Research Article

Hayati Olgar*

On completeness of weak eigenfunctions for multi-interval Sturm-Liouville equations with boundary-interface conditions

https://doi.org/10.1515/dema-2022-0210
received August 2, 2022; accepted February 9, 2023

Abstract: The goal of this study is to analyse the eigenvalues and weak eigenfunctions of a new type of multi-interval Sturm-Liouville problem (MISLP) which differs from the standard Sturm-Liouville problems (SLPs) in that the Strum-Liouville equation is defined on a finite number of non-intersecting subintervals and the boundary conditions are set not only at the endpoints but also at finite number internal points of interaction. For the self-adjoint treatment of the considered MISLP, we introduced some self-adjoint linear operators in such a way that the considered multi-interval SLPs can be interpreted as operator-pencil equation. First, we defined a concept of weak solutions (eigenfunctions) for MISLPs with interface conditions at the common ends of the subintervals. Then, we found some important properties of eigenvalues and corresponding weak eigenfunctions. In particular, we proved that the spectrum is discrete and the system of weak eigenfunctions forms a Riesz basis in appropriate Hilbert space.

Keywords: multi-interval Sturm-Liouville problems, boundary and interface conditions, weak eigenfunctions, completeness

MSC 2020: 34L10, 34L15, 34B08, 34B24

1 Introduction

One of the main sources of inspiration for differential operators and their important branch, the Sturm-Liouville theory, is boundary value problems (BVPs) for two-order ordinary differential equations that arise in various types of mathematical physics problems. The foundations of this theory were laid down by Sturm and Liouville in the mid-nineteenth century while studying heat conduction problems. The Sturm-Liouville theory, initially applied to heat conduction problems, was later found to be applicable to many concrete problems appearing in physics, biology, engineering, finance etc. and has maintained its usefulness until today. As it is known, BVPs which consist of second-order ordinary linear differential equations and self-adjoint boundary conditions are generally known as Sturm-Liouville problems (SLPs). In the classical SLPs, the boundary conditions do not depend on the eigenvalue parameter. For detailed information about classical SLPs, [1] can be consulted. Many mathematicians have studied some spectral properties (such as the basis properties, orthogonality and eigenfunction expansions) of the eigenfunctions of the classical SLPs involving spectral parameter not only in its equation but also in boundary conditions (e.g. [2,3] and references cited therein).

Some problems in different fields of applied sciences arise as singular BVPs of the Sturm-Liouville type, which include interface conditions at the internal singular points. Such problems are called boundary value

* **Corresponding author: Hayati Olgar,** Department of Mathematics, Faculty of Science and Arts, Tokat Gaziosmanpasa University, 60250 Tokat, Turkey, e-mail: hayati.olgar@gop.edu.tr

interface problems (BVIPs) of the Sturm-Liouville type. Adding interface conditions to classical SLPs presents some difficulties. The first of these difficulties is that it is not clear how to extend the classical methods to a problem with additional interface conditions. Another major difficulty lies in the completeness of the eigenfunctions, since SLPs with interface conditions may not have infinitely many eigenvalues [4–6]. SLPs under complementary interface conditions at some internal points arise as the mathematical modelling of some systems and processes in the fields of physics, engineering, chemistry, biology, quantum computing, mathematical finance, aerodynamics, electrodynamics of electrical circuits, fluid dynamics, diffusion, string theory and magnetism, etc. as a result of using the method of separation of variables (see [7–10]).

In recent years, there has also been a remarkable revival of interest in investigating the properties of SLPs with interface conditions (see e.g. [11–18] and references cited therein). In 2015, Zhang et al. investigated an SLP with interface conditions and obtained that the eigenvalues depend not only continuously but also smoothly on the coefficient functions, boundary conditions and interface conditions [19].

In this work, we shall study the weak eigenfunctions of a new type of SLP defined on finite number of non-intersecting intervals with eigenparameter-dependent boundary conditions and additional interface conditions at the common ends of these intervals. The concept of weak eigenfunctions allows us to reduce the eigenvalue problem to an operator-pencil equation (see [20,21]). The famous work of Keldysh [22] contains the first important results in the spectral theory of operator pencils in which the concepts of multiplicity of an eigenvalue-associated vectors and multiple completeness of the root vectors (i.e., eigen- and associated vectors) were introduced. In the article published in 2019, Olğar [16] proved that this operator pencil is self-adjoint and positive definite for sufficiently large negative values of the eigenparameter. Olğar et al. [23] have investigated weak eigenfunctions of a new type of many-interval BVP consisting of a two-interval Sturm-Liouville equation together with interface conditions and with eigenvalue parameter depending on boundary conditions. In these studies, they defined a new concept of the so-called weak (generalized) eigenfunction for the considered problem and proved that the spectrum is discrete and the set of weak eigenfunctions forms a Riesz basis of the suitable Hilbert space.

Let us consider the following MISL equation:

$$z''(k) + (\mu j(k) - h(k))z(k) = 0,\ k \in \cup_{s=1}^{m}(t_s, t_{s+1}) \tag{1}$$

with the eigenparameter-dependent boundary conditions, given by

$$\cos\theta z(t_1) + \sin\theta z'(t_1) = 0, \quad \theta \in [0, \pi), \quad c_1 z(t_{m+1}) - \mu c_2 z'(t_{m+1}) = 0, \tag{2}$$

and with the interface conditions at the points of interaction t_s $(s = 2, 3, \dots, m)$ given by

$$z(t_s - 0) - z(t_s + 0) = 0, \quad z'(t_s + 0) - z'(t_s - 0) = \delta_{s-1}, \quad s = 2, 3, \dots, m. \tag{3}$$

We will assume that the following assumptions are fulfilled.

Assumption 1.1.

i. The real-valued functions $h(k)$ and $j(k)$ are measurable and bounded on $\cup_{s=1}^{m}(t_s, t_{s+1})$,
ii. The function $j(k)$ is positively definite,
iii. The function $h(k)$ is continuous in the intervals (t_s, t_{s+1}), $s = 1, 2, 3, \dots, m$ and has finite limit values $h(t_1 + 0)$, $h(t_s \pm 0)$ $(s = 2, 3, \dots, m)$ and $h(t_{m+1} - 0)$,
iv. $\mu \in \mathbb{C}$ is an eigenvalue parameter,
v. $\delta_s (s = 1, 2, \dots, m - 1)$ are real numbers and $\delta_s > 0$ for each $s = 1, 2, \dots, m - 1$.

Note that the sign of the parameters δ_s $(s = 1, 2, \dots, m - 1)$ plays an essential role since the case $\delta_s > 0$ $(s = 1, 2, \dots, m - 1)$ allows us to interpret the considered problem (1)–(3) as a spectral problem for a self-adjoint operator in a suitable Hilbert space.

The main goal of this work is to provide an operator-pencil realization of MISLP (1)–(3) to show that the system of weak eigenfunctions of the considered problem forms a Riesz basis in a suitable Hilbert space.

Remark 1.2. Note that MISLPs of type (1)–(3) are of interest not only for pure mathematics but also for applied mathematics as well as for physics and engineering. MISLPs arise as a rule in solving many physical transfer problems by the separation of variable method. To show this, consider the non-homogeneous Laplace's equation

$$-\Delta U(x) = f(x), \quad x \in \oplus\Omega := \bigcup_{i=1}^{3} \Omega_i \tag{4}$$

together with boundary condition of Dirichlet type

$$U(x) = 0 \quad \text{on } \partial(\oplus\Omega) \tag{5}$$

and additional interface conditions

$$I_k(U) = 0, \quad \text{across } C_k := \overline{\Omega_k} \cap \overline{\Omega_{k+1}}, \quad k = 1, 2, \tag{6}$$

$$-I_k\left(\frac{\partial U}{\partial n}\right) = a_0(\Delta_t)_k U \quad \text{on } C_k, \quad k = 1, 2, \tag{7}$$

where Ω_i ($i = \overline{1-3}$) are bounded regular domains in $\mathbb{R}^3$, $I_k(U) = U_k - U_{k+1}$ denotes the jump of the function $U(x)$ across the section C_k, $I_k(\frac{\partial U}{\partial n})$ denotes the jump of the normal derivatives across the section C_k, $a_0 > 0$ is a constant and $(\Delta_t)_k$ denotes the tangential Laplacian on C_k. Such type of BVPs with additional interface conditions arises often from the problems of hydraulic fracturing and from some problems of electrostatics and magnetostatics (for other BVPs with interface conditions we refer to [24–26] and corresponding references cited therein). It is easy to see that the method of separation of variables applied to problem (4)–(7) leads to spectral problem of type (1)–(3).

2 Preliminaries

Let us present briefly the main definitions and results that will be used in what follows.

Definition 2.1. [27] The linear space consisting of all functions of the space $L_2(a, b)$ having generalized derivatives $f', f'', \ldots, f^{(s)} \in L_2(a, b)$ and equipped with an inner product

$$\langle z, w \rangle_{W_2^m(a,b)} := \sum_{s=0}^{m} \int_a^b z^{(s)}(k)\overline{w(k)}^{(s)} \mathrm{d}k$$

and with the corresponding norm $\|z\|^2_{W_2^m(a,b)} = \langle z, z \rangle_{W_2^m(a,b)}$ is denoted by $W_2^m(a, b)$ and is called the Sobolev space.

Here, as usual by $L_2(a, b)$ we mean the classical Lebesgue space which consists of all square-integrable functions defined on the interval $[a, b]$ and equipped with the inner product $\langle z, w \rangle_{L_2(a,b)} := \int_a^b z(k)\overline{w(k)}\mathrm{d}k$ and norm $\|z\|^2_{L_2(a,b)} = \langle z, z \rangle_{L_2(a,b)}$.

Definition 2.2. [28] A system of elements $\{z_m\}$ in a Hilbert space $\mathbb{H}$ with the property that for an arbitrary $z \in \mathbb{H}$ there is a unique sequence of scalars $\{d_s\}$, such that

$$\left\| z - \sum_{s=1}^{m} d_s z_s \right\|_{\mathbb{H}} \to 0 \quad \text{as } m \to \infty$$

is called a basis (or Schauder basis) for $\mathbb{H}$.

A basis $\{z_m\}$ in $\mathbb{H}$ is said to be an orthogonal basis if $\langle z_m, z_k\rangle_{\mathbb{H}} = 0$ for $m \neq k$ and orthonormal basis if $\langle z_m, z_k\rangle_{\mathbb{H}} = \delta_{mk}$, where δ_{mk} is the Kronecker delta, i.e. $\delta_{mk} = \begin{cases} 1, & \text{for } m = k \\ 0, & \text{for } m \neq k \end{cases}$.

Definition 2.3. [29] A basis $\{z_m\}$ in a Hilbert space $\mathbb{H}$ is called the Riesz basis of this space if the series $\sum_{m=0}^{\infty} d_m z_m$ is convergent when and only when $\sum_{m=0}^{\infty} d_m^2 < \infty$.

Obviously, any orthonormal basis forms a Riesz basis.

For investigation of the MISLPs (1)–(3), we use the direct sum space $\oplus L_2 := \oplus_{s=1}^{n} L_2(t_s, t_{s+1})$ with the inner product

$$\langle z, w\rangle_0 := \sum_{s=1}^{m} \int_{t_s+0}^{t_{s+1}-0} z(k)\overline{w(k)}\mathrm{d}k.$$

We will use the new Hilbert space $\oplus W_2^{0,1}$ corresponding to our problem (1)–(3), which is defined as follows. The linear space consisting of all elements $z \in \oplus_{s=1}^{m} W_2^1(t_s, t_{s+1})$ satisfying the interface conditions

$$z(t_s - 0) = z(t_s + 0), \quad s = 2, 3, \ldots, m \tag{8}$$

and equipped with the inner product

$$\langle z, w\rangle_1 := \sum_{s=1}^{m} \int_{t_s+0}^{t_{s+1}-0} (z'(k)\overline{w(k)}' + z(k)\overline{w(k)})\mathrm{d}k$$

and corresponding norm $\|z\|_1^2 = \langle z, z\rangle_1$ is denoted by $\oplus W_2^{0,1}$.

Taking in view Assumption 1.1 in the Hilbert space $\oplus W_2^{0,1}$ we can introduce a new, but equivalent inner product by

$$\langle z, w\rangle_{1,h} := \sum_{s=1}^{m} \int_{t_s+0}^{t_{s+1}-0} \{z'(k)\overline{w(k)}' + h(k)z(k)\overline{w(k)}\}\mathrm{d}k$$

with the corresponding norm $\|z\|_{1,h}^2 = \langle z, z\rangle_{1,h}$. Obviously,

$$\langle z, w\rangle_{1,h} := \langle z, hw\rangle_0 + \langle z', w'\rangle_0.$$

It is easy to see that there are constants $K_1 > 0$ and $K_2 > 0$, such that

$$K_1 \quad \|z\|_1 < \|z\|_{1,h} < K_2 \quad \|z\|_1$$

for all $z \in \oplus W_2^{0,1}$.

By using the well-known embedding theorems (see e.g. [21]) it is easy to show that for any $\varepsilon > 0$, small enough, the inequalities

$$|z(t_1 + 0)|^2 \leq \varepsilon||z'||_0^2 + \frac{2}{\varepsilon}||z||_0^2, \quad |z(t_s \pm 0)|^2 \leq \varepsilon||z'||_0^2 + \frac{2}{\varepsilon} \quad ||z||_0^2, \quad s = 2, 3, \ldots, m \tag{9}$$

and

$$|z(t_{m+1} - 0)|^2 \leq \varepsilon||z'||_0^2 + \frac{2}{\varepsilon}||z||_0^2$$

and

$$|z(\xi)| \leq D(\xi) \quad ||z||_1 \tag{10}$$

hold for all $z \in \oplus W_2^{0,1}$, where $\xi \in \cup_{s=1}^{m}(t_s, t_{s+1})$, $\quad D(\xi)$ is the constant independent of the function z and dependent only on ξ.

We will also use the Hilbert space $\mathfrak{H} := \oplus W_2^{0,1} \oplus \mathbb{C}$ equipped with the scalar product

$$\langle Z, W\rangle_{\mathfrak{H}} := \langle z, w\rangle_1 + z_1\overline{w_1}$$

for $Z = \begin{pmatrix} z(k) \\ z_1 \end{pmatrix}$, $W = \begin{pmatrix} w(k) \\ w_1 \end{pmatrix} \in \mathfrak{H}$, where $z(k), w(k) \in \oplus W_2^{0,1}$, $z_1, w_1 \in \mathbb{C}$.

3 Operator-polynomial realization of the MISLP (1)–(3)

Let us introduce to the consideration the concept of a weak eigenfunction which is fundamental to this work. Let $v \in \oplus W_2^{0,1}$ be an arbitrary function. Multiply equation (1) by $\bar{v}$ and then integrate by parts over (t_s, t_{s+1}) $(s = 1, 2, \dots, m)$ we find that

$$\begin{aligned}\mu \sum_{s=1}^{m} \int_{t_{s+0}}^{t_{s+1}-0} j(k)z(k)\overline{v(k)}\mathrm{d}k &= \sum_{s=1}^{m} \int_{t_{s+0}}^{t_{s+1}-0} \{-z''(k) + h(k)z(k)\}\overline{v(k)}\mathrm{d}k \\ &= \sum_{s=1}^{m} \int_{t_{s+0}}^{t_{s+1}-0} \{z'(k)\overline{v'(k)} + h(k)z(k)\overline{v(k)}\}\mathrm{d}k + z'(t_1)\overline{v(t_1)} \\ &\quad - z'(t_{m+1})\overline{v(t_{m+1})} + [z'(t_2+0)\overline{v(t_2+0)} - z'(t_2-0)\overline{v(t_2-0)}] \\ &\quad + [z'(t_3+0)\overline{v(t_3+0)} - z'(t_3-0)\overline{v(t_3-0)}] + \dots \\ &\quad + [z'(t_{m-1}+0)\overline{v(t_{m-1}+0)} - z'(t_{m-1}-0)\overline{v(t_{m-1}-0)}] \\ &\quad + [z'(t_m+0)\overline{v(t_m+0)} - z'(t_m-0)\overline{v(t_m-0)}].\end{aligned}$$

Since for functions belonging to the space $\oplus W_2^{0,1}$ the normal derivative is generally not defined, it should be excluded from this identity. To this end, we will assume that the function $v(k)$ satisfies the conditions (8).

Recalling that $z(k)$ satisfies all boundary-interface conditions (2) and (3), we obtain

$$\langle z, v\rangle_{1,h} - \cot\theta z(t_1)\overline{v(t_1)} + \sum_{s=2}^{m} \delta_{s-1} z(t_s)\overline{v(t_s)} + \frac{\kappa}{c_2}\overline{v(t_{m+1})} = \mu\langle jz, v\rangle_0, \tag{11}$$

$$\frac{z(t_{m+1})}{c_2} = \mu\frac{\kappa}{\rho}, \tag{12}$$

where $\kappa := -c_2 z'(t_{m+1})$.

Thus, the MISLP (1)–(3) is transformed into the system of equalities (11)–(12) for all terms of which are defined for the $z, v \in \oplus W_2^{0,1}$.

Based on the above transformations, we will define the concept of weak eigenfunction for the considered MISLP (1)–(3).

Definition 3.1. The vector-function $Z = \begin{pmatrix} z(k) \\ \kappa \end{pmatrix} \in \oplus W_2^{0,1}$ is said to be a weak (or generalized) eigenfunction of the MISLP (1)–(3) if the relations (11)–(12) are satisfied for any $v(k) \in \oplus W_2^{0,1}$.

Now we shall introduce to the consideration the following functionals:

$$\tau_0(z, v) := -\cot\theta z(t_1)\overline{v(t_1)} + \sum_{s=2}^{m} \delta_{s-1} z(t_s)\overline{v(t_s)}, \tag{13}$$

$$\tau_1(z, v) := \langle j(k)z(k), v(k)\rangle_0 := \sum_{s=1}^{m} \int_{t_s+0}^{t_{s+1}-0} j(k)z(k)\overline{v(k)}\mathrm{d}k, \tag{14}$$

$$\tau_2(\kappa, v) := \frac{\kappa}{c_2}\overline{v(t_{m+1})}, \tag{15}$$

where $z(k) \in \oplus W_2^{0,1}$, $\kappa(k) \in \mathbb{C}$.

The reduction of relations (11)–(12) to an operator-polynomial is based on the following theorem.

Theorem 3.2. *There are linear bounded operators* $B_0, B_1 : \oplus W_2^{0,1} \to \oplus W_2^{0,1}$ *and* $B_2 : \mathbb{C} \to \oplus W_2^{0,1}$ *such that*

$$\tau_i(z, \nu) = \langle B_i z, \nu\rangle_{1,h}, \quad i = 0, 1, \tag{16}$$
$$\tau_2(\kappa, \nu) = \langle B_2 \kappa, \nu\rangle_{1,h}$$

for all $z(k), \nu(k) \in \oplus W_2^{0,1}$ *and* $\kappa(k) \in \mathbb{C}$.

Proof. Let $z(k) \in \oplus W_2^{0,1}$ be any function. From (13) to (15), it follows immediately that

$$|\tau_0(z, \nu)| \le M_1 \left\{ |z(t_1)||\nu(t_1)| + \sum_{s=2}^{m} |z(t_s)||\nu(t_s)| \right\},$$
$$|\tau_1(z, \nu)| \le M_2 \|z\|_0 \|\nu\|_0,$$
$$|\tau_2(\kappa, \nu)| \le M_3 |\kappa||\nu(t_{m+1})|.$$

Here and below, the symbols M_j $(j = 1, 2, \dots, m)$ are used to denote different constants whose exact values are not important for the proof.

By using the interpolation inequalities (9) and (10), we have the following inequalities:

$$\|z\| \le M_4 \|z\|_{1,h}, \; |z(\xi)| \le M_5 ||z||_{1,h} \quad \text{for any } \xi \in \cup_{s=1}^{m}(t_s, t_{s+1}).$$

Hence, the functionals τ_i $(i = 0, 1, 2)$ allow the following estimates

$$|\tau_0(z, \nu)| \le M_6 \|z\|_0 \|\nu\|_0 \le M_6 M_1 \|z\|_{1,h} \|\nu\|_0 \le M_6 M_1^2 \|z\|_{1,h} \|\nu\|_{1,h},$$
$$|\tau_1(z, \nu)| \le M_7 \|z\|_0 \|\nu\|_0 \le M_7 M_2 \|z\|_0 \|\nu\|_{1,h} \le M_7 M_2^2 \|z\|_{1,h} \|\nu\|_{1,h},$$
$$|\tau_2(\kappa, \nu)| \le M_8 |\kappa||\overline{\nu(t_{m+1})}| \le M_9 |\kappa| \|\nu\|_{1,h}.$$

So the linear forms $\tau_i : \oplus W_2^{0,1} \to \oplus W_2^{0,1} (i = 0, 1)$ and $\tau_2 : \mathbb{C} \to \oplus W_2^{0,1}$ are continuous with respect to the second argument. The proof is complete. □

Theorem 3.3. *The operators* $B_0, B_1 : \oplus W_2^{0,1} \to \oplus W_2^{0,1}$ *are self-adjoint and the operator* $B_1 : \oplus W_2^{0,1} \to \oplus W_2^{0,1}$ *is positive.*

Proof. Let $z(k), \nu(k) \in \oplus W_2^{0,1}$ be arbitrary functions. By (13) and (16) we have that

$$\begin{aligned}
\langle z, B_0 \nu\rangle_{1,h} &= \overline{\langle B_0 \nu, z\rangle_{1,h}} = \overline{\tau_0(\nu, z)} \\
&= \overline{-\cot\theta \nu(t_1)\overline{z(t_1)} + \delta_1 \nu(t_2)\overline{z(t_2)} + \delta_2 \nu(t_3)\overline{z(t_3)} + \ldots + \delta_{m-1} \nu(t_m)\overline{z(t_m)}} \\
&= -\cot\theta z(t_1)\overline{\nu(t_1)} + \delta_1 z(t_2)\overline{\nu(t_2)} + \delta_2 z(t_3)\overline{\nu(t_3)} + \ldots + \delta_{m-1} z(t_m)\overline{\nu(t_m)} \\
&= -\cot\theta z(t_1)\overline{\nu(t_1)} + \sum_{s=2}^{m} \delta_{s-1} z(t_s)\overline{\nu(t_s)} \\
&= \langle \tau_0 z, \nu\rangle_{1,h} = \langle B_0 z, \nu\rangle_{1,h}.
\end{aligned}$$

Consequently, the linear operator B_0 is self-adjoint in the Hilbert space $\oplus W_2^{0,1}$. Similarly, we can show that the linear operator B_1 is also self-adjoint in the same Hilbert space $\oplus W_2^{0,1}$.

To show the positivity of B_1, let $z(k) \in \oplus W_2^{0,1}$ be arbitrary function. Then, by (14) and (16) we have

$$\langle B_1 z, z\rangle_{1,h} = \langle jz, z\rangle_0 := \sum_{s=1}^{m} \int_{t_s+0}^{t_{s+1}-0} j(k) z(k)\overline{z(k)} \mathrm{d}k = \sum_{s=1}^{m} \int_{t_s+0}^{t_{s+1}-0} j(k)|z(k)|^2 \mathrm{d}k. \tag{17}$$

Since the function $j(k)$ is positive definitely, the right hand side of (17) is greater than zero. Consequently, the operator B_1 is positive. □

Theorem 3.4. *The operators* $B_i : \oplus W_2^{0,1} \to \oplus W_2^{0,1}$ $(i = 0, 1)$, $B_2 : \mathbb{C} \to \oplus W_2^{0,1}$ *and* $B_2^* : \oplus W_2^{0,1} \to \mathbb{C}$ *are compact, where* B_2^* *is the adjoint of* B_2.

Proof. The compactness of the linear operators B_0 and B_1 follows immediately from the well-known embedding theorems (see e.g. [21]). Let us prove the compactness of the linear operators B_2 and B_2^*.

It is obvious that the adjoint operator of B_2 is defined on whole $\oplus W_2^{0,1}$ with equality $B_2^* z = \frac{1}{c_2} z(t_{m+1})$. By applying the well-known embedding theorems (see e.g. [21]), we have that the adjoint operator for B_2^* from $\oplus W_2^{0,1}$ to $\mathbb{C}$ is bounded, i.e. there exists $M_{10} > 0$ such that $\forall z \in \oplus W_2^{0,1}$

$$|B_2^* z| \le M_{10} \|z\|_{1,h}.$$

Moreover, since the range of B_2^* is the finite-dimensional inner product space, it follows immediately that B_2^* is compact. Therefore, the linear operator B_2 is also compact (see e.g. [30]). □

4 Riesz basis property of the weak eigenfunctions

It is easy to show that the MISLP (1)–(3) can be reduced as the operator-polynomial equation in the space $\mathfrak{H}$ as follows:

$$\mathbb{A}(\mu) Z = 0, \quad \mathbb{A}(\mu) = \mathbb{A}_1 - \mu \mathbb{A}_2.$$

Here the linear operators $\mathbb{A}_1$ and $\mathbb{A}_2$ are defined by the equalities

$$\mathbb{A}_1(z, \kappa) = (z + B_0 z + B_2 \kappa, \quad B_2^* z), \quad \mathbb{A}_2(z, \kappa) = \left(B_1 z, \frac{\kappa}{\rho}\right), \tag{18}$$

respectively.

Thus, we have the following result.

Lemma 4.1. If $\widetilde{Z}(k, \mu) := \begin{pmatrix} z(k, \mu) \\ \kappa \end{pmatrix}$ *is the weak eigenelement of the MISLP* (1)–(3), *then the operator-polynomial equation* $\mathbb{A}(-\mu)\widetilde{Z}(.,\mu) = 0$ *is hold in the Hilbert space* $\mathfrak{H}$.

By applying the well-known polar identity (see e.g. [31]), we have the following lemma.

Lemma 4.2. $\pm 2\mathrm{Re}\left(\int z\overline{v}\,\mathrm{d}x\right) \ge -\|z\|^2 - \|v\|^2$.

Proof. By using the well-known Hölder's integral inequality, we have

$$\begin{aligned} \left|\mathrm{Re}\left(\int z\overline{v}\,\mathrm{d}x\right)\right| \le \left|\int z\overline{v}\,\mathrm{d}x\right| &\le \left(\int |z|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int |v|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\ &\le \frac{1}{2}\left(\int |z|^2 \mathrm{d}x + \int |v|^2 \mathrm{d}x\right) = \frac{1}{2}(\|z\|^2 + \|v\|^2). \end{aligned}$$

Consequently,

$$\pm 2\mathrm{Re}\left(\int z\overline{v}\,\mathrm{d}x\right) \ge -\|z\|^2 - \|v\|^2.$$

The proof is complete. □

The last lemma is used below to show that an operator $\mathbb{A}(-\mu_0)$ is positive for sufficiently large μ_0. By using this lemma and applying [16, Theorem 4.2], we have the next result.

Theorem 4.3. *There is $\alpha > 0$ such that for all $\mu > \alpha$ the operator polynomial $\mathbb{A}(-\mu)$ is positive definite.*

Corollary 4.4. *The eigenvalues of the operator-polynomial $\mathbb{A}(-\mu)$ are positive for sufficiently large positive μ.*

Theorem 4.5. *The operator-polynomial $\mathbb{A}(-\mu) = \mathbb{A}_1 + \mu\mathbb{A}_2$ is self-adjoint and compact for any real number μ.*

Proof. Since the operators B_0, B_1, B_2, and B_2^* in the Hilbert space $\oplus W_2^{0,1}$ are compact and self-adjoint, the operators $\mathbb{A}_1$ and $\mathbb{A}_2$ in $\mathfrak{H}$ defined by (18) are compact and self-adjoint. Therefore, the operator-polynomial $\mathbb{A}(-\mu) = \mathbb{A}_1 + \mu\mathbb{A}_2$ in $\mathfrak{H}$, which is the linear combination of the operators $\mathbb{A}_1$ and $\mathbb{A}_2$, is also self-adjoint and compact. □

Since the linear operator $(\mathbb{A}_1 + \mu_0\mathbb{A}_2)$ is self-adjoint and positive definite for sufficiently large $\mu_0 > 0$, we can consider the transformation $(\mathbb{A}_1 + \mu_0\mathbb{A}_2)^{\frac{1}{2}}Z := W$, where $(\mathbb{A}_1 + \mu_0\mathbb{A}_2)^{\frac{1}{2}}$ is the positive square root of the operator $(\mathbb{A}_1 + \mu_0\mathbb{A}_2)$. Since $(\mathbb{A}_1 + \mu_0\mathbb{A}_2)$ is positive defined, the operator $(\mathbb{A}_1 + \mu_0\mathbb{A}_2)^{\frac{1}{2}}$ is invertible. Therefore, the operator-polynomial

$$\bigvee(\mu_0) := (\mathbb{A}_1 + \mu_0\mathbb{A}_2)^{-\frac{1}{2}}\mathbb{A}_2(\mathbb{A}_1 + \mu_0\mathbb{A}_2)^{-\frac{1}{2}}$$

is well-defined in the Hilbert space $\mathfrak{H}$.

Theorem 4.6. *The operator-polynomial $\vee(\mu_0)$ is positive, self-adjoint and compact in the Hilbert space $\mathfrak{H}$ for sufficiently large $\mu_0 > 0$.*

Proof. From self-adjointness of B_1 in $\oplus W_2^{0,1}$ follows easily the self-adjointness of $\mathbb{A}_2$ in $\mathfrak{H}$. From Assumption 1.1 ii. and the representations (14), (16), and (18), we have

$$\langle\mathbb{A}_2 Z, Z\rangle_{\mathfrak{H}} = \langle B_1 z, z\rangle_{1,h} + \frac{1}{\rho}|\kappa|^2 = \sum_{s=1}^{m}\int_{t_s+0}^{t_{s+1}-0} j(k)|z(k)|^2 \mathrm{d}k + \frac{1}{\rho}|\kappa|^2 \geq 0$$

for all $Z = (z(k), \kappa) \in \mathfrak{H}$, so $\mathbb{A}_2$ is positive in $\mathfrak{H}$.

Moreover, since the operator $(\mathbb{A}_1 + \mu_0\mathbb{A}_2)$ is positive defined and self-adjoint, the operator $(\mathbb{A}_1 + \mu_0\mathbb{A}_2)^{-\frac{1}{2}}$ is positive, self-adjoint and bounded. Consequently, by well-known theorems of functional analysis, the operator $\vee(\mu_0)$ is positive, self-adjoint and compact operator in the Hilbert space $\mathfrak{H}$. □

It is evident that if (μ_m, Z_m) is any eigenpair of the MISLP (1)–(3), then μ_m is the eigenvalue of the operator $\vee(\mu_0)$ with corresponding eigenelement $W_m = (\mathbb{A}_1 + \mu_0\mathbb{A}_2)^{\frac{1}{2}}Z_m$, i.e. the functions (W_m) satisfy the operator equation

$$W_m - \mu_m \vee (\mu_0)W_m = 0.$$

It is easy to see that the operator

$$\bigvee(\mu_0) = (\mathbb{A}_1 + \mu_0\mathbb{A}_2)^{-\frac{1}{2}} \quad \mathbb{A}_2(\mathbb{A}_1 + \mu_0\mathbb{A}_2)^{-\frac{1}{2}} \tag{19}$$

for sufficiently large μ_0 is positive, self-adjoint and compact operator in the Hilbert space $\mathfrak{H}$.

Corollary 4.7. *There is $\alpha > 0$, such that for all $\mu_0 > \alpha$ the operator $\vee(\mu_0)$ has denumerable many real eigenvalues $\{\eta_m\}_{m=1}^{\infty}$ with $\eta_m \to 0$ as $m \to \infty$ and the corresponding system of eigenfunctions $\{W_m\}$ forms an orthonormal basis of the Hilbert space $\mathfrak{H}$.*

Theorem 4.8. *There exists infinitely many real eigenvalues of the MISLP (1)–(3) $\lambda_1 \leq \lambda_2 \leq \ldots$, with $\lambda_m \to \infty$ and the corresponding orthonormal system of eigenfunctions forms a basis of the Hilbert space $\mathfrak{H}$.*

Theorem 4.9. *Every invertible linear bounded operator transforms any orthonormal basis of a Hilbert space* $\mathfrak{H}$ *into another basis of* $\mathfrak{H}$, *a so-called Riesz basis* (*see e.g.*, [28]).

Taking in view this theorem, Theorem 4.5, Corollary 4.7 and equality (19), we have the needed important result:

Theorem 4.10. *The system of the weak eigenfunctions of the MISLP* (1)–(3) *forms a Riesz basis of* $\mathfrak{H}$, *and the corresponding eigenfunction expansion converges in the* $\mathfrak{H}$*-norm.*

5 Conclusion

In this article, we study a new type of MISLP that differs from classical SLPs in that it is defined on finite number non-intersecting intervals with points of interaction. Furthermore, the finite number additional conditions are given at these points of interaction, the so-called interface conditions. Spectral analysis of such type of SLP is much more complicated to analyse than classical SLPs, because it is not clear how to apply the classical methods to the many-interval BVPs. Developing a new technique, we have defined some self-adjoint linear operators in such a way that the BVIP under consideration can be formulated in the form of an operator-polynomial equation. Then, using the spectral theory of operator-polynomials, we have proved that spectrum is discrete and the system of weak (generalized) eigenfunctions forms a Riesz basis.

Acknowledgments: The author wishes to express sincere thanks to the referees and the editorial team for their valuable comments, suggestions and contributions for the improvement of the article.

Funding information: This research received no external funding.

Author contributions: The author has accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The author states no conflict of interest.

Ethical approval: The conducted research is not related to either human or animal use.

References

[1] E. C. Titchmarsh, *Eigenfunctions Expansion Associated with Second Order Differential Equations I*, 2nd edn. Oxford University Press, London, 1962.
[2] C. T. Fulton, *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions*, Proc. Roy. Soc. Edin. **77A** (1977), 293–308.
[3] N. J. Guliyev, *Schrödinger operators with distributional potentials and boundary conditions dependent on the eigenvalue parameter*, J. Math. Phys. **60** (2019), no. 6, 063501, 23.
[4] K. Aydemir and O. S. Mukhtarov, *Completeness of one two-interval boundary value problem with transmission conditions*, Miskolc Math Notes **15** (2014), no. 2, 293–303.
[5] O. Sh. Mukhtarov, H. Olğar, and K. Aydemir, *Resolvent operator and spectrum of new type boundary value problems*, Filomat **29** (2015), no. 7, 1671–1680.
[6] O. Sh. Mukhtarov and S. Yakubov, *Problems for differential equations with transmission conditions*, Applicable Anal. **81** (2002), 1033–1064.
[7] B. P. Belinskiy, J. W. Hiestand, and J. V. Matthews, *Piecewise uniform optimal design of a bar with an attached mass*, Electronic J. Differential Equations **2015** (2015), no. 206, 1–17.

[8] A. Kawano, A. Morassi, and R. Zaera, *Detecting a prey in a spider orb-web from in-plane vibration*, SIAM J. Appl. Math. **81** (2021), no. 6, 2297–2322.
[9] A. V. Likov and Y. A. Mikhalilov, *The Theory of Heat and Mass Transfer*, Qosenergaizdat, 1963 (In Russian).
[10] A. Parra-Rodriguez, E. Rico, E. Solano, and I. L. Egusquiza, *Quantum networks in divergence-free circuit QED*, Quantum Sci. Technol. **3** (2018), no. 2, 024012.
[11] O. Akcay, *Uniqueness theorems for inverse problems of discontinuous Sturm-Liouville operator*, Bull. Malays. Math. Sci. Soc. **44** (2021), 1927–1940.
[12] B. P. Allahverdiev and H. Tuna, *Eigenfunction expansion for singular Sturm-Liouville problems with transmission conditions*, Electron. J. Differential Equations **3** (2019), 1–10.
[13] S. Çavuşoğlu and O. Sh. Mukhtarov, *A new finite difference method for computing approximate solutions of boundary value problems including transition conditions*, Bulletin Karaganda Univ. Math. Series **2** (2021), no. 102, 54–61.
[14] Y. A. Küçükevcilioğlu, E. Bayram, and G. G. Özbey, *On the spectral and scattering properties of eigenparameter dependent discreteimpulsive Sturm-Liouville equations*, Turk. J. Math. **45** (2021), no. 2, 988–1000.
[15] O. Sh. Mukhtarov, M. Yücel, and K. Aydemir, *A new generalization of the differential transform method for solving boundary value problems*, J. New Results Sci. **10** (2021), no. 2, 49–58.
[16] H. Olğar, *Self adjointness and positiveness of the differential operators generated by new type Sturm-Liouville problems*, Cumhuriyet Sci. J. **40** (2019), no. 1, 24–34.
[17] E. Şen, M. Açıkgöz, and S. Aracı, *Spectral problem for Sturm-Liouville operator with retarded argument which contains a spectral parameter in the boundary condition*, Ukrainian Math. J. **68** (2017), no. 8, 1263–1277.
[18] E. Uğurlu and K. Taş, *A new method for dissipative dynamic operator with transmission conditions*, Compl. Anal. Oper. Theory **12** (2018), no. 4, 1027–1055.
[19] M. Z. Zhang and Y. C. Wang, *Dependence of eigenvalues of Sturm-Liouville problems with interface conditions*, Appl. Math. Comput. **265** (2015), 31–39.
[20] B. P. Belinskiy and J. P. Dauer, *Eigenoscillations of mechanical systems with boundary conditions containing the frequency*, Quarterly Appl Math. **56** (1998), 521–541.
[21] O. A. Ladyzhenskaia, *The Boundary Value Problems of Mathematical Physics*, Springer-Verlag, New York, 1985.
[22] M. V. Keldysh, *On the eigenvalues and eigenfunctions of certain classes of non-self-adjoint equations*, Dokl Akad. Nauk SSSR (in Russian) **77** (1951), 11–14; English transl. in this volume.
[23] H. Olğar, O. S. Mukhtarov, F. S. Muhtarov, and K. Aydemir, *The weak eigenfunctions of boundary-value problem with symmetric discontinuities*, J. Appl. Anal. **28** (2022), no. 2, 275–283, DOI: https://doi.org/10.1515/jaa-2021-2079.
[24] J. R. Cannon and G. H. Meyer, *On a diffusion in a fractured medium*, SIAM J. Appl. Math. **3** (1971), 434–448.
[25] R. Dautray and J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, vol. 2, Springer-Verlag, Berlin, 1988.
[26] M. R. Lancia and M. A. Vivaldi, *On the regularity of the solutions for transmission problems*, Adv. Math. Sci. Appl. **13** (2002), 455–466.
[27] J. P. Keener, *Principles of Applied Mathematics: Transformation and Approximation*, Addison-Wesley Publishing Company, Redwood City, California, 1988.
[28] I. C. Gohberg and M. G. Krein, *Introduction to The Theory of LinearNon-Selfadjoint Operators*, Translation of Mathematical Monographs, vol. 18, American Mathematical Society, Providence, Rhode Island, 1969.
[29] N. K. Bary, *Biorthogonal systems and bases in Hilbert space*, Učenye zapiski Moskovskogo Gos. Universiteta **148** (1951), Matematika 4, 69–107 (in Russian).
[30] E. Kreyszig, *Introductory Functional Analysis With Applications*, Wiley & Sons, New York, 1978.
[31] J. B. Conway, *A Course in Functional Analysis*, Springer-Verlag, New York, 1985.