Research Article

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L-Fuzzy fixed point results in $\mathcal{F}$-metric spaces with applications

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Abstract: Jleli and Samet in [*On a new generalization of metric spaces*, J. Fixed Point Theory Appl. **20** (2018), 128 (20 pages)] introduced the notion of $\mathcal{F}$-metric space as a generalization of traditional metric space and proved Banach contraction principle in the setting of this generalized metric space. The objective of this article is to use $\mathcal{F}$-metric space and establish some common fixed point theorems for (β-ψ)-contractions. Our results expand, generalize, and consolidate several known results in the literature. As applications of the main result, the solution for fuzzy initial-value problems in the background of a generalized Hukuhara derivative was discussed.

Keywords: $\mathcal{F}$-metric space, fixed point, β-admissible, Hukuhara derivative

MSC 2020: 47H10, 54H25, 65Q10, 65Q30

1 Introduction

The theory of fixed points is contemplated to be the most delightful and energetic field of investigations in the evolution of mathematical analysis. In this scope, the familiar Banach contraction principle [1] is the first result for mathematicians in the last few years. This result plays a noteworthy and remarkable contribution in solving different problems in mathematics. Zadeh [2] introduced the notion of fuzzy set which is powerful, effective and efficient means to represent and handle imprecise information in 1960. Later on, Goguen [3] extended this notion of fuzzy set to L-fuzzy set. He replaced the interval $[0, 1]$ by L that is completely distributive lattice.

Heilpern [4] gave the notion of fuzzy mappings and presented fixed point results in the metric linear space. On the other side, Estruch and Vidal [5] set up fuzzy fixed point results for fuzzy mappings in the background of complete metric space (CMS). The concept of Estruch and Vidal [5] is extended by many mathematicians in various metric spaces involving generalized contractions. Rashid et al. [6] gave the conception of $\beta_{\mathcal{F}_L}$-admissible for two L-fuzzy mappings and used this notion to establish some results for these mappings. For more details in the direction of fixed and common fixed point results of L-fuzzy mappings, we refer the investigators to [7–15].

Furthermore, Jleli and Samet [16] gave a contemporary metric space, which is known as $\mathcal{F}$-metric space, to extend the classical metric space in 2018. Later on, Alnaser et al. [17] used this concept and proved fixed point results for rational expression. Recently, Alansari et al. [18] studied some results in this $\mathcal{F}$-metric space and highlighted some open problems such as common fixed point theorems for L-fuzzy mappings.

In this research, we establish some common L-fuzzy fixed point theorems for (β-ψ)-contraction in the context of $\mathcal{F}$-metric space to generalize certain results of literature. We also provide a nontrivial example to validate the main result.

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2 Preliminaries

Definition 2.1. [3] Let $L \neq \emptyset$ and $\precsim_L$ be a partial order set.
(i) If $\kappa \vee \omega \in L$, $\kappa \wedge \omega \in L$ for any $\kappa, \omega \in L$, then L is a lattice.
(ii) If $\vee\Omega \in L$, $\wedge\Omega \in L$ for any $\Omega \subseteq L$, then L is a complete lattice.
(iii) If $\kappa \vee (\omega \wedge \varpi) = (\kappa \vee \omega) \wedge (\kappa \vee \varpi)$, $\kappa \wedge (\omega \vee \varpi) = (\kappa \wedge \omega) \vee (\kappa \wedge \varpi)$ for any $\kappa, \omega, \varpi \in L$, then L is a distributive lattice.

Definition 2.2. [3] An L-fuzzy set Ω on a nonempty set $\mathcal{W}$ is a function $\Omega : \mathcal{W} \to L$, where L satisfies (iii) with 1_L (top element) and 0_L (bottom element 0_L).

The α_L-level set of Ω is designated by Ω_{α_L} and is defined by:

$$\Omega_{\alpha_L} = \{\kappa : \alpha_L \precsim_L \Omega(\kappa)\} \quad \text{if } \alpha_L \in L \backslash \{0_L\},$$

$$\Omega_{0_L} = \overline{\{\kappa : 0_L \precsim_L \Omega(\kappa)\}}.$$

Here, $\mathcal{F}_L(\mathcal{W}_2)$ and $\mathrm{cl}(\Omega)$ represent L-fuzzy set on $\mathcal{W}_2$ and closure of Ω, respectively.

Define

$$\chi_{L_A} := \begin{cases} 0_L & \text{if } \kappa \notin \Omega \\ 1_L & \text{if } \kappa \in \Omega, \end{cases}$$

the characteristic function χ_{L_A} of a L-fuzzy set Ω.

We refer the researchers to [2–15] for further analysis in the direction of L-fuzzy mappings results.

In 2018, Jleli and Samet [16] gave a fascinating metric space, which is known as $\mathcal{F}$-metric space in this manner.

Let $f : (0, +\infty) \to \mathbb{R}$ and $\mathcal{F}$ denotes the set of functions f satisfying:
($\mathcal{F}1$) $0 < \kappa < \iota = \Rightarrow f(\kappa) \leq f(\iota)$,
($\mathcal{F}2$) for $\{\kappa_n\} \subseteq R^+$, $\lim_{n\to\infty}\kappa_n = 0 \Leftrightarrow \lim_{n\to\infty} f(\kappa_n) = -\infty$.

Definition 2.3. [16] Let $\mathcal{W} \neq \emptyset$, and let $d_{\mathcal{F}} : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$. Assume that $\exists (f, h) \in \mathcal{F} \times [0, +\infty)$ such that
(D_1) $(\kappa, \omega) \in \mathcal{W} \times \mathcal{W}$, $d_{\mathcal{F}}(\kappa, \omega) = 0 \Leftrightarrow \kappa = \omega$,
(D_2) $d_{\mathcal{F}}(\kappa, \omega) = d_{\mathcal{F}}(\omega, \kappa)$, for all $(\kappa, \omega) \in \mathcal{W} \times \mathcal{W}$,
(D_3) for every $(\kappa, \omega) \in \mathcal{W} \times \mathcal{W}$, for every $N \in \mathbb{N}$, $N \geq 2$, and for each $(u_i)_{i=1}^N \subset \mathcal{W}$, with $(u_1, u_N) = (\kappa, \omega)$, we obtain

$$d_{\mathcal{F}}(\kappa, \omega) > 0 \Rightarrow f(d_{\mathcal{F}}(\kappa, \omega)) \leq f\left(\sum_{i=1}^{N-1} d_{\mathcal{F}}(\kappa_i, \kappa_{i+1})\right) + h.$$

Then, $d_{\mathcal{F}}$ is called an $\mathcal{F}$-metric on $\mathcal{W}$ and $(\mathcal{W}, d_{\mathcal{F}})$ is claimed as an $\mathcal{F}$-metric space.

Example 2.1. [16] The function $d_{\mathcal{F}} : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$

$$d_{\mathcal{F}}(\kappa, \omega) = \begin{cases} (\kappa - \omega)^2 & \text{if } (\kappa, \omega) \in [0, 3] \times [0, 3] \\ |\kappa - \omega| & \text{if } (\kappa, \omega) \notin [0, 3] \times [0, 3], \end{cases}$$

with $f(\iota) = \ln(\iota)$ and $h = \ln(3)$, is an $\mathcal{F}$-metric.

Definition 2.4. [16] Let $(\mathcal{W}, d_{\mathcal{F}})$ be an $\mathcal{F}$-metric space.
(i) Let $\{\kappa_n\} \subseteq \mathcal{W}$. The sequence $\{\kappa_n\}$ is called $\mathcal{F}$-convergent to $\kappa \in \mathcal{W}$ if $\{\kappa_n\}$ is convergent to κ regarding $\mathcal{F}$-metric $d_{\mathcal{F}}$.
(ii) The sequence $\{\kappa_n\}$ is said to be $\mathcal{F}$-Cauchy, iff

$$\lim_{n,m\to\infty} d_{\mathcal{F}}(\kappa_n, \kappa_m) = 0.$$

(iii) If each $\mathcal{F}$-Cauchy sequence in $\mathcal{W}$ is $\mathcal{F}$-convergent to a point in $\mathcal{W}$, then $(\mathcal{W}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete.

Theorem 2.1. [16] *Let $(\mathcal{W}, d_{\mathcal{F}})$ be $\mathcal{F}$-complete $\mathcal{F}$-metric space and $O : \mathcal{W} \to \mathcal{W}$. If $\exists \lambda \in (0, 1)$ such that*

$$d_{\mathcal{F}}(O(\kappa), O(\omega)) \leq \lambda d_{\mathcal{F}}(\kappa, \omega).$$

Then, $\exists \kappa^* \in \mathcal{W}$ such that $O\kappa^* = \kappa^*$. Furthermore, for any $\kappa_0 \in \mathcal{W}$, the sequence $\{\kappa_n\} \subset \mathcal{W}$ defined by

$$\kappa_{n+1} = O(\kappa_n), \quad n \in \mathbb{N},$$

is $\mathcal{F}$-convergent to κ^*.

Later on, Alnaser et al. [17] used $\mathcal{F}$-metric space and proved some results for (α, φ) rational contraction. Al-Mezel et al. [19] introduced $(\alpha\beta, \psi)$-contractions in $\mathcal{F}$-metric space and presented some generalized results. Recently, Alansari et al. [18] studied some results in this $\mathcal{F}$-metric space and highlighted some open problems such as common fixed point theorems for L-fuzzy mappings. For more characteristics, we refer the researchers to [20–25].

Lemma 2.1. [18] *Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be nonempty closed and compact subsets of an $\mathcal{F}$-metric space $(\mathcal{W}, d_{\mathcal{F}})$. If $\kappa \in \mathcal{W}_1$, then $d_{\mathcal{F}}(\kappa, \mathcal{W}_2) \leq H_{\mathcal{F}}(\mathcal{W}_1, \mathcal{W}_2)$.*

Lemma 2.2. *Let $(\mathcal{W}, d_{\mathcal{F}})$ be a $\mathcal{F}$-metric space and $\mathcal{W}_1$ be nonempty, closed subsets of $\mathcal{W}$ and $q > 1$. Then, for each $\kappa \in \mathcal{W}$ with $d_{\mathcal{F}}(\kappa, \mathcal{W}_1) > 0$ and $q > 1$, $\exists \omega \in \mathcal{W}_1$ such that*

$$d_{\mathcal{F}}(\kappa, \omega) < q d_{\mathcal{F}}(\kappa, \mathcal{W}_1).$$

In this work, we prove some common fixed point results for L-fuzzy mappings in the background of $\mathcal{F}$-metric spaces to extend and generalize various results of in the literature. We also provide an example to support our main result.

3 Main results

In 2012, Samet et al. [21] initiated the concepts of β-ψ-contractive and β-admissible mappings and established some fixed point results for these mappings in CMSs.

Consistent with Samet et al. [21], we represent the set of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(\iota) < +\infty$ for all $\iota > 0$, where ψ^n is the n-th iterate of ψ by Ψ.

This lemma is very useful in the proof of main result.

Lemma 3.1. *If $\psi \in \Psi$, then these conditions hold*:

(i) $(\psi^n(\iota))_{n\in\mathbb{N}} \to 0$ *as* $n \to \infty$, $\forall \iota \in (0, +\infty)$,

(ii) $\psi(\iota) < \iota$, $\forall \iota > 0$,

(iii) $\psi(\iota) = 0$ *iff* $\iota = 0$.

Definition 3.1. [21] Let $O : \mathcal{W} \to W$ and $\alpha : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$. Then, O is called a β-admissible mapping if

$$\kappa, \omega \in \mathcal{W}, \quad \beta(\kappa, \omega) \geq 1 \;\Rightarrow\; \beta(O\kappa, O\omega) \geq 1.$$

Theorem 3.1. [21] *Let $(\mathcal{W}, d)$ be a CMS and O be β-admissible mapping. Suppose that*

$$\beta(\kappa, \omega) d(O\kappa, O\omega) \leq \psi(d(\kappa, \omega))$$

for all $\kappa, \omega \in \mathcal{W}$, where $\psi \in \Psi$. Also, assume that

(*i*) $\exists \kappa_0 \in \mathcal{W}$ *such that* $\beta(\kappa_0, O\kappa_0) \geq 1$,

(*ii*) *either O is continuous or for $\{\kappa_n\}$ in $\mathcal{W}$ with $\beta(\kappa_n, \kappa_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\kappa_n \to \kappa$ as $n \to +\infty$, we have* $\beta(\kappa_n, \kappa) \geq 1$, $\forall n \in \mathbb{N}$.

Then, $\exists \kappa^* \in \mathcal{W}$ *such that* $O\kappa^* = \kappa^*$.

Definition 3.2. Let $(\mathcal{W}, d_{\mathcal{F}})$ be an $\mathcal{F}$-metric space, $\alpha : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ and let O_1, O_2 be fuzzy mapping from $\mathcal{W}$ into $\mathcal{F}_L(\mathcal{W})$. The pair (O_1, O_2) is called an $\alpha_{\mathcal{F}}$-admissible if

(i) for each $\kappa \in \mathcal{W}$ and $\omega \in [O_1\kappa]_{\alpha_{O_1}(\kappa)}$, where $\alpha_{O_1}(\kappa) \in (0, 1]$, with $\alpha(\kappa, \omega) \geq 1$, we have $\alpha(\omega, z) \geq 1$, $\forall z \in [O_2\omega]_{\alpha_{O_2}(\omega)} \neq \emptyset$, where $\alpha_{O_2}(\omega) \in (0, 1]$,

(ii) for each $\kappa \in \mathcal{W}$ and $\omega \in [O_2\kappa]_{\alpha_{O_2}(\kappa)}$, where $\alpha_{O_2}(\kappa) \in (0, 1]$, with $\alpha(\kappa, \omega) \geq 1$, we have $\alpha(\omega, z) \geq 1$, $\forall z \in [O_1\omega]_{\alpha_{O_1}(\omega)} \neq \emptyset$, where $\alpha_{O_1}(\omega) \in (0, 1]$.

From now onward we denote $(\mathcal{W}, d_{\mathcal{F}})$ as an $\mathcal{F}$-metric space.

Theorem 3.2. *Let $\beta : (\mathcal{W}, d_{\mathcal{F}}) \times (\mathcal{W}, d_{\mathcal{F}}) \to [0, \infty)$ and let O_1, O_2 be L-fuzzy mappings from $(\mathcal{W}, d_{\mathcal{F}})$ into $\mathcal{F}_L(\mathcal{W})$ satisfying these assertions*:

(*i*) $(\mathcal{W}, d_{\mathcal{F}})$ *be an $\mathcal{F}$-complete,*

(*ii*) *for* $\kappa_0 \in \mathcal{W}$, $\exists \alpha_L(\kappa) \in L\backslash\{0_L\}$ *such that* $\kappa_1 \in [O_1\kappa_0]_{\alpha_L(\kappa_0)}$ or $\kappa_1 \in [O_2\kappa_0]_{\alpha_L(\kappa_0)}$ *with* $\beta(\kappa_0, \kappa_1) \geq 1$,

(*iii*) $\exists \psi \in \Psi$ *such that*

$$\max\{\beta(\kappa, \omega), \beta(\omega, \kappa)\} H_{\mathcal{F}}([O_1\kappa]_{\alpha_L(\kappa)}, [O_2\omega]_{\alpha_L(\omega)}) \leq \psi(d_{\mathcal{F}}(\kappa, \omega)) \quad \forall \kappa, \omega \in \mathcal{W}, \tag{1}$$

(*iv*) (O_1, O_2) *is $\beta_{\mathcal{F}}$-admissible,*

(*v*) *if $\{\kappa_n\}$ is a sequence in $\mathcal{W}$ such that $\beta(\kappa_n, \kappa_{n+1}) \geq 1$ for all n and $\kappa_n \to \kappa$, then* $\beta(\kappa_n, \kappa) \geq 1$, $\forall n$.

Then, there exists some $\kappa^* \in [O_1\kappa^*]_{\alpha_L(\kappa^*)} \cap [O_2\kappa^*]_{\alpha_L(\kappa^*)}$.

Proof. For $\kappa_0 \in \mathcal{W}$, there exists $\alpha_L(\kappa_0) \in L\backslash\{0_L\}$ such that $[O_1\kappa_0]_{\alpha_L(\kappa_0)} \neq \emptyset$ and $\exists \kappa_1 \in [O_1\kappa_0]_{\alpha_L(\kappa_0)}$ with $\beta(\kappa_0, \kappa_1) \geq 1$. Now for κ_1, there exists $\alpha_L(\kappa_1) \in L\backslash\{0_L\}$ such that $[O_1\kappa_0]_{\alpha_L(\kappa_0)}$ and $[O_2\kappa_1]_{\alpha_L(\kappa_1)} \in CB(\mathcal{W})$. By Lemma 2.1 and Inequality 1, we obtain that

$$\begin{aligned} 0 &< d_{\mathcal{F}}\Big(\kappa_1, [O_2\kappa_1]_{\alpha_L(\kappa_1)}\Big) \\ &\leq H_{\mathcal{F}}([O_1\kappa_0]_{\alpha_L(\kappa_0)}, [O_2\kappa_1]_{\alpha_L(\kappa_1)}) \\ &\leq \beta(\kappa_0, \kappa_1) H_{\mathcal{F}}([O_1\kappa_0]_{\alpha_L(\kappa_0)}, [O_2\kappa_1]_{\alpha_L(\kappa_1)}) \\ &\leq \max\{\beta(\kappa_0, \kappa_1), \beta(\kappa_1, \kappa_0)\} H_{\mathcal{F}}([O_1\kappa_0]_{\alpha_L(\kappa_0)}, [O_2\kappa_1]_{\alpha_L(\kappa_1)}) \\ &\leq \psi(d_{\mathcal{F}}(\kappa_0, \kappa_1)). \end{aligned} \tag{2}$$

For given $q > 1$ and by Lemma 2.2, $\exists \kappa_2 \in [O_2\kappa_1]_{\alpha_L(\kappa_1)}$ such that

$$0 < d_{\mathcal{F}}(\kappa_1, \kappa_2) < q d_{\mathcal{F}}\Big(\kappa_1, [O_2\kappa_1]_{\alpha_L(\kappa_1)}\Big). \tag{3}$$

Thus by (2) and (3), we have

$$0 < d_{\mathcal{F}}(\kappa_1, \kappa_2) \leq q\psi(d_{\mathcal{F}}(\kappa_0, \kappa_1)).$$

It is clear that $\kappa_2 \neq \kappa_1$, as $d_{\mathcal{F}}(\kappa_1, \kappa_2) < q\psi(d_{\mathcal{F}}(\kappa_0, \kappa_1))$. Since ψ is strictly increasing, so $\psi(d_{\mathcal{F}}(\kappa_1, \kappa_2)) < \psi(q\psi(d_{\mathcal{F}}(\kappa_0, \kappa_1)))$. Put $q_1 = \frac{\psi(q\psi(d_{\mathcal{F}}(\kappa_0, \kappa_1)))}{\psi(d_{\mathcal{F}}(\kappa_1, \kappa_2))}$. Then, $q_1 > 1$. Now for $\kappa_2 \in \mathcal{W}$, there exists $\alpha_L(\kappa_2) \in L\backslash\{0_L\}$ such that $[O_1\kappa_2]_{\alpha_L(\kappa_2)}$ is nonempty. So, we assume that $\kappa_2 \notin [O_1\kappa_2]_{\alpha_L(\kappa_2)}$. As $\beta(\kappa_0, \kappa_1) \geq 1$ and the pair (O_1, O_2) is $\beta_{\mathcal{F}_L}$-admissible, so $\beta(\kappa_1, \kappa_2) \geq 1$. Again by Lemma 2.1 and Inequality 1, we obtain that

$$\begin{aligned}0 &< d_{\mathcal{F}}\Big(\kappa_2, [O_1\kappa_2]_{\alpha_L(\kappa_2)}\Big)\\ &\le H_{\mathcal{F}}([O_2\kappa_1]_{\alpha_L(\kappa_1)}, [O_1\kappa_2]_{\alpha_L(\kappa_2)})\\ &= H_{\mathcal{F}}([O_1\kappa_2]_{\alpha_L(\kappa_2)}, [O_2\kappa_1]_{\alpha_L(\kappa_1)})\\ &\le \beta(\kappa_1, \kappa_2)H_{\mathcal{F}}([O_1\kappa_2]_{\alpha_L(\kappa_2)}, [O_2\kappa_1]_{\alpha_L(\kappa_1)})\\ &\le \max\{\beta(\kappa_1, \kappa_2), \beta(\kappa_2, \kappa_1)\}H_{\mathcal{F}}([O_1\kappa_2]_{\alpha_L(\kappa_2)}, [O_2\kappa_1]_{\alpha_L(\kappa_1)})\\ &\le \psi(d_{\mathcal{F}}(\kappa_2, \kappa_1)) = \psi(d_{\mathcal{F}}(\kappa_1, \kappa_2)).\end{aligned} \tag{4}$$

For given $q_1 > 1$ and by Lemma 2.2, $\exists \kappa_3 \in [O_1\kappa_2]_{\alpha_L(\kappa_2)}$ such that

$$0 < d_{\mathcal{F}}(\kappa_2, \kappa_3) < q_1 d_{\mathcal{F}}\Big(\kappa_2, [O_1\kappa_2]_{\alpha_L(\kappa_2)}\Big). \tag{5}$$

Thus by (4) and (5), we have

$$0 < d_{\mathcal{F}}(\kappa_2, \kappa_3) \le q_1\psi(d_{\mathcal{F}}(\kappa_1, \kappa_2)) = \psi(q\psi(d_{\mathcal{F}}(\kappa_0, \kappa_1))).$$

It is clear that $\kappa_3 \ne \kappa_2$, as $d_{\mathcal{F}}(\kappa_2, \kappa_3) < \psi(q\psi(d_{\mathcal{F}}(\kappa_0, \kappa_1)))$. Since ψ is strictly increasing, so $\psi(d_{\mathcal{F}}(\kappa_2, \kappa_3)) < \psi^2(q\psi(d_{\mathcal{F}}(\kappa_0, \kappa_1)))$. Put $q_2 = \frac{\psi^2(q\psi(d_{\mathcal{F}}(\kappa_0, \kappa_1)))}{\psi(d_{\mathcal{F}}(\kappa_2, \kappa_3))}$. Then, $q_2 > 1$. Now for $\kappa_3 \in \mathcal{W}$, there exists $\alpha_L(\kappa_3) \in L\backslash\{0_L\}$ such that $[O_2\kappa_3]_{\alpha_L(\kappa_3)}$ is nonempty. So, we assume that $\kappa_3 \notin [O_2\kappa_3]_{\alpha_L(\kappa_3)}$. As $\beta(\kappa_1, \kappa_2) \ge 1$ and the pair (O_1, O_2) is $\beta_{\mathcal{F}_L}$-admissible, so $\beta(\kappa_2, \kappa_3) \ge 1$. Again by Lemma 2.1 and Inequality 1, we obtain that

$$\begin{aligned}0 &< d_{\mathcal{F}}\Big(\kappa_3, [O_2\kappa_3]_{\alpha_L(\kappa_3)}\Big)\\ &\le H_{\mathcal{F}}([O_1\kappa_2]_{\alpha_L(\kappa_2)}, [O_2\kappa_3]_{\alpha_L(\kappa_3)})\\ &\le \beta(\kappa_2, \kappa_3)H_{\mathcal{F}}([O_1\kappa_2]_{\alpha_L(\kappa_2)}, [O_2\kappa_3]_{\alpha_L(\kappa_3)})\\ &\le \max\{\beta(\kappa_2, \kappa_3), \beta(\kappa_3, \kappa_2)\}H_{\mathcal{F}}([O_1\kappa_2]_{\alpha_L(\kappa_2)}, [O_2\kappa_3]_{\alpha_L(\kappa_3)})\\ &\le \psi(d_{\mathcal{F}}(\kappa_2, \kappa_3)).\end{aligned} \tag{6}$$

For given $q_2 > 1$ and by Lemma 2.2, $\exists \kappa_4 \in [O_2\kappa_3]_{\alpha_L(\kappa_3)}$ such that

$$0 < d_{\mathcal{F}}(\kappa_3, \kappa_4) < q_2 d_{\mathcal{F}}\Big(\kappa_3, [O_2\kappa_3]_{\alpha_L(\kappa_3)}\Big). \tag{7}$$

Thus by (6) and (7), we have

$$0 < d_{\mathcal{F}}(\kappa_3, \kappa_4) \le \psi^2(q\psi(d_{\mathcal{F}}(\kappa_0, \kappa_1))).$$

Pursuing in this way by induction, we can establish a sequence $\{\kappa_n\}$ in $\mathcal{W}$ such that $\kappa_{2n+1} \in [O_1\kappa_{2n}]_{\alpha_L(\kappa_{2n})}$, $\kappa_{2n+2} \in [O_2\kappa_{2n+1}]_{\alpha_L(\kappa_{2n+1})}$, and $\beta(\kappa_{n-1}, \kappa_n) \ge 1$

$$d_{\mathcal{F}}(\kappa_{2n+1}, \kappa_{2n+2}) \le \psi^{2n}(q\psi(d_{\mathcal{F}}(\kappa_0, \kappa_1))) \tag{8}$$

and

$$d_{\mathcal{F}}(\kappa_{2n+2}, \kappa_{2n+3}) \le \psi^{2n+1}(q\psi(d_{\mathcal{F}}(\kappa_0, \kappa_1))) \tag{9}$$

for all n. It follows from (8) and (9), we obtain

$$d_{\mathcal{F}}(\kappa_n, \kappa_{n+1}) \le \psi^{n-1}(q\psi(d_{\mathcal{F}}(\kappa_0, \kappa_1))), \tag{10}$$

which yields

$$\sum_{i=n}^{m-1} d_{\mathcal{F}}(\kappa_i, \kappa_{i+1}) \le \sum_{i=n}^{m-1} \psi^{i-1}(q\psi(d_{\mathcal{F}}(\kappa_0, \kappa_1))), \tag{11}$$

for $m > n$. Fix $\varepsilon > 0$ and let $n(\varepsilon) \in \mathbb{N}$ such that $\sum_{n\ge n(\delta)}\psi^{i-1}(q\psi(d_{\mathcal{F}}(\kappa_0, \kappa_1))) < \varepsilon$. Next, let $(f, h) \in \mathcal{F} \times [0, +\infty)$ be such that (D_3) holds. Let $\varepsilon > 0$ be fixed. By $(\mathcal{F}_2)$, $\exists \delta > 0$ such that

$$0 < \iota < \delta \Rightarrow f(\iota) < f(\delta) - h. \tag{12}$$

Hence, by (10), (11), and $(\mathcal{F}_1)$, we have

$$f\left(\sum_{i=n}^{m-1} d_{\mathcal{F}}(\kappa_i, \kappa_{i+1})\right) \le f\left(\sum_{i=n}^{m-1} \psi^{i-1}(q\psi(d_{\mathcal{F}}(\kappa_0, \kappa_1)))\right) \le f\left(\sum_{n\ge n(\delta)} \psi^{i-1}(q\psi(d_{\mathcal{F}}(\kappa_0, \kappa_1)))\right) < f(\varepsilon) - h \tag{13}$$

$\forall m > n \ge N$. By (D_3) and (13), we obtain

$$d_{\mathcal{F}}(\kappa_n, \kappa_m) > 0,$$

which implies

$$f(d_{\mathcal{F}}(\kappa_n, \kappa_m)) \le f\left(\sum_{i=n}^{m-1} d_{\mathcal{F}}(\kappa_i, \kappa_{i+1})\right) + \alpha < f(\varepsilon).$$

It follows by $(\mathcal{F}_1)$ that $d_{\mathcal{F}}(\kappa_n, \kappa_m) < \varepsilon, m > n \ge N$. It shows that $\{\kappa_n\}$ is $\mathcal{F}$-Cauchy. As $(\mathcal{W}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete, $\exists \kappa^* \in \mathcal{W}$ such that $\{\kappa_n\}$ is $\mathcal{F}$-convergent to κ^*, i.e.,

$$\lim_{n\to\infty} d_{\mathcal{F}}(\kappa_n, \kappa^*) = 0. \tag{14}$$

Now, we prove that $\kappa^* \in [O_1\kappa^*]_{\alpha_L(\kappa^*)}$, so we assume that $d_{\mathcal{F}}(\kappa^*, [O_1\kappa^*]_{\alpha_L(\kappa^*)}) > 0$. By condition (iv), we have $\beta(\kappa_{2n-1}, \kappa^*) \ge 1$ for all $n \in \mathbb{N}$.

Thus, by the definition of f and (D_3), we have

$$\begin{aligned}
f(d_{\mathcal{F}}(\kappa^*, [O_1\kappa^*]_{\alpha_L(\kappa^*)})) &\le f(d_{\mathcal{F}}(\kappa^*, \kappa_{2n}) + d_{\mathcal{F}}(\kappa_{2n}, [O_1\kappa^*]_{\alpha_L(\kappa^*)})) + h \\
&\le f(d_{\mathcal{F}}(\kappa^*, \kappa_{2n}) + H_{\mathcal{F}}([O_2\kappa_{2n-1}]_{\alpha_L(\kappa_{2n-1})}, [O_1\kappa^*]_{\alpha_L(\kappa^*)})) + h \\
&\le f(d_{\mathcal{F}}(\kappa^*, \kappa_{2n}) + \beta(\kappa_{2n-1}, \kappa^*)H_{\mathcal{F}}([O_2\kappa_{2n-1}]_{\alpha_L(\kappa_{2n-1})}, [O_1\kappa^*]_{\alpha_L(\kappa^*)})) + h \\
&\le f(d_{\mathcal{F}}(\kappa^*, \kappa_{2n}) + \max\{\beta(\kappa_{2n-1}, \kappa^*), \beta(\kappa^*, \kappa_{2n-1})\}H_{\mathcal{F}}([O_2\kappa_{2n-1}]_{\alpha_L(\kappa_{2n-1})}, [O_1\kappa^*]_{\alpha_L(\kappa^*)})) + h \\
&= f(d_{\mathcal{F}}(\kappa^*, \kappa_{2n}) + \max\{\beta(\kappa_{2n-1}, \kappa^*), \beta(\kappa^*, \kappa_{2n-1})\}H_{\mathcal{F}}([O_1\kappa^*]_{\alpha_L(\kappa^*)}, [O_2\kappa_{2n-1}]_{\alpha_L(\kappa_{2n-1})})) + h \\
&\le f(d_{\mathcal{F}}(\kappa^*, \kappa_{2n}) + \psi(d_{\mathcal{F}}(\kappa^*, \kappa_{2n-1}))) + h \\
&\le f(d_{\mathcal{F}}(\kappa^*, \kappa_{2n}) + d_{\mathcal{F}}(\kappa^*, \kappa_{2n-1})) + h.
\end{aligned}$$

Taking the limit in the aforementioned inequality and using (F_2) and equation (14), we have

$$\lim_{n\to\infty} f(d_{\mathcal{F}}(\kappa^*, [O_1\kappa^*]_{\alpha_L(\kappa^*)})) = \lim_{n\to\infty} f(d_{\mathcal{F}}(\kappa^*, \kappa_{2n}) + d_{\mathcal{F}}(\kappa^*, \kappa_{2n-1})) + h = -\infty,$$

which is a contradiction. Therefore, we have $d_{\mathcal{F}}(\kappa^*, [O_1\kappa^*]_{\alpha_L(\kappa^*)}) = 0$, that is, $\kappa^* \in [O_1\kappa^*]_{\alpha_L(\kappa^*)}$. Similarly by condition (iv), we have $\beta(\kappa_{2n}, \kappa^*) \ge 1$ for all $n \in \mathbb{N}$. Thus, by the definition of f, we obtain

$$\begin{aligned}
f(d_{\mathcal{F}}(\kappa^*, [O_2\kappa^*]_{\alpha_L(\kappa^*)})) &\le f(d_{\mathcal{F}}(\kappa^*, \kappa_{2n+1}) + d_{\mathcal{F}}(\kappa_{2n+1}, [O_2\kappa^*]_{\alpha_L(\kappa^*)})) + h \\
&\le f(d_{\mathcal{F}}(\kappa^*, \kappa_{2n+1}) + H_{\mathcal{F}}([O_1\kappa_{2n}]_{\alpha_L(\kappa_{2n})}, [O_2\kappa^*]_{\alpha_L(\kappa^*)})) + h \\
&\le f(d_{\mathcal{F}}(\kappa^*, \kappa_{2n+1}) + \beta(\kappa_{2n}, \kappa^*)H_{\mathcal{F}}([O_1\kappa_{2n}]_{\alpha_L(\kappa_{2n})}, [O_2\kappa^*]_{\alpha_L(\kappa^*)})) + h \\
&\le f(d_{\mathcal{F}}(\kappa^*, \kappa_{2n+1}) + \max\{\beta(\kappa_{2n}, \kappa^*), \beta(\kappa^*, \kappa_{2n})\}H_{\mathcal{F}}([O_1\kappa_{2n}]_{\alpha_L(\kappa_{2n})}, [O_2\kappa^*]_{\alpha_L(\kappa^*)})) + h \\
&\le f(d_{\mathcal{F}}(\kappa^*, \kappa_{2n+1}) + \psi(d_{\mathcal{F}}(\kappa_{2n}, \kappa^*))) + h \\
&\le f(d_{\mathcal{F}}(\kappa^*, \kappa_{2n+1}) + d_{\mathcal{F}}(\kappa_{2n}, \kappa^*)) + h.
\end{aligned}$$

Taking the limit in the aforementioned inequality and using $(\mathcal{F}_2)$ and equation (14), we have

$$\lim_{n\to\infty} f(d_{\mathcal{F}}(\kappa^*, [O_2\kappa^*]_{\alpha_L(\kappa^*)})) = \lim_{n\to\infty} f(d_{\mathcal{F}}(\kappa^*, \kappa_{2n}) + d_{\mathcal{F}}(\kappa^*, \kappa_{2n-1})) + h = -\infty,$$

which is a contradiction. Therefore, we have $d_{\mathcal{F}}(\kappa^*, [O_2\kappa^*]_{\alpha_L(\kappa^*)}) = 0$, that is, $\kappa^* \in [O_2\kappa^*]_{\alpha_L(\kappa^*)}$. □

Example 3.1. Let $\mathcal{W} = [0, 1]$, define $\mathcal{F}$-metric $d_{\mathcal{F}} : \mathcal{W} \times \mathcal{W} \to \mathbb{R}_0^+$ by $d_{\mathcal{F}}(\kappa, \omega) = |\kappa - \omega|$, whenever $\kappa, \omega \in \mathcal{W}$ and $f(\iota) = \ln(\iota)$ for $\iota > 0$ and $h = 0$. Then, $(\mathcal{W}, d_{\mathcal{F}})$ is a $\mathcal{F}$-complete $\mathcal{F}$-metric space. Let $L = \{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$, with $\upsilon_1 \preccurlyeq_L \upsilon_2 \preccurlyeq_L \upsilon_4$ and $\upsilon_1 \preccurlyeq_L \upsilon_3 \preccurlyeq_L \upsilon_4$, where υ_2 and υ_3 are not comparable. Consider $O_1, O_2 : \mathcal{W} \to \mathcal{F}_L(\mathcal{W})$ as follows:

$$O_1(\kappa)(\iota) = \begin{cases} \upsilon_4 & \text{if } 0 \le \iota \le \frac{\kappa}{6} \\ \upsilon_2 & \text{if } \frac{\kappa}{6} < \iota \le \frac{\kappa}{3} \\ \upsilon_3 & \text{if } \frac{\kappa}{3} < \iota \le \frac{\kappa}{2} \\ \upsilon_1 & \text{if } \frac{\kappa}{2} < \iota \le 1, \end{cases}$$

$$O_2(\kappa)(\iota) = \begin{cases} \upsilon_4 & \text{if } 0 \le \iota \le \frac{\kappa}{12} \\ \upsilon_1 & \text{if } \frac{\kappa}{12} < \iota \le \frac{\kappa}{8} \\ \upsilon_2 & \text{if } \frac{\kappa}{8} < \iota \le \frac{\kappa}{4} \\ \upsilon_3 & \text{if } \frac{\kappa}{4} < \iota \le 1. \end{cases}$$

Then, $\forall \kappa \in \mathcal{W}$, $\exists \alpha_L(\kappa) = \upsilon_4$, such that

$$[O_1\kappa]_{\alpha_L(\kappa)} = \left[0, \frac{\kappa}{6}\right] \quad \text{and} \quad [O_2\omega\kappa]_{\alpha_L(\kappa)} = \left[0, \frac{\kappa}{12}\right]$$

and all assertions of Theorem 3.2 with $\psi(\iota) = \frac{1}{2}\iota$, for $\iota > 0$ are satisfied. Thus, $\exists 0 \in [0,1]$ such that $0 \in [O_1 0]_{\alpha_L(0)} \cap [O_2 0]_{\alpha_L(0)}$.

Corollary 3.1. *Let $\beta : (\mathcal{W}, d_{\mathcal{F}}) \times (\mathcal{W}, d_{\mathcal{F}}) \to [0, \infty)$ and let O be an L-fuzzy mapping from $(\mathcal{W}, d_{\mathcal{F}})$ into $\mathcal{F}_L(\mathcal{W})$ satisfying these assertions*:

(*i*) $(\mathcal{W}, d_{\mathcal{F}})$ *be an $\mathcal{F}$-complete*,

(*ii*) *for $\kappa_0 \in \mathcal{W}$, $\exists \alpha_L(\kappa) \in L \backslash \{0_L\}$ such that $\kappa_1 \in [O\kappa_0]_{\alpha_L(\kappa_0)}$, with $\beta(\kappa_0, \kappa_1) \ge 1$*,

(*iii*) $\exists \psi \in \Psi$ *such that*

$$\max\{\beta(\kappa, \omega), \beta(\omega, \kappa)\} H_{\mathcal{F}}([O\kappa]_{\alpha_L(\kappa)}, [O\omega]_{\alpha_L(\omega)}) \le \psi(d_{\mathcal{F}}(\kappa, \omega)) \quad \forall \kappa, \omega \in \mathcal{W},$$

(*iv*) *O is $\beta_{\mathcal{F}_L}$-admissible*,

(*v*) *if $\{\kappa_n\}$ is a sequence in $\mathcal{W}$ such that $\beta(\kappa_n, \kappa_{n+1}) \ge 1$ for all n and $\kappa_n \to \kappa$, then $\beta(\kappa_n, \kappa) \ge 1$, $\forall n$*,

then, there exists some $\kappa^* \in [O\kappa^*]_{\alpha_L(\kappa^*)}$.

Proof. Taking one L-fuzzy mapping from $\mathcal{W}$ into $\mathcal{F}_L(\mathcal{W})$ in Theorem 3.2. □

Corollary 3.2. *Let $O_1, O_2 : (\mathcal{W}, d_{\mathcal{F}}) \to \mathcal{F}_L(\mathcal{W})$. Assume that these conditions hold*:

(*i*) $(\mathcal{W}, d_{\mathcal{F}})$ *be an $\mathcal{F}$-complete*,

(*ii*) *for $\kappa_0 \in \mathcal{W}$, $\exists \alpha_L(\kappa) \in L \backslash \{0_L\}$ such that $\kappa_1 \in [O_1\kappa_0]_{\alpha_L(\kappa_0)}$* or *$\kappa_1 \in [O_2\kappa_0]_{\alpha_L(\kappa_0)}$*,

(*iii*) $\exists \psi \in \Psi$ *such that*

$$H_{\mathcal{F}}([O_1\kappa]_{\alpha_L(\kappa)}, [O_2\omega]_{\alpha_L(\omega)}) \le \psi(d_{\mathcal{F}}(\kappa, \omega)) \quad \forall \kappa, \omega \in \mathcal{W},$$

then, there exists some $\kappa^* \in [O_1\kappa^*]_{\alpha_L(\kappa^*)} \cap [O_2\kappa^*]_{\alpha_L(\kappa^*)}$.

Proof. Taking $\beta : \mathcal{W} \times \mathcal{W} \to [0, \infty)$ by $\beta(\kappa, \omega) = 1$, for all $\kappa, \omega \in \mathcal{W}$. □

Corollary 3.3. *Let $O : (\mathcal{W}, d_{\mathcal{F}}) \to \mathcal{F}_L(\mathcal{W})$. Assume that these conditions hold*:

(*i*) $(\mathcal{W}, d_{\mathcal{F}})$ *be an $\mathcal{F}$-complete*,

(*ii*) *for $\kappa_0 \in \mathcal{W}$, $\exists \alpha_L(\kappa) \in L \backslash \{0_L\}$ such that $\kappa_1 \in [O\kappa_0]_{\alpha_L(\kappa_0)}$*,

(*iii*) $\exists \psi \in \Psi$ *such that*

$$H_{\mathcal{F}}([O\kappa]_{\alpha_L(\kappa)}, [O\omega]_{\alpha_L(\omega)}) \le \psi(d_{\mathcal{F}}(\kappa, \omega)) \quad \forall \kappa, \omega \in \mathcal{W},$$

then, there exists some $\kappa^* \in [O\kappa^*]_{\alpha_L(\kappa^*)}$.

Proof. Taking one L-fuzzy mapping from $\mathcal{W}$ into $\mathcal{F}_L(\mathcal{W})$ in Corollary 3.2. □

Theorem 3.3. *Let* $\beta : (\mathcal{W}, d_{\mathcal{F}}) \times (\mathcal{W}, d_{\mathcal{F}}) \to [0, \infty)$ *and let* $O_1, O_2 : (\mathcal{W}, d_{\mathcal{F}}) \to \mathcal{F}(\mathcal{W})$. *Assume that these conditions hold*:

(*i*) $(\mathcal{W}, d_{\mathcal{F}})$ *be an* $\mathcal{F}$*-complete,*
(*ii*) *for* $\kappa_0 \in \mathcal{W}$, *there exists* $\alpha(\kappa) \in (0, 1]$ *and* $\kappa_1 \in [O_1\kappa_0]_{\alpha(\kappa_0)}$ *or* $\kappa_1 \in [O_2\kappa_0]_{\alpha(\kappa_0)}$, *with* $\beta(\kappa_0, \kappa_1) \geq 1$,
(*iii*) $\exists \psi \in \Psi$ *such that*

$$\max\{\beta(\kappa, \omega), \beta(\omega, \kappa)\} H_{\mathcal{F}}([O_1\kappa]_{\alpha(\kappa)}, [O_2\omega]_{\alpha(\omega)}) \leq \psi(d_{\mathcal{F}}(\kappa, \omega))$$

for all $\kappa, \omega \in \mathcal{W}$,
(*iv*) (O_1, O_2) *is* $\beta_{\mathcal{F}}$*-admissible,*
(*v*) *if* $\{\kappa_n\}$ *is a sequence in* $\mathcal{W}$ *such that* $\beta(\kappa_n, \kappa_{n+1}) \geq 1$ *for all* n *and* $\kappa_n \to \kappa$, *then* $\beta(\kappa_n, \kappa) \geq 1$ *for all* n,

then, there exists some $\kappa^* \in [O_1\kappa^*]_{\alpha(\kappa^*)} \cap [O_2\kappa^*]_{\alpha(\kappa^*)}$.

Proof. Consider an L-fuzzy mappings $A, B : \mathcal{W} \to \mathcal{F}_L(\mathcal{W})$ defined by:

$$A\kappa = \chi_{L_{O_1\kappa}}$$

and

$$B\kappa = \chi_{L_{O_2\kappa}}.$$

Then, for $\alpha_L \in L \backslash \{0_L\}$, we have

$$[A\kappa]_{\alpha_L} = O_1\kappa \quad \text{and} \quad [B\kappa]_{\alpha_L} = O_2\kappa.$$

Thus, by Theorem 3.2, we obtain the conclusion. □

Corollary 3.4. *Let* $\beta : (\mathcal{W}, d_{\mathcal{F}}) \times (\mathcal{W}, d_{\mathcal{F}}) \to [0, \infty)$ *and let* $O : (\mathcal{W}, d_{\mathcal{F}}) \to \mathcal{F}_L(\mathcal{W})$. *Assume that these conditions hold*:

(*i*) $(\mathcal{W}, d_{\mathcal{F}})$ *be an* $\mathcal{F}$*-complete,*
(*ii*) *for* $\kappa_0 \in \mathcal{W}$, *there exists* $\kappa_1 \in [O\kappa_0]_{\alpha(\kappa_0)}$, *with* $\beta(\kappa_0, \kappa_1) \geq 1$,
(*iii*) $\exists \psi \in \Psi$ *such that*

$$\max\{\beta(\kappa, \omega), \beta(\omega, \kappa)\} H_{\mathcal{F}}([O\kappa]_{\alpha(\kappa)}, [O\omega]_{\alpha(\omega)}) \leq \psi(d_{\mathcal{F}}(\kappa, \omega))$$

for all $\kappa, \omega \in \mathcal{W}$,
(*iv*) O *is* $\beta_{\mathcal{F}_L}$*-admissible,*
(*v*) *if* $\{\kappa_n\}$ *is a sequence in* $\mathcal{W}$ *such that* $\beta(\kappa_n, \kappa_{n+1}) \geq 1$ *for all* n *and* $\kappa_n \to \kappa$, *then* $\beta(\kappa_n, \kappa) \geq 1$ *for all* n,

then, there exists some $\kappa^* \in [O\kappa^*]_{\alpha(\kappa^*)}$.

Proof. Taking one fuzzy mapping from $\mathcal{W}$ into $\mathcal{F}(\mathcal{W})$ in Corollary 3.3. □

Theorem 3.4. *Let* $\beta : (\mathcal{W}, d_{\mathcal{F}}) \times (\mathcal{W}, d_{\mathcal{F}}) \to [0, \infty)$ *and let* $\mathcal{R}_1, \mathcal{R}_2 : (\mathcal{W}, d_{\mathcal{F}}) \to CB(\mathcal{W})$. *Assume that these conditions hold*:

(*i*) $(\mathcal{W}, d_{\mathcal{F}})$ *be an* $\mathcal{F}$*-complete,*
(*ii*) *for each* $\kappa_0 \in \mathcal{W}$, $\exists \kappa_1 \in \mathcal{R}_1\kappa_0$, *with* $\beta(\kappa_0, \kappa_1) \geq 1$,
(*iii*) $\exists \psi \in \Psi$ *such that*

$$\max\{\beta(\kappa, \omega), \beta(\omega, \kappa)\} H_{\mathcal{F}}(\mathcal{R}_1\kappa, \mathcal{R}_2\omega) \leq \psi(d_{\mathcal{F}}(\kappa, \omega)) \quad \forall \kappa, \omega \in \mathcal{W},$$

(*iv*) $(\mathcal{R}_1, \mathcal{R}_2)$ *is* β*-admissible,*
(*v*) *if* $\{\kappa_n\}$ *is a sequence in* $\mathcal{W}$ *such that* $\beta(\kappa_n, \kappa_{n+1}) \geq 1$ *for all* n *and* $\kappa_n \to \kappa$, *then* $\beta(\kappa_n, \kappa) \geq 1$ *for all* n,

then, there exists some $\kappa^* \in \mathcal{R}_1\kappa^* \cap \mathcal{R}_2\kappa^*$.

Proof. Define L-fuzzy mappings O_1, O_2 from $\mathcal{W}$ into $\mathcal{F}_L(\mathcal{W})$ as, for some $\alpha_L \in L\backslash\{0_L\}$ by

$$O_1(\kappa)(\iota) = \begin{cases} \alpha_L, & \text{if } \iota \in \mathcal{R}_1\kappa, \\ 0, & \text{if } \iota \notin \mathcal{R}_1\kappa \end{cases}$$

and

$$O_2(\kappa)(\iota) = \begin{cases} \alpha_L, & \text{if } \iota \in \mathcal{R}_2\kappa, \\ 0, & \text{if } \iota \notin \mathcal{R}_2\kappa. \end{cases}$$

Then,

$$[O_1\kappa]_{\alpha_L} = \mathcal{R}_1\kappa \quad \text{and} \quad [O_2\kappa]_{\alpha_L} = \mathcal{R}_2\kappa.$$

Hence,

$$H_{\mathcal{F}}([O_1\kappa]_{\alpha_L(\kappa)}, [O_2\omega]_{\alpha_L(\omega)}) = H_{\mathcal{F}}(\mathcal{R}_1\kappa, \mathcal{R}_2\omega)$$

for all $\kappa, \omega \in \mathcal{W}$ and by Theorem 3.2, $\exists\kappa^* \in \mathcal{W}$ such that

$$\kappa^* \in [O_1\kappa^*]_{\alpha_L(\kappa^*)} \cap [O_2\kappa^*]_{\alpha_L(\kappa^*)} = \mathcal{R}_1\kappa^* \cap \mathcal{R}_2\kappa^*.$$ □

Corollary 3.5. *Let* $\mathcal{R} : (\mathcal{W}, d_{\mathcal{F}}) \to CB(\mathcal{W})$. *Assume that these conditions hold*:
(*i*) $(\mathcal{W}, d_{\mathcal{F}})$ *be an* $\mathcal{F}$*-complete,*
(*ii*) $\exists\psi \in \Psi$ *such that*

$$H_{\mathcal{F}}(\mathcal{R}\kappa, \mathcal{R}\omega) \leq \psi(d_{\mathcal{F}}(\kappa, \omega))$$

for all $\kappa, \omega \in \mathcal{W}$.

Then, there exists some $\kappa^* \in \mathcal{R}\kappa^*$.

Proof. Taking one multivalued mapping from $\mathcal{W}$ into $CB(\mathcal{W})$ in Corollary 3.4. □

4 Applications

Fuzzy differential equations and fuzzy integral equations play a significant role in modeling dynamic systems in which apprehension or indeterminate concepts bloom. These ideas have been constructed in visible theoretical ways, and various applications in feasible problems have been explored. Several structures for exploring fuzzy differential equations have been given. The fundamental and the pretty submission is using the Hukuhara differentiability for fuzzy functions (see [26–30]). Therefore, the study of fuzzy integral equations was given by Kaleva [31] and Seikkala [32]. In this theory, various mathematicians have used noticeable fixed point results to solve the uniqueness and existence of fuzzy differential and integral equations. In 1996, Subrahmanyam and Sudarsanam [33] set up the existence and uniqueness result for few Volterra integral equations concerning fuzzy multivalued mappings by using the familiar Banach's fixed point result. In 2015, Villamizar-Roa et al. [34] investigated the solution of fuzzy initial-value problem by using generalized Hukuhara differentiability. The researchers can see [27,31] for more details in this direction.

We symbolize $\aleph_c$ the family of all nonempty, convex, and compact subsets of $\mathbb{R}$. Then, the Hausdorff metric H in $\aleph_c$ is defined in this way.

$$H(\Xi_1, \Xi_2) = \max\left\{\sup_{\ell_1 \in \Xi_1} \inf_{\ell_2 \in \Xi_2} \|\ell_1 - \ell_2\|_{\mathbb{R}}, \sup_{\ell_2 \in \Xi_2} \inf_{\ell_1 \in \Xi_1} \|\ell_1 - \ell_2\|_{\mathbb{R}}\right\},$$

where $\Xi_1, \Xi_2 \in \aleph_c$. Therefore, $(\aleph_c, H)$ is considered as CMS (see [35]).

Definition 4.1. A mapping $\varrho : (-\infty, +\infty) \to [0, 1]$ is called an fuzzy number if it satisfies
(a) ϱ is normal,
(b) for $0 \le \lambda \le 1$,

$$\varrho(\lambda \iota_1 + (1 - \lambda)\iota_2) \ge \min\{\varrho(\iota_1), \varrho(\iota_2)\}$$

$\forall \iota_1, \iota_2 \in (-\infty, +\infty)$,
(c) ϱ is upper semicontinuous,
(d) $[\varrho]^0 = \mathrm{cl}\{\iota \in \mathbb{R} : \varrho(\iota) > 0\}$ is compact.

Therefore, $\beth^1$ will be used to show the set of fuzzy number in $(-\infty, +\infty)$, which satisfies (a)–(d).

For $\alpha \in (0, 1]$, $[\varrho]^\alpha = \{\iota \in \mathbb{R} : \varrho(\iota) > \alpha\} = [\varrho_l^\alpha, \varrho_r^\alpha]$ be α-cut of ϱ. For $\varrho \in \beth^1$, and $[\varrho]^\alpha \in \aleph_c$ for each $\alpha \in [0, 1]$. The metric on $\beth^1$ is given as:

$$d_\infty(\varrho_1, \varrho_2) = \sup_{\alpha \in [0,1]} \max\{|\varrho_{1,l}^\alpha - \varrho_{2,l}^\alpha|, |\varrho_{1,r}^\alpha - \varrho_{2,r}^\alpha|\},$$

for every $\varrho_1, \varrho_2 \in \beth^1$, where $\varrho_r^\alpha - \varrho_l^\alpha = \mathrm{diam}([\varrho])$ is claimed to be the diameter of $[\varrho]$. We represent the family of all continuous fuzzy functions defined on $[\ell_1, \ell_2]$, for $\rho > 0$ as $C([\ell_1, \ell_2], \beth^1)$.

From [36], it is familiar that $C([\ell_1, \ell_2], \beth^1)$ is a CMS regarding

$$d(\varrho_1, \varrho_2) = \sup_{\iota \in J} d_\infty(\varrho_1(\iota), \varrho_2(\iota)), \quad \varrho_1, \varrho_2 \in C([\ell_1, \ell_2], \beth^1).$$

Lemma 4.1. [31] *Let* $\varrho_1, \varrho_2 : [\ell_1, \ell_2] \to \beth^1$ *and* $\eta \in \mathbb{R}$. *Then,*
(*i*) $\int_{\ell_1}^{\ell_2} (\varrho_1 + \varrho_2)(\iota)\mathrm{d}\iota = \int_{\ell_1}^{\ell_2} \varrho_1(\iota)\mathrm{d}\iota + \int_{\ell_1}^{\ell_2} \varrho_2(\iota)\mathrm{d}\iota$,
(*ii*) $\int_{\ell_1}^{\ell_2} \eta\varrho_1(\iota)\mathrm{d}\iota = \eta\int_{\ell_1}^{\ell_2} \varrho_1(\iota)\mathrm{d}\iota$,
(*iii*) $d_\infty(\varrho_1(\iota), \varrho_2(\iota))$ *is integrable,*
(*iv*) $d_\infty\left(\int_{\ell_1}^{\ell_2} \varrho_1(\iota)\mathrm{d}\iota, \int_{\ell_1}^{\ell_2} \varrho_2(\iota)\mathrm{d}\iota\right) \le \int_{\ell_1}^{\ell_2} d_\infty(\varrho_1(\iota), \varrho_2(\iota))\mathrm{d}\iota$.

for $\iota \in [\ell_1, \ell_2]$.

Definition 4.2. [34] Let $\beth^n$ denote the family of fuzzy numbers in $\mathbb{R}^n$ and for $\varrho, \varkappa, \hbar \in \beth^n$. An element $\hbar$ is called the Hukuhara difference of two points ϱ and $\varkappa$, if $\varrho = \varkappa + \hbar$ is satisfied. Now, $\varrho\Theta_H\varkappa$ (or $\varrho - \varkappa$) denotes the Hukuhara difference of the points ϱ and $\varkappa$. This is obvious that $\varrho\Theta_H\varrho = \{0\}$, and if $\varrho\Theta_H\varkappa$ exists, then this is unique.

Definition 4.3. [34] Let $\zeta : (\ell_1, \ell_2) \to \beth^n$. The mapping ζ is called GH-differentiable at $\iota_0 \in (\ell_1, \ell_2)$, if $\exists \zeta_G'(\iota_0) \in \beth^n$ such that

$$\zeta(\iota_0 + \delta)\Theta_H\zeta(\iota_0), \zeta(\iota_0)\Theta_H\zeta(\iota_0 - \delta)$$

and

$$\lim_{\delta \to 0^+} \frac{\zeta(\iota_0 + \delta)\Theta_H\zeta(\iota_0)}{\delta} = \lim_{\delta \to 0^+} \frac{\zeta(\iota_0)\Theta_H\zeta(\iota_0 - \delta)}{\delta} = \zeta_G'(\iota_0).$$

Consider

$$\begin{cases} \varrho'(\iota) = \zeta(\iota, \varrho(\iota)), & \iota \in J = [\ell_1, \rho] \\ \varrho(0) = \varrho_0, \end{cases} \tag{15}$$

where ϱ' is appropriated as GH-differentiable and $\zeta : J \times ℶ^1 \to ℶ^1$ is continuous. The initial data ϱ_0 is supposed in $ℶ^1$. We represent the set of all $\zeta : J \to ℶ^1$ with continuous derivative as $C^1(J, ℶ^1)$.

Lemma 4.2. *A function $\varrho \in C^1(J, ℶ^1)$ is a solution of (15) iff it satisfies*

$$\varrho(\iota) = \varrho_0 \Theta_H (-1) \int_{\ell_1}^{\iota} \zeta(s, \varrho(s)) \mathrm{d}s, \quad \iota \in J = [\ell_1, \rho].$$

Theorem 4.1. *Let $\zeta : J \times ℶ^1 \to ℶ^1$ be continuous such that*

(*i*) *$\zeta(\iota, \varrho) < \zeta(\iota, \varkappa)$, for $\varrho < \varkappa$,*

(*ii*) *there exist some constants $\tau > 0$ such that $\lambda \in (0, \frac{1}{2(\rho - \ell_1)})$ such that*

$$\|\zeta(\iota, \varrho(\iota)) - \zeta(\iota, \varkappa(\iota))\|_{\mathbb{R}} \leq \tau \max_{\iota \in J}\{d_\infty(\varrho, \varkappa) e^{-\tau(\iota - \ell_1)}\},$$

for $\varrho, \varkappa \in ℶ^1$, with $\varrho < \varkappa$ and $\iota \in J$. Then, (15) has a solution in $C^1(J, ℶ^1)$.

Proof. Consider $C^1(J, ℶ^1)$ equipped with

$$d_\tau(\varrho, \varkappa) = \sup_{\iota \in J}\{d_\infty(\varrho(\iota), \varkappa(\iota)) e^{-\tau(\iota - \ell_1)}\},$$

for $\varrho, \varkappa \in C^1(J, ℶ^1)$ and $\tau > 0$. Then, with $\zeta(\varrho) = \ln(\varrho)$, $\varrho > 0$ and $h = 0$, $(C^1(J, ℶ^1), d_\tau)$ is $\mathcal{F}$-CMS.

Let $M, Q : C^1(J, ℶ^1) \to (0, 1]$. For $\varrho \in C^1(J, ℶ^1)$, take

$$L_\varrho(\iota) = \varrho_0 \Theta_H (-1) \int_{\ell_1}^{\iota} \zeta(s, \varrho(s)) \mathrm{d}s.$$

Assume $\varrho < \varkappa$. Then, it follows by assumption (a) that

$$L_\varrho(\iota) = \varrho_0 \Theta_H (-1) \int_{\ell_1}^{\iota} \zeta(s, \varrho(s)) \mathrm{d}s < \varrho_0 \Theta_H (-1) \int_{\ell_1}^{\iota} \zeta(s, \varkappa(s)) \mathrm{d}s = R_\varkappa(\iota).$$

Therefore, $L_\varrho(\iota) \neq R_\varkappa(\iota)$. Define $O_1, O_2 : C^1(J, ℶ^1) \to ℶ^{C^1(J, ℶ^1)}$ by

$$\mu_{O_1\varrho}(r) = \begin{cases} M(\varrho), & \text{if } r(\iota) = L_\varrho(\iota) \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu_{O_2\varkappa}(r) = \begin{cases} Q(\varkappa), & \text{if } r(\iota) = R_\varkappa(\iota) \\ 0, & \text{otherwise.} \end{cases}$$

Now if $\alpha_{O_1}(\varrho) = M(\varrho)$ and $\alpha_{O_2}(\varkappa) = Q(\varkappa)$, we obtain

$$[O_1\varrho]_{\alpha_{O_1}(\varrho)} = \{r \in X : (O_1\varrho)(\iota) \geq M(\varrho)\} = \{L_\varrho(\iota)\},$$

and likewise $[O_2\varkappa]_{\alpha_{O_1}(\varkappa)} = \{R_\varkappa(\iota)\}$. Hence,

$$
\begin{aligned}
H([O_1\varrho]_{\alpha_{O_1}(\varrho)}, [O_2\varkappa]_{\alpha_{O_1}(\varkappa)}) &= \max\left\{\begin{array}{c}\sup\limits_{\varrho\in[O_1\varrho]_{\alpha_{O_1}(\varrho)},\varkappa\in[O_2\varkappa]_{\alpha_{O_1}(\varkappa)}} \inf\|\varrho-\varkappa\|_{\mathbb{R}},\\ \sup\limits_{\varkappa\in[O_2\varkappa]_{\alpha_{O_1}(\varkappa)},\varrho\in[O_1\varrho]_{\alpha_{O_1}(\varrho)}} \inf\|\varrho-\varkappa\|_{\mathbb{R}}\end{array}\right\}\\
&\le \max\left\{\sup_{\iota\in J}\|L_{\varrho}(\iota)-R_{\varkappa}(\iota)\|_{\mathbb{R}}\right\}\\
&= \sup_{\iota\in J}\|L_{\varrho}(\iota)-R_{\varkappa}(\iota)\|_{\mathbb{R}}\\
&= \sup_{\iota\in J}\left\|\int_{\ell_1}^{\iota}\zeta(s,\varrho(s))\mathrm{d}s-\int_{0}^{\iota}\zeta(s,\varkappa(s))\mathrm{d}s\right\|_{\mathbb{R}}\\
&\le \sup_{\iota\in J}\left\{\int_{\ell_1}^{\iota}\|\zeta(s,\varrho(s))-\zeta(s,\varkappa(s))\|\mathrm{d}s\right\}\\
&\le \sup_{\iota\in J}\left\{\int_{\ell_1}^{\iota}\mathrm{d}u\lambda\max\{D_{\infty}(\varrho,\varkappa)e^{-\tau(\iota-\ell_1)}\}ds\right\}\\
&\le \lambda\sup_{\iota\in J}\{(\iota-\ell_1)\max\{D_{\infty}(\varrho,\varkappa)e^{-\tau(\iota-\ell_1)}\}\}\\
&\le \lambda(\rho-\ell_1)d_{\mathcal{F}}(\varrho,\varkappa)\\
&\le \frac{1}{2}d_{\mathcal{F}}(\varrho,\varkappa)\\
&= \psi(d_{\mathcal{F}}(\varrho,\varkappa)).
\end{aligned}
$$

Thus, all the assumptions of Theorem 3.3 are fulfilled with $\psi(\iota)=\frac{1}{2}\iota$, for $\iota>0$. Thus, ϱ^* is a solution of (15). □

5 Conclusion

In this work, we established some significant common fixed point theorems for L-fuzzy mappings with respect to (α,ψ)-contractions in the background of complete $\mathcal{F}$-metric spaces. The established theorems improved and generalized different conventional theorems in fuzzy fixed point theory. As an application, we explored the solution for fuzzy initial-value problems. Our theorems are new and important contribution to the existing results in fuzzy fixed point theory. Our results can be extended and improved for intuitionistic fuzzy mappings as a future work.

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