In this article, we follow [6] to introduce and study the concept of graded weakly 1-absorbing primary ideal of a graded commutative ring R. A proper graded ideal P of R is said to be a graded weakly 1-absorbing primary ideal of R if whenever nonunit elements $a, b, c \in h(R)$ such that $0 \neq abc \in P$, then $ab \in P$ or $c \in \text{Grad}(P)$. It is recognizable that a graded 1-absorbing primary ideal of R is a graded weakly 1-absorbing primary ideal of R. However, since $\{0\}$ is always graded weakly 1-absorbing primary, a graded weakly 1-absorbing primary ideal of R needs not be a graded 1-absorbing primary ideal of R (see Example 2.2).

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Among several results, we show that if P is a graded weakly 1-absorbing primary ideal of R and $\mathrm{Grad}(\{0\}) = \{0\}$, then $\mathrm{Grad}(P)$ is a graded weakly prime ideal of R (Proposition 2.3). We prove that if P is a graded weakly 1-absorbing primary ideal of R, then $(P : z)$ is a graded weakly primary ideal of R for every nonunit $z \in h(R) - P$ (Proposition 2.17). We show that if $R = R_1 \times R_2 \times \cdots \times R_n$, where $R_1, R_2, \ldots, R_n$ are graded rings, then every proper graded ideal of R is a graded weakly 1-absorbing primary ideal of R if and only if $n = 2$ and R_1, R_2 are graded fields (Proposition 2.19). In Proposition 2.22, we study graded weakly 1-absorbing primary ideals over graded homomorphisms.

1.1 Motivation

Studying graded prime ideals and their generalizations is important for several reasons. First, graded prime ideals are a natural generalization of prime ideals in commutative algebra and algebraic geometry, which are fundamental concepts in mathematics. They have many applications in areas such as algebraic geometry, algebraic number theory, and commutative algebra. Additionally, graded prime ideals are related to other important mathematical concepts such as graded rings and modules, which are important in areas such as algebraic geometry and algebraic topology. Understanding these concepts can help to provide a deeper understanding of the underlying structures in these areas of mathematics. Finally, the study of graded prime ideals and their generalizations can also have practical applications in areas such as computer science and engineering. For example, the theory of graded prime ideals can be used to study error-correcting codes and cryptography. Overall, studying graded prime ideals and their generalizations is important for both theoretical and practical reasons, and can help to deepen our understanding of many areas of mathematics and its application to other fields. For recent generalizations on graded prime ideals, see [7–9]. Also, for recent applications, one can look at [10–15].

1.2 Preliminaries

Throughout this article, G will be a group with identity e and R a commutative ring with a nonzero unity 1. R is said to be G-graded if $R = \oplus_{g\in G} R_g$ with $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Also, we set $h(R) = \cup_{g\in G} R_g$. Moreover, it has been proved in [16] that R_e is a subring of R and $1 \in R_e$. Let I be an ideal of a graded ring R. Then, I is said to be graded ideal if $I = \oplus_{g\in G}(I \cap R_g)$, i.e., for $x \in I$, $x = \sum_{g\in G} x_g$ where $x_g \in I$ for all $g \in G$. An ideal of a graded ring need not be graded. Let R be a G-graded ring and I be a graded ideal of R. Then, R/I is G-graded by $(R/I)_g = (R_g + I)/I$ for all $g \in G$. If R and S are G-graded rings, then $R \times S$ is a G-graded ring by $(R \times S)_g = R_g \times S_g$ for all $g \in G$.

Lemma 1.1. ([17], Lemma 2.1) *Let R be a G-graded ring.*
(1) *If I and J are graded ideals of R, then $I + J$, IJ, and $I \cap J$ are graded ideals of R.*
(2) *If $x \in h(R)$, then Rx is a graded ideal of R.*

Let P be a proper graded ideal of R. Then, the graded radical of P is $\mathrm{Grad}(P)$ and is defined to be the set of all $r \in R$ such that for each $g \in G$, there exists a positive integer n_g satisfies $r_g^{n_g} \in P$. One can see that if $r \in h(R)$, then $r \in \mathrm{Grad}(P)$ if and only if $r^n \in P$ for some positive integer n.

2 Graded weakly 1-absorbing primary ideals

In this section, we introduce and study the concept of graded weakly 1-absorbing primary ideals.

Definition 2.1. A proper graded ideal P of a graded ring R is said to be a graded weakly 1-absorbing primary ideal of R if whenever nonunit elements $x, y, z \in h(R)$ such that $0 \neq xyz \in P$, then $xy \in P$ or $z \in \mathrm{Grad}(P)$.

It is recognizable that a graded 1-absorbing primary ideal of R is a graded weakly 1-absorbing primary ideal of R. However, since $\{0\}$ is always graded weakly 1-absorbing primary, a graded weakly 1-absorbing primary ideal of R needs not be a graded 1-absorbing primary ideal of R, see the following example:

Example 2.2. Consider $R = \mathbb{Z}_6[i]$ and $G = \mathbb{Z}_2$. Then, R is G-graded by $R_0 = \mathbb{Z}_6$ and $R_1 = i\mathbb{Z}_6$. Now, $P = \{0\}$ is a graded weakly 1-absorbing primary ideal of R. In contrast, $2, 3 \in h(R)$ such that $2.2.3 \in P$ with neither $2.2 \in P$ nor $3 \in \mathrm{Grad}(P)$. Hence, P is not a graded 1-absorbing primary ideal of R.

Proposition 2.3. *Let R be a graded ring and P be a graded weakly 1-absorbing primary ideal of R. If* $\mathrm{Grad}(\{0\}) = \{0\}$, *then* $\mathrm{Grad}(P)$ *is a graded weakly prime ideal of R.*

Proof. Let $x, y \in h(R)$ such that $0 \neq xy \in \mathrm{Grad}(P)$. We may assume that x and y are nonunit elements. Then, there exists an even positive integer $k = 2s$ ($s \geq 1$) such that $(xy)^k \in P$. Since $\mathrm{Grad}(\{0\}) = \{0\}$, we have that $(xy)^k \neq 0$. So, $0 \neq x^s x^s y^k \in P$. Since P is a graded weakly 1-absorbing primary ideal of R, we conclude that either $x^s x^s = x^k \in P$ or $y^k \in \mathrm{Grad}(P)$, which implies that $x \in \mathrm{Grad}(P)$ or $y \in \mathrm{Grad}(P)$. Hence, $\mathrm{Grad}(P)$ is a graded weakly prime ideal of R. □

Remark 2.4. If $\mathrm{Grad}(P)$ is a graded maximal ideal of R, then by ([18], Proposition 1.11), P is a graded primary ideal of R, and hence, P is a graded 1-absorbing primary ideal of R.

Definition 2.5. Let R be a G-graded ring and P be a graded ideal of R. Assume that $g \in G$ such that $P_g \neq R_g$. Then, P is said to be a g-weakly 1-absorbing primary ideal of R if whenever nonunit elements $x, y, z \in R_g$ such that $0 \neq xyz \in P$, then $xy \in P$ or $z \in \mathrm{Grad}(P)$. Also, P is said to be a g-weakly primary ideal if whenever $x, y \in R_g$ such that $0 \neq xy \in P$, then either $x \in P$ or $y \in \mathrm{Grad}(P)$.

Proposition 2.6. *Let R be a G-graded ring, $g \in G$ such that R_g has no zero divisors and P be a g-weakly 1-absorbing primary ideal of R. If for every nonzero $p \in P$, there exists a nonunit $w \in R_g$ such that $wp \neq 0$ and $w + u$ is a nonunit element of R_g for some unit $u \in R_g$. Then, P is a g-weakly primary ideal of R.*

Proof. Let $x, y \in R_g$ such that $0 \neq xy \in P$ and $y \notin \mathrm{Grad}(P)$. We may assume that x, y are nonunit elements of R. Then, there is a nonunit $w \in R_g$ such that $wxy \neq 0$ and $w + u$ is a nonunit element of R_g for some unit $u \in R_g$. Since $0 \neq wxy \in P$ and $y \notin \mathrm{Grad}(P)$ and P is a g-weakly 1-absorbing primary ideal of R, we conclude that $wx \in P$. Since $0 \neq (w + u)xy \in P$ and P is a g-weakly 1-absorbing primary ideal of R and $y \notin \mathrm{Grad}(P)$, we conclude that $(w + u)x = wx + ux \in P$. Since $wx \in P$ and $wx + ux \in P$, we conclude that $ux \in P$. Since u is a unit, we have $x \in P$. Hence, P is a g-weakly primary ideal of R. □

Definition 2.7. Let R be a graded ring.
(1) For $a, b \in h(R)$, we say that a divides b (we write $a|b$) if $b = ax$ for some $x \in h(R)$.
(2) R is said to be a graded chained ring if for every $a, b \in h(R)$, we have either $a|b$ or $b|a$.
(3) R is said to be a graded weakly divided ring if for every graded weakly prime ideal P of R and for every $a \in h(R) - P$, we have $a|p$ for every $p \in P$.

Clearly, every graded chained ring is a graded weakly divided ring.

Proposition 2.8. *Let R be a graded weakly divided ring and P be a proper graded ideal of R. If* $\mathrm{Grad}(\{0\}) = \{0\}$, *then P is a graded weakly 1-absorbing primary ideal of R if and only if P is a graded weakly primary ideal of R.*

Proof. Suppose that P is a graded weakly 1-absorbing primary ideal of R. Let $x, y \in h(R)$ such that $0 \neq xy \in P$ and $y \notin \text{Grad}(P)$. We may assume that x, y are nonunit elements of R. Since $\text{Grad}(P)$ is a graded weakly prime ideal of R by Proposition 2.3, we conclude that $x \in \text{Grad}(P)$. Since R is a graded weakly divided ring, we conclude that $y|x$. Thus, $x = yc$ for some $c \in h(R)$. Note that c is a nonunit element of R as $y \notin \text{Grad}(P)$ and $x \in \text{Grad}(P)$. Since $0 \neq xy = ycy \in P$ and P is a graded weakly 1-absorbing primary ideal of R, and $y \notin \text{Grad}(P)$, we conclude that $yc = x \in P$. Hence, P is a graded weakly primary ideal of R. The converse is clear. □

Corollary 2.9. *Let R be a graded chained ring and P be a proper graded ideal of R. If $\text{Grad}(\{0\}) = \{0\}$, then P is a graded weakly 1-absorbing primary ideal of R if and only if P is a graded weakly primary ideal of R.*

Let R be a G-graded ring and P be a graded ideal of R. Then, it has been proved in ([19], Theorem 2.17), $(P : a) = \{x \in R : xa \in P\}$ is a graded ideal of R for every $a \in h(R)$. Similarly, one can prove that $(P : K)$ is a graded ideal of R for every graded ideal K of R.

Proposition 2.10. *Let R be a G-graded ring, $g \in G$ and P be graded ideal of R with $P_g \neq R_g$. Consider the following statements:*

(1) *P is a g-weakly 1-absorbing primary ideal of R.*

(2) *For every nonunit elements $x, y \in R_g$ with $xy \notin P$, $(P :_{R_g} xy) = (0 :_{R_g} xy)$ or $(P :_{R_g} xy) \subseteq \text{Grad}(P)$.*

(3) *For every nonunit element $x \in R_g$ and every graded ideal K of R with $K \nsubseteq \text{Grad}(P)$, if $(P :_{R_g} xK) \neq R_g$, then $(P :_{R_g} xK) = (0 :_{R_g} xK)$ or $(P :_{R_g} xK) \subseteq (P :_{R_g} x)$.*

(4) *For every graded ideals K and J of R with $K \nsubseteq \text{Grad}(P)$, if $(P :_{R_g} KJ) \neq R_g$, then $(P :_{R_g} KJ) = (0 :_{R_g} KJ)$ or $(P :_{R_g} KJ) \subseteq (P :_{R_g} J)$.*

Then, (1) ⇒ (2) ⇒ (3) ⇒ (4).

Proof. (1) ⇒ (2): Suppose that $x, y \in R_g$ are nonunit elements such that $xy \notin P$. Let $c \in (P :_{R_g} xy)$. Since $xy \notin P$; c is nonunit. If $xyc = 0$, then $c \in (0 :_{R_g} xy)$. Assume that $0 \neq xyc \in P$. Since P is a g-weakly 1-absorbing primary ideal of R, we have $c \in \text{Grad}(P)$. Hence, $(P :_{R_g} xy) \subseteq (0 :_{R_g} xy) \cup \text{Grad}(P)$ and so we obtain $(P :_{R_g} xy) \subseteq (0 :_{R_g} xy)$ or $(P :_{R_g} xy) \subseteq \text{Grad}(P)$. If $(P :_{R_g} xy) \subseteq (0 :_{R_g} xy)$, then $(P :_{R_g} xy) = (0 :_{R_g} xy)$ since the reverse inclusion always holds. So, we obtain $(P :_{R_g} xy) = (0 :_{R_g} xy)$ or $(P :_{R_g} xy) \subseteq \text{Grad}(P)$.

(2) ⇒ (3): Let $x \in R_g$ be a nonunit element and K be a graded ideal of R with $K \nsubseteq \text{Grad}(P)$. If $xK \subseteq P$, then nothing to prove. Suppose that $xK \nsubseteq P$ and let $c \in (P :_{R_g} xK)$. It is clear that c is nonunit. Then, $xcK \subseteq P$. Now $K \subseteq (P :_{R_g} xc)$. If $xc \in P$, then $c \in (P :_{R_g} x)$. Suppose that $xc \notin P$. Hence, $(P :_{R_g} xc) = (0 :_{R_g} xc)$ or $(P :_{R_g} xc) \subseteq \text{Grad}(P)$ by (2). Thus, $K \subseteq (0 :_{R_g} xc)$ or $K \subseteq \text{Grad}(P)$. Since $K \nsubseteq \text{Grad}(P)$, we conclude that $K \subseteq (0 :_{R_g} xc)$; that is $c \in (0 :_{R_g} xK)$. Thus, $(P :_{R_g} xK) \subseteq (0 :_{R_g} xK) \cup (P :_{R_g} x)$, and then we have $(P :_{R_g} xK) = (0 :_{R_g} xK)$ or $(P :_{R_g} xK) \subseteq (P :_{R_g} x)$.

(3) ⇒ (4): Let K and J be graded ideals of R with $K \nsubseteq \text{Grad}(P)$. Let $c \in (P :_{R_g} KJ)$. Then, $J \subseteq (P :_{R_g} cK)$. Since $(P :_{R_g} KJ) \neq R_g$, c is nonunit. So, $J \subseteq (0 :_{R_g} cK)$ or $J \subseteq (P :_{R_g} c)$ by (3). If $J \subseteq (0 :_{R_g} cK)$, then $c \in (P :_{R_g} KJ)$. If $J \subseteq (P :_{R_g} c)$, then $c \in (P :_{R_g} J)$. So, $(P :_{R_g} KJ) \subseteq (0 :_{R_g} KJ) \cup (P :_{R_g} J)$ which implies that $(P :_{R_g} KJ) = (0 :_{R_g} KJ)$ or $(P :_{R_g} KJ) \subseteq (P :_{R_g} J)$; as required. □

Remark 2.11. Let R be a graded ring and P be a graded ideal of R. If P is a graded weakly 1-absorbing primary ideal of R that is not a graded 1- absorbing primary ideal of R, then it is clear that there exist nonunit elements $x, y, z \in h(R)$ such that $xyz = 0$ with $xy \notin P$ and $z \notin \text{Grad}(P)$. In fact, these x, y, z satisfy several properties as we see in the next results.

Let R be a G-graded ring, $g \in G$ and P be a graded ideal of R such that $P_g \neq R_g$. Then, P is said to be a g-1-absorbing primary ideal if whenever $x, y, z \in R_g$ such that $xyz \in P$, then either $xy \in P$ or $z \in \text{Grad}(P)$.

Proposition 2.12. *Let R be a G-graded ring, $g \in G$ and P be a graded ideal of R. If P is a g-weakly 1-absorbing primary ideal of R that is not a g-1- absorbing primary ideal of R, then there exist nonunit elements $x, y, z \in R_g$ such that $xyz = 0$ with $xy \notin P$ and $z \notin \mathrm{Grad}(P)$ and the following hold:*
(1) $xyP_g = \{0\}$.
(2) *If $x, y \notin (P :_{R_g} z)$, then $yzP_g = xzP_g = xP_g^2 = yP_g^2 = zP_g^2 = \{0\}$.*
(3) *If $x, y \notin (P :_{R_g} z)$, then $P_g^3 = \{0\}$.*

Proof.
(1) Suppose that $xyP_g \neq \{0\}$. Then, $xya \neq 0$ for some nonunit $a \in P_g$. So, $0 \neq xy(z + a) \in P$. Since $xy \notin P$; $(z + a)$ is a nonunit element of R_g. Since P is a g-weakly 1-absorbing primary ideal of R and $xy \notin P$, we conclude that $(z + a) \in \mathrm{Grad}(P)$. Since $a \in P$, we have $z \in \mathrm{Grad}(P)$, which is a contradiction. So, $xyP_g = \{0\}$.
(2) Suppose that $yzP_g \neq \{0\}$. Then, $yzb \neq 0$ for some nonunit element $b \in P_g$. Hence, $0 \neq yzb = y(x + b)z \in P$. Since $y \notin (P :_{R_g} z)$, we conclude that $x + b$ is a nonunit element of R_g. Since P is a g-weakly 1-absorbing primary ideal of R and $xy \notin P$ and $yb \in P$, we conclude that $y(x + b) \notin P$, and hence, $z \in \mathrm{Grad}(P)$, which is a contradiction. Thus, $yzP_g = \{0\}$. Suppose that $xzP_g \neq \{0\}$. Then, $xzb \neq 0$ for some nonunit element $b \in P_g$. Hence, $0 \neq xzb = x(y + b)z \in P$. Since $x \notin (P :_{R_g} z)$, we conclude that $y + b$ is a nonunit element of R_g. Since P is a g-weakly 1-absorbing primary ideal of R and $xy \notin P$ and $xb \in P$, we conclude that $x(y + b) \notin P$, and hence, $z \in \mathrm{Grad}(P)$, which is a contradiction. Thus, $xzP_g = \{0\}$. Suppose that $xP_g^2 \neq \{0\}$. Then, $xab \neq 0$ for some $a;b \in P_g$. Since $xyP_g = \{0\}$ by (1) and $xzP_g = \{0\}$ by (2), $0 \neq xab = x(y + a)(z + b) \in P$. Since $xy \notin P$, we conclude that $z + b$ is a nonunit element of R_g. Since $x \notin (P :_{R_g} z)$, we conclude that $y + a$ is a nonunit element of R_g. Since P is a g-weakly 1-absorbing primary ideal of R, we have $x(y + a) \in P$ or $(z + b) \in \mathrm{Grad}(P)$. Since $a;b \in P$, we conclude that $xy \in P$ or $z \in \mathrm{Grad}(P)$, which is a contradiction. Thus, $xP_g^2 = \{0\}$. Suppose that $yP_g^2 \neq \{0\}$. Then, $yab \neq 0$ for some $a;b \in P_g$. Since $xyP_g = \{0\}$ by (1) and $yzP_g = \{0\}$ by (2), $0 \neq yab = y(x + a)(z + b) \in P$. Since $xy \notin P$, we conclude that $z + b$ is a nonunit element of R_g. Since $y \notin (P :_{R_g} z)$, we conclude that $x + a$ is a nonunit element of R_g. Since P is a g-weakly 1-absorbing primary ideal of R, we have $y(x + a) \in P$ or $(z + b) \in \mathrm{Grad}(P)$. Since $a;b \in P$, we conclude that $xy \in P$ or $z \in \mathrm{Grad}(P)$, which is a contradiction. Thus, $yP_g^2 = \{0\}$. Suppose that $zP_g^2 \neq \{0\}$. Then, $zab \neq 0$ for some $a;b \in P_g$. Since $xzP_g = yzP_g = \{0\}$ by (2), $0 \neq zab = (x + a)(y + b)z \in P$. Since $x;y \notin (P :_{R_g} z)$, we conclude that $x + a$ and $y + b$ are nonunit elements of R_g. Since P is a g-weakly 1-absorbing primary ideal of R, we have $(x + a)(y + b) \in P$ or $z \in \mathrm{Grad}(P)$. Since $a;\ b \in P$, we conclude that $xy \in P$ or $z \in \mathrm{Grad}(P)$, which is a contradiction. Thus, $zP_g^2 = \{0\}$.
(3) Suppose that $P_g^3 \neq \{0\}$. Then, $abc \neq 0$ for some $a, b, c \in P_g$. Then, $0 \neq abc = (x + a)(y + b)(z + c) \in P$ by (1) and (2). Since $xy \notin P$, we conclude $z + c$ is a nonunit element of R_g. Since $x;y \notin (P :_{R_g} z)$, we conclude that $x + a$ and $y + b$ are nonunit elements of R_g. Since P is a g-weakly 1-absorbing primary ideal of R, we have $(x + a)(y + b) \in P$ or $z + c \in \mathrm{Grad}(P)$. Since $a;\ b;\ c \in P$, we conclude that $xy \in P$ or $z \in \mathrm{Grad}(P)$, which is a contradiction. Thus, $P_g^3 = \{0\}$. □

Corollary 2.13. *Let R be a G-graded ring, $g \in G$ and P be a graded ideal of R. If P is a g-weakly 1-absorbing primary ideal of R that is not a g-1-absorbing primary ideal of R, then there exist nonunit elements $x, y, z \in R_g$ such that $xyz = 0$ with $xy \notin P$ and $z \notin \mathrm{Grad}(P)$. If $x, y \notin (P :_{R_g} z)$ and $\mathrm{Grad}(\{0\}) = \{0\}$, then $P_g = \{0\}$.*

Proof. By Proposition 2.12 (3), we have $P_g^3 = \{0\}$, and since $\mathrm{Grad}(\{0\}) = \{0\}$, we conclude that $P_g = \{0\}$. □

Definition 2.14. Let R be a graded ring. Then, $x \in h(R)$ is said to be a homogeneous reducible element of R if $x = yz$ for some nonunit elements $y, z \in h(R)$. Otherwise, x is called a homogeneous irreducible element of R.

Proposition 2.15. *Let P be a graded weakly 1-absorbing primary ideal of a graded ring R. If P is not a graded weakly primary ideal of R, then there exist a homogeneous irreducible element $a \in R$ and a nonunit element $b \in h(R)$ such that $ab \in P$, but neither $a \in P$ nor $b \in \mathrm{Grad}(P)$. Moreover, if $xy \in P$ for some nonunit elements $x, y \in h(R)$ such that neither $x \in P$ nor $y \in \mathrm{Grad}(P)$, then x is a homogeneous irreducible element of R.*

Proof. Since P is not a graded weakly primary ideal of R, we conclude that there exist nonunit elements $a;b \in h(R)$ such that $0 \neq ab \in P$ with $a \notin P$ and $b \notin \mathrm{Grad}(P)$. Suppose that a is a homogeneous reducible element of R. Then, $a = cd$ for some nonunit elements c; $d \in h(R)$. Since $0 \neq ab = cdb \in P$ and P is a graded weakly 1-absorbing primary ideal of R and $b \notin \mathrm{Grad}(P)$, we conclude that $cd = a \in P$, which is a contradiction. Hence, a is a homogeneous irreducible element of R. □

Proposition 2.16. *Let R be a graded ring and $P_1, P_2, \ldots, P_n$ be graded weakly 1-absorbing primary ideals of R. If $\mathrm{Grad}(P_i) = \mathrm{Grad}(P_j) = Q$ for every i, j, then $P = \bigcap_{i=1}^{n} P_i$ is a graded weakly-1-absorbing primary ideal of R.*

Proof. Suppose that $x, y, z \in h(R)$ are nonunit elements such that $0 \neq xyz \in P$. Suppose that $xy \notin P$. Then, $xy \notin P_k$ for some $1 \leq k \leq n$. Since P_k is a graded weakly 1-absorbing primary ideal of R and $0 \neq xyz \in P_k$ and $xy \notin P_k$, we have that $z \in \mathrm{Grad}(P_k) = Q = \mathrm{Grad}(P)$. Hence, P is a graded weakly 1-absorbing primary ideal of R. □

Proposition 2.17. *Let P be a graded weakly 1-absorbing primary ideal of a graded ring R. Then, $(P : z)$ is a graded weakly primary ideal of R for every nonunit $z \in h(R) - P$.*

Proof. Let $z \in h(R) - P$ be a nonunit element. Then, $(P : z)$ is a graded ideal of R. Let $x, y \in h(R)$ such that $0 \neq xy \in (P : z)$, assume that $x \notin (P : z)$. Hence, y is a nonunit element of R. If x is an unit element of R, then $y \in (P : z) \subseteq \mathrm{Grad}((P : z))$ and we are done. Assume that x is a nonunit element of R. Since $0 \neq xyz = xzy \in P$ and $xz \notin P$ and P is a graded weakly-1-absorbing primary ideal of R, we conclude that $y \in \mathrm{Grad}(P) \subseteq \mathrm{Grad}((P : z))$. Thus, $(P : z)$ is a graded weakly primary ideal of R. □

A G-graded ring R is said to be a cross product if R_g contains a unit element for all $g \in G$ [16].

Proposition 2.18. *Let $R = S \times T$, where S and T are G-graded commutative rings with a nonzero unity 1 that are not graded fields. Suppose that R and S are cross products. Assume that P is a nonzero proper graded ideal of R. Then, the following assertions are equivalent:*
(1) *P is a graded weakly 1-absorbing primary ideal of R.*
(2) *$P = I \times T$ for some graded primary ideal I of S or $P = S \times J$ for some graded primary ideal J of T.*
(3) *P is a graded primary ideal of R.*
(4) *P is a graded 1-absorbing primary ideal of R.*

Proof. (1) ⇒ (2): Now, P is of the form $I \times J$ for some graded ideals I and J of S and T, respectively. Assume that both I and J are proper. Since P is a nonzero ideal of R, we conclude that I or J is nonzero. We may assume that I is nonzero. Let $0 \neq z \in I$. Then, since I is graded, $0 \neq z_g \in I$ for some $g \in G$. Since T is a cross product, choose a unit homogeneous element $t_g \in T_g$. Then, $0 \neq (1, 0)(1, 0)(z_g, t_g) = (z_g, 0) \in P$. It implies that $(1, 0)(1, 0) \in P$ or $(z_g, t_g) \in \mathrm{Grad}(P) = \mathrm{Grad}(I) \times \mathrm{Grad}(J)$, that is $I = S$ or $J = T$, a contradiction. Thus, $I = S$ or $J = T$. Without loss of generality, assume that $P = I \times T$ for some proper graded ideal I of S. We show that I is a graded primary ideal of S. Let $xy \in I$ for some $x, y \in h(S)$, where $\deg(x) = g$ and $\deg(y) = h$. We can assume that x and y are nonunit elements of S. Since T is not a graded field, there exists a nonunit nonzero element $a \in h(T)$, where $\deg(a) = j$. Since S and T are cross products, there exist homogeneous unit elements $s \in S$ and $t, t' \in T$, where $\deg(s) = j$, $\deg(t) = g$ and $\deg(t') = h$. Then, $0 \neq (x, t)(s, a)(y, t') = (sxy, tat') \in I \times T$, which implies that either $(x, t)(s, a) \in I \times T$ or $(y, t') \in \mathrm{Grad}(I \times T) = \mathrm{Grad}(I) \times \mathrm{Grad}(T)$, that is, $x \in I$ or $y \in \mathrm{Grad}(I)$.

(2) ⇒ (3): It is well-known.

(3) ⇒ (4) and (4) ⇒ (1) are clear. □

Proposition 2.19. *Let $R = R_1 \times R_2 \times \cdots \times R_n$, where $R_1, R_2, \ldots, R_n$ are G-graded commutative rings with a nonzero unity 1. Suppose that R_i is a cross product, for every $i = 1, 2, \ldots, n$. Then, every proper graded ideal of R is a graded weakly 1-absorbing primary ideal of R if and only if $n = 2$ and R_1, R_2 are graded fields.*

Proof. Suppose that every proper graded ideal of R is a graded weakly 1-absorbing primary ideal of R. Without loss of generality, we may assume that $n = 3$. Then, $P = R_1 \times \{0\} \times \{0\}$ is a graded weakly 1-absorbing primary ideal of R. Choose a nonzero $x \in h(R_1)$, where $\deg(x) = g$. Since R_2 is a cross product, there exists a unit homogeneous element $t \in h(R_2)$, where $\deg(t) = g$. Then, $0 \neq (1,0,1)(1,0,1)(x,t,0) = (x,0,0) \in P$, but $(1,0,1)(1,0,1) \in P$ nor $(x,t,0) \in Grad(P)$, a contradiction. Thus, $n = 2$. Assume that R_1 is not a graded field. Then, there exists a nonzero proper graded ideal I of R_1. Hence, $P = I \times \{0\}$ is a graded weakly 1-absorbing primary ideal of R. For a nonzero $x \in I$, $0 \neq x_g \in I$ for some $g \in G$. Since R_2 is a cross product, there exists a unit homogeneous element $u \in h(R_2)$, where $\deg(u) = g$. Then, we have $0 \neq (1,0)(1,0)(x_g, u) = (x_g, 0) \in P$, but $(1,0)(1,0) \in P$ nor $(x_g, u) \in Grad(P)$, a contradiction. Similarly, R_2 is a graded field. Conversely, R has exactly three proper graded ideals; $\{(0,0)\}$, $\{0\} \times R_2$ and $R_1 \times \{0\}$, and all of them are graded weakly 1-absorbing primary ideals of R. □

First strongly graded rings have been introduced and studied in [20]; a G-graded ring R is said to be first strong if $1 \in R_g R_{g^{-1}}$ for all $g \in supp(R, G)$, where $supp(R, G) = \{g \in G : R_g \neq \{0\}\}$. In fact, it has been proved that R is first strongly G-graded if and only if $supp(R, G)$ is a subgroup of G and $R_g R_h = R_{gh}$ for all $g, h \in supp(R, G)$. We introduce the following:

Lemma 2.20. *Every G-graded field is first strongly graded.*

Proof. Let R be a G-graded field. Suppose that $g \in supp(R, G)$. Then, $R_g \neq \{0\}$, and then there exists $0 \neq x \in R_g$. Since R is a graded field, we conclude that there exists $y \in h(R)$ such that $xy = 1$. Since $y \in h(R)$, $y \in R_h$ for some $h \in G$, and then $1 = xy \in R_g R_h \subseteq R_{gh}$. So, $0 \neq 1 \in R_{gh} \cap R_e$, which implies that $gh = e$, that is $h = g^{-1}$. Hence, $1 = xy \in R_g R_{g^{-1}}$, and thus, R is first strongly graded. □

Proposition 2.21. *Let $R = R_1 \times R_2$, where R_1 and R_2 are G-graded commutative rings with a nonzero unity 1. Suppose that R_i is a cross product, for every $i = 1, 2$. If every proper graded ideal of R is a graded weakly 1-absorbing primary ideal of R, then R_1 and R_2 are first strongly graded.*

Proof. Apply Proposition 2.19 and Lemma 2.20. □

Let R and S be two G-graded rings. A ring homomorphism $f : R \to S$ is said to be graded homomorphism if $f(R_g) \subseteq S_g$ for all $g \in G$.

Proposition 2.22. *Let R and S be G-graded commutative rings and $f : R \to S$ is a graded homomorphism with $f(1_R) = 1_S$. Then, the following hold:*

(1) *Suppose that $f(x)$ is a nonunit element of S for every nonunit element $x \in R$ and I is a graded weakly 1-absorbing primary ideal of S. Then, $f^{-1}(I)$ is a graded weakly 1-absorbing primary ideal of R.*
(2) *If f is surjective and P is a graded weakly 1-absorbing primary ideal of R such that* $Ker(f) \subseteq P$, *then $f(P)$ is a graded weakly 1-absorbing primary ideal of S.*

Proof.

(1) Clearly, $f^{-1}(I)$ is a graded ideal of R. Let $x, y, z \in h(R)$ be nonunit elements such that $0 \neq xyz \in f^{-1}(I)$. Then, $f(x), f(y), f(z) \in h(S)$ are nonunit elements such that $0 \neq f(x)f(y)f(z) = f(xyz) \in I$. Since I is a graded weakly 1-absorbing primary ideal of S, we have that $f(xy) = f(x)f(y) \in I$ or $f(z) \in Grad(I)$, which implies that $xy \in f^{-1}(I)$ or $z \in f^{-1}(Grad(I)) = Grad(f^{-1}(I))$. Thus, $f^{-1}(I)$ is a graded weakly 1-absorbing primary ideal of R.

(2) Clearly, $f(P)$ is a graded ideal of S. Let $a, b, c \in h(S)$ be nonunit elements such that $0 \neq abc \in f(P)$. Then, since f is surjective, there exist nonunit elements $x, y, z \in h(R)$ such that $f(x) = a, f(y) = b$ and $f(z) = c$. Now, $f(xyz) = f(x)f(y)f(z) = abc \in f(P)$. Since $\mathrm{Ker}(f) \subseteq P$, we have that $0 \neq xyz \in P$. Since P is a graded weakly 1-absorbing primary ideal of R, we have that $xy \in P$ or $z \in \mathrm{Grad}(P)$, which implies that $ab = f(x)f(y) = f(xy) \in f(P)$ or $c = f(z) \in f(\mathrm{Grad}(P)) = \mathrm{Grad}(f(P))$ as f is surjective and $\mathrm{Ker}(f) \subseteq P$. Hence, $f(P)$ is a graded weakly 1-absorbing primary ideal of S. □

Corollary 2.23. *Let P be a proper graded ideal of a graded ring R. Then, the following hold*:

(1) *If K is a proper graded ideal of R with $K \subseteq P$ and P is a graded weakly 1-absorbing primary ideal of R, then P/K is a graded weakly 1-absorbing primary ideal of R/K.*

(2) *Let K be a proper graded ideal of R with $K \subseteq P$ such that $U(R/K) = \{x + K : x \in U(R)\}$. If K is a graded 1-absorbing primary ideal of R and P/K is a graded weakly 1-absorbing primary ideal of R/K, then P is a graded 1-absorbing primary ideal of R.*

(3) *If $\{0\}$ is a graded 1-absorbing primary ideal of R and P is a graded weakly 1-absorbing primary ideal of R, then P is a graded 1-absorbing primary ideal of R.*

(4) *Let K be a proper graded ideal of R with $K \subseteq P$ such that $U(R/K) = \{x + K : x \in U(R)\}$. If K is a graded weakly 1-absorbing primary ideal of R and P/K is a graded weakly 1-absorbing primary ideal of R/K, then P is a graded weakly 1-absorbing primary ideal of R.*

Proof.

(1) Let $f : R \to R/K$ such that $f(x) = x + K$. Then, f is surjective graded homomorphism and $f(1_R) = 1_{R/K}$. Suppose that P is a graded weakly 1-absorbing primary ideal of R. Since f is surjective and $\mathrm{Ker}(f) = K \subseteq P$, by Proposition 2.22(2), we have that $f(P) = P/K$ is a graded weakly 1-absorbing primary ideal of R/K.

(2) Suppose that $xyz \in P$ for some nonunit elements $x, y, z \in h(R)$. If $xyz \in K$, then $xy \in K \subseteq P$ or $z \in \mathrm{Grad}(K)$ $\subseteq \mathrm{Grad}(P)$ as K is a graded 1-absorbing primary ideal of R. Now, assume that $xyz \notin K$. Then, $K \neq (x + K)(y + K)(z + K) \in P/K$, where $x + K, y + K, z + K$ are nonunit elements of $h(R/K)$ by assumption. Thus, $(x + K)(y + K) \in P/K$ or $(z + K) \in \mathrm{Grad}(P/K)$. Hence, $xy \in P$ or $z \in \mathrm{Grad}(P)$.

(3) Apply (2).

(4) Suppose that $0 \neq xyz \in P$ for some nonunit elements $x, y, z \in h(R)$. If $xyz \in K$, then $xy \in K \subseteq P$ or $z \in \mathrm{Grad}(K) \subseteq \mathrm{Grad}(P)$ as K is a graded weakly 1-absorbing primary ideal of R. Now, assume that $xyz \notin K$. Then, $K \neq (x + K)(y + K)(z + K) \in P/K$, where $x + K, y + K, z + K$ are nonunit elements of $h(R/K)$ by assumption. Thus, $(x + K)(y + K) \in P/K$ or $(z + K) \in \mathrm{Grad}(P/K)$. Hence, $xy \in P$ or $z \in \mathrm{Grad}(P)$. □

3 Conclusion

In this article, we follow [6] to introduce and study the concept of graded weakly 1-absorbing primary ideal of a graded commutative ring R. It is recognizable that a graded 1-absorbing primary ideal of R is a graded weakly 1-absorbing primary ideal of R. However, since $\{0\}$ is always graded weakly 1-absorbing primary, a graded weakly 1-absorbing primary ideal of R needs not be a graded 1-absorbing primary ideal of R (see Example 2.2). Among several results, we show that if P is a graded weakly 1-absorbing primary ideal of R and $\mathrm{Grad}(\{0\}) = \{0\}$, then $\mathrm{Grad}(P)$ is a graded weakly prime ideal of R (Proposition 2.3). We prove that if P is a graded weakly 1-absorbing primary ideal of R, then $(P : z)$ is a graded weakly primary ideal of R for every nonunit $z \in h(R) - P$ (Proposition 2.17). We show that if $R = R_1 \times R_2 \times \cdots \times R_n$, where $R_1, R_2, \ldots, R_n$ are graded rings, then every proper graded ideal of R is a graded weakly 1-absorbing primary ideal of R if and only if $n = 2$ and R_1, R_2 are graded fields (Proposition 2.19). In Proposition 2.22, we study graded weakly 1-absorbing primary ideals over graded homomorphisms. Concerning the weakness of the developed work, we should recover it by finding several applications in several areas, and that will be studied as a proposal for a future work.

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References

[1] S. E. Atani, *On graded weakly prime ideals*, Turkish J. Math. **30** (2006), 351–358.

[2] S. E. Atani, *On graded weakly primary ideals*, Quasigroups Related Syst. **13** (2005), 185–191.

[3] K. Al-Zoubi, R. Abu-Dawwas, and S. Ceken, *On graded 2-absorbing and graded weakly 2-absorbing ideals*, Hacettepe J. Math. Stat. **48** (2019), no. 3, 724–731.

[4] F. Soheilnia and A. Y. Darani, *On graded 2-absorbing and graded weakly 2-absorbing primary ideals*, Kyungpook Math. J. **57** (2017), no. 4, 559–580.

[5] R. Abu-Dawwas and M. Bataineh, *Graded 1-absorbing primary ideals*, Conference: Turkish Journal of Mathematics - Studies on Scientific Developments in Geometry, Algebra, and Applied Mathematics, February 1–3, Istanbul - Turkey, 2022.

[6] A. Badawi and E. Y. Celikel, *On weakly 1-absorbing primary ideals of commutative rings*, Algebra Colloquium **29** (2022), no. 4, 189–202.

[7] R. Abu-Dawwas, E. Yildiz, U. Tekir, and S. Koc, *On graded 1-absorbing prime ideals*, Sao Paulo J. Math. Sci. **15** (2021), no. 1, 450–462.

[8] S. Koc, U. Tekir, and E. Yildiz, *On weakly 1-absorbing prime ideals*, Ricerche di Matematica, (2021), 1–16. DOI: https://doi.org/10.1007/s11587-020-00550-4.

[9] R. N. Uregen, U. Tekir, K. P. Shum, and S. Koc, *On graded 2-absorbing quasi primary ideals*, Southeast Asian Bull. Math. **43** (2019), no. 4, 601–613.

[10] T. Senapati, *T-fuzzy KU-ideals of KU-algebras*, Afrika Matematika **29** (2018), no. 3–4, 591–600.

[11] T. Senapati, Y. B. Jun, A. Iampan, and R. Chinram, *Cubic intuitionistic structure applied to commutative ideals of BCK-algebras*, Thai J Math. **20** (2022), no. 2, 877–887.

[12] T. Senapati, Y. B. Jun, and K. P. Shum, Cubic intuitionistic implicative ideals of BCK-algebras, *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, vol. 91, 2021, pp. 273–282.

[13] T. Senapati, Y. B. Jun, and K. P. Shum, *Cubic intuitionistic subalgebras and closed cubic intuitionistic ideals of B-algebras*, J. Intell. Fuzzy Syst. **36** (2019), no. 2, 1563–1571.

[14] T. Senapati, Y. B. Jun, and K. P. Shum, *Cubic intuitionistic structure of KU-algebras*, Afrika Matematika **31** (2020), no. 2, 237–248.

[15] T. Senapati and K. P. Shum, *Atanassov's intuitionistic fuzzy bi-normed KU-ideals of a KU-algebra*, J. Intell. Fuzzy Syst. **30** (2016), 1169–1180.

[16] C. Nastasescu and F. van Oystaeyen, *Methods of graded rings*, Lecture Notes in Mathematics, 1836, Springer-Verlag, Berlin, 2004.

[17] F. Farzalipour and P. Ghiasvand, *On the union of graded prime submodules*, Thai J. Math. **9** (2011), no. 1, 49–55.

[18] M. Refai and K. Al-Zoubi, *On graded primary ideals*, Turkish J. Math. **28** (2004), no. 3, 217–229.

[19] R. Abu-Dawwas and M. Bataineh, *Graded r-ideals*, Iranian J. Math. Sci. Inform. **14** (2019), no. 2, 1–8.

[20] M. Refai, *Various types of strongly graded rings*, Abhath Al-Yarmouk J. (Pure Sci. Eng. Ser.) **4** (1995), no. 2, 9–19.