Research Article

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On some conformable boundary value problems in the setting of a new generalized conformable fractional derivative

https://doi.org/10.1515/dema-2022-0212
received October 17, 2021; accepted February 16, 2023

Abstract: The fundamental objective of this article is to investigate about the boundary value problem with the uses of a generalized conformable fractional derivative introduced by Zarikaya et al. (*On generalized the conformable calculus*, TWMS J. App. Eng. Math. **9** (2019), no. 4, 792–799, http://jaem.isikun.edu.tr/web/images/articles/vol.9.no.4/11.pdf). In the development of the this article, by using classical methods of fractional calculus, we find a definition of the generalized fractional Wronskian according to the fractional differential operator defined by Zarikaya, a fractional version of the Sturm-Picone theorem, and in addition, the stability criterion given by the Hyers-Ulam theorem is studied with the use of the aforementioned fractional derivatives.

Keywords: conformable fractional derivatives, boundary value problems, Sturm-Picone theorem

MSC 2020: 26A33, 34B08

1 Introduction

In 1965, L'Hopital gave the preliminary definition of the idea of fractional derivative. Since then, several related new definitions have been proposed. The most common ones are the Riemann-Liouville and Caputo definitions. For more information about the most known fractional definitions, we refer to [1–3].

The so-called fractional calculus has had a wide expansion, both from the theoretical and the applied point of view. In either case, the classical (global) fractional derivative has been used in differential equations, but in the case of local fractional derivatives, this type of research is very limited.

It is known that from 1960, certain differential operators have appeared which are called local fractional derivatives. It is not until 2014 that Khalil et al. introduced in [4] a local derivative (conformable)

$$T_\alpha f(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

and in 2015 Abdeljawad [5] introduced a slight modification

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$$T_a^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon}.$$

In 2018, Nápoles Valdés et al. [6] introduced a definition of a nonconformable fractional derivative, denoted by N_F^α, with very good properties, and defined by

$$N_F^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon F(t,\alpha)) - f(t)}{\varepsilon},$$

where $F(t, \alpha)$ is an absolutely continuous function depending on $t > 0$ and $\alpha \in (0, 1]$. Also in 2019, Abreu-Blaya et al. [7] introduced a generalized conformable fractional derivative

$$G_T^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^{\lceil \alpha \rceil}} \sum_{k=0}^{\lceil \alpha \rceil} (-1)^k \binom{\lceil \alpha \rceil}{k} f(t - khT(t,\alpha)),$$

and in 2020, Fleitas et al. [8] gave a note on this generalized conformable derivative. These definitions have properties suitable to that of the classical Riemman derivative with a better behavior than the classical fractional derivatives when used in different fields of application. To solve a given fractional problem, the question arises as to what type of fractional operator should be considered, since there are several different definitions of fractional derivative in the literature and the choice depends on the problem under consideration.

It can be seen from those articles that use the Riemann-Liouville or Caputo fractional derivative and the corresponding definitions of the conformable derivatives that there is a quantitative and qualitative difference between the two types of operators, local and global [9]. Conformable fractional derivatives are new tools that have demonstrated their usefulness and potential in the modeling of different processes and phenomena.

As a result, several important elements of the mathematical analysis of functions of a real variable have been formulated, such as chain rule, fractional power series expansion and fractional integration by parts formulas, Rolle's theorem, and mean value theorem [10]. The conformable partial derivative of the order $\alpha \in (0, 1]$ of the real-valued functions of several variables and conformable gradient vector are also defined. In addition, a conformable version of Clairaut's theorem for partial derivative is investigated in [11]. In [12], the conformable version of Euler's theorem on homogeneous equations is introduced. Furthermore, in a short time, various research studies have been conducted on the theory and applications of fractional differential equations and fractional integral inequalities in the context of this newly introduced fractional derivative [13–28].

In the literature, some problems related to the classical and fractional differential equations and stability criteria have been published [29–34].

With the motivation given by the aforementioned works, in this research article, we focus on the boundary value problems using a new definition of conformable fractional derivative. We have organized our present document in a subsection of preliminary knowledge, a section of main results where we define the Wronskian from the perspective of the conformable derivative defined in the preliminaries, some basic properties, and we proceed to deal with a conformable version of the conformable Sturm-Picone second-order conformable identity, establish generalized conformable Sturm-Liouville comparison and separation theorems, construct the Green's function and study its properties, and then prove the generalized Hyers-Ulam stability of conformable nonhomogeneous linear differential equations with homogeneous boundary conditions. Also we include conclusions respect to the obtained results.

1.1 Preliminaries

In [35, Definition 2.1], a generalized conformable fractional derivative was defined, and some properties are given.

Definition 1.1. Let $f : [a, b] \to \mathbb{R}$ be a function and $0 \le a < b$. Then the (α, a)-conformable derivative of f of order α is defined by

$$D_a^\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{-\alpha}(t-a)) - f(t)}{\varepsilon(1 - at^{-\alpha})} \tag{1}$$

for all $t > a$, $\alpha \in (0, 1)$. If this limit exists, then it will be said that the function f is (α, a)-differentiable at the point t.

Remark 1.1. Note that if $a = 0$, then this generalized conformable derivative coincides with that proposed in [4], i.e., $D_0^\alpha(f)(t) = T_\alpha f(t)$.

Theorem 1.1. *Let $\alpha \in (0, 1]$ and h, g be α-differentiable at a point $t > a$. Then,*
(1) $D_a^\alpha(uh + vg)(t) = uD_a^\alpha h(t) + vD_a^\alpha g(t)$ *for all* $u, v \in \mathbb{R}$,
(2) $D_a^\alpha(hg)(t) = h(t)D_a^\alpha g(t) + g(t)D_a^\alpha h(t)$,
(3) $D_a^\alpha\left(\frac{h}{g}\right)(t) = \frac{h(t)D_a^\alpha g(t) - g(t)D_a^\alpha h(t)}{g^2(t)}$,
(4) $D_a^\alpha(c) = 0$ *for all constant functions* $h(t) = c$,
(5) $D_a^\alpha(h \circ g)(t) = h'(g(t))D_a^\alpha g(t)$, if *$h$ is differentibale at $g(t)$.*
(6) *If, in addition, h is differentiable then $D_a^\alpha h(t) = \frac{t-a}{t^\alpha - a}h'(t)$, for $t^\alpha \ne a$.*

Also some (α, a)-fractional conformable derivatives for several classic functions are established.

Theorem 1.2. *Let $\alpha \in (0, 1]$, $t > a$, $t^\alpha \ne a$, and $c, n \in \mathbb{R}$. Then we have the following results*:
(1) $D_a^\alpha(t^n) = \frac{t-a}{t^\alpha - a} nt^{n-1}$,
(2) $D_a^\alpha(e^{ct}) = \frac{c(t-a)}{t^\alpha - a} e^{ct}$,
(3) $D_a^\alpha(\sin(ct)) = \frac{c(t-a)}{t^\alpha - a} \cos(ct)$,
(4) $D_a^\alpha(\cos(ct)) = \frac{-c(t-a)}{t^\alpha - a} \sin(ct)$.

In the recently cited work, some important results for the calculation were also established for (α, a)-conformable differentiable functions: the continuity of a function at a point from its conformable differentiability in it, Rolle's theorem, and the mean value and extended mean value theorems.

Also it was introduced a definition of (α, a)-conformable fractional integral and some properties related.

Definition 1.2. Let $\alpha \in (0, 1)$ and $0 \le a < b$. A function $f : [a, b] \to R$ is (α, a)-conformable fractional integrable on $[a, b]$ if the integral

$$\int_a^b f(x) d_\alpha^a x = \int_a^b \frac{x^\alpha - a}{x - a} f(x) \mathrm{d}x \tag{2}$$

exists and is finite. The set of all (α, a)-conformable fractional integrable functions is denoted by $L^1_{(\alpha,a)}([a, b])$. The (α, a)-conformable fractional integral operator is defined by

$$\mathcal{I}^{(\alpha,a)} f(t) = \int_a^t f(x) d_\alpha^a x = \int_a^t \frac{x^\alpha - a}{x - a} f(x) \mathrm{d}x,$$

where the integral is the usual Riemann improper integral. When the lower bound of the integral is any number $c > a$ then we use the notation

$$\mathcal{I}_c^{(\alpha,a)}f(t)=\int_c^t f(x)d_a^\alpha x=\int_c^t \frac{x^\alpha-a}{x-a}f(x)\mathrm{d}x.$$

It was observed [35, Theorem 3.1 and 3.2] that

$$D_a^\alpha(\mathcal{I}^{(\alpha,a)}f)(t)=f(t)$$

and

$$\mathcal{I}^{(\alpha,a)}(D_a^\alpha f)(t)=f(t)-f(a).$$

Theorem 1.3. *Let $\alpha\in(0,1]$ and $0\le a<b$. Let $f,g:[a,b]\to\mathbb{R}$ be continuous functions. Then*
(1) $\mathcal{I}^{(\alpha,a)}(\lambda f\pm\gamma g)(t)=\lambda\mathcal{I}^{(\alpha,a)}f(t)\pm\gamma\mathcal{I}^{(\alpha,a)}g(t)$ *for* $\lambda,\gamma\in\mathbb{R}$,
(2) $\mathcal{I}^{(\alpha,a)}f(a)=0$,
(3) *if* $f(t)\ge 0$ *for all* $t\in[a,b]$, *then* $\mathcal{I}^{(\alpha,a)}f(b)\ge 0$,
(4) $\mathcal{I}^{(\alpha,a)}f(b)=\mathcal{I}^{(\alpha,a)}f(c)+\mathcal{I}_c^{(\alpha,a)}f(b)$ *for any* $c\in(a,b)$,
(5) $\int_b^a f(x)d_a^\alpha x=-\mathcal{I}^{(\alpha,a)}f(b)$,
(6) $|\mathcal{I}^{(\alpha,a)}f(b)|\le\mathcal{I}^{(\alpha,a)}|f|(b)$ *for* $x^\alpha>a$.

2 Main results

Next, we give the following definition of an (α,a)-Wronskian and (α,a)-conformable partial derivative.

Definition 2.1. Let f,g be two (α,a)-differentiable functions on $[a,b]$ with $\alpha\in(0,1]$. Then we set the function:

$$W_a^\alpha(f,g)(t)=f(t)D_a^\alpha g(t)-g(t)D_a^\alpha f(t).$$

Definition 2.2. Let $f:D\subset\mathbb{R}^n\to\mathbb{R}$ be a real valued function defined on an open set $D\subset\mathbb{R}^n$ and $\mathbf{c}=(c_1,\cdots,c_n)\in D$. If the following limit exists

$$\lim_{\varepsilon\to 0}\frac{f(c_1,\cdots,c_i+\varepsilon c_i^{-\alpha}(c_i-a_i),\cdots,c_n)-f(c_1,\cdots,c_n)}{\varepsilon(1-a_ic_i^{1-\alpha})},$$

then it is denoted by

$$\frac{\partial^{(\alpha,a)}f}{\partial t_i^\alpha}(\mathbf{c})$$

and is called the (α,a)-conformable fractional partial derivative of f at $\mathbf{c}$.

2.1 Generalized fractional conformable Sturm-Picone's theorem

We will focus on the following second-order fractional differential equation given by

$$D_a^\alpha(D_a^\alpha f(t))+p(t)D_a^\alpha f(t)+q(t)f(t)=0, \tag{3}$$

where p and q are continuous functions, $\alpha\in(0,1]$. Let us remember that two functions φ_1 and φ_2 are linearly dependent if there exists $c_1,c_2\in\mathbb{R}$ with $|c_1|+|c_2|>0$ such that $c_1\varphi_1+c_2\varphi_2\equiv 0$; otherwise, they are linearly independent.

Lemma 2.1. *Let φ_1 and φ_2 be two solutions of the fractional differential equation* (3) *in the interval I and $W_a^\alpha(\varphi_1, \varphi_2)$ is a differentiable function on I. Then the fractional (α, a)-Wronskian of φ_1, φ_2 of order $\alpha \in (0, 1]$ has the form*

$$W_a^\alpha(\varphi_1, \varphi_2)(t) = e^{-\int_{t_0}^t p(s) d_a^\alpha s} W_a^\alpha(\varphi_1, \varphi_2)(t_0)$$

for all $t_0 \in I$.

Proof. Let φ_1 and φ_2 be two solutions of (3), and some $t_0 \in [a, b]$. Then, by an application of the operator D_a^α to $W_a^\alpha(\varphi_1, \varphi_2)$, we obtain

$$\begin{aligned} D_a^\alpha(W_a^\alpha(\varphi_1, \varphi_2))(t) &= D_a^\alpha(\varphi_1(t) D_a^\alpha \varphi_2(t) - \varphi_2(t) D_a^\alpha \varphi_1(t)) \\ &= D_a^\alpha \varphi_1(t) D_a^\alpha \varphi_2(t) + \varphi_1(t) D_a^\alpha(D_a^\alpha \varphi_2(t)) - D_a^\alpha \varphi_2(t) D_a^\alpha \varphi_1(t) - \varphi_2(t) D_a^\alpha(D_a^\alpha \varphi_1(t)) \\ &= \varphi_1(t) D_a^\alpha(D_a^\alpha \varphi_2(t)) - \varphi_2(t) D_a^\alpha(D_a^\alpha \varphi_1(t)). \end{aligned}$$

By using (3), we have

$$D_a^\alpha(D_a^\alpha \varphi_1(t)) = -p(t) D_a^\alpha \varphi_1(t) - q(t)\varphi_1(t)$$

and

$$D_a^\alpha(D_a^\alpha \varphi_2(t)) = -p(t) D_a^\alpha \varphi_2(t) - q(t)\varphi_2(t).$$

Therefore,

$$\begin{aligned} D_a^\alpha(W_a^\alpha(\varphi_1, \varphi_2))(t) &= \varphi_1(t)(-p(t) D_a^\alpha \varphi_2(t) - q(t)\varphi_2(t)) - \varphi_2(t)(-p(t) D_a^\alpha \varphi_1(t) - q(t)\varphi_1(t)) \\ &= -\varphi_1(t) p(t) D_a^\alpha \varphi_2(t) + \varphi_2(t) p(t) D_a^\alpha \varphi_1(t) \\ &= -p(t) W_a^\alpha(\varphi_1, \varphi_2)(t), \end{aligned}$$

i.e.,

$$\frac{D_a^\alpha(W_a^\alpha(\varphi_1, \varphi_2))(t)}{W_a^\alpha(\varphi_1, \varphi_2)(t)} = -p(t).$$

By using the fact of $D_a^\alpha f(t) = \frac{t-a}{t^\alpha - a} f'(t)$, then we have

$$\frac{t-a}{t^\alpha - a} \frac{(W_a^\alpha)'}{(\varphi_1, \varphi_2)(t)} W_a^\alpha(\varphi_1, \varphi_2)(t) = -p(t),$$

therefore,

$$\begin{aligned} \ln\left(\frac{W_a^\alpha(\varphi_1, \varphi_2)(t)}{W_a^\alpha(\varphi_1, \varphi_2)(t_0)}\right) &= \int_{t_0}^t \frac{-(s^\alpha - a)p(s)}{s-a} \mathrm{ds} \\ &\Rightarrow W_a^\alpha(\varphi_1, \varphi_2)(t) = e^{-\int_{t_0}^t \frac{(s^\alpha - a)p(s)}{s-a}\mathrm{ds}} W_a^\alpha(\varphi_1, \varphi_2)(t_0) \\ &\Rightarrow W_a^\alpha(\varphi_1, \varphi_2)(t) = e^{-\int_{t_0}^t p(s) d_a^\alpha s} W_a^\alpha(\varphi_1, \varphi_2)(t_0) \square \end{aligned}$$

□

The following equivalent condition of linear independence can be obtained from Lemma 2.1 using the classical method.

Theorem 2.1. *Two solutions φ_1 and φ_2 of the fractional differential equation* (3) *defined on an interval I are linearly independent if and only if $W_a^\alpha(\varphi_1, \varphi_2)(t) \neq 0$ for all $t \in I$.*

To continue this study, we introduce the following self-adjoint fractional differential equation of Sturm-Liouville-type:

$$-D_a^\alpha[p_1(t)D_a^\alpha x(t)] + p_0(t)x(t) = 0, \tag{4}$$

$$-D_a^\alpha[q_1(t)D_a^\alpha y(t)] + q_0(t)y(t) = 0, \tag{5}$$

where $p_1, p_0, q_1, q_0, D_a^\alpha x$, and $D_a^\alpha y$ are continuous functions on some closed interval $I \subset [0, +\infty)$, and p_1 and q_1 are positive on I.

Theorem 2.2. *If x, y, and $p_1(t)D_a^\alpha x(t)$, $q_1(t)D_a^\alpha y(t)$ are D_a^α-differentiable for $t \in I$ and $y(t) \neq 0$, then we obtain*

$$\begin{aligned} &D_a^\alpha\left[\frac{x(t)}{y(t)}(p_1(t)y(t)D_a^\alpha x(t) - q_1(t)x(t)D_a^\alpha y(t))\right] \\ &= x(t)D_a^\alpha(p_1(t)D_a^\alpha x(t)) - \frac{x^2(t)}{y(t)}D_a^\alpha(q_1(t)D_a^\alpha y(t)) + (p_1(t) - q_1(t))(D_a^\alpha x(t))^2 + q_1(t)\left(D_a^\alpha x(t) - \frac{x(t)}{y(t)}D_a^\alpha y(t)\right)^2. \end{aligned}$$

Proof. After a straightforward D_a^α-differentiation, it follows the desired result. □

Theorem 2.3. *Let a and b with $0 \leq a < b$ be two consecutive zeroes of a nontrivial solution $\varphi(t)$ of (4). Suppose that*

$$(i) \quad 0 < q_1(t) \leq p_1(t) \quad \text{and} \quad (ii) \quad q_0(t) \leq p_0(t)$$

for all $t \in [a, b]$. Then, every solution $\chi(t)$ of (5) has at least one zero in $[a, b]$.

Proof. If $\varphi(t)$ and $\chi(t)$ are solutions of (4) and (5), respectively, and $\chi(t) \neq 0$ for all. Then by substitution of these solutions and an applications of the algebraic properties of D_a^α, we have the Picone's identity

$$\begin{aligned} D_a^\alpha\left[\frac{\varphi(t)}{\chi(t)}(p_1(t)\chi(t)D_a^\alpha\varphi(t) - q_1(t)\varphi(t)D_a^\alpha\chi(t))\right] &= (p_0(t) - q_0(t))(x(t))^2 + (p_1(t) - q_1(t))(D_a^\alpha\varphi(t))^2 \\ &+ q_1(t)\left(D_a^\alpha\varphi(t) - \frac{\varphi(t)}{\chi(t)}D_a^\alpha\chi(t)\right)^2. \end{aligned} \tag{6}$$

Then, by taking the (α, a)-integrating over $[a, b]$, we have

$$\begin{aligned} &\int_a^b\left[(p_0(t) - q_0(t))(\varphi(t))^2 + (p_1(t) - q_1(t))(D_a^\alpha\varphi(t))^2 + q_1(t)\left(D_a^\alpha\varphi(t) - \frac{\varphi(t)}{\chi(t)}D_a^\alpha\chi(t)\right)^2\right]d_a^\alpha t \\ &= \frac{\varphi(t)}{\chi(t)}(p_1(t)\chi(t)D_a^\alpha\varphi(t) - q_1(t)\varphi(t)D_a^\alpha\chi(t))\Big|_a^b. \end{aligned} \tag{7}$$

Since $\varphi(a) = \varphi(b) = 0$ and $\chi(t) \neq 0$ in $[a, b]$, then the right-hand side side of 7 equals to zero. Also, since $q_1(t) > 0$, then the third term in the integral is nonnegative, so we must have either

$$(i) \quad D_a^\alpha\varphi(t) - \frac{\varphi(t)}{\chi(t)}D_a^\alpha\chi(t) \equiv 0$$

or

$$(ii) \quad \int_a^b\left[\varphi(t)D_a^\alpha(p_1(t)D_a^\alpha\varphi(t)) - \frac{\varphi^2(t)}{\chi(t)}D_a^\alpha(q_1(t)D_a^\alpha\chi(t)) + (p_1(t) - q_1(t))(D_a^\alpha\varphi(t))^2\right]d_a^\alpha t < 0.$$

In case (ii), we have contradiction because $q_1(t) \leq p_1(t)$ and $q_0(t) \leq p_0(t)$. From case (i), we observe that

$$D_a^\alpha\varphi(t) - \frac{\varphi(t)}{\chi(t)}D_a^\alpha\chi(t) = 0$$

implies that

$$\chi(t)D_a^\alpha\varphi(t) - \varphi(t)D_a^\alpha(t) = 0.$$

This means that $\varphi(t) = k\chi(t)$ for some $k \neq 0$ on $[a, b]$, which implies that $\chi(a) = \chi(b) = 0$; thus, obtaining a contradiction. □

Theorem 2.4. *Let* $0 \le a < b$ *be two consecutive zeros of a nontrivial solution* $\varphi(t)$ *of equation* (4)*. Let* $\chi(t)$ *be any other solution of equation* (4)*, which is linearly independent of* $\varphi(t)$*. Then,* $\chi(t)$ *has exactly one zero in the interval* (a, b)*. In other words, the zeros of any two linearly independent solutions of* (4) *are interlaced.*

Proof. Suppose that $\chi(t) \neq 0$ for all $t \in (a, b)$. Since φ and χ are linearly independent, we have that $\chi(a) \neq 0$, and otherwise, we would have

$$W_a^\alpha(\varphi, \chi)(t) = \varphi(t)D_a^\alpha\chi(t) - \chi(t)D_a^\alpha\varphi(t) = 0,$$

and therefore, φ and χ would be linearly dependent, contrary to our supposition. For the same reason, $\chi(d) \neq 0$.

If $q_1(t) \equiv p_1(t)$ and $q_0(t) \equiv p_0(t)$ from (6), we have

$$\int_a^b p_1(t)\left(D_a^\alpha\varphi(t) - \frac{\varphi(t)}{\chi(t)}D_a^\alpha\chi(t)\right)^2 d_a^\alpha t = \frac{\varphi(t)}{\chi(t)}(p_1(t)\chi(t)D_a^\alpha\varphi(t) - q_1(t)\varphi(t)D_a^\alpha\chi(t))\Big|_a^b. \tag{8}$$

Since a and b are zeroes of φ, $\chi(a) \neq 0$, and $\chi(b) \neq 0$, then the right-hand side of (8) evaluates to zero. Also we have that $p_1(t) > 0$ and the kernel $(t^\alpha - a)/(t - a) > 0$, then it must be that

$$D_a^\alpha\varphi(t) - \frac{\varphi(t)}{\chi(t)}D_a^\alpha\chi(t) \equiv 0$$

for all $t \in [a, b]$, from which we obtain that

$$W_a^\alpha(\varphi, \chi)(t) \equiv 0 \quad \text{for all } t \in [a, b].$$

Hence, φ and χ are be linearly dependent on (a, b) contrary to the supposition. □

2.2 Green's function study

In this section, we consider the conformable Sturm-Liouville system

$$\begin{cases} D_a^\alpha(p(t)D_a^\alpha f(t)) + (\lambda\rho(t) - q(t))f(t) = 0 \\ \beta_1 f(a) + \beta_2 D_a^\alpha f(a) = 0 \\ \gamma_1 f(b) + \gamma_2 D_a^\alpha f(b) = 0, \end{cases} \tag{9}$$

with $|\beta_1| + |\beta_2| \neq 0$, $|\gamma_1| + |\gamma_2| \neq 0$, p, q, and ρ continuous functions on $[a, b]$, where $0 \le a < b$, such that $\rho(t), p(t) > 0$ for all $t \in [a, b]$.

Definition 2.3. Let Q denote the square $[a, b] \times [a, b]$ in the $t\varepsilon$-plane. A function $G^\alpha(t, \varepsilon)$ defined in Q is called a conformable Green's function of the Sturm-Liouville system given by (9), if it has the following properties:

(1) The function $G^\alpha(t, \varepsilon)$ is continuous in Q.
(2) Let $\varepsilon \in (a, b)$ be fixed. Then $G^\alpha(t, \varepsilon)$ has continuous (α, a)-conformable partial derivatives of first and second order with respect to the variable x, if $t \neq \varepsilon$, and it satisfies

$$\frac{\partial^{(\alpha,a)}}{\partial t^\alpha}G^\alpha(\varepsilon^+, \varepsilon) - \frac{\partial^{(\alpha,a)}}{\partial t^\alpha}G^\alpha(\varepsilon^-, \varepsilon) = -\frac{1}{p(\varepsilon)}.$$

(3) Let $\varepsilon \in (a, b)$ be fixed. Then $G^\alpha(t, \varepsilon)$ have left and right conformable partial derivatives:

$$\frac{\partial^{(\alpha,a)}}{\partial t^\alpha}(p(t)D_a^\alpha G^\alpha(t,\varepsilon)) + (\lambda\rho(t) - q(t))G^\alpha(t,\varepsilon) = 0.$$

(4) Let $\varepsilon \in (a, b)$ be fixed. Then $G^\alpha(t, \varepsilon)$ satisfies the initial conditions in (9).

Lemma 2.2. *Let φ_1 and φ_2 be two solutions of* (9) *that verify the first initial condition. Then, φ_1 and φ_2 are linearly dependent.*

Proof. Since $|\beta_1| + |\beta_2| \neq 0$, we have

$$\beta_1\varphi_1(a) + \beta_2 D_a^\alpha\varphi_1(a) = 0,$$
$$\beta_1\varphi_2(a) + \beta_2 D_a^\alpha\varphi_2(a) = 0,$$

and therefore, $W_a^\alpha(\varphi_1, \varphi_2) = 0$. □

Lemma 2.3. *Let φ_1 and φ_2 be two solutions of* (9) *that verify the second condition. Then, φ_1 and φ_2 are linearly dependent.*

Proof. Similar to the proof of Lemma 2.2. □

Theorem 2.5. *The system given by* (9) *has no Green's function if λ is an eigenvalue.*

Proof. Let φ_1 an eigenfunction of the system given by (9). Let φ_2 be a solution of the fractional differential equation linearly independent of φ_1. From Lemmas 2.2 and 2.3 we have that φ_2 does not satisfy the initial conditions in the system.

We know that $G^\alpha(t, \varepsilon)$ satisfy the fractional differential equation in (9) over the intervals $[a, \varepsilon)$ and $(\varepsilon, b]$, and so, it has the form

$$G^\alpha(t,\varepsilon) = \begin{cases} A_1(\varepsilon)\varphi_1(t) + A_2(\varepsilon)\varphi_2(t), & t \in [a, \varepsilon) \\ B_1(\varepsilon)\varphi_1(t) + B_2(\varepsilon)\varphi_2(t), & t \in (\varepsilon, b], \end{cases}$$

and also the function $G^\alpha(t, \varepsilon)$ fulfills the condition 4 in Definition 2.3, so

$$\beta_1(A_1(\varepsilon)\varphi_1(a) + A_2(\varepsilon)\varphi_2(a)) + \beta_2(A_1(\varepsilon)D_a^\alpha\varphi_1(a) + A_2(\varepsilon)D_a^\alpha\varphi_2(a)) = 0$$
$$\gamma_1(B_1(\varepsilon)\varphi_1(b) + B_2(\varepsilon)\varphi_2(b)) + \gamma_2(B_1(\varepsilon)D_a^\alpha\varphi_1(b) + B_2(\varepsilon)D_a^\alpha\varphi_2(b)) = 0.$$

Since φ_1 fulfill the initial conditions, then

$$A_2(\varepsilon)(\beta_1\varphi_2(a) + \beta_2 D_a^\alpha\varphi_2(a)) = 0$$
$$B_2(\varepsilon)(\gamma_1\varphi_2(b) + \gamma_2 D_a^\alpha\varphi_2(b)) = 0.$$

On the contrary, if

$$\beta_1\varphi_2(a) + \beta_2 D_a^\alpha\varphi_2(a) \neq 0$$
$$\gamma_1\varphi_2(b) + \gamma_2 D_a^\alpha\varphi_2(b) \neq 0,$$

so $A_2(\varepsilon) = 0$ in $[a, \varepsilon)$ and $B_2(\varepsilon) = 0$ in $(\varepsilon, b]$. From here, we can write

$$G^\alpha(t,\varepsilon) = \begin{cases} A_1(\varepsilon)\varphi_1(t), & t \in [a, \varepsilon) \\ B_1(\varepsilon)\varphi_1(t), & t \in (\varepsilon, b]. \end{cases}$$

Since $G^\alpha(t, \varepsilon)$ is a continuous function, we have

$$\lim_{t\to\varepsilon^-} G^\alpha(t,\varepsilon) = A_1(\varepsilon)\varphi_1(\varepsilon) = \lim_{t\to\varepsilon^+} G^\alpha(t,\varepsilon) = B_1(\varepsilon)\varphi_1(\varepsilon),$$

which implies that $A_1(\varepsilon) = B_1(\varepsilon)$ in (c, d); therefore,

$$\frac{\partial^{(\alpha,a)}}{\partial t^\alpha} G^\alpha(\varepsilon^+, \varepsilon) - \frac{\partial^{(\alpha,a)}}{\partial t^\alpha} G^\alpha(\varepsilon^-, \varepsilon) = 0,$$

which contradicts condition 2 in Definition 2.3. □

Theorem 2.6. *System given by* (9) *has one and only one Green's Function if λ is not an eigenvalue.*

Proof. Let φ_1 and φ_2 be two solutions of the considered system such that

$$\varphi_1(a) = \beta_2, \quad D_a^\alpha \varphi_1(a) = -\beta_1, \quad \varphi_2(b) = \gamma_2, \quad D_a^\alpha \varphi_2(b) = -\gamma_1.$$

Since $|\beta_1| + |\beta_2| \neq 0$, $|\gamma_1| + |\gamma_2| \neq 0$, $\varphi_1(t)$, and $\varphi_2(t)$ are no null, they also satisfy the initial conditions, respectively.

These solutions are linearly independent, since otherwise it would be

$$\varphi_1(t) = \delta \varphi_2(t), \quad \text{for some } \delta \neq 0.$$

Therefore, we have

$$\gamma_1 \varphi_1(b) + \gamma_2 D_a^\alpha \varphi_1(b) = \delta(\gamma_1 \varphi_2(b) + \gamma_2 D_a^\alpha \varphi_2(b)) = 0,$$

which would imply that φ_1 fulfills the initial conditions, but this is not possible because φ_1 is not an eigenfunction.

Reasoning as in the proof of Theorem 2.5, we have that

$$G^\alpha(t, \varepsilon) = \begin{cases} A_1(\varepsilon)\varphi_1(t) + A_2(\varepsilon)\varphi_2(t), & t \in [a, \varepsilon) \\ B_1(\varepsilon)\varphi_1(t) + B_2(\varepsilon)\varphi_2(t), & t \in (\varepsilon, b], \end{cases}$$

and knowing that $G^\alpha(t, \varepsilon)$ fulfill the condition 4 in Definition 2.3, it follows that

$$\begin{aligned} \beta_1(A_1(\varepsilon)\varphi_1(a) + A_2(\varepsilon)\varphi_2(a)) + \beta_2(A_1(\varepsilon)D_a^\alpha\varphi_1(a) + A_2(\varepsilon)D_a^\alpha\varphi_2(a)) = 0, \\ \gamma_1(B_1(\varepsilon)\varphi_1(b) + B_2(\varepsilon)\varphi_2(b)) + \gamma_2(B_1(\varepsilon)D_a^\alpha\varphi_1(b) + B_2(\varepsilon)D_a^\alpha\varphi_2(b)) = 0, \end{aligned}$$

and it can be reduced to

$$\begin{aligned} A_2(\varepsilon)(\beta_1\varphi_2(a) + \beta_2 D_a^\alpha \varphi_2(a)) = 0, \\ B_1(\varepsilon)(\gamma_1\varphi_1(b) + \gamma_2 D_a^\alpha \varphi_1(b)) = 0, \end{aligned}$$

and since φ_1 and φ_2 are not eigenfunctions, we have that

$$\begin{aligned} \beta_1\varphi_2(a) + \beta_2 D_a^\alpha \varphi_2(a) \neq 0, \\ \gamma_1\varphi_1(b) + \gamma_2 D_a^\alpha \varphi_1(b) \neq 0, \end{aligned}$$

and then $A_2(\varepsilon) = 0$ and $B_1(\varepsilon) = 0$ in (a, b).

By conditions 1 and 2 in Definition 2.3, we have

$$\begin{aligned} A_1(\varepsilon)\varphi_1(\varepsilon) + B_2(\varepsilon)\varphi_2(\varepsilon) = 0, \\ A_1(\varepsilon)D_a^\alpha\varphi_1(\varepsilon) + B_2(\varepsilon)D_a^\alpha\varphi_2(\varepsilon) = \frac{1}{p(\varepsilon)}, \end{aligned}$$

which allows us to calculate the following:

$$\begin{aligned} A_1(\varepsilon) = \frac{-\varphi_2(\varepsilon)}{p(\varepsilon)[\varphi_1(\varepsilon)D_a^\alpha\varphi_2(\varepsilon) - \varphi_2(\varepsilon)D_a^\alpha\varphi_1(\varepsilon)]}, \\ B_2(\varepsilon) = \frac{\varphi_1(\varepsilon)}{p(\varepsilon)[\varphi_1(\varepsilon)D_a^\alpha\varphi_2(\varepsilon) - \varphi_2(\varepsilon)D_a^\alpha\varphi_1(\varepsilon)]}. \end{aligned}$$

Note that the expression $\varphi_1(\varepsilon)D_a^\alpha\varphi_2(\varepsilon) - \varphi_2(\varepsilon)D_a^\alpha\varphi_1(\varepsilon)$ is the α-Wronskian of two linearly independent solutions of (9), so it is not zero.

Now, given the following

$$D_a^\alpha(p(t)D_a^\alpha\varphi_1(t)) + (\lambda\rho(t) - q(t))\varphi_1(t) = 0,$$
$$D_a^\alpha(p(t)D_a^\alpha\varphi_2(t)) + (\lambda\rho(t) - q(t))\varphi_2(t) = 0,$$

by multiplying the first equation by φ_2, the second by φ_1, and subtracting, we have

$$\varphi_2(t)D_a^\alpha(p(t)D_a^\alpha\varphi_1(t)) - \varphi_1(t)D_a^\alpha(p(t)D_a^\alpha\varphi_2(t)) = 0. \tag{10}$$

Note that

$$\varphi_2(t)D_a^\alpha(p(t)D_a^\alpha\varphi_1(t)) + (p(t)D_a^\alpha\varphi_1)D_a^\alpha\varphi_2 t - (p(t)D_a^\alpha\varphi_1)D_a^\alpha\varphi_2 t - \varphi_1(t)D_a^\alpha(p(t)D_a^\alpha\varphi_2(t)) + (p(t)D_a^\alpha\varphi_2)D_a^\alpha\varphi_1 t - (p(t)D_a^\alpha\varphi_2)D_a^\alpha\varphi_2 t = 0,$$

it follows that

$$D_a^\alpha(p(t)\varphi_2 D_a^\alpha(t)\varphi_1(t)) - D_a^\alpha(p(t)\varphi_1 D_a^\alpha(t)\varphi_2(t)) = 0,$$

so,

$$D_a^\alpha(p(t)\varphi_2 D_a^\alpha(t)\varphi_1(t) - p(t)\varphi_1 D_a^\alpha(t)\varphi_2(t)) = 0,$$

and hence, $p(\varepsilon)[\varphi_2(\varepsilon)D_a^\alpha\varphi_1(\varepsilon) - \varphi_1(\varepsilon)D_a^\alpha\varphi_2(\varepsilon)]$ is a constant K that does not depend on ε. Then we can define

$$G^\alpha(x, y) = \begin{cases} \frac{1}{K}\varphi_1(t)\varphi_2(\varepsilon), & a \le t < \varepsilon \\ \frac{1}{K}\varphi_1(\varepsilon)\varphi_2(t), & \varepsilon < t \le b. \end{cases}$$

This conformable Green's function satisfies the conditions 1–4 in Definition 2.3. The uniqueness of this function is easily deduced from the method that we have followed to determine $G^\alpha(x, y)$. □

2.3 The applicability of conformable Green's function

In this section, we consider the system

$$\begin{cases} D_a^\alpha(p(t)D_a^\alpha f(t)) - q(t)f(t) = 0 \\ \beta_1 f(a) + \beta_2 D_a^\alpha f(a) = 0 \\ \gamma_1 f(b) + \gamma_2 D_a^\alpha f(b) = 0 \end{cases} \tag{11}$$

obtained from (9) for $\lambda = 0$. We now propose to solve the nonhomogeneous system:

$$\begin{cases} D_a^\alpha(p(t)D_a^\alpha f(t)) - q(t)f(t) = -h(t) \\ \beta_1 f(a) + \beta_2 D_a^\alpha f(a) = 0 \\ \gamma_1 f(b) + \gamma_2 D_a^\alpha f(b) = 0, \end{cases} \tag{12}$$

where $h(t)$ is a real continuous function in the interval $[a, b]$ for some $0 \le a < b$.

Theorem 2.7. *If the given homogeneous system* (11) *has the identically null function as its only solution, then the system given by* (12) *has only one solution, which is given by*

$$f(t) = \mathcal{I}_{(\alpha,a)}(G^\alpha(t, \cdot)h)(b) = \int_a^b \frac{(\varepsilon^\alpha - a)}{\varepsilon - a} G^\alpha(t, \varepsilon)h(\varepsilon)\mathrm{d}\varepsilon,$$

where $G^\alpha(t, \varepsilon)$ is the conformable Green's function of (11).

Proof. Since the homogeneous system (11) has the identically null function as its only solution, then $\lambda = 0$ is not an eigenvalue of (9); therefore, there exists the conformable Green's function of (11).

Let φ_1 and φ_2 be two linearly independent solutions of (11) that verify the initial conditions, respectively. Let us apply the generalized conformable version of the method of variation of the parameters to solve the fractional differential equation in (11). Then, with

$$f(t) = A(t)\varphi_1(t) + B(t)\varphi_2(t),$$

we have

$$D_a^\alpha(p(t)D_a^\alpha(A(t)\varphi_1(t) + B(t)\varphi_2(t))) - q(t)(A(t)\varphi_1(t) + B(t)\varphi_2(t)) = -h(t),$$

and by applying the internal fractional operator, we obtain

$$D_a^\alpha(p(t)[\varphi_1(t)D_a^\alpha A(t) + A(t)D_a^\alpha\varphi_1(t) + \varphi_2 D_a^\alpha B(t) + B(t)D_a^\alpha\varphi_2(t)]) - q(t)(A(t)\varphi_1(t) + B(t)\varphi_2(t)) = -h(t),$$

now, by using the linearity property of the fractional differential operator, we obtain

$$\begin{aligned} &D_a^\alpha(p(t)[\varphi_1(t)D_a^\alpha A(t) + \varphi_2 D_a^\alpha B(t)]) + D_a^\alpha(p(t)A(t)D_a^\alpha\varphi_1(t)) + D_a^\alpha(B(t)D_a^\alpha\varphi_2(t)) - q(t)(A(t)\varphi_1(t) + B(t)\varphi_2(t)) \\ &\quad = -h(t), \end{aligned}$$

and if we apply the fractional differential's product rule to the second and third term, we obtain

$$\begin{aligned} &D_a^\alpha(p(t)[\varphi_1(t)D_a^\alpha A(t) + \varphi_2 D_a^\alpha B(t)]) + A(t)D_a^\alpha(p(t)D_a^\alpha\varphi_1(t)) + p(t)D_a^\alpha\varphi_1(t)D_a^\alpha A(t) + B(t)D_a^\alpha(p(t)D_a^\alpha\varphi_2(t)) \\ &\quad + p(t)D_a^\alpha\varphi_2(t)D_a^\alpha B(t) - q(t)(A(t)\varphi_1(t) + B(t)\varphi_2(t)) = -h(t), \end{aligned}$$

that is, to say

$$\begin{aligned} &A(t)(D_a^\alpha(p(t)D_a^\alpha\varphi_1(t)) - q(t)\varphi_1(t)) + B(t)(D_a^\alpha(p(t)D_a^\alpha\varphi_2(t)) - q(t)\varphi_2(t)) \\ &\quad + p(t)(D_a^\alpha A(t)D_a^\alpha\varphi_1(t) + D_a^\alpha B(t)D_a^\alpha\varphi_2(t)) + D_a^\alpha(p(t)(\varphi_1(t)D_a^\alpha A(t) + \varphi_2(t)D_a^\alpha B(t))) = -h(t). \end{aligned}$$

Since φ_1 and φ_2 are two linearly independent solutions of (11), it follows that

$$p(t)(D_a^\alpha A(t)D_a^\alpha\varphi_1(t) + D_a^\alpha B(t)D_a^\alpha\varphi_2(t)) + D_a^\alpha(p(t)(\varphi_1(t)D_a^\alpha A(t) + \varphi_2(t)D_a^\alpha B(t))) = -h(t).$$

From

$$\varphi_1(t)D_a^\alpha A(t) + \varphi_2(t)D_a^\alpha B(t) = 0,$$

we have

$$p(t)(D_a^\alpha A(t)D_a^\alpha\varphi_1(t) + D_a^\alpha B(t)D_a^\alpha\varphi_2(t)) = -h(t),$$

so

$$\begin{aligned} D_a^\alpha A(t) &= \frac{-\varphi_2(t)h(t)}{p(t)[\varphi_2(t)D_a^\alpha\varphi_1(t) - \varphi_1(t)D_a^\alpha\varphi_2(t)]} \\ D_a^\alpha B(t) &= \frac{-\varphi_1(t)h(t)}{p(t)[\varphi_1(t)D_a^\alpha\varphi_2(t) - \varphi_2(t)D_a^\alpha\varphi_1(t)]}. \end{aligned}$$

We know, from the proof of Theorem 2.6, that $p(t)[\varphi_1(t)D_a^\alpha\varphi_2(t) - \varphi_2(t)D_a^\alpha\varphi_1(t)]$ is a constant, and it is equal to K.

Also, by using the initial conditions, we have

$$\begin{aligned} \beta_1 f(a) + \beta_2 D_a^\alpha f(a) &= \beta_1(A(a)\varphi_1(a) + B(a)\varphi_2(a)) + \beta_2(\varphi_1(c)D_a^\alpha A(a) + \varphi_2(a)D_a^\alpha B(a) + A(c)D_a^\alpha\varphi_1(a) \\ &\quad + B(a)D_a^\alpha\varphi_2(a)) \\ &= A(a)(\beta_1\varphi_1(a) + \beta_2 D_a^\alpha\varphi_1(a)) + B(a)(\beta_1\varphi_2(a) + \beta_2 D_a^\alpha\varphi_2(a)) \\ &= B(a)(\beta_1\varphi_2(a) + \beta_2 D_a^\alpha\varphi_2(a)) = 0, \end{aligned}$$

and since φ_2 is not an eigenfunction of (11), it turns out that

$$\beta_1\varphi_2(a) + \beta_2 D_a^\alpha\varphi_2(a) \neq 0,$$

it follows that $B(a) = 0$.

Similarly, if $\gamma_1 f(b) + \gamma_2 D_a^\alpha f(b) = 0$, then we obtain that $A(b) = 0$.

So we have

$$A(t) = -\int_a^t \frac{(\varepsilon^\alpha - a)}{\varepsilon - a}\frac{\varphi_2(\varepsilon)h(\varepsilon)}{K}\mathrm{d}\varepsilon$$

and since $A(b) = 0$, then

$$A(t) = -\int_a^t \frac{(\varepsilon^\alpha - a)}{\varepsilon - a}\frac{\varphi_2(\varepsilon)h(\varepsilon)}{K}\mathrm{d}\varepsilon + \int_a^b \frac{(\varepsilon^\alpha - a)}{\varepsilon - a}\frac{\varphi_2(\varepsilon)h(\varepsilon)}{K}\mathrm{d}\varepsilon = \int_t^b \frac{(\varepsilon^\alpha - a)}{\varepsilon - a}\frac{\varphi_2(\varepsilon)h(\varepsilon)}{K}\mathrm{d}\varepsilon.$$

Analogously

$$B(t) = \int_a^t \frac{(\varepsilon^\alpha - a)}{\varepsilon - a}\frac{\varphi_1(\varepsilon)h(\varepsilon)}{K}\mathrm{d}\varepsilon.$$

Thus, we obtain that

$$\begin{aligned} f(t) &= A(t)\varphi_1(t) + B(t)\varphi_2(t) \\ &= \int_t^b \frac{(\varepsilon^\alpha - a)}{\varepsilon - a}\frac{\varphi_2(\varepsilon)\varphi_1(t)h(\varepsilon)}{K}\mathrm{d}\varepsilon + \int_a^t \frac{(\varepsilon^\alpha - a)}{\varepsilon - a}\frac{\varphi_1(\varepsilon)h(\varepsilon)\varphi_2(t)}{K}\mathrm{d}\varepsilon \\ &= \mathcal{I}_{a+}(G^\alpha(t,\varepsilon)f)(b), \end{aligned}$$

where the Green's function is

$$G^\alpha(t,\varepsilon) = \begin{cases} \frac{1}{K}\varphi_1(\varepsilon)\varphi_2(t), & a \le \varepsilon < t \\ \frac{1}{K}\varphi_1(t)\varphi_2(\varepsilon), & t < \varepsilon \le b. \end{cases}$$

□

Finally, we investigate the generalized Hyers-Ulam stability of the conformable linear nonhomogeneous differential equation of second order (12) in the class of twice continuously D_a^α-differentiable functions.

Theorem 2.8. *Let $p, q : [a,b] \to R$ be continuous functions and let p be D_a^α-differentiable function on $[a,b]$. Assume that the conformable homogeneous differential equation in* (11) *has the only null solution. If a twice continuously D_a^α-differentiable function $f : [a,b] \to R$ satisfies the inequality*

$$|D_a^\alpha(p(t)D_a^\alpha f(t)) - q(t)f(t) + f(t)| \le g(t) \tag{13}$$

for all $t \in [a,b]$, where $g : [a,b] \to [0,\infty)$ is given such that of the following integrals exists, then there exists a solution $f_0 : [a,b] \to R$ of (12) *such that*

$$|f(t) - f_0(t)| \le \frac{1}{|K|}\left(|\varphi_1(t)|\int_t^b \frac{(\varepsilon^\alpha - a)}{\varepsilon - a}\frac{|\varphi_2(\varepsilon)|g(\varepsilon)}{K}\mathrm{d}\varepsilon + |\varphi_2(t)|\int_c^t \frac{(\varepsilon^\alpha - a)}{\varepsilon - a}\frac{|\varphi_1(\varepsilon)|g(\varepsilon)}{K}\mathrm{d}\varepsilon\right),$$

where K is a nonzero constant and $\varphi_1(t)$ and $\varphi_2(t)$ are two linearly independent solutions of (11) (*Theorem* 2.7)

Proof. If we define a continuous function $s : [a,b] \to R$ by

$$s(t) = D_a^\alpha(p(t)D_a^\alpha f(t)) - q(t)f(t) \tag{14}$$

for all $t \in [a, b]$, then from (13), it follows that

$$|s(t) + f(t)| \leq g(t). \tag{15}$$

for all $t \in [a, b]$.

From Theorems 13 and 14, we have

$$f(t) = \mathcal{I}^{\alpha}_{a+}(G^{\alpha}(t, .)f)(b) = \int_t^b \frac{(\varepsilon^{\alpha} - a)}{\varepsilon - a} \frac{\varphi_2(\varepsilon)\varphi_1(t)s(\varepsilon)}{K} d\varepsilon + \int_a^t \frac{(\varepsilon^{\alpha} - a)}{\varepsilon - a} \frac{\varphi_1(\varepsilon)s(\varepsilon)\varphi_2(t)}{K} d\varepsilon, \tag{16}$$

where K is a nonzero constant and $\varphi_1(t)$ and $\varphi_2(t)$ are two linearly independent solutions of (11) .

We now define a function $f_0 : [a, b] \to R$ by

$$f_0(t) = \int_t^b \frac{(\varepsilon^{\alpha} - a)}{\varepsilon - a} \frac{\varphi_2(\varepsilon)\varphi_1(t)f(\varepsilon)}{K} d\varepsilon + \int_a^t \frac{(\varepsilon^{\alpha} - a)}{\varepsilon - a} \frac{\varphi_1(\varepsilon)f(\varepsilon)\varphi_2(t)}{K} d\varepsilon \tag{17}$$

for all $t \in [a, b]$. According to Theorem 2.7, it is obvious that f_0 is a solution of the system (12). Moreover, it follows from (15)–(17) that

$$\begin{aligned} |f(t) - f_0(t)| &\leq \left| \int_t^b \frac{(\varepsilon^{\alpha} - a)}{\varepsilon - a} \frac{\varphi_2(\varepsilon)\varphi_1(t)(s + f)(\varepsilon)}{K} d\varepsilon + \int_a^t \frac{(\varepsilon^{\alpha} - a)}{\varepsilon - a} \frac{\varphi_1(\varepsilon)(s + f)(\varepsilon)\varphi_2(t)}{K} d\varepsilon \right| \\ &\leq \frac{1}{K} \left(|\varphi_1(t)| \int_t^b \frac{(\varepsilon^{\alpha} - a)}{\varepsilon - a} \frac{|\varphi_2(\varepsilon)| g(\varepsilon)}{K} d\varepsilon + |\varphi_2(t)| \int_a^t \frac{(\varepsilon^{\alpha} - a)}{\varepsilon - a} \frac{|\varphi_1(\varepsilon)| g(\varepsilon)}{K} d\varepsilon \right) \end{aligned}$$

for all $t \in [a, b]$. □

3 Conclusion

In the development of the present article, fractional versions of the Sturm-Picone Theorem and the study of the problem of boundary value determined with the use of a generalization of fractional derivatives introduced by Zarikaya et al. in [35] were established. In addition, the stability criterion given by the Hyers-Ulam Theorem was studied with the use of the aforementioned fractional derivatives.

Acknowledgments: The authors are gratefully to the Dirección de Investigación from Pontificia Universidad Católica del Ecuador for the technical support given to this work.

Funding information: This article received financial support from Dirección de Investigación fro Pontificia Universidad Católica del Ecuador under the project entitled “Resultados Cualitativos de ecuaciones diferenciales fraccionarias locales y desigualdades integrales,” code 070-UIO-2022.

Author contributions: All authors contributed equally to and approved the final manuscript.

Conflict of interest: The authors declare that there is no conflict of interest regarding the publications of this article.

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