Research Article

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Petri net analysis of a queueing inventory system with orbital search by the server

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Abstract: In this article, a queueing inventory system with finite sources of demands, retrial demands, service time, lead time, (s, S) replenishment policy, and demands search from the orbit was studied. When the lead time is exponentially distributed (resp. lead time is generally distributed), generalized stochastic Petri net (GSPN) (resp. Markov regenerative stochastic Petri net [MRSPN]) is proposed for this inventory system. The quantitative analysis of this stochastic Petri net model was obtained by continuous time Markov chain for the GSPN model (resp. the supplementary variable method for the MRSPN model). The probability distributions are obtained, witch allowed us to compute performance measures and the expected cost rate of the studied system.

Keywords: inventory models, stochastic Petri nets, Markov chain, orbital search, (s, S) policy

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1 Introduction

Stochastic management inventory systems have many practical applications and have been studied extensively in the literature (see [1–4]). These systems play an important role in real-life situations (in manufacture, warehouse, supply chains, etc.). The complexity of stochastic inventory systems depends on various characteristics:

- Single product inventory and multi-product inventory
- Sources (finite, infinite)
- Arrival demand (batch arrival, generally distributed, etc.)
- Service time and lead time (positive/instantaneous exponential/non-exponential, etc.)
- Ordering policies: (s, S), (s, Q), etc.
- Retrial, vacation, breakdown, etc.

The study of queueing inventory systems with retrial was initiated by Artalejo et al. [5]. The authors have assumed Poisson demand, exponential retrial time, and exponential lead time. Retrial queueing inventory models are characterized by the feature that a customer finding the level stock equal to zero is obliged to join the orbit and to repeat his demand after some random periods of time [6]. In most of the works on retrial

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queueing systems, the authors assume that after completion of each service, the server will remain unoccupied until the arrival of the next primary or secondary customer. In order to reduce the idle time of the server, Neuts introduced the notion of search for customers, at a service completion epoch, in the queueing systems [7]. Thus, the $M/G/1$ queue with retrials and search for orbital customers was introduced by Artalejo [8]. Other works in this area of research have been done in addition to the work of Artalejo (the readers can refer to [9] and references therein). Krishnamoorthy et al. have studied an infinite inventory system with retrial and orbital search [10]. Recently, continuous review infinite queueing inventory system with customer search from the orbit and registration has been investigated by Gui and Wang [11].

In most works on the inventory systems, the authors assume that the population of demands is very large. However, in several practical situations, the number of demands who access the system is finite [2,12–15]. Thus, when a source is free at time moment t, it may generate primary demands during interval $(t, t+h)$ with probability $\lambda h + 0(h)$. In [2], a continuous review (s, S) inventory system with retrial, service facility, and finite sources of demands is analyzed.

Several research articles have dealt with the inventory systems with positive or zero service time/lead time. Artalejo et al. studied the retrial inventory systems with immediate service [5]. The notion of inventory model with positive service time was initiated by Sigman and Simchi-Levi [16]. Shajin and Krishnamoorthy obtained the product-form solution for $M/M/1$ retrial queueing inventory system with positive service time [17]. An extensive survey on inventory systems with positive service time could be found in [18].

The lead time may actually be uncertain in duration due to variability in shipping times, material availability, and supplier processing times. The inventory systems with instantaneous lead time were studied by Liu and Yang [19], Berman and Sapna [20], etc. The previous works have extended to include exponential lead time (see [5,21,22]). The inventory models with positive lead time are complex to analyze. Still more complex are the models in which the lead time has a general distribution [23–25], the reason for which is that there have not been significant studies on retrial inventory systems with arbitrary distributed lead time.

The applicability of Petri nets for stochastic analysis of inventory queueing system has received few attention [15,26]. In [15], the authors have analyzed priority multi-server retrial inventory queues with Markovian arrival process generated by single or dual sources and exponential service times where both queueing and orbit spaces are assumed finite using the generalized stochastic Petri net (GSPN) formalism. In [26], Chen et al. studied a finite source inventory system using the batch deterministic and stochastic Petri nets (SPNs).

The dynamic [27] evolution of inventory systems proceeds from one discrete state to another at arbitrary moments in time. The steady-state of stochastic inventory systems can be obtained either by discrete event simulation or by stochastic process theory. Most works on quantitative evaluation of these systems have addressed models with exponentially distributed durations, which always satisfy the Markov property. In this case, the underlying stochastic process of the model is a continuous time Markov chain (CTMC), and standard numerical algorithms can be used to compute the state probabilities. From these systems, the interesting performance measures of the system can then be derived. For multi-dimensional Markov chains underlying to the inventory systems, an appropriate numbering of the states yields a repetitive structure infinitesimal generator matrix, which can be formulated as birth-and-death process, quasi-birth-and-death process [28,29], $M/G/1$-type, etc. This repetitive structure of the infinitesimal generator matrix allows the calculation of the state probabilities by approaches known as the matrix-geometric method, matrix analytic method [30], etc.

However, in several application contexts, the inventory systems are characterized by deterministic timers or non-exponential durations (e.g., arrival pattern, the service pattern of servers, lead time, and lifetime). In this case, the underlying stochastic process of these systems falls in more complex classes of the so-called non-Markovian processes but solution may still be viable if the model satisfies the Markov property at specific times (regeneration points). The traditional approaches to solve non-Markovian models use phase-type expansions, apply the method of supplementary variables [31], or construct an embedded Markov chain [32]. The supplementary variable method is known to be originated by Cox [33] and has become one of the most frequently used approaches for both the continuous and discrete-time queuing systems. The framework of this method in stochastic models leads to the notion of the partial differential equations, ordinary differential equations, and integral equations [34]. The supplementary variable method is known as an efficient way of deriving the steady-state solution for the stochastic process underlying a Markov regenerative stochastic Petri net (MRSPN) formalism [31].

This article is an extension of the work [35] presented in the conference "IEEE International Conference on Recent Advances in Mathematics and Informatics (ICRAMI'2021)," where we have presented an approach for modeling and analyzing the inventory system with finite sources of demands, retrial demands, positive service time, exponential lead time, (s, S) replenishment policy, and demand search from the orbit. Immediately after a service completion, the server with probability p makes an instantaneous search for an orbital demand for the next service or remains idle with the complimentary probability $(1 - p)$. The notion of orbital search for inventory system was introduced with the hope that it would decrease the length of server idle period and not neglected the unsatisfied (orbital) demands. We gave an extensive analysis of this system using GSPN formalism. This high-level modeling formalism allows us to generate the underlying diagram state and the CTMC of the studied system. However, the state space of CTMC underlying our GSPN increases exponentially as a function of the source size, the maximum inventory level, and the inventory level threshold. So, for real applications, the models may have a very large state space (state space explosion). Hence, by exploiting the repetitive structure of the transition rate matrix of our GSPN, we develop a recursive algorithm for automatically calculating the steady-state probability vector and the performance indices, without generating the reachability graph or the CTMC.

In this extended work, we added some numerical results to original work [35] in order to highly illustrate the influence of the system parameters on some performance characteristics. Numerically, we investigate the sensitivity analysis of the search probability, the maximum inventory level, the inventory level thresholds, and the number of sources over the total expected cost rate. Most of the aforementioned works assume that the lead time is instantaneous or exponentially distributed. However, this basic lead time assumption is far from the real situation. For this, we assume that the lead time is generally distributed. So the underlying process to the inventory system considered is a Markov regenerative process. Under this assumption, we suggested to use the MRSPN formalism in order to establish the appropriate model for our system. This formalism allows us to capture easily the interaction between the components of this system, to better understand its behavior, to generate its underlying diagram state, and to compute its performances. To obtain the quantitative analysis of this MRSPN model, we used the supplementary variable method. In addition, we established an algorithm that comprises steady-state solution and computes performance measures. The numerical results for performance measures and the total expected cost rate are presented under the assumption that the lead time follows the following family of distributions: exponential, Erlang, hypoexponential, and hyperexponential.

2 SPNs

SPNs are defined by associating a random variable called firing time for transitions [36]. A random firing time elapses after a transition is enabled until it fires. Many stochastic variants of SPN have been proposed, which can be used as high-modeling formalism for performances evaluation and reliability of systems, like GSPN [37], deterministic and SPNs, MRSPN [38]. According to the type of distribution allowed for the firing time, the SPN may correspond to a wide range of stochastic processes.

The GSPN class contains two types of transitions: immediate transitions and exponential transitions. The marking process $(\mathcal{M}_t)_{t\geq 0}$ generated by the GSPN class is a semi-Markovian process; this process includes two types of markings: tangible markings and vanishing markings. The quantitative analysis of the GSPN can be summarized as follows:

- Generating the reachability graph of the GSPN,
- Eliminating the vanishing markings in order to obtain the reduced reachability graph,
- Converting the reduced reachability graph to CTMC with state space H (set of tangible markings),
- Constructing the transition rate matrix $\mathbf{A}$ of the CTMC,
- Computing the steady-state probability vector v of the CTMC by solving the linear system of balance equations:

$$\begin{cases} v\mathbf{A} = \mathbf{0}, \\ v\mathbf{e} = 1, \end{cases}$$

where $\mathbf{e}$ is a column vector of 1's of appropriate dimension and $\mathbf{0}$ is a vector of zeros of appropriate dimension.
- Computing the characteristics of the GSPN model (mean number of customers in places, throughput of transitions, probability of event, etc.).

The MRSPN class is an extension of SPNs in which the transition firing time is immediate, exponentially distributed, or generally distributed. This MRSPN class describes a stochastic process $(\mathcal{M}_t)_{t\geq 0}$, and this marking process cannot be mapped into a CTMC. Since the stochastic behavior of timed transitions with non-exponentially distributed firing delay is not memoryless, the evolution of the underlying stochastic process depends on the history. The inclusion of supplementary variable into the state description leads to a continuous-state Markov process for which the Kolmogorov-state equations can be derived and be numerically solved. This variable represents the time elapsed since the GEN transition was enabled. The marking process $(\mathcal{M}_t)_{t\geq 0}$ is described by three matrices $\mathbf{A}$, $\overline{\mathbf{A}}$, and $\mathbf{\Delta}$, where:
- The generator matrix $\mathbf{A}$ contains all exponentially distributed-state transitions that do not preempt a GEN transition.
- The exponential-state transitions that preempt a GEN transition are denoted by the generator matrix $\overline{\mathbf{A}}$.
- The probability that the firing of a GEN transition leads to state "j," given that the transition is fired in state i, is represented by the matrix of branching probabilities $\mathbf{\Delta}$.

The steady-state solution based on the supplementary variable method of the MRSPN is obtained by the following steps:
- **Step 01.** Obtain the two matrices Ω and Ψ:

$$\Omega = \sum_{g\in G} I^{\text{GEN}} \int_0^\infty e^{\mathbf{A}^{\text{GEN}}x} \mathrm{d}F^{\text{GEN}}(x),$$

$$\Psi = \sum_{g\in G} I^{\text{GEN}} \int_0^\infty e^{\mathbf{A}^{\text{GEN}}x} [1 - F^{\text{GEN}}(x)]\mathrm{d}x,$$

where $F^{\text{GEN}}(x)$ is the firing time distribution of the GEN transition g, G denotes the set of all GEN transitions of the MRSPN, $\mathbf{A}^{\text{GEN}}$ is the generator matrix of the subordinated CTMC of the GEN transition g, I^{GEN} is the diagonal matrix whose ith element is equal to one if $i \in G$ and equal to zero otherwise.
- **Step 02.** Compute the vector x by solving the linear system:

$$\begin{cases} xS = \mathbf{0}, \\ xL\mathbf{e} = 1, \end{cases}$$

where, $S = \mathbf{A}^{\text{EXP}} + \Omega\mathbf{\Delta} + \Psi\overline{\mathbf{A}} - I^{\text{GEN}}$ is the generator matrix of a CTMC, we refer to it as the embedded CTMC, $L = I^{\text{EXP}} + \Psi$ is the conversion matrix, I^{EXP} is the diagonal matrix whose ith element is equal to one if the state i is exponential and equal to zero otherwise and $\mathbf{A}^{\text{EXP}}$ filtered matrix is defined by: $\mathbf{A}^{\text{EXP}} = I^{\text{EXP}}\mathbf{A}$.
- **Step 03.** Compute the steady-state probability distributions v of the MRSPN by the formula:

$$v = xL.$$

3 The proposed GSPN model

We consider a continuous review inventory system with finite sources of demands ($2 \leq N < \infty$), retrial demands, positive service time, positive lead time, (s, S) replenishment policy, and demand search from the orbit. The following assumptions and notations are considered:

- The arrival process of primary demands is quasi-random with parameter $\lambda(>0)$.
- The maximum inventory level is S units. The policy used is (s, S), in which an order is placed for a quantity (“$Q = S - s > s + 1$”) units up to S whenever the inventory level falls to the threshold s or below.
- The lead time distribution is exponential with parameter $\alpha(>0)$.
- When the inventory level is zero or the server is occupied, any arriving primary demand joins a virtual room called orbit. These orbiting demands compete for their demands according to an exponential distribution with parameter $\gamma(>0)$. We consider the constant retrial policy (i.e., the retrial rate is independent of the number of customers in the orbit).
- Immediately after a service completion, the server takes a demand from the orbit with probability p, when there are demands in the orbit and the stock is not empty, and with probability $(1 - p)$, the server remains idle.
- We suppose that the service time following exponential distribution with rate $\mu(>0)$ and the search time is negligible.

The GSPN describing this system is given in Figure 1.

Our model contains six places p_i, $i = \overline{1, 6}$ (noted by circles) and nine transitions t_i, $i = \overline{1, 9}$. The white rectangular boxes represent the exponential transitions (t_1, t_3, t_7, and t_9), and the thin bars represent the immediate transitions (t_2, t_4, t_5, t_6, and t_8). The interpretation of the places and transitions of our model is explained in Table 1.

The markings (ordinary states) of our GSPN are given by:

$$M_i = (\#p_1, \#p_2, \#p_3, \#p_4, \#p_5, \#p_6),$$

and its initial marking is $M_0 = (N, 0, 0, S, 0, 0)$. According to the equation “$\#p_1 + \#p_3 + \#p_6 = N$,” we deduce that the vectors (micro-states) $M_i = (\#p_3, \#p_4, \#p_6)$ provide all information needed for states description of this GSPN model. So, the state space of this GSPN is defined as:

$$\begin{aligned} H = &\{(0, 0, k),\ k = 0, \ldots, N\} \cup \{(1, j, k),\ j = 1, \ldots, S,\ k = 0, \ldots, N - 1\} \\ &\cup \{(0, j, k),\ j \neq Q,\ j = 1, \ldots, S,\ k = 0, \ldots, N - 1\} \cup \{(0, Q, k),\ k = 0, \ldots, N\}, \end{aligned}$$

and the total number of its states is equal to “$N(2S + 1) + 2$.”

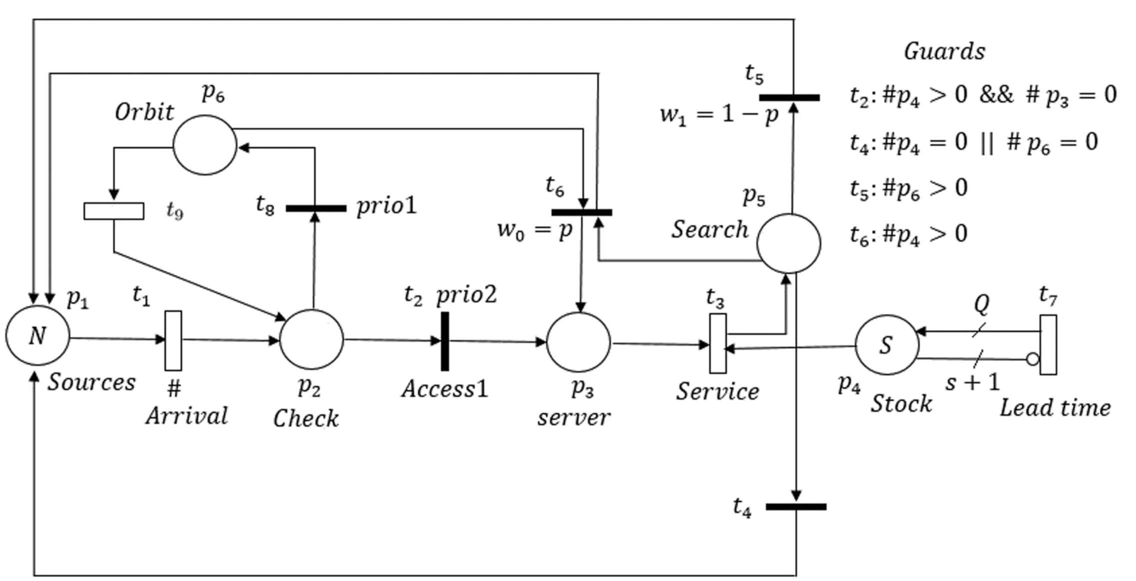

Figure 1: GSPN model of the inventory system with finite sources of demands, retrial demands, service time, lead time, and demand search from the orbit.

Table 1: Interpretation of the places and the transitions in the GSPN model

p_1	Contains the free sources
p_2	Contains the primary or repeated demands
p_3	Contains the demand in service
p_4	Represents the level stock
p_5	Represents the service completion
p_6	Represents the orbit
t_1	Represents the arrival of the primary demands
t_2	Represents the access to the service
t_3	Represents the service of demand
t_4, t_5, and t_6	Represent the feedback to the sources
t_7	Represents the lead time
t_8	Represents the access to the orbit
t_9	Represents the repeated demands

The marking process $(\mathcal{M}_t)_{t\geq 0}$ underlying the GSPN depicted in Figure 1 is isomorphic to a three-dimensional CTMC that expresses server status ($\#p_3 = 0$ if the server is idle and $\#p_3 = 1$ if the server is busy), $\#p_4$ the number of demands in the orbit, and $\#p_6$ the inventory level. We arranged the state space of this CTMC as:

$$\{\mathbf{00}, \mathbf{1j}, \mathbf{0j}, j = 1, \ldots, S\},$$

where $\mathbf{00} = \ll 00k \gg$, $k = 0, \ldots, N$, $\mathbf{1j} = \ll 1jk \gg$, $j = 1, \ldots, S$, $k = 0, \ldots, N-1$, $\mathbf{0j} = \ll 0jk \gg$, $j = 1, \ldots, S$, $k = 0, \ldots, N-1$, $\mathbf{0Q} = \ll 0Qk \gg$, $k = 0, \ldots, N$.

The infinitesimal generator $\mathbf{A}$ can be expressed in the block form:

$$\mathbf{A} = \begin{array}{c|cccccccccccccccc}
 & \mathbf{00} & \mathbf{11} & \mathbf{01} & \ldots & \mathbf{1s} & \mathbf{0s} & \mathbf{1s+1} & \mathbf{0s+1} & \ldots & \mathbf{1Q} & \mathbf{0Q} & \mathbf{1Q+1} & \mathbf{0Q+1} & \ldots & \mathbf{1S} & \mathbf{0S} \\
\hline
\mathbf{00} & A_0 & & & & & & & & & & C_Q & & & & & \\
\mathbf{11} & B_0 & A_1 & & & & & & & & & & C & & & & \\
\mathbf{01} & & D & A_2 & & & & & & & & & & C & & & \\
\mathbf{12} & & E & B & & & & & & & & & & & & & \\
\vdots & & & & \ddots & & & & & & & & & & \ddots & & \\
\mathbf{1s} & & & & & A_1 & & & & & & & & & & C & \\
\mathbf{0s} & & & & & D & A_2 & & & & & & & & & & C \\
\mathbf{1s+1} & & & & & E & B & A_3 & & & & & & & & & \\
\mathbf{0s+1} & & & & & & & D & A_4 & & & & & & & & \\
\mathbf{1s+2} & & & & & & & E & B & & & & & & & & \\
\vdots & & & & & & & & & \ddots & & & & & & & \\
\mathbf{1Q} & & & & & & & & & & A_3 & & & & & & \\
\mathbf{0Q} & & & & & & & & & & D_Q & A_Q & & & & & \\
\mathbf{1Q+1} & & & & & & & & & & E & B_Q & A_3 & & & & \\
\mathbf{0Q+1} & & & & & & & & & & & & D & A_4 & & & \\
\mathbf{1Q+2} & & & & & & & & & & & & E & B & & & \\
\vdots & & & & & & & & & & & & & & \ddots & & \\
\mathbf{1S} & & & & & & & & & & & & & & & A_3 & \\
\mathbf{0S} & & & & & & & & & & & & & & & D & A_4
\end{array}.$$

where

$$A_{0(k,l)} = \begin{cases} -[(N-k)\lambda + \alpha], & k = 0, \ldots, N,\ l = k, \\ (N-k)\lambda, & k = 0, \ldots, N-1,\ l = k+1, \\ 0, & \text{otherwise.} \end{cases} \quad B_{0(k,l)} = \begin{cases} \mu, & k = 0, \ldots, N-1,\ l = k, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{1(k,l)} = \begin{cases} -[(N-k-1)\lambda + \alpha + \mu], & k = 0, \ldots, N-1,\ l = k, \\ (N-k-1)\lambda, & k = 0, \ldots, N-2,\ l = k+1, \\ 0, & \text{otherwise.} \end{cases}$$

$$D_{(k,l)} = \begin{cases} (N-k)\lambda, & k = 0, \ldots, N-1,\ l = k, \\ \gamma, & k = 1, \ldots, N-1,\ l = k-1, \\ 0, & \text{otherwise.} \end{cases}$$

$$E_{(k,l)} = \begin{cases} p\mu, & k = 1, \ldots, N-1,\ l = k-1, \\ 0, & \text{otherwise.} \end{cases} \quad A_{2(k,l)} = \begin{cases} -N\lambda + \alpha, & k = 0,\ l = 0, \\ -[(N-k)\lambda + \alpha + \gamma], & k = 1, \ldots, N-1,\ l = k, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{(k,l)} = \begin{cases} \mu, & k = 0,\ l = 0, \\ (1-p)\mu, & k = 1, \ldots N-1,\ l = k, \\ 0, & \text{otherwise.} \end{cases} \quad A_{3(k,l)} = \begin{cases} -[(N-k-1)\lambda + \mu], & k = 0, \ldots, N-1,\ l = k, \\ (N-k-1)\lambda, & k = 0, \ldots, N-2,\ l = k+1, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{4(k,l)} = \begin{cases} -N\lambda, & k = 0,\ l = 0, \\ -[(N-k)\lambda + \gamma], & k = 1, \ldots, N-1,\ l = k, \\ 0, & \text{otherwise.} \end{cases} \quad D_{Q(k,l)} = \begin{cases} (N-k)\lambda, & k = 0, \ldots, N-1,\ l = k, \\ \gamma, & k = 1, \ldots, N,\ l = k-1, \\ 0, & \text{otherwise.} \end{cases}$$

$$C_{Q(k,l)} = \begin{cases} \alpha, & k = 0, \ldots, N,\ l = k, \\ 0, & \text{otherwise.} \end{cases} \quad A_{Q(k,l)} = \begin{cases} -N\lambda, & k = 0, l = 0, \\ -[(N-k)\lambda + \gamma], & k = 1, \ldots, N,\ l = k, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{Q(k,l)} = \begin{cases} \mu, & k = 0,\ l = 0, \\ (1-p)\mu, & k = 1, \ldots, N-1,\ l = k, \\ 0, & \text{otherwise.} \end{cases} \quad C_{(k,l)} = \begin{cases} \alpha, & k = 0, \ldots, N-1,\ l = k, \\ 0, & \text{otherwise.} \end{cases}$$

The dimensions of the entries (sub-matrices) of A are given in Table 2.

4 The steady-state solution

The GSPN model given in Figure 1 is bounded and admits the initial marking M_0 as home state. Then, the steady-state probability distribution $v = v^{\ll i,j,k \gg}$ of this GSPN exists and is unique. The elements $v^{\ll i,j,k \gg}$ are given by:

$$v^{\ll i,j,k \gg} = \lim_{t \to +\infty} (\#p_3(t) = i,\ \#p_4(t) = j,\ \#p_6(t) = k | \#p_3(0) = i_0,\ \#p_4(0) = j_0,\ \#p_6(0) = k_0).$$

Using the block-structured matrix $\mathbf{A}$, the vector v can be represented by:

$$v = (v^{\mathbf{00}}, v^{\mathbf{1j}}, v^{\mathbf{0j}}), j = 1, \ldots, S,$$

where

$$\begin{aligned} v^{\mathbf{00}} &= (v^{\ll 00k \gg}), \quad k = 0, \ldots, N, \\ v^{\mathbf{1j}} &= (v^{\ll 1jk \gg}), \quad j = 1, \ldots, S, k = 0, \ldots, N-1, \\ v^{\mathbf{0j}} &= (v^{\ll 0jk \gg}), \quad j \neq Q, j = 1, \ldots, S, k = 0, \ldots, N-1, \\ v^{\mathbf{0Q}} &= (v^{\ll 0Qk \gg}), \quad k = 0, \ldots, N. \end{aligned}$$

Table 2: Dimension of sub-matrices of A

$A_{0(N+1,N+1)}$	$B_{0(N,N+1)}$	$A_{1(N,N)}$	$D_{(N,N)}$	$E_{(N,N)}$	$A_{2(N,N)}$	$B_{(N,N)}$
$A_{3(N,N)}$	$A_{4(N,N)}$	$D_{Q(N+1,N)}$	$C_{Q(N+1,N+1)}$	$A_{Q(N+1,N+1)}$	$B_{Q(N,N+1)}$	$C_{(N,N)}$

The linear system $v\mathbf{A} = \mathbf{0}$ yields the following global balance equations:

- $$v^{\mathbf{00}}A_0 + v^{\mathbf{11}}B_0 = \mathbf{0}.$$
- For $j = 1, \ldots, s$,
$$\begin{cases} v^{\mathbf{1j}}A_1 + v^{\mathbf{0j}}D + v^{\mathbf{1j+1}}E = \mathbf{0}, \\ v^{\mathbf{0j}}A_2 + v^{\mathbf{1j+1}}B = \mathbf{0}. \end{cases}$$
- For $j = s + 1, \ldots, Q - 1$,
$$\begin{cases} v^{\mathbf{1j}}A_3 + v^{\mathbf{0j}}D + v^{\mathbf{1j+1}}E = \mathbf{0}, \\ v^{\mathbf{0j}}A_4 + v^{\mathbf{1j+1}}B = \mathbf{0}. \end{cases} \tag{1}$$
- For $j = Q$,
$$\begin{cases} v^{\mathbf{1j}}A_3 + v^{\mathbf{0j}}D_Q + v^{\mathbf{1j+1}}E = \mathbf{0}, \\ v^{\mathbf{00}}C_Q + v^{\mathbf{0j}}A_Q + v^{\mathbf{1j+1}}B_Q = \mathbf{0}. \end{cases}$$
- For $j = Q + 1, \ldots, S - 1$,
$$\begin{cases} v^{\mathbf{1j-Q}}C + v^{\mathbf{1j}}A_3 + v^{\mathbf{0j}}D + v^{\mathbf{1j+1}}E = \mathbf{0}, \\ v^{\mathbf{0j-Q}}C + v^{\mathbf{0j}}A_4 + v^{\mathbf{1j+1}}B = \mathbf{0}. \end{cases}$$
- For $j = S$,
$$\begin{cases} v^{\mathbf{1s}}C + v^{\mathbf{1S}}A_3 + v^{\mathbf{0S}}D = \mathbf{0}, \\ v^{\mathbf{0s}}C + v^{\mathbf{0S}}A_4 = \mathbf{0}. \end{cases}$$

The aforementioned global balance equations (except the first equation in (1) for $j = s + 1$) can be recursively solved to obtain

$$\begin{cases} v^{\mathbf{0j}} = v^{\mathbf{0S}}\mathcal{H}(0, j), \quad j = 0, \ldots, S - 1, \\ v^{\mathbf{1j}} = v^{\mathbf{0S}}\mathcal{H}(1, j), \quad j = 1, \ldots, S, \end{cases}$$

where the matrices $\mathcal{H}(0, j)$ and $\mathcal{H}(1, j)$ are given by:

$$\mathcal{H}(0, j) = \begin{cases} -A_4C^{-1}(D - A_2B^{-1}E)A_1^{-1}[(BA_2^{-1}D - E)A_1^{-1}]^{s-1}B_0A_0^{-1}, \quad j = 0, \\ -A_4C^{-1}(D - A_2B^{-1}E)A_1^{-1}[(BA_2^{-1}D - E)A_1^{-1}]^{s-j-1}BA_2^{-1}, \quad j = 1, \ldots, s - 1, \\ -A_4C^{-1}, \quad j = s, \\ I, \quad j = s + 1, \\ -[\mathcal{H}(0, 0)C_QA_Q^{-1}D_QA_3^{-1}[(BA_4^{-1}D - E)A_3^{-1}]^{Q-j-1} + \mathcal{H}(1, Q + 1)(BA_5^{-1}D_0 - E)A_3^{-1}[(BA_4^{-1}D \\ \qquad - E)A_3^{-1}]^{Q-j-1}]BA_4^{-1}, \quad j = s + 2, \ldots, Q - 1, \\ -[\mathcal{H}(0, 0)C_QA_Q^{-1} + \mathcal{H}(1, Q + 1)B_QA_Q^{-1}], \quad j = Q; \\ \mathcal{H}(0, j - Q)CA_4^{-1} + \left[\sum_{l=j+1}^{S}\mathcal{H}(1, l - Q)CA_3^{-1}[(BA_4^{-1}D - E)A_3^{-1}]^{l-j-1} + \sum_{l=j+1}^{S}\mathcal{H}(0, l - Q)CA_3^{-1}[(BA_4^{-1}D \right. \\ \qquad \left. - E)A_3^{-1}]^{l-j-1} + DA_3^{-1}[(BA_4^{-1}D - E)A_3^{-1}]^{S-j-1}\right]BA_4^{-1}, \; j = Q + 1, \ldots, S - 1 \end{cases}$$

and

$$\mathcal{H}(1,j)=\begin{cases}-A_4C^{-1}(D-A_2B^{-1}E)A_1^{-1}[(BA_2^{-1}D-E)A_1^{-1}]^{s-j}, \quad j=1,\dots,s,\\ -A_4C^{-1}A_2B^{-1}, \quad j=s+1,\\ \mathcal{H}(0,0)C_QA_Q^{-1}D_QA_3^{-1}[(BA_4^{-1}D-E)A_3^{-1}]^{Q-j}+\mathcal{H}(1,Q+1)(B_QA_Q^{-1}D_Q-E)A_3^{-1}[(BA_4^{-1}D-E)A_3^{-1}]^{Q-j}, \quad j=s+2,\dots,Q,\\ -\left[\sum_{l=j}^{S}\mathcal{H}(1,l-Q)CA_3^{-1}[(BA_4^{-1}D-E)A_3^{-1}]^{l-j}+\sum_{l=j}^{S-1}\mathcal{H}(0,l-Q)CA_4^{-1}DA_3^{-1}[(BA_4^{-1}D-E)A_3^{-1}]^{l-j}\right.\\ \left.\quad+DA_3^{-1}[(BA_4^{-1}D-E)A_3^{-1}]^{S-j}\right], \quad j=Q+1,\dots,S.\end{cases}$$

And I is an identity matrix.

The vector $v^{\mathbf{0S}}$ can be obtained by solving the two equations:

$$\begin{cases}v^{\mathbf{0S}}[\mathcal{H}(1,s+1)A_3+D+\mathcal{H}(1,s+2)E]=\mathbf{0},\\ v^{\mathbf{0S}}[\sum_{j=0}^{S-1}\mathcal{H}(0,j)+\sum_{j=1}^{S}\mathcal{H}(1,j)+I]\mathbf{e}=1.\end{cases}$$

5 The proposed MRSPN model

In this section, we consider that the lead time is of general distribution with cumulative distribution function $F^{\mathrm{GEN}}(x)$. Thus, the suggested model is depicted in Figure 2. The black rectangular box represents the general transition t_7 of lead time.

The matrix **A** is given by:

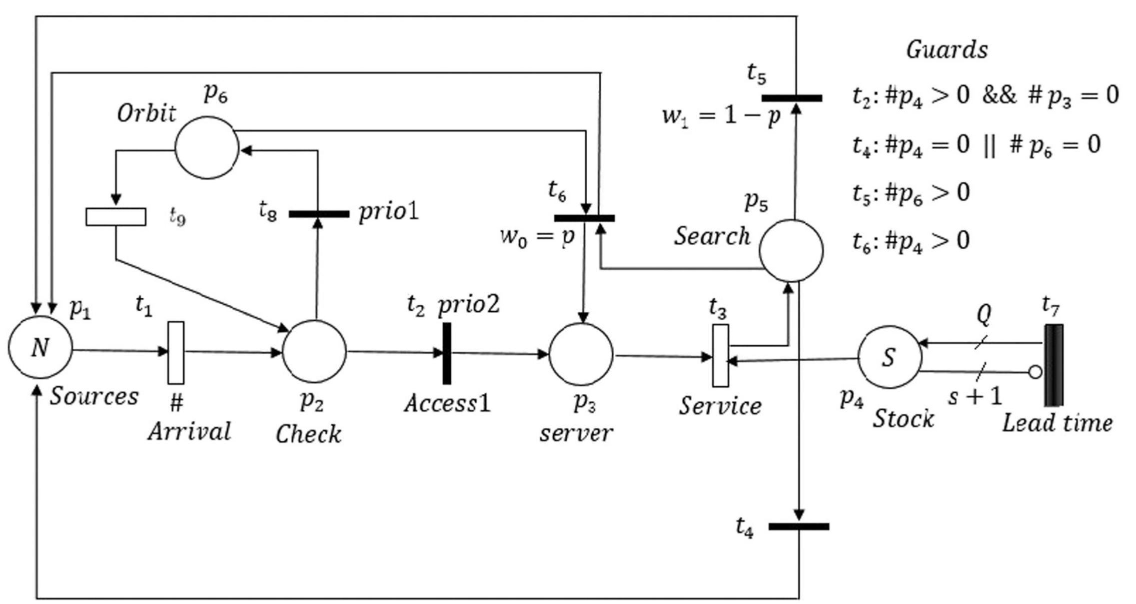

Figure 2: MRSPN model of the inventory system with finite sources of demands, retrial demands, service time, lead time and demand search from the orbit.

$$
\mathbf{A}=\begin{array}{c}
\\ \mathbf{00}\\ \mathbf{11}\\ \mathbf{01}\\ \mathbf{12}\\ \vdots\\ \mathbf{1s}\\ \mathbf{0s}\\ \mathbf{1s+1}\\ \mathbf{0s+1}\\ \mathbf{1s+2}\\ \vdots\\ \mathbf{1Q}\\ \mathbf{0Q}\\ \mathbf{1Q+1}\\ \mathbf{0Q+1}\\ \mathbf{1Q+2}\\ \vdots\\ \mathbf{1S}\\ \mathbf{0S}
\end{array}
\begin{array}{cccccccccccccccc}
\mathbf{00} & \mathbf{11} & \mathbf{01} & \ldots & \mathbf{1s} & \mathbf{0s} & \mathbf{1s+1} & \mathbf{0s+1} & \ldots & \mathbf{1Q} & \mathbf{0Q} & \mathbf{1Q+1} & \mathbf{0Q+1} & \ldots & \mathbf{1S} & \mathbf{0S}\\
A_0 &&&&&&&&&&&&&&&\\
B_0 & A_1 &&&&&&&&&&&&&&\\
& D & A_2 &&&&&&&&&&&&&\\
& E & B &&&&&&&&&&&&&\\
&&& \ddots &&&&&&&&&&&&\\
&&&& A_1 &&&&&&&&&&&\\
&&&& D & A_2 &&&&&&&&&&\\
&&&& E & B & A_3 &&&&&&&&&\\
&&&&&& D & A_4 &&&&&&&&\\
&&&&&& E & B &&&&&&&&\\
&&&&&&&& \ddots &&&&&&&\\
&&&&&&&&& A_3 &&&&&&\\
&&&&&&&&& D_Q & A_Q &&&&&\\
&&&&&&&&& E & B_Q & A_3 &&&&\\
&&&&&&&&&&& D & A_4 &&&\\
&&&&&&&&&&& E & B &&&\\
&&&&&&&&&&&&& \ddots &&\\
&&&&&&&&&&&&&& A_3 &\\
&&&&&&&&&&&&&& D & A_4
\end{array}.
$$

where

$$
A_{0(k,l)}=\begin{cases}-[(N-k)\lambda], & k=0,\ldots,N,\ l=k,\\ (N-k)\lambda, & k=0,\ldots,N-1,\ l=k+1,\\ 0, & \text{otherwise.}\end{cases}
$$

$$
A_{1(k,l)}=\begin{cases}-[(N-k-1)\lambda+\mu], & k=0,\ldots,N-1,\ l=k,\\ (N-k-1)\lambda, & k=0,\ldots,N-2,\ l=k+1,\\ 0, & \text{otherwise.}\end{cases}
$$

$$
A_{2(k,l)}=\begin{cases}-N\lambda, & k=0,\ l=0,\\ -[(N-k)\lambda+\gamma], & k=1,\ldots N-1,\ l=k,\\ 0, & \text{otherwise.}\end{cases}
$$

The other matrices: A_3, A_4, A_Q, B, B_0, B_Q, D, D_Q, and E are the same to those obtained for the previous GSPN model.

The matrix of branching probabilities $\boldsymbol{\Delta}$ is given by:

$$
\Delta=\begin{array}{c}
\\ \mathbf{00}\\ \mathbf{11}\\ \mathbf{01}\\ \mathbf{12}\\ \vdots\\ \mathbf{1s}\\ \mathbf{0s}\\ \mathbf{1s+1}\\ \mathbf{0s+1}\\ \mathbf{1s+2}\\ \vdots\\ \mathbf{1Q}\\ \mathbf{0Q}\\ \mathbf{1Q+1}\\ \mathbf{0Q+1}\\ \mathbf{1Q+2}\\ \vdots\\ \mathbf{1S}\\ \mathbf{0S}
\end{array}
\begin{array}{cccccccccccccccc}
\mathbf{00} & \mathbf{11} & \mathbf{01} & \ldots & \mathbf{1s} & \mathbf{0s} & \mathbf{1s+1} & \mathbf{0s+1} & \ldots & \mathbf{1Q} & \mathbf{0Q} & \mathbf{1Q+1} & \mathbf{0Q+1} & \ldots & \mathbf{1S} & \mathbf{0S}\\
&&&&&&&&&& I_{N+1} &&&&&\\
&&&&&&&&&&& I_N &&&&\\
&&&&&&&&&&&& I_N &&&\\
&&&&&&&&&&&&&&&\\
&&&&&&&&&&&&& \ddots &&\\
&&&&&&&&&&&&&& I_N &\\
&&&&&&&&&&&&&&& I_N\\
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\end{array}.
$$

The matrix $\overline{\mathbf{A}}$ is a zero matrix (i.e., there is not an exponential-state transition that preempts a GEN transition t_7).

6 Performance indices

The performance measures of our inventory systems and the total expected cost rate in the steady-state can be defined as follows:

- The mean inventory level, n_l,

$$n_l = Qv^{\ll 0QN \gg} + \sum_{j=1}^{S} j \sum_{k=0}^{N-1} [v^{\ll 0jk \gg} + v^{\ll 1jk \gg}].$$

- The expected number of customers in the orbit, n_o,

$$n_o = Nv^{\ll 0QN \gg} + \sum_{k=1}^{N} kv^{\ll 00k \gg} + \sum_{j=1}^{S} \sum_{k=1}^{N-1} k[v^{\ll 0jk \gg} + v^{\ll 1jk \gg}].$$

- The expected reorder level, n_r,

$$n_r = \sum_{k=0}^{N-1} \mu v^{\ll 1s+1k \gg}.$$

- The probability that server is busy, n_B,

$$n_B = \sum_{j=1}^{S} \sum_{k=0}^{N-1} v^{\ll 1jk \gg}.$$

- The effective search rate, n_{ES},

$$n_{\mathrm{ES}} = \sum_{j=2}^{S} \sum_{k=1}^{N-1} p\mu v^{\ll 1jk \gg}.$$

- The expected number of successful retrials, n_{ESR},

$$n_{\mathrm{ESR}} = \gamma[v^{\ll 0QN \gg} + \sum_{j=1}^{S} \sum_{k=1}^{N-1} v^{\ll 0jk \gg}].$$

- The probability that inventory level is zero, p_0,

$$p_0 = \sum_{k=0}^{N} v^{\ll 00k \gg}.$$

- The probability that inventory level is greater than s, p_s,

$$p_s = v^{\ll 0QN \gg} + \sum_{j=s+1}^{S} \sum_{k=0}^{N-1} [v^{\ll 0jk \gg} + v^{\ll 1jk \gg}].$$

- The mean generation of primary calls, λ_e,

$$\lambda_e = \sum_{k=0}^{N-1} (N-k) v^{\ll 00k \gg} + \sum_{j=1}^{S} \sum_{k=0}^{N-1} (N-k) v^{\ll 0jk \gg} + \sum_{j=1}^{S} \sum_{k=0}^{N-2} (N-k-1) v^{\ll 1jk \gg}.$$

- The mean waiting time, ω,

$$\omega = \frac{n_o}{\lambda_e}.$$

- The mean response time, ϖ,

$$\varpi = \frac{n_o + n_B}{\lambda_e}.$$

- The total expected cost rate TC in the steady-state is given by:

$$TC = C_h\left(Qv^{\ll 0QN\gg} + \sum_{j=1}^{S} j\sum_{k=0}^{N-1}[v^{\ll 0jk\gg} + v^{\ll 1jk\gg}]\right) + C_s\left(\sum_{k=0}^{N-1}\mu v^{\ll 1s+1k\gg}\right) + C_w\left(Nv^{\ll 0QN\gg} + \sum_{k=1}^{N} kv^{\ll 00k\gg} + \sum_{j=1}^{S}\sum_{k=1}^{N-1} k[v^{\ll 0jk\gg} + v^{\ll 1jk\gg}]\right),$$

where C_h, C_s, and C_w are, respectively, the inventory carrying cost per unit item per unit time, setup cost per order, and the waiting cost for an orbiting demand per unit time.

7 Numerical application

We construct two programs based on the formulas obtained in previous sections in order to compute numerically the probability distributions of the two models (GSPN and MRSPN). We show the influence of the system parameters on the system performance measures and the total expected cost rate. The numerical results for:

- the GSPN model are given in Tables 3–5 and illustrated in Figures 3–6.
- the MRSPN model is given in Table 6.

Table 3 presents the effect of search probability p on various performance measures for $\lambda = 0.8$, $\mu = 1$, $\gamma = 0.25$, $\alpha = 0.6$, $N = 10$, $S = 9$, $s = 3$, $C_h = 1$, $C_s = 5$, and $C_w = 3$. Table 3 shows that an increase in the search probability p makes an increase in measures such as expected reorder level n_r, probability that server is busy n_b, effective search rate n_{ES}, probability that inventory level is zero p_0, mean generation of primary calls λ_e; however, mean inventory level n_l, expected number of customer in the orbit n_o, expected number of successful retrial n_{ESR}, probability that inventory level is greater than s, p_s, mean waiting time ω, mean response time ϖ, and total expected cost rate TC decrease. In Tables 4 and 5, the total expected cost rate TC for various combinations of maximum inventory level S, level stock s, the search probability p, and the number of sources N are given. In Table 4, the numerical values show that TC is a convex function in S and s and the minimum occurs at $(s, S) = (6, 22)$, which equals to 230.0767. We observe that TC is a decreasing function on p.

From Table 5, we observe that TC is an increasing function on N. Also, it may be observed that TC is more sensitive to changes in N than to changes in S and s. Figures 3 and 4 show the influence of arrival rate λ, retrial rate γ, and the search probability p on the mean response time ϖ. We note that the mean response time ϖ of the inventory system with orbital search mechanism is maximum. The location and the amplitude of this maximum depend on the retrial rate γ and the search probability p. For higher values of the search probability p or lower values of retrial rate γ, the maximum becomes less dominant. The arrival rate λ has a significant influence on the mean response time when the retrial rate and the search probability p are weak. The effect of the generation of primary demands λ and the retrial rate γ on the total expected cost rate TC is shown in Figure 5. It shows that TC increases when λ increases and TC decreases when γ increases. Figure 6 shows the influence of the lead time rate α and the service time rate μ on the total expected cost rate TC. We observe that TC decreases when the mean lead time rate α increases. Table 6 the numerical results obtained by using the supplementary variables method to analyze the MRSPN that models the inventory system considered are given. It presents numerical results when the lead time follows different distributions, namely, exponential $Exp(\mu)$, Erlang $E_2(\mu)$, Hypoexponential $\mathrm{Hypo}_2(\mu_1;\mu_2)$, and hyperexponential $H_2(q;\ \mu_1;\ \mu_2)$, assuming that the expected lead time of these distributions has the same value $\frac{1}{\mu} = 2, 5$. This table shows that there is a difference in the values of the performance indices between exponential and non-exponential distributed cases. Note that Erlang distributed lead time gives the lowest TC (resp. n_o),

Table 3: Effect of search probability p on selected performance measures for GSPN that models our inventory system for $\lambda = 0.8$, $\mu = 1$, $\gamma = 0.25$, $\alpha = 0.6$, $N = 10$, $S = 9$, $s = 3$, $C_h = 1$, $C_s = 5$, and $C_w = 3$

Measures	$p = 0$	10^{-2}	0.2	0.5	0.7	0.9	1
n_I	5.6070	5.6017	5.4962	5.3145	5.1850	5.0507	4.9824
n_o	8.8107	8.8040	8.6721	8.4471	8.2875	8.1220	8.0374
n_r	0.0881	0.0886	0.0984	0.1150	0.1269	0.1391	0.1454
n_B	0.5286	0.5316	0.5902	0.6902	0.7611	0.8347	0.8723
n_{ES}	0.0000	0.0052	0.1159	0.3361	0.5158	0.7228	0.8368
n_{ESR}	0.1151	0.1143	0.0981	0.0699	0.0496	0.0282	0.0171
p_0	0.0110	0.0112	0.0174	0.0301	0.0405	0.0525	0.0592
p_s	0.8532	0.8523	0.8361	0.8083	0.7886	0.7681	0.7577
λ_e	0.5286	0.5316	0.5902	0.6902	0.7611	0.8347	0.8723
ω	16.6687	16.5628	14.6941	12.2391	10.8889	9.7305	9.2144
ϖ	17.6687	17.5628	15.6941	13.2391	11.8889	10.7305	10.2144
TC	32.4796	32.4567	32.0043	31.2309	30.6818	30.1122	29.8214

Table 4: Effect of maximum inventory level S, inventory level s, and search probability p on the total expected cost rate TC for $\lambda = 1.5$, $\mu = 5$, $\gamma = 0.2$, $\alpha = 2$, $N = 14$, $C_h = 1$, $C_s = 25$, and $C_w = 20$

p	S / s	2	3	4	5	6	7	8
0.2	20	261.7116	261.4102	261.6048	262.0612	262.6660	263.3668	264.1426
	21	262.0044	261.7159	261.9072	262.3490	262.9300	263.5980	264.3312
	22	262.3169	262.0416	262.2313	262.6616	263.2234	263.8651	264.5642
	23	262.6465	262.3844	262.5738	262.9949	263.5410	264.1613	264.8329
	24	262.9910	262.7418	262.9319	263.3457	263.8789	264.4814	265.1304
	25	263.3485	263.1119	263.3033	263.7114	264.2338	264.8215	265.4518
0.5	20	248.9849	247.7635	247.2975	247.3271	247.6887	248.2841	249.0591
	21	249.0862	247.8965	247.4416	247.4638	247.8013	248.3563	249.0732
	22	249.2240	248.0661	247.6239	247.6423	247.9617	248.4851	249.1565
	23	249.3934	248.2670	247.8386	247.8559	248.1615	248.6599	249.2951
	24	249.5905	248.4949	248.0808	248.0989	248.3943	248.8728	249.4787
	25	249.8120	248.7463	248.3469	248.3672	248.6549	249.1173	249.6994
0.8	20	234.3789	232.2309	230.9601	230.3532	230.2514	230.5429	231.1547
	21	234.2088	232.1021	230.8539	230.2486	230.1263	230.3738	230.9153
	22	234.0975	232.0336	230.8107	230.2117	230.0767	230.2918	230.7797
	23	234.0375	232.0170	230.8210	230.2319	230.0897	230.2809	230.7275
	24	234.0225	232.0456	230.8772	230.3004	230.1552	230.3288	230.7432
	25	234.0474	232.1136	230.9737	230.4103	230.2651	230.4259	230.8152

Table 5: Effect of the number of source N and maximum inventory level S on the total expected cost rate TC for $\lambda = 1.2$, $\mu = 4$, $\gamma = 0.5$, $\alpha = 1.8$, $p = 0.8$, $s = 4$, $C_h = 0.5$, $C_s = 9$, and $C_w = 13$

S / N	6	7	8	9	10	11	12
16	41.2537	53.4579	66.1565	79.0579	92.0295	105.0223	118.0206
17	41.1877	53.3705	66.0591	78.9565	91.9268	104.9192	117.9174
18	41.1654	53.3295	66.0093	78.9031	91.8723	104.8643	117.8624
19	41.1782	53.3260	65.9979	78.8886	91.8567	104.8484	117.8464
20	41.2198	53.3530	66.0180	78.9059	91.8730	104.8644	117.8624
21	41.2851	53.4054	66.0642	78.9496	91.9158	104.9070	117.9049
21	41.3704	53.4791	66.1323	79.0155	91.9809	104.9718	117.9696

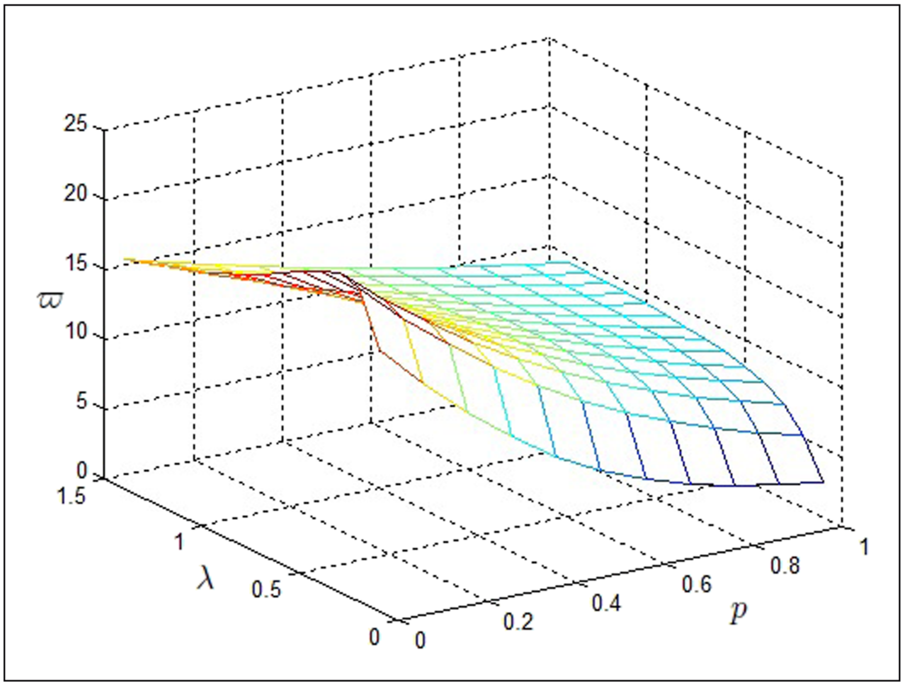

Figure 3: Effect of arrival rate λ and the search probability p on the mean response time ϖ; $N = 12$, $S = 15$, $s = 5$, $\lambda = 0.1, \ldots, 1.5$, $\mu = 1.2$, $\gamma = 0.15$, $\alpha = 1$, and $p = 0, \ldots, 1$.

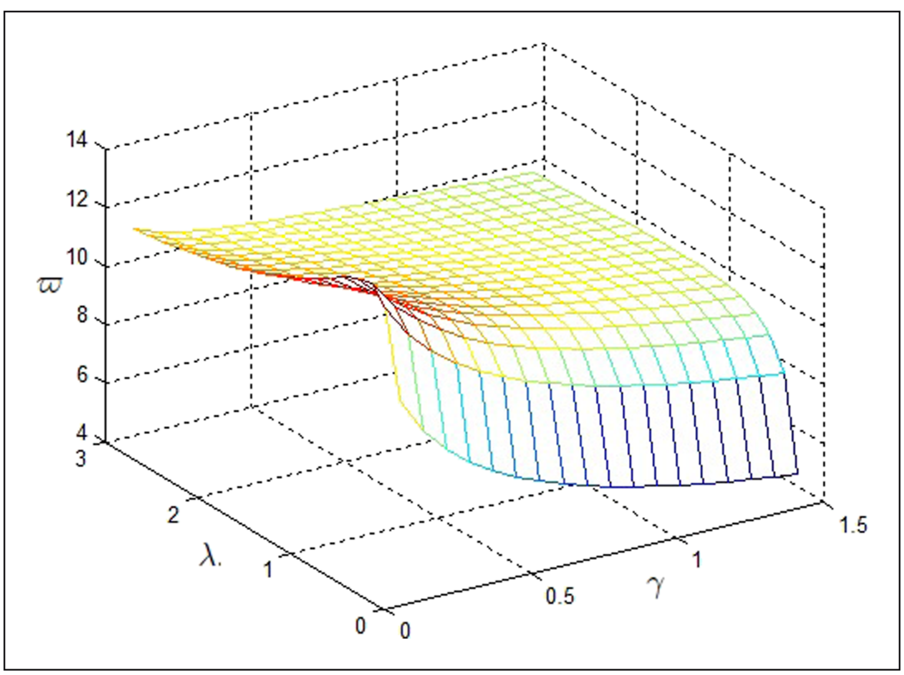

Figure 4: Effect of arrival rate λ and the retrial rate γ on the mean response time ϖ; $N = 16$, $S = 14$, $s = 6$, $\lambda = 0.15, \ldots, 3$, $\mu = 2$, $\gamma = 0.15, \ldots, 1.5$, $\alpha = 1$, and $p = 0.4$.

whereas hyperexponential distribution gives the highest value. Thus, the hyperexponential distributed lead time is the pessimistic since it overestimates the cost rate TC. In this case, the approximation of our MRSPN model by the GSPN model one, it causes little loss in terms of cost. So, under this situation, the GSPN model

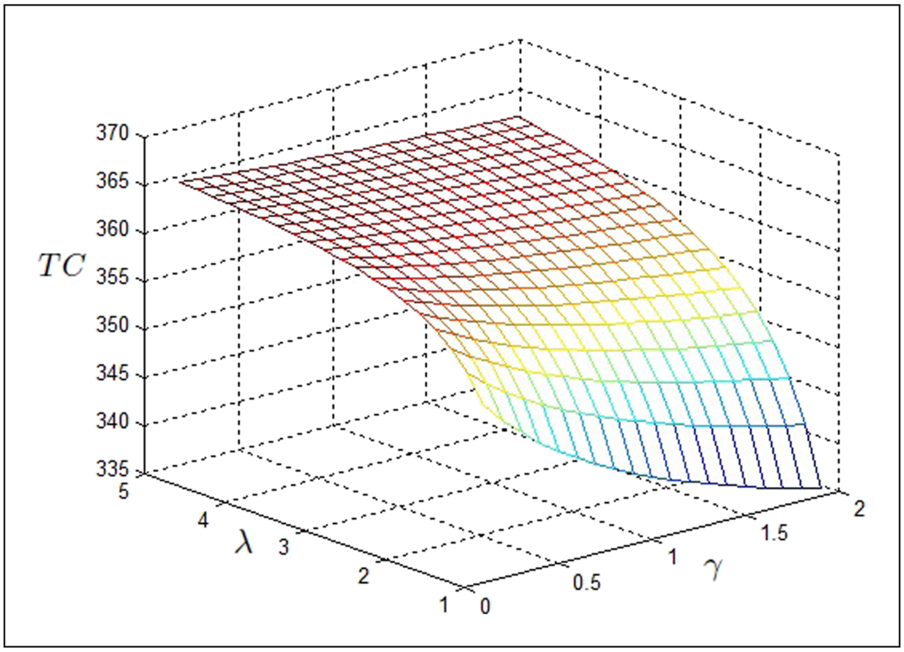

Figure 5: Effect of arrival rate λ and the retrial rate γ on the total expected cost rate TC; $N = 20$, $S = 25$, $s = 8$, $\lambda = 1, \ldots, 5$, $\mu = 3.5$, $\gamma = 0.1, \ldots, 2$, $\alpha = 2$, $p = 0.3$, $C_h = 2.8$, $C_s = 10$, and $C_w = 17$.

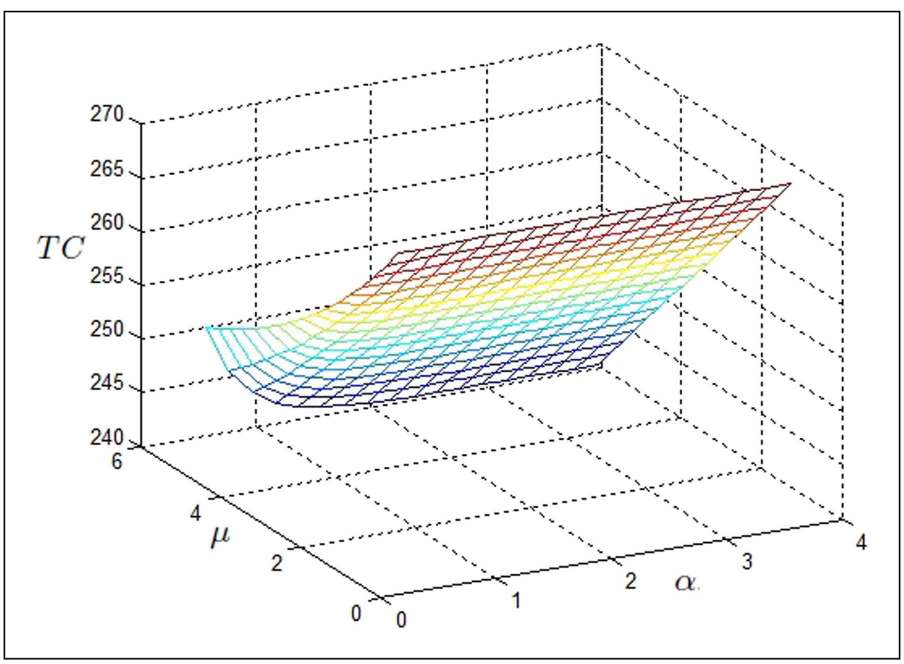

Figure 6: Effect of the replenishment rate α and the service time rate μ on the total expected cost rate TC; $N = 18$, $S = 25$, $s = 10$, $\lambda = 1.5$, $\mu = 1, \ldots, 6$, $\gamma = 0.5$, $\alpha = 0.5, \ldots, 4$, $p = 0.6$, $C_h = 0.2$, $C_s = 7$, and $C_w = 16$.

is valid to estimate the cost rate TC of our inventory system. Also, we observe that, with exponentially distributed lead time, TC (resp. n_o) lies in between that of Erlang and hyperexponential cases. In addition, the mean waiting time ω is low when the lead time follows the Erlang distribution and it is high when the lead time follows the hyperexponential distribution.

Table 6: Some performance measures for the MRSPN models of the inventory system with four types of lead time distributions $N = 4$, $S = 3$, $s = 2$, $\lambda = 0.15$, $\mu = 1.2$, $\gamma = 0.2$, $p = 0.6$, $C_h = 1.5$, $C_s = 4$, and $C_w = 2.5$

Measures	GSPN	Exp(0.4)	$E_2(0.4)$	$Hypo_2(0.5; 2)$	$H_2(0.25; 0.5; 0.375)$
n_I	1.2837	1.2837	1.2669	1.2734	1.2841
n_o	1.8280	1.8280	1.7137	1.7510	1.8313
n_r	0.0808	0.0808	0.0629	0.0690	0.0814
n_B	0.2614	0.2614	0.2725	0.2690	0.2612
n_{ES}	0.0982	0.0982	0.0947	0.0964	0.0984
n_{ESR}	0.0762	0.0762	0.0800	0.0783	0.0760
p_0	0.2877	0.2877	0.2515	0.2647	0.2890
p_s	0.1537	0.1537	0.1223	0.1329	0.1547
λ_e	0.2866	0.2866	0.3021	0.2970	0.2861
ω	6.3785	6.3785	5.6733	5.8954	6.4008
ϖ	7.2908	7.2908	6.5753	6.8010	7.3133
TC	6.8187	6.8187	6.4361	6.5635	6.8301

8 Conclusion

In this article, the queueing inventory model with finite sources of demands, $(s;S)$ replenishment policy, service time, retrials demands, lead time, and demand search from the orbit has been studied by the SPNs formalism. We analyze this inventory system for two different lead time scenarios: exponential and non-exponential. On the one hand, when the lead time is exponentially distributed, the GSPN model is proposed for this inventory system and the probability distribution obtained by using the CTMC. On the other hand, when the lead time is generally distributed, the MRSPN model is proposed for this inventory system and the probability distribution is obtained by the supplementary variable method. Both extensions GSPN and MRSPN gave us a graphical representation, which allowed us to generate the diagram states and to calculate the performance indices and the total expected cost rate of the inventory systems studied. Furthermore, numerical results are established in order to see the influence of the system parameters on some performance characteristics and to highlight the convexity of the total expected cost rate.

This work could be extended in different directions. One among these is the introduction of arbitrarily distributed research time. It is also interesting to study this inventory system when the lead time distribution is unknown. The kernel method is used to estimate the distribution of the lead time from real data.

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Conflict of interest: The authors state no conflict of interest.

Ethical approval: The conducted research is not related to either human or animal use.

Data availability statement: Data sharing is not applicable to this article as no data sets were generated or analyzed during this study.

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