Research Article

Jehad Alzabut*, Mohamed Houas, and Mohamed I. Abbas

Application of fractional quantum calculus on coupled hybrid differential systems within the sequential Caputo fractional *q*-derivatives

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Abstract: In the current manuscript, we combine the q-fractional integral operator and q-fractional derivative to investigate a coupled hybrid fractional q-differential systems with sequential fractional q-derivatives. The existence and uniqueness of solutions for the proposed system are established by means of Leray-Schauder's alternative and the Banach contraction principle. Furthermore, the Ulam-Hyers and Ulam-Hyers-Rassias stability results are discussed. Finally, two illustrative examples are given to highlight the theoretical findings.

Keywords: fractional q-calculus, existence and uniqueness of solutions, boundary conditions, Ulam-Hyers-stability, fixed point

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1 Introduction and preliminaries

In recent years, the fractional differential equations have been applied in various areas of engineering, mathematics, physics, and other applied sciences because of their ability to describe memory effects (see, e.g., [1–3]).

The discrete versions of continuous-type problems in science can be made from the point of view of the so-called q-calculus. The Caputo q-fractional derivative has been introduced on the base of the fractional q-integral and fractional q-derivative, always with the lower limit of integration equal to 0. However, in some considerations, such as solving q-differential equation of fractional order with initial values at a nonzero point, it is of interest to allow that the lower limit of integration is variable. Recently, many researchers got much interested in looking at fractional q-differential equations as new model equations for many physical problems. For example, some researchers obtained q-analogue of the integral and differential fractional operators properties, such as the q-Laplace transform, q-Taylor's formula, and q-Mittage-Leffler function. For some fundamental results in the theory of fractional calculus and fractional differential equations, see, e.g., [4–9]. In addition, many scholars have paid much attention to the fractional quantum calculus, which has lots of applications in different areas of mathematics, such as

* **Corresponding author: Jehad Alzabut,** Department of Mathematics and Sciences, Prince Sultan University, 11586 Riyadh, Saudi Arabia; Department of Industrial Engineering, OSTİM Technical University, Ankara 06374, Türkiye, e-mail: jalzabut@psu.edu.sa, jehad.alzabut@ostimteknik.edu.tr
Mohamed Houas: Laboratory FIMA, UDBKM, Khemis Miliana University, Algeria, e-mail: m.houas.st@univ-dbkm.dz
Mohamed I. Abbas: Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, Alexandria 21511, Egypt, e-mail: miabbas@alexu.edu.eg

combinatorics, number theory, basic hypergeometric functions, and other sciences. Recently, considerable attention has been given to the existence and stability of solutions for differential equations and systems with fractional quantum calculus. For more details, we refer the reader to the monographs [10–12] and the references cited therein. Hybrid differential equations have been the object of many researchers, see [13–15]. The hybrid differential equation of first order

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\xi(\varsigma)}{\phi(\varsigma,\xi(\varsigma))}\right) = \psi(\varsigma,\xi(\varsigma)), \quad \varsigma \in [0,T], \\ \xi(\varsigma_0) = \xi_0 \in \mathbb{R}, \end{cases}$$

was studied in [16] under the hypotheses that the functions $\phi \in C([0,T]\times\mathbb{R},\mathbb{R}\backslash\{0\})$ and $\psi \in C([0,T]\times\mathbb{R},\mathbb{R})$. The hybrid differential equations with fractional derivative have been addressed extensively by several researchers, for which the reader can consult [17–20] and the references cited therein. On the other hand, several papers including the hybrid differential equations with fractional q-derivative have been raised, for example [21–23] and the references therein. In [21], Ahmad and Ntouyas studied the existence of solutions for the fractional hybrid q-differential equation as follows:

$$\begin{cases} {}^{C}D_q^{\delta}\left(\dfrac{\xi(\varsigma)}{\phi(\varsigma,\xi(\varsigma))}\right) = \psi(\varsigma,\xi(\varsigma)), \varsigma \in [0,1], 1<\delta\le 2, 0<q<1, \\ \xi(0)=0, \quad \xi(1)=0, \end{cases}$$

where ${}^{C}D_q^{\delta}$ is the Caputo fractional q-derivative of order δ and the functions $\phi \in C([0,1]\times\mathbb{R},\mathbb{R}\backslash\{0\})$, and $\psi \in C([0,1]\times\mathbb{R},\mathbb{R})$. In [22], the author studied the existence, uniqueness, and Ulam-Hyers-Rassias stability for a class of hybrid Caputo fractional q-differential pantograph equations described as follows:

$$\begin{cases} {}^{C}D_q^{\delta}\left[\dfrac{\xi(\varsigma)}{\sum_{i=1}^{k}\phi_i(\varsigma,\xi(\varsigma),\xi(\lambda\varsigma))}\right] \\ = \displaystyle\sum_{i=1}^{k}\varphi_i\left(\varsigma,\xi(\varsigma),\xi(\eta\varsigma),{}^{C}D_q^{\delta}\left[\dfrac{\xi(\varsigma)}{\sum_{i=1}^{k}\phi_i(\varsigma,\xi(\varsigma),\xi(\lambda\varsigma))}\right]\right), \\ \xi(0)+\psi(\xi)=\xi_0, \xi_0\in\mathbb{R}, \end{cases}$$

where $0<q,\delta<1$, $0<\lambda,\mu<1$, ${}^{C}D_q^{\delta}$ is the Caputo fractional q-derivative, $\varsigma\in[0,T]$, $\phi_i \in C(J\times\mathbb{R}^2,\mathbb{R}-\{0\})$, and $\varphi_i \in C(J\times\mathbb{R}^3,\mathbb{R})$, $i=1,\ldots,k$, $k\in\mathbb{N}^*$. Samei and Ranjbar [23] discussed the existence and uniqueness of solutions for the fractional hybrid q-differential inclusions with the boundary conditions of the form

$$\begin{cases} {}^{C}D_q^{\delta}\left[\dfrac{\xi}{\phi(\varsigma,\xi(\varsigma),I_q^{\alpha_1}\xi(\varsigma),I_q^{\alpha_2}\xi(\varsigma),\ldots,I_q^{\alpha_n}\xi(\varsigma))}\right] \\ \in \varphi(\varsigma,\xi(\varsigma),I_q^{\gamma_1}\xi(\varsigma),I_q^{\gamma_2}\xi(\varsigma),\ldots,I_q^{\gamma_m}\xi(\varsigma)), \varsigma\in[0,1], \\ \xi(0)=\xi_0, \xi(1)=\xi_1, \quad \xi_0,\xi_1\in\mathbb{R}, \end{cases}$$

where $1<\delta\le 2$, $q\in(0,1)$, I_q^{β} denotes the Riemann-Liouville-type q-integral of order $\beta>0$, $\beta\in\{\alpha_i,\gamma_j\}$, $i=1,\ldots,n$, $j=1,\ldots,m$, $n,m\in\mathbb{N}$, ${}^{C}D_q^{\delta}$ denotes Caputo-type q-derivative of order δ, $\phi:[0,1]\times\mathbb{R}^n\to(0,\infty)$ is continuous, and $\varphi:[0,1]\times\mathbb{R}^m\to P(\mathbb{R})$ is a multifunction. Recently, sequential hybrid fractional differential equations have also been studied by several scholars, see, e.g., [24–27] and the references cited therein. In [24], the authors considered the fractional sequential type of the hybrid differential equation

$$\begin{cases} {}^{C}D^{\delta}\left[\dfrac{{}^{C}D^{\gamma}\xi(\varsigma)-\sum_{i=1}^{k}I^{\gamma_i}\phi_i(\varsigma,\xi(\varsigma))}{\psi(\varsigma,\xi(\varsigma))}\right] = \varphi(\varsigma,\xi(\varsigma),I^{\alpha}\xi(\varsigma)), \varsigma\in[0,T], \\ a\dfrac{\xi(0)}{\psi(\varsigma,\xi(\varsigma))}+b\dfrac{\xi(\xi)}{\psi(\varsigma,\xi(\varsigma))}=c, {}^{C}D^{\gamma}\xi(0)=0, \end{cases}$$

where $0 < \delta, \gamma \leq 1, 1 < \delta + \gamma \leq 2, I^{\beta}$ is the Riemann-Liouville fractional integral of order $\beta > 0, \beta \in \{\alpha, \gamma_i\}$, $\phi_i \in C([0, T] \times \mathbb{R}, \mathbb{R}), i = 1, 2, \ldots, k, \psi \in C([0, T] \times \mathbb{R}, \mathbb{R}\backslash\{0\})$, and $\varphi \in C([0, T] \times \mathbb{R}^2, \mathbb{R})$ and a, b, c are real constants with $a + b \neq 0$. The existence and uniqueness results were obtained by applying a generalization of Krasnoselskii's fixed point theorem. In [25], the authors studied the existence, uniqueness, and stability analysis for a class of sequential hybrid fractional differential equations described as follows:

$$\begin{cases} {}^{C}D^{\delta}\left[\dfrac{{}^{C}D^{\gamma}\xi(\varsigma) - \sum_{i=1}^{k} I^{\lambda_i}\phi_i(\varsigma, \xi(\varsigma), {}^{C}D^{\gamma-1}\xi(\varsigma))}{\psi(\varsigma, \xi(\varsigma), {}^{C}D^{\gamma-1}\xi(\varsigma))}\right] \\ = \varphi(\varsigma, \xi(\varsigma), I^{\alpha}\xi(\varsigma)), \varsigma \in [0, 1], \\ {}^{C}D^{\gamma}\xi(0) = 0, \xi(0) = g_1(\xi(\varepsilon)), \xi(1) = g_2(\xi(\varepsilon)), \end{cases}$$

where $0 < \delta \leq 1, 1 < \gamma \leq 2, 0 < \varepsilon < 1, I^{\beta}$ is the Riemann-Liouville fractional integral of order $\beta > 0, \beta \in \{\alpha, \lambda_i\}$, the functions $\psi : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}\backslash\{0\}$, $\phi_i : [0, 1] \times \mathbb{R} \to \mathbb{R}, i = 1, 2, \ldots, k$, and $\varphi \in C([0, 1] \times \mathbb{R}^2, \mathbb{R})$ satisfy the Carathéodory conditions, the boundary functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are non linear, and $\mathbb{R}$ represents the set of real numbers. To the best of our knowledge, there is no article discussing the coupled system of fractional hybrid q-differential equations in the literature. The objective of this article is to study the sequential coupled system of fractional hybrid q-differential equations of the following form:

$$\begin{cases} {}^{C}D_q^{\delta_1}\left({}^{C}D_q^{\theta_1}\left(\dfrac{\kappa(\varsigma)}{\phi_1(\varsigma, \kappa(\varsigma), \nu(\varsigma))}\right)\right) = \psi_1(\varsigma, \kappa(\varsigma), \nu(\varsigma)), \varsigma \in [0, 1], \\ {}^{C}D_q^{\delta_2}\left({}^{C}D_q^{\theta_2}\left(\dfrac{\nu(\varsigma)}{\phi_2(\varsigma, \kappa(\varsigma), \nu(\varsigma))}\right)\right) = \psi_2(\varsigma, \kappa(\varsigma), \nu(\varsigma)), \varsigma \in [0, 1], \\ \kappa(0) = \kappa(1) = 0, \quad \nu(0) = \nu(1) = 0, \\ 0 < q < 1, \quad 0 < \delta_i, \theta_i \leq 1, \quad i = 1, 2, \end{cases} \tag{1}$$

where ${}^{C}D_q^{\alpha}$ is the Caputo fractional q-derivative of order α, where $\alpha \in \{\delta_i, \theta_i\}$, $\phi_i : [0, 1] \times \mathbb{R}^2 \to \mathbb{R} - \{0\}$ and $\psi_i : [0, 1] \times \mathbb{R}^2 \to \mathbb{R},\ i = 1, 2$ are continuous functions. The operator ${}^{C}D_q^{\alpha}$ is the fractional q-derivative in the sense of the Caputo, which is define as follows:

$$\begin{cases} {}^{C}D_q^{\alpha}\nu(\varsigma) = I_q^{n-\alpha}D_q^{n}\nu(\varsigma), \quad \alpha > 0, \\ D_q^{0}\nu(\varsigma) = \nu(\varsigma), \end{cases}$$

where n is the smallest integer greater than or equal to α. The fractional q-integral of the Riemann-Liouville-type [28–30] is given as follows:

$$\begin{cases} I_q^{\alpha}\nu(\varsigma) = \dfrac{1}{\Gamma_q(\alpha)}\displaystyle\int_0^{\varsigma}(\varsigma - qs)^{(\alpha-1)}\nu(s)d_qs, \alpha > 0, \\ I_q^{0}\nu(\varsigma) = \nu(\varsigma), \end{cases}$$

where the q-gamma function is defined by $\Gamma_q(\alpha) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}}$, $\alpha \in \mathbb{R}\backslash\{0, -1, -2, \ldots\}$ and satisfies

$$\Gamma_q(\alpha + 1) = [\alpha]_q\Gamma_q(\alpha), \quad [a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}.$$

We need the following essential lemmas.

Lemma 1. [31] *For $\alpha, \beta \geq 0$ and the function ν defined in $[0, 1]$, the following formulas hold*:

$$I_q^{\alpha}I_q^{\beta}\nu(\varsigma) = I_q^{\alpha+\beta}\nu(\varsigma) \text{ and } D_q^{\alpha}I_q^{\alpha}\nu(\varsigma) = \nu(\varsigma).$$

Lemma 2. [31] *Let $\alpha \in \mathbb{R}_+$. Then, the following equality holds*:

$$I_q^{\alpha} D_q^{\alpha} \nu(\varsigma) = \nu(\varsigma) - \sum_{j=0}^{\sigma-1} \frac{\varsigma^j}{\Gamma_q(j+1)} D_q^j \nu(0).$$

Lemma 3. [31] *For $\alpha \in \mathbb{R}_+$ and $\beta > -1$, we have*

$$I_q^{\alpha}[(\varsigma - x)^{(\beta)}] = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)}(\varsigma - x)^{(\alpha+\beta)}.$$

Furthermore, the following auxiliary result is crucial.

Lemma 4. *For $i = 1, 2$, let $\phi_i \in C([0, 1] \times \mathbb{R}^2, \mathbb{R} - \{0\})$ and $\varphi_i \in C([0, 1], \mathbb{R})$. Then, the solution of the problem*

$$\begin{cases} {}^{C}D_q^{\delta_1}\left({}^{C}D_q^{\theta_1}\left(\dfrac{\kappa(\varsigma)}{\phi_1(\varsigma, \kappa(\varsigma), \nu(\varsigma))}\right)\right) = \varphi_1(\varsigma), \ \varsigma \in [0, 1] \\ {}^{C}D_q^{\delta_2}\left({}^{C}D_q^{\theta_2}\left(\dfrac{\kappa(\varsigma)}{\phi_2(\varsigma, \kappa(\varsigma), \nu(\varsigma))}\right)\right) = \varphi_2(\varsigma), \ \varsigma \in [0, 1], \\ \kappa(0) = \kappa(1) = 0, \nu(0) = \nu(1) = 0, \\ 0 < q < 1, \quad 0 < \delta_i, \theta_i < 1, \quad i = 1, 2, \end{cases} \tag{2}$$

is given as follows:

$$\kappa(\varsigma) = \frac{\phi_1(\varsigma, \kappa(\varsigma), \nu(\varsigma))}{\Gamma_q(\delta_1+\theta_1)} \int_0^{\varsigma} (\varsigma - qs)^{(\delta_1+\theta_1-1)} \varphi_1(s) d_q s - \frac{\phi_1(\varsigma, \kappa(\varsigma), \nu(\varsigma))\varsigma^{\theta_1}}{\Gamma_q(\delta_1+\theta_1)} \int_0^1 (1 - qs)^{(\delta_1+\theta_1-1)} \varphi_1(s) d_q s \tag{3}$$

and

$$\nu(\varsigma) = \frac{\phi_2(\varsigma, \kappa(\varsigma), \nu(\varsigma))}{\Gamma_q(\delta_2+\theta_2)} \int_0^{\varsigma} (\varsigma - qs)^{(\delta_2+\theta_2-1)} \varphi_2(s) d_q s - \frac{\phi_2(\varsigma, \kappa(\varsigma), \nu(\varsigma))\varsigma^{\theta_2}}{\Gamma_q(\delta_2+\theta_2)} \int_0^1 (1 - qs)^{(\delta_2+\theta_2-1)} \varphi_2(s) d_q s. \tag{4}$$

Proof. Applying the Riemann-Liouville fractional q-integral operators $I_q^{\delta_1}$ and $I_q^{\delta_2}$ on both sides of equations in (2) and using Lemma 2, we obtain

$${}^{C}D_q^{\theta_1}\left(\frac{\kappa(\varsigma)}{\phi_1(\varsigma, \kappa(\varsigma), \nu(\varsigma))}\right) = I_q^{\delta_1}\varphi_1(\varsigma) + c_1, \ {}^{C}D_q^{\theta_2}\left(\frac{\kappa(\varsigma)}{\phi_2(\varsigma, \kappa(\varsigma), \nu(\varsigma))}\right) = I_q^{\delta_2}\varphi_2(t) + d_1, \tag{5}$$

where $c_1, d_1 \in \mathbb{R}$. Next, applying the Riemann-Liouville fractional q-integral operator $I_q^{\theta_1}$ and $I_q^{\theta_2}$ on both sides and using Lemma 2, we obtain

$$\kappa(\varsigma) = \phi_1(\varsigma, \kappa(\tau), \nu(\varsigma))\left[I_q^{\delta_1+\theta_1}\varphi_1(\varsigma) + \frac{c_1}{\Gamma_q(\theta_1+1)}\varsigma^{\theta_1} + c_2\right], \tag{6}$$

$$\nu(\varsigma) = \phi_2(\varsigma, \kappa(\varsigma), \nu(\varsigma))\left[I_q^{\delta_2+\theta_2}\varphi_2(\varsigma) + \frac{\mathrm{d}_1}{\Gamma_q(\theta_2+1)}\varsigma^{\theta_2} + d_2\right], \tag{7}$$

where $c_2, d_2 \in \mathbb{R}$. Now, using the conditions $\kappa(0) = \kappa(1) = 0$ and $\nu(0) = \nu(1) = 0$, we can obtain

$$\begin{aligned} c_1 &= -\Gamma_q(\theta_1+1) I_q^{\delta_1+\theta_1}\varphi_1(1), \quad c_2 = 0, \\ d_1 &= -\Gamma_q(\theta_2+1) I_q^{\delta_2+\theta_2}\varphi_2(1), \quad d_2 = 0. \end{aligned} \tag{8}$$

Substituting the values of $c_i, d_i, i = 1, 2$ in (6) and (7), we obtain (3) and (4).

Conversely, applying the operators ${}^{C}D_q^{\theta_1}$ and ${}^{C}D_q^{\theta_2}$ on (3) and (4), respectively, it follows that

$$\begin{aligned} {}^{C}D_q^{\theta_1}\left(\frac{\kappa(\varsigma)}{\phi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma))}\right) &= I_q^{\delta_1}\varphi_1(\varsigma) - \frac{\Gamma_q(\theta_1+1)}{\Gamma_q(\delta_2+\theta_1)} I_q^{\delta_1+\theta_1}\varphi_1(1), \\ {}^{C}D_q^{\theta_2}\left(\frac{\nu(\varsigma)}{\phi_2(\varsigma,\kappa(\varsigma),\nu(\varsigma))}\right) &= I_q^{\delta_2}\varphi_2(\varsigma) - \frac{\Gamma_q(\theta_2+1)}{\Gamma_q(\delta_2+\theta_2)} I_q^{\delta_2+\theta_2}\varphi_2(1). \end{aligned} \tag{9}$$

Next, applying the operators ${}^{C}D_q^{\delta_1}$ and ${}^{C}D_q^{\delta_2}$, we obtain

$${}^{C}D_q^{\delta_1}\left({}^{C}D_q^{\theta_1}\left(\frac{\kappa(\varsigma)}{\phi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma))}\right)\right) = \varphi_1(\varsigma), \quad {}^{C}D_q^{\delta_2}\left({}^{C}D_q^{\theta_2}\left(\frac{\nu(\varsigma)}{\phi_2(\varsigma,\kappa(\varsigma),\nu(\varsigma))}\right)\right) = \varphi_2(\varsigma). \tag{10}$$

From (3) and (4), it is easy to verify that the boundary conditions $\kappa(0) = \kappa(1) = 0$ and $\nu(0) = \nu(1) = 0$ are satisfied. This establishes the equivalence between (2) and (3)–(4). This completes the proof. □

The rest of the article is organized as follows. In Section 2, we establish sufficient conditions for the existence and uniqueness of solutions for the main system. The stability of solutions is discussed in Section 4. We present numerical examples to illustrate and validate the effectiveness of the main results in Section 5.

2 Existence and uniqueness results

Theorem 5. *Let us now define the space*

$$W \times Z = \{(\kappa,\nu) : \kappa,\nu \in C([0,1],\mathbb{R})\},$$

endowed with the norm $\|(\kappa,\nu)\|_{W\times Z} = \|\kappa\| + \|\nu\|$, *where*

$$\|\kappa\| = \sup\{|\kappa(\varsigma)| : \varsigma \in [0,1]\} \text{ and } \|\nu\| = \sup\{|\nu(\varsigma)| : \varsigma \in [0,1]\}.$$

It is clear that $(W \times Z, \|.\|_{W\times Z})$ *is a Banach space.*

In view of Lemma 4, we can define operator $T : W \times Z \to W \times Z$ as follows:

$$T(\kappa,\nu)(\varsigma) = (T_1(\kappa,\nu)(\varsigma), \quad T_2(\kappa,\nu)(\varsigma)), \tag{11}$$

where

$$\begin{aligned} T_1(\kappa,\nu)(\varsigma) = {} & \frac{\phi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma))}{\Gamma_q(\delta_1+\theta_1)} \int_0^{\varsigma} (\varsigma - qs)^{(\delta_1+\theta_1-1)}\psi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma))d_qs \\ & - \frac{\phi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma))\varsigma^{\theta_1}}{\Gamma_q(\delta_1+\theta_1)} \int_0^1 (1-qs)^{(\delta_1+\theta_1-1)}\psi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma))d_qs \end{aligned} \tag{12}$$

and

$$\begin{aligned} T_2(\kappa,\nu)(\varsigma) = {} & \frac{\phi_2(\varsigma,\kappa(\varsigma),\nu(\varsigma))}{\Gamma_q(\delta_2+\theta_2)} \int_0^{\varsigma} (\varsigma - qs)^{(\delta_2+\theta_2-1)}\psi_2(\varsigma,\kappa(\varsigma),\nu(\varsigma))d_qs \\ & - \frac{\phi_2(\varsigma,\kappa(\varsigma),\nu(\varsigma))\varsigma^{\theta_2}}{\Gamma_q(\delta_2+\theta_2)} \int_0^1 (1-qs)^{(\delta_2+\theta_2-1)}\psi_2(\varsigma,\kappa(\varsigma),\nu(\varsigma))d_qs. \end{aligned} \tag{13}$$

We impose the following hypotheses:

(H1) $\psi_i : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist constants $\eta_i > 0$ such that for all $\varsigma \in J$ and $\kappa_i, \nu_i \in \mathbb{R}$, $i = 1, 2$,

$$|\psi_1(\varsigma, \kappa_1, \nu_1) - \psi_1(\varsigma, \kappa_2, \nu_2)| \le \eta_1(|\kappa_1 - \kappa_2| + |\nu_1 - \nu_2|),$$
$$|\psi_2(\varsigma, \kappa_1, \nu_1) - \psi_2(\varsigma, \kappa_2, \nu_2)| \le \eta_2(|\kappa_1 - \kappa_2| + |\nu_1 - \nu_2|).$$

(H2) $\phi_i : [0,1] \times \mathbb{R}^2 \to \mathbb{R} - \{0\}$ are continuous functions and there exist positive constants Λ_i, $i = 1, 2$, such that for all $\varsigma \in J$ and $\kappa, \nu \in \mathbb{R}$,

$$|\phi_1(\varsigma, \kappa, \nu)| \le \Lambda_1, \quad |\phi_2(\varsigma, \kappa, \nu)| \le \Lambda_2.$$

In the following, we present the existence and uniqueness of solutions of problem (1) using Banach's fixed point theorem.

Theorem 6. *Assume that* (H_1) *and* (H_2) *hold and that*

$$\frac{\Lambda_1\eta_1}{\Gamma_q(\delta_1 + \theta_1 + 1)} < \frac{1}{4}, \quad \frac{\Lambda_2\eta_2}{\Gamma_q(\delta_2 + \theta_2 + 1)} < \frac{1}{4}, \tag{14}$$

where $M_i := \sup_{\varsigma \in [0,1]} |\psi_i(\varsigma, 0, 0)| < \infty$, $i = 1, 2$. *Then, the problem* (1) *has a unique solution on* $[0, 1]$.

Proof. Define the set $B_\sigma = \{(\kappa, \nu) \in W \times Z : \|(\kappa, \nu)\|_{W\times Z} < \sigma\}$, where $\sigma \in \mathbb{R}$ satisfies the following inequality:

$$\max\left\{\frac{\frac{2\Lambda_1 M_1}{\Gamma_q(\delta_1+\theta_1+1)}}{\frac{1}{4} - \frac{\Lambda_1\eta_1}{\Gamma_q(\delta_1+\theta_1+1)}}, \frac{\frac{2\Lambda_2 M_2}{\Gamma_q(\delta_2+\theta_2+1)}}{\frac{1}{4} - \frac{\Lambda_2\eta_2}{\Gamma_q(\delta_2+\theta_2+1)}}\right\} \le \sigma.$$

We first show that $TB_\sigma \subset B_\sigma$. For all $\varsigma \in J$ and $(\kappa, \nu) \in B_\sigma$, we have

$$\begin{aligned} |\psi_1(\varsigma, \kappa(\varsigma), \nu(\varsigma))| &\le |\psi_1(\varsigma, \kappa(\varsigma), \nu(\varsigma)) - \psi_1(\varsigma, 0, 0)| + |\psi_1(\varsigma, 0, 0)| \\ &\le \eta_1(|\kappa(\varsigma)| + |\nu(\varsigma)|) + M_1 \\ &\le \eta_1(\|\kappa\| + \|\nu\|) + M_1 \le \eta_1\sigma + M_1 \end{aligned} \tag{15}$$

and

$$\begin{aligned} |\psi_2(\varsigma, \kappa(\varsigma), \nu(\varsigma))| &\le |\psi_2(\varsigma, \kappa(\varsigma), \nu(\varsigma)) - \psi_2(\varsigma, 0, 0)| + |\psi_2(\varsigma, 0, 0)| \\ &\le \eta_2(|\kappa(\varsigma)| + |\nu(\varsigma)|) + M_2 \\ &\le \eta_2(\|\kappa\| + \|\nu\|) + M_2 \le \eta_2\sigma + M_2. \end{aligned} \tag{16}$$

By (15), we obtain

$$\begin{aligned} |T_1(\kappa, \nu)(\varsigma)| &\le \Lambda_1 \sup_{\varsigma \in [0,1]} \left\{ \frac{1}{\Gamma_q(\delta_1 + \theta_1)} \int_0^\varsigma (\varsigma - qs)^{(\delta_1+\theta_1-1)} |\psi_1(\varsigma, \kappa(\varsigma), \nu(\varsigma))| d_q s \right. \\ &\quad \left. + \frac{\varsigma^{\theta_1}}{\Gamma_q(\delta_1 + \theta_1)} \int_0^1 (1 - qs)^{(\delta_1+\theta_1-1)} |\psi_1(\varsigma, \kappa(\varsigma), \nu(\varsigma))| d_q s \right\} \\ &\le \frac{2\Lambda_1\eta_1}{\Gamma_q(\delta_1 + \theta_1 + 1)}\sigma + \frac{2\Lambda_1 M_1}{\Gamma_q(\alpha + \beta + 1)}, \end{aligned} \tag{17}$$

which implies that

$$\|T_1(\kappa, \nu)\| \le \frac{2\Lambda_1\eta_1}{\Gamma_q(\delta_1 + \theta_1 + 1)}\sigma + \frac{2\Lambda_1 M_1}{\Gamma_q(\delta_1 + \theta_1 + 1)} \le \frac{\sigma}{2}. \tag{18}$$

Now, using (16), we obtain

$$\|T_2(\kappa,\nu)\| \le \frac{2\Lambda_2\eta_2}{\Gamma_q(\delta_2+\theta_2+1)}\sigma + \frac{2\Lambda_2 M_2}{\Gamma_q(\delta_2+\theta_2+1)} \le \frac{\sigma}{2}. \tag{19}$$

Consequently, we obtain

$$\begin{aligned}\|T(\kappa,\nu)\|_{W\times Z} &= \|T_1(\kappa,\nu)\| + \|T_2(\kappa,\nu)\| \\ &\le 2\left(\frac{\Lambda_1\eta_1}{\Gamma_q(\delta_1+\theta_1+1)} + \frac{\Lambda_2\eta_2}{\Gamma_q(\delta_2+\theta_2+1)}\right)\sigma + \frac{2\Lambda_1 M_1}{\Gamma_q(\delta_1+\theta_1+1)} + \frac{2\Lambda_2 M_2}{\Gamma_q(\delta_2+\theta_2+1)} \\ &\le \sigma,\end{aligned} \tag{20}$$

which implies that $TB_\sigma \subset B_\sigma$.

For $(\kappa_i,\nu_i) \in W\times Z$, $i=1,2$, and for each $\varsigma\in[0,1]$, we have

$$\begin{aligned}&|T_1(\kappa_1,\nu_1)(\varsigma) - T_1(\kappa_2,\nu_2)(\varsigma)| \\ &\le M_1 \sup_{\varsigma\in[0,1]}\left\{\frac{1}{\Gamma_q(\delta_1+\theta_1)}\int_0^\varsigma (\varsigma-qs)^{(\delta_1+\theta_1-1)}|\psi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma)) - \psi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma))|d_qs\right. \\ &\left. + \frac{\varsigma^{\theta_1}}{\Gamma_q(\delta_1+\theta_1)}\int_0^1 (1-qs)^{(\delta_1+\theta_1-1)}\int_0^1 (1-s)^{\alpha+\beta-1}|\psi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma)) - \psi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma))|d_qs\right\}.\end{aligned} \tag{21}$$

Thanks to (H_1), we can write

$$\|T_1(\kappa_1,\nu_1) - T_1(\kappa_2,\nu_2)\| \le \frac{2\Lambda_1\eta_1}{\Gamma_q(\delta_1+\theta_1+1)}\|\kappa_1-\nu_2,\nu_1-\nu_2\|_{W\times Z}. \tag{22}$$

Analogously, we can obtain

$$\|T_2(\kappa_1,\nu_1) - T_2(\kappa_2,\nu_2)\| \le \frac{2\Lambda_2\eta_2}{\Gamma_q(\delta_2+\theta_2+1)}\|\kappa_1-\kappa_2,\nu_1-\nu_2\|_{W\times Z}. \tag{23}$$

From the definition of $\|(.)\|_{W\times Z}$, we have

$$\begin{aligned}\|T_1(\kappa_1,\nu_1) - T_1(\kappa_2,\nu_2)\|_{W\times Z} &= \|T_1(\kappa_1,\nu_1) - T_1(\kappa_2,\nu_2)\| + \|T_2(\kappa_1,\nu_1) - T_2(\kappa_2,\nu_2)\| \\ &\le \left[\frac{2\Lambda_1\eta_1}{\Gamma_q(\delta_1+\theta_1+1)} + \frac{2\Lambda_2\eta_2}{\Gamma_q(\delta_2+\theta_2+1)}\right]\|\kappa_1-\kappa_2,\nu_1-\nu_2\|_{W\times Z}.\end{aligned} \tag{24}$$

Thanks to (14), we conclude that T is a contraction mapping. Hence, by Banach fixed point theorem, there exists a unique fixed point, which is a solution of system (1). This completes the proof. □

Now, we prove the existence of solutions of problem (1) by applying the Leray-Schauder nonlinear alternative.

Lemma 7. [33] (Leray-Schauder alternative). *Let $F : E \to E$ be a completely continuous operator (i.e., a map that is restricted to any bounded set in E is compact). Let*

$$\Theta(F) = \{u\in E : u = \lambda F(u) \quad \text{for some } 0<\lambda<1\}.$$

Then, either the set $\Theta(F)$ is unbounded or F has at least one fixed point.

For the forthcoming result, we suppose that

(H_3) $\psi_i : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist constants $\vartheta_i, \mu_i \ge 0$, $i=1,2$, and $\vartheta_0>0$, $\mu_0>0$ such that for all $\varsigma\in J$ and $\kappa,\nu\in\mathbb{R}$, we have:

$$\psi_1(\varsigma,\kappa,\nu) \le \vartheta_0 + \vartheta_1|\kappa| + \vartheta_2|\nu|$$

and

$$\psi_2(\varsigma,\kappa,\nu) \le \mu_0 + \mu_1|\kappa| + \mu_2|\nu|.$$

Theorem 8. *Assume that hypotheses* (H_2) *and* (H_3) *are satisfied. Furthermore, assume that*

$$\frac{\Lambda_1\vartheta_1}{\Gamma_q(\delta_1+\theta_1+1)}+\frac{\Lambda_2\mu_1}{\Gamma_q(\delta_2+\theta_2+1)}<\frac{1}{2},\ \frac{\Lambda_1\vartheta_2}{\Gamma_q(\delta_1+\theta_1+1)}+\frac{\Lambda_2\mu_2}{\Gamma_q(\delta_2+\theta_2+1)}<\frac{1}{2}. \tag{25}$$

Then, system (1) *has at least one solution on* $[0, 1]$.

Proof. In the first step, we show that the operator $T : W\times Z \to W\times Z$ is completely continuous. By continuity of the functions ϕ_i, ψ_i, $i = 1, 2$, it follows that the operator T is continuous.

Let $\Omega \subset W\times Z$ be bounded. Then, we can find positive constants A and B such that

$$|\psi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma))|\le A,\ |\psi_2(\varsigma,\kappa(\varsigma),\nu(\varsigma))|\le B,\quad \text{for all } (\kappa,\nu)\in\Omega.$$

Then, for any $(\kappa, \nu) \in \Omega$, we have

$$\begin{aligned}\|T_1(\kappa,\nu)\| &\le \Lambda_1\Bigg[\frac{1}{\Gamma_q(\delta_1+\theta_1)}\int_0^\varsigma(\varsigma-qs)^{(\delta_1+\theta_1-1)}|\psi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma))|d_qs\\ &\quad+\frac{\varsigma^{\theta_1}}{\Gamma_q(\delta_1+\theta_1)}\int_0^\varsigma(\varsigma-qs)^{(\delta_1+\theta_1-1)}|\psi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma))|d_qs\Bigg]\\ &\le\frac{2\Lambda_1A}{\Gamma_q(\delta_1+\theta_1+1)},\end{aligned} \tag{26}$$

which yields

$$\|T_1(\kappa,\nu)\|\le\frac{2\Lambda_1A}{\Gamma_q(\delta_1+\theta_1+1)}<+\infty. \tag{27}$$

We also have

$$\|T_2(\kappa,\nu)\|\le\frac{2\Lambda_2B}{\Gamma_q(\delta_2+\theta_2+1)}<+\infty. \tag{28}$$

It follows from (27) and (28) that

$$\|T(\kappa,\nu)\|_{W\times Z}\le\frac{2\Lambda_1A}{\Gamma_q(\delta_1+\theta_1+1)}+\frac{2\Lambda_2B}{\Gamma_q(\delta_2+\theta_2+1)}. \tag{29}$$

Thus,

$$\|T(w,z)\|_{W\times Z}<+\infty.$$

Next, we show that T is equicontinuous. For all $0 \le \varsigma_2 < \varsigma_1 \le 1$, we have

$$|T_1(\kappa,\nu)(\varsigma_1)-T_1(\kappa,\nu)(\varsigma_2)|\le\Lambda_1A\left(\frac{[(\varsigma_1-\varsigma_2)^{(\delta_1+\theta_1)}+|\varsigma_1^{(\delta_1+\theta_1)}-\varsigma_2^{(\delta_1+\theta_1)}|]}{\Gamma_q(\delta_1+\theta_1+1)}+\frac{|\varsigma_1^{\theta_1}-\varsigma_2^{\theta_1}|}{\Gamma_q(\delta_1+\theta_1+1)}\right) \tag{30}$$

and

$$|T_2(\kappa,\nu)(\varsigma_1)-T_2(\kappa,\nu)(\varsigma_2)|\le\Lambda_2B\left(\frac{[(\varsigma_1-\varsigma_2)^{(\delta_2+\theta_2)}+|\varsigma_1^{(\delta_2+\theta_2)}-\varsigma_2^{(\delta_2+\theta_2)}|]}{\Gamma_q(\delta_2+\theta_2+1)}+\frac{|\varsigma_1^{\theta_2}-\varsigma_2^{\theta_2}|}{\Gamma_q(\delta_2+\theta_2+1)}\right). \tag{31}$$

Thanks to (30) and (31), we can state that $\|T(\kappa,\nu)(\varsigma_1)-T(\kappa,\nu)(\varsigma_2)\|_{W\times Z}\to 0$ as $\varsigma_2\to\varsigma_1$. Thus, by using the Arzela-Ascoli theorem, one can conclude that the operator $T : W\times Z\to W\times Z$ is completely continuous.

Finally, it will be verified that the set $\Sigma = \{(\kappa,\nu)\in W\times Z, (\kappa,\nu)=\varepsilon T(\kappa,\nu), 0<\varepsilon<1\}$ is bounded. Let $(\kappa,\nu)\in\Sigma$. Then, for each $\varsigma\in[0,1]$, we can write

$$\kappa(\varsigma)=\varepsilon T_1(\kappa,\nu)(\varsigma),\quad \nu(\varsigma)=\varepsilon T_2(\kappa,\nu)(\varsigma).$$

Then,

$$|\kappa(\varsigma)| \leq \frac{2\Lambda_1}{\Gamma_q(\delta_1+\theta_1+1)}(\vartheta_0+\vartheta_1|\kappa(\varsigma)|+\vartheta_2|\nu(\varsigma)|) \tag{32}$$

and

$$|\nu(\varsigma)| \leq \frac{2\Lambda_2}{\Gamma_q(\delta_2+\theta_2+1)}(\mu_0+\mu_1|\kappa(\varsigma)|+\mu_2|\nu(\varsigma)|). \tag{33}$$

Hence, we have

$$\|\kappa\| \leq \frac{2\Lambda_1}{\Gamma_q(\delta_1+\theta_1+1)}(\vartheta_0+\vartheta_1\|\kappa\|+\vartheta_2\|\nu\|) \tag{34}$$

and

$$\|\nu\| \leq \frac{2\Lambda_2}{\Gamma_q(\delta_2+\theta_2+1)}(\mu_0+\mu_1\|\kappa\|+\mu_2\|\nu\|), \tag{35}$$

which imply that

$$\begin{aligned}\|\kappa\|+\|\nu\| \leq{}& \frac{2\Lambda_1}{\Gamma_q(\delta_1+\theta_1+1)}\vartheta_0+\frac{2\Lambda_2}{\Gamma_q(\delta_2+\theta_2+1)}\mu_0+\left(\frac{2\Lambda_1}{\Gamma_q(\delta_1+\theta_1+1)}\vartheta_1+\frac{2\Lambda_2}{\Gamma_q(\delta_2+\theta_2+1)}\mu_1\right)\|\kappa\|\\ &+\left(\frac{2\Lambda_1}{\Gamma_q(\delta_1+\theta_1+1)}\vartheta_2+\frac{2\Lambda_2}{\Gamma_q(\delta_2+\theta_2+1)}\mu_2\right)\|\nu\|.\end{aligned} \tag{36}$$

Consequently,

$$\|(\kappa,\nu)\|_{W\times Z} \leq \frac{\frac{2\Lambda_1\vartheta_0}{\Gamma_q(\delta_1+\theta_1+1)}+\frac{2\Lambda_2\mu_0}{\Gamma_q(\delta_2+\theta_2+1)}}{\Pi} \tag{37}$$

for all $\varsigma \in [0,1]$, where

$$\Pi := \min\left\{1-\left(\frac{2\Lambda_1\vartheta_1}{\Gamma_q(\delta_1+\theta_1+1)}+\frac{2\Lambda_2\mu_1}{\Gamma_q(\delta_2+\theta_2+1)}\right), 1-\left(\frac{2\Lambda_1\vartheta_2}{\Gamma_q(\delta_1+\theta_1+1)}+\frac{2\Lambda_2\mu_2}{\Gamma_q(\delta_2+\theta_2+1)}\right)\right\}.$$

This shows that the set Σ is bounded. Hence, all the conditions of Lemma 6 are satisfied, and consequently, the operator has at least one fixed point, which corresponds to a solution of the system (1). This completes the proof. □

3 Ulam-Hyers-Rassias-stability results

In the following section, we consider Ulam's-type stability of the q-fractional problem (1). For $\varsigma \in [0,1]$, we provide the following inequalities:

$$\begin{cases}\left|{}^{C}D_q^{\delta_1}\left({}^{C}D_q^{\theta_1}\left(\frac{\kappa_1(\varsigma)}{\phi_1(\varsigma,\kappa_1(\varsigma),\nu_1(\varsigma))}\right)\right)-\psi_1(\varsigma,\kappa_1(\varsigma),\nu_1(\varsigma))\right| \leq \lambda_1,\\ \left|{}^{C}D_q^{\delta_2}\left({}^{C}D_q^{\theta_2}\left(\frac{\nu_1(\varsigma)}{\phi_2(\varsigma,\kappa_1(\varsigma),\nu_1(\varsigma))}\right)\right)-\psi_2(\varsigma,\kappa_1(\varsigma),\nu_1(\varsigma))\right| \leq \lambda_2,\end{cases} \tag{38}$$

and

$$\begin{cases} \left| {}^{C}D_q^{\delta_1}\left({}^{C}D_q^{\theta_1}\left(\frac{\kappa_1(\varsigma)}{\phi_1(\varsigma,\kappa_1(\varsigma),\nu_1(\varsigma))}\right)\right) - \psi_1(\varsigma,\kappa_1(\varsigma),\nu_1(\varsigma)) \right| \le \lambda_1 h_1(\varsigma), \\ \left| {}^{C}D_q^{\delta_2}\left({}^{C}D_q^{\theta_2}\left(\frac{\nu_1(\varsigma)}{\phi_2(\varsigma,\kappa_1(\varsigma),\nu_1(\varsigma))}\right)\right) - \psi_2(\varsigma,\kappa_1(\varsigma),\nu_1(\varsigma)) \right| \le \lambda_2 h_2(\varsigma), \end{cases} \tag{39}$$

where λ_i are positive real numbers and $h_i : [0, 1] \to \mathbb{R}^+$, $i = 1, 2$ are continuous functions.

Definition 9. System (1) is Ulam-Hyers stable if there exists a real number $\varpi_{\psi_1,\psi_2} = (\varpi_{\psi_1}, \varpi_{\psi_2}) > 0$ such that for each $\lambda = (\lambda_1, \lambda_2) > 0$ and for each solution $(\kappa_1, \nu_1) \in W \times Z$ of the inequality (38), there exists a solution $(\kappa, \nu) \in W \times Z$ of the problem (1) with

$$|\kappa_1(\varsigma) - \kappa(\varsigma), \nu_1(\varsigma) - \nu(\varsigma)| \le \varpi_{\psi_1\psi_2}\lambda, \quad \varsigma \in [0, 1].$$

Definition 10. System (1) is Ulam-Hyers-Rassias stable with respect to $h = (h_2, h_2) \in C([0, 1], \mathbb{R})$ if there exists a real number $\varpi_{\psi_1,\psi_2,h} = (\varpi_{\psi_1,h}, \varpi_{\psi_2,h}) > 0$ such that for each $\lambda = (\lambda_1, \lambda_2) > 0$ and for each solution $(\kappa_1, \nu_1) \in W \times Z$ of the inequality (39), there exists a solution $(\kappa, \nu) \in W \times Z$ of the problem (1) with

$$|\kappa_1(\varsigma) - \kappa(\varsigma), \nu_1(\varsigma) - \nu(\varsigma)| \le \varpi_{\psi_1\psi_2,h}\lambda h(\varsigma), \quad \varsigma \in [0, 1].$$

Theorem 11. *Assume that* (H_1) *and* (H_2) *hold. If*

$$\Lambda_1\eta_1 < \Gamma_q(\delta_1 + \theta_1 + 1) \quad \text{and} \quad \Lambda_2\eta_2 < \Gamma_q(\delta_2 + \theta_2 + 1), \tag{40}$$

then the problem (1) *is Ulam-Hyers stable.*

Proof. Let $(\kappa_1, \nu_1) \in W \times Z$ is a solution of the inequality (38) and let $(\kappa, \nu) \in W \times Z$ be the unique solution of the problem

$$\begin{cases} {}^{C}D_q^{\delta_1}\left({}^{C}D_q^{\theta_1}\left(\frac{\kappa(\varsigma)}{\phi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma))}\right)\right) = \psi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma)), \varsigma \in [0,1], \\ \kappa_1(0) = \kappa_2(0), \quad \kappa_1(1) = \kappa_2(1), \\ {}^{C}D_q^{\delta_2}\left({}^{C}D_q^{\theta_2}\left(\frac{\nu(\varsigma)}{\phi_2(\varsigma,\kappa(\varsigma),\nu(\varsigma))}\right)\right) = \psi_2(\varsigma,\kappa(\varsigma),\nu(\varsigma)), \varsigma \in [0,1], \\ \nu_1(0) = \nu_2(0), \quad \nu_1(1) = \nu_2(1). \end{cases}$$

By Lemma 7, we have

$$\kappa(\varsigma) = \phi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma))\left(I_q^{\delta_1+\theta_1}\varphi_1^{\kappa}(\varsigma) + \frac{c_1}{\Gamma_q(\theta_1+1)}\varsigma^{\theta_1} + c_2\right),$$
$$\nu(\varsigma) = \phi_2(\varsigma,\kappa(\varsigma),\nu(\varsigma))\left(I_q^{\delta_2+\theta_2}\varphi_2^{\nu}(\varsigma) + \frac{d_1}{\Gamma_q(\theta_2+1)}\varsigma^{\theta_2} + d_2\right),$$

where

$$\varphi_1^{\kappa}(\varsigma) = \psi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma)), \varphi_2^{\nu}(\varsigma) = \psi_2(\varsigma,\kappa(\varsigma),\nu(\varsigma)), \varsigma \in [0,1].$$

By integration of the (40), we obtain

$$\left| \kappa_1(\varsigma) - \phi_1(\varsigma,\kappa(\varsigma),\nu(\varsigma))\left(I_q^{\delta_1+\theta_1}\varphi_1^{\kappa}(\varsigma) + \frac{c_3}{\Gamma_q(\theta_1+1)}\varsigma^{\theta_1} + c_4\right) \right| \le \frac{\lambda\varsigma^{\delta_1+\theta_1}}{\Gamma_q(\delta_1+\theta_1+1)} \le \frac{\lambda}{\Gamma_q(\delta_1+\theta_1+1)}, \tag{41}$$

and

$$\left| \nu_1(\varsigma) - \phi_2(\varsigma, \kappa(\varsigma), \nu(\varsigma))\left(I_q^{\delta_2+\theta_2}\varphi_2^{\nu}(\varsigma) + \frac{d_3}{\Gamma_q(\theta_2+1)}\varsigma^{\theta_2} + d_4 \right) \right| \le \frac{\lambda\varsigma^{\delta_2+\theta 2}}{\Gamma_q(\delta_2+\theta_2+1)} \le \frac{\lambda}{\Gamma_q(\delta_2+\theta_2+1)}. \tag{42}$$

From hypotheses (H_1) and (H_2), we have

$$\begin{aligned} |\kappa_1(\varsigma) - \kappa(\varsigma)| &\le \left| \kappa_1(\varsigma) - \phi_1(\varsigma, \kappa(\varsigma), \nu(\varsigma))\left(I_q^{\delta_1+\theta_1}\varphi_1^{\kappa}(\varsigma) + \frac{c_3}{\Gamma_q(\theta_1+1)}\varsigma^{\theta_1} + c_4 \right) \right| \\ &\quad + ||\phi_1(\varsigma, \kappa(\varsigma), \nu(\varsigma))||I_q^{\delta_1+\theta_1}|\varphi_1^{\kappa_1}(\varsigma) - \varphi_1^{\kappa}(\varsigma)| \\ &\le \frac{\lambda}{\Gamma_q(\delta_1+\theta_1+1)} + \frac{\Lambda_1\eta_1}{\Gamma_q(\delta_1+\theta_1+1)}(|\kappa_1(\varsigma) - \kappa(\varsigma)| + |\nu_1(\varsigma) - \nu(\varsigma)|), \end{aligned} \tag{43}$$

which implies that

$$|\kappa_1(\varsigma) - \kappa(\varsigma)| \le \frac{\lambda_1}{\Gamma_q(\delta_1+\theta_1+1)} + \frac{\Lambda_1\eta_1}{\Gamma_q(\delta_1+\theta_1+1)}(|\kappa_1(\varsigma) - \kappa(\varsigma)| + |\nu_1(\varsigma) - \nu(\varsigma)|). \tag{44}$$

In addition, we have

$$|\nu_1(\varsigma) - \nu(\varsigma)| \le \frac{\lambda_2}{\Gamma_q(\delta_2+\theta_2+1)} + \frac{\Lambda_2\eta_2}{\Gamma_q(\delta_2+\theta_2+1)}(|\kappa_1(\varsigma) - \kappa(\varsigma)| + |\nu_1(\varsigma) - \nu(\varsigma)|). \tag{45}$$

Thus,

$$\begin{aligned} |(\kappa_1(\varsigma), \nu_1(\varsigma)) - (\kappa(\varsigma), \nu(\varsigma))| &= |\kappa_1(\varsigma) - \kappa(\varsigma)| + |\nu_1(\varsigma) - \nu(\varsigma)| \\ &\le \frac{\left(\frac{1}{\Gamma_q(\delta_1+\theta_1+1)} + \frac{1}{\Gamma_q(\delta_2+\theta_2+1)}\right)}{\min\left(1 - \frac{\Lambda_1\eta_1}{\Gamma_q(\delta_1+\theta_1+1)}, 1 - \frac{\Lambda_2\eta_2}{\Gamma_q(\delta_2+\theta_2+1)}\right)}\lambda := \varpi_{\psi_1\psi_2}\lambda, \end{aligned} \tag{46}$$

where $\lambda = \max\{\lambda_1, \lambda_2\}$. Hence, problem (1) is Ulam-Hyers stable. □

Theorem 12. *Assume that* (H_1), (H_2), *and* (39) *hold. Suppose there exist* $\gamma_{h_i} > 0$, $i = 1, 2$, *such that*

$$I_q^{\delta_i+\theta_i}h_i(t) \le \gamma_{h_i}h_i(t), \ t \in [0,1], \quad i = 1, 2, \tag{47}$$

where $h \in C([0,1], \mathbb{R}_+)$, $i = 1, 2$, *are nondecreasing. Then, system* (1) *is Ulam-Hyers-Rassias stable.*

Proof. Let $(\kappa_1, \nu_1) \in W \times Z$ is a solution of the inequality (39) and let us assume that $(\kappa, \nu) \in W \times Z$ is a solution of system (1). Thus, we have

$$\begin{aligned} \kappa(\varsigma) &= \phi_1(\varsigma, \kappa(\varsigma), \nu(\varsigma))\left(I_q^{\delta_1+\theta_1}\varphi_1^{\kappa}(\varsigma) + \frac{c_1}{\Gamma_q(\theta_1+1)}\varsigma^{\theta_1} + c_2 \right), \\ \nu(\varsigma) &= \phi_2(\varsigma, \kappa(\varsigma), \nu(\varsigma))\left(I_q^{\delta_2+\theta_2}\varphi_2^{\nu}(\varsigma) + \frac{d_1}{\Gamma_q(\theta_2+1)}\varsigma^{\theta_2} + d_2 \right). \end{aligned}$$

From inequality (41), we have

$$\left| \kappa_1(\varsigma) - \phi_1(\varsigma, \kappa_1(\varsigma), \nu_1(\varsigma))\left(I_q^{\delta_1+\theta_1}\varphi_1^{\kappa_1}(\varsigma) + \frac{c_3}{\Gamma_q(\theta_1+1)}\varsigma^{\theta_1} + c_4 \right) \right| \le \lambda I_q^{\delta_1+\theta_1}h(\varsigma) \le \lambda\gamma_{h_1}h(\varsigma), \quad \varsigma \in [0,1] \tag{48}$$

and

$$\left| \nu_1(\varsigma) - \phi_2(\varsigma, \kappa_1(\varsigma), \nu_1(\varsigma))\left(I_q^{\delta_2+\theta_2}\varphi_2^{\nu_1}(\varsigma) + \frac{d_3}{\Gamma_q(\theta_2+1)}\varsigma^{\theta_2} + d_4 \right) \right| \le \lambda_1 I_q^{\delta_2+\theta_2}h_1(\varsigma) \le \lambda_1\gamma_{h_1}h_1(\varsigma), \quad \varsigma \in [0,1]. \tag{49}$$

Now, using (H_1) and (H_2), we can write

$$\begin{aligned}|\kappa_1(\varsigma)-\kappa(\varsigma)| \le &\left|\kappa_1(\varsigma)-\phi_1(\varsigma,\kappa_1(\varsigma),\nu_1(\varsigma))\left(I_q^{\delta_1+\theta_1}\varphi_1^{\kappa_1}(\varsigma)+\frac{c_3}{\Gamma_q(\theta_1+1)}\varsigma^{\theta_1}+c_4\right)\right| \\ &+ \|\phi_1(\varsigma,\kappa_1(\varsigma),\nu_1(\varsigma))\| I_q^{\delta_1+\theta_1}|\varphi_1^{\kappa_1}(\varsigma)-\varphi_1^{\kappa}(\varsigma)| \\ &\le \lambda_1\vartheta_{\gamma_1}h_1(t)+\frac{\Lambda_1\eta_1}{\Gamma_q(\delta_1+\theta_1+1)}(|\kappa_1(\varsigma)-\kappa(\varsigma)|+|\nu_1(\varsigma)-\nu(\varsigma)|),\end{aligned} \tag{50}$$

which implies that

$$|\kappa_1(t)-\kappa(t)| \le \lambda_1\vartheta_{\gamma_1}h_1(t)+\frac{\Lambda_1\eta_1}{\Gamma_q(\delta_1+\theta_1+1)}(|\kappa_1(\varsigma)-\kappa(\varsigma)|+|\nu_1(\varsigma)-\nu(\varsigma)|). \tag{51}$$

On the other hand,

$$|\nu_1(\varsigma)-\nu(\varsigma)| \le \lambda_2\vartheta_{\gamma_2}h_2(t)+\frac{\Lambda_2\eta_2}{\Gamma_q(\delta_2+\theta_2+1)}(|\kappa_1(\varsigma)-\kappa(\varsigma)|+|\nu_1(\varsigma)-\nu(\varsigma)|). \tag{52}$$

Thanks to (51) and (52), we obtain

$$\begin{aligned}|(\kappa_1(\varsigma),\nu_1(\varsigma))-(\kappa(\varsigma),\nu(\varsigma))| &= |\kappa_1(\varsigma)-\kappa(\varsigma)|+|\nu_1(\varsigma)-\nu(\varsigma)|, \\ &\le \frac{\gamma_{h_1}+\gamma_{h_2}}{\min\left(1-\frac{\Lambda_1\eta_1}{\Gamma_q(\delta_1+\theta_1+1)},1-\frac{\Lambda_2\eta_2}{\Gamma_q(\delta_2+\theta_2+1)}\right)}\lambda h(\varsigma) \\ &:= \varpi_{\psi_1\psi_2,h}\lambda h(\varsigma).\end{aligned} \tag{53}$$

Thus, system (1) is Ulam-Hyers-Rassias stable. □

4 Examples

Example 13. Consider the following system of hybrid q-fractional differential equations:

$$\begin{cases} {}^C D_{\frac{1}{4}}^{\frac{3}{4}}\left({}^C D_{\frac{1}{4}}^{\frac{1}{2}}\left(\frac{\kappa(\varsigma)}{\frac{|\cos(\kappa(\varsigma)+\nu(\varsigma))|+1}{31}}\right)\right)=\frac{1}{60}\sin^2\kappa(\varsigma)\frac{|\nu(\varsigma)|}{15(t+2)(|\nu(\varsigma)|+1)}+\frac{2\varsigma+1}{3}, \varsigma\in[0,1], \\ {}^C D_{\frac{1}{4}}^{\frac{2}{3}}\left({}^C D_{\frac{1}{4}}^{\frac{5}{6}}\left(\frac{\nu(\varsigma)}{\frac{|\sin\kappa(\varsigma)|+|\cos\nu(\varsigma)|+1}{37}}\right)\right)=\frac{1}{2(\varsigma+4)^2}\frac{|\kappa(\varsigma)|}{(|\kappa(\varsigma)|+1)}+\frac{\cos\nu(\varsigma)}{32e^{\varsigma+1}}+\frac{\varsigma+1}{2}, \varsigma\in[0,1], \\ \kappa(0)=\kappa(1)=0, \quad \nu(0)=\nu(1)=0. \end{cases} \tag{54}$$

and the following $\frac{1}{4}$-fractional inequalities:

$$\begin{cases} \left|{}^C D_{\frac{1}{4}}^{\frac{3}{4}}\left({}^C D_{\frac{1}{4}}^{\frac{1}{2}}\left(\frac{\kappa(\varsigma)}{\frac{|\cos(\kappa(\varsigma)+\nu(\varsigma))|+1}{31}}\right)\right)-\frac{1}{60}\sin^2\kappa(\varsigma)-\frac{|\nu(\varsigma)|}{15(\varsigma+2)(|\nu(\varsigma)|+1)}-\frac{2\varsigma+1}{3}\right| \le \lambda, \\ \left|{}^C D_{\frac{1}{4}}^{\frac{2}{3}}\left({}^C D_{\frac{1}{4}}^{\frac{5}{6}}\left(\frac{\nu(\varsigma)}{\frac{|\sin\kappa(\varsigma)|+|\cos\nu(\varsigma)|+1}{37}}\right)\right)=\frac{1}{2(\varsigma+4)^2}\frac{|\kappa(\varsigma)|}{(|\kappa(\varsigma)|+1)}-\frac{\cos\nu(\varsigma)}{32e^{\varsigma+1}}-\frac{\varsigma+1}{2}\right| \le \lambda, \end{cases}$$

and

$$\begin{cases} \left| {}^{C}D_{\frac{1}{4}}^{\frac{3}{4}}\left({}^{C}D_{\frac{1}{4}}^{\frac{1}{2}}\left(\frac{\kappa(\varsigma)}{\frac{|\cos(\kappa(\varsigma)+\nu(\varsigma))|+1}{31}}\right)\right) - \frac{1}{60}\sin^2\kappa(\varsigma) - \frac{|\nu(\varsigma)|}{15(\varsigma+2)(|\nu(\varsigma)|+1)} - \frac{2\varsigma+1}{3} \right| \le \lambda h(\varsigma), \\ \left| {}^{C}D_{\frac{1}{4}}^{\frac{2}{3}}\left({}^{C}D_{\frac{1}{4}}^{\frac{5}{6}}\left(\frac{\nu(\varsigma)}{\frac{|\sin\kappa(\varsigma)|+|\cos\nu(\varsigma)|+1}{37}}\right)\right) = \frac{1}{2(\varsigma+4)^2}\frac{|\kappa(\varsigma)|}{(|\kappa(\varsigma)|+1)} - \frac{\cos\nu(\varsigma)}{32e^{\varsigma+1}} - \frac{\varsigma+1}{2} \right| \le \lambda h(\varsigma). \end{cases}$$

Here, $\delta_1 = \frac{3}{4}$, $\theta_1 = \frac{1}{2}$, $\delta_2 = \frac{2}{3}$, $\theta_2 = \frac{5}{6}$, $q = \frac{1}{4}$, and

$$\begin{aligned} \psi_1(\varsigma,\kappa,\nu) &= \frac{1}{60}\sin^2\kappa(\varsigma) + \frac{|\nu(\varsigma)|}{15(\varsigma+2)(|\nu(\varsigma)|+1)} + \frac{2\varsigma+1}{3}, \\ \psi_2(\varsigma,\kappa,\nu) &= \frac{1}{2(\varsigma+4)^2}\frac{|\kappa(\varsigma)|}{(|\kappa(\varsigma)|+1)} + \frac{\cos\nu(\varsigma)}{32e^{\varsigma+1}} + \frac{\varsigma+1}{2}, \\ \phi_1(\varsigma,\kappa,\nu) &= \frac{|\cos(\kappa(\varsigma)+\nu(\varsigma))|+1}{31}, \ \phi_2(\varsigma,\kappa,\nu) = \frac{|\sin\kappa(\varsigma)|+|\cos\nu(\varsigma)|+1}{37}. \end{aligned}$$

For $(w_i, z_i) \in \mathbb{R}^2$, $i = 1, 2$, and $t \in [0, 1]$, we have

$$\begin{aligned} |\psi_1(\varsigma,\kappa_1,\nu_1) - \psi_1(\varsigma,\kappa_2,\nu_2)| &\le \frac{1}{30}(|\kappa_1-\kappa_2| + |\nu_1-\nu_2|), \\ |\psi_2(\varsigma,\kappa_1,\nu_1) - \psi_2(\varsigma,\kappa_2,\nu_2)| &\le \frac{1}{32}(|\kappa_1-\kappa_2| + |\nu_1-\nu_2|), \end{aligned}$$

and

$$|\varphi_1(\varsigma,\kappa,\nu)| \le \frac{2}{31}, \quad |\varphi_2(\varsigma,\kappa,\nu)| \le \frac{3}{37}.$$

Hence, condition (H_1) is satisfied with $\eta_1 = \frac{1}{30}$ and $\eta_2 = \frac{1}{32}$, respectively, and condition (H_2) is satisfied with $\Lambda_1 = \frac{2}{31}$ and $\Lambda_2 = \frac{3}{37}$, respectively.

Thus, conditions

$$\frac{\Lambda_1\eta_1}{\Gamma_q(\delta_1+\theta_1+1)} = 4.8957\times10^{-3} < \frac{1}{4}, \quad \frac{\Lambda_2\eta_2}{\Gamma_q(\delta_2+\theta_2+1)} = 4.9873\times10^{-3} < \frac{1}{4}$$

are satisfied. It follows from Theorem 5 that system (54) has a unique solution and is Ulam-Hyers stable with

$$|(\kappa_1(\varsigma), \nu_1(\varsigma)) - (\kappa(\varsigma), \nu(\varsigma))| \le 9.9378\times10^{-3}\lambda.$$

Let $h_1(\varsigma) = h_2(\varsigma) = \varsigma^2$, then

$$I_{\frac{1}{4}}^{\frac{3}{4}+\frac{1}{2}}h_1(\varsigma) = I_{\frac{1}{4}}^{\frac{3}{4}+\frac{1}{2}}(\varsigma^2) \le \frac{2}{\Gamma_{\frac{1}{4}}\left(\frac{17}{4}\right)}\varsigma^2 = \upsilon_{h_1}h_1(\varsigma)$$

and

$$I_{\frac{1}{4}}^{\frac{2}{3}+\frac{5}{6}}h_2(\varsigma) = I_{\frac{1}{4}}^{\frac{2}{3}+\frac{5}{6}}(\varsigma^2) \le \frac{2}{\Gamma_{\frac{1}{4}}\left(\frac{9}{2}\right)}\varsigma^2 = \upsilon_{h_2}h_2(\varsigma).$$

Thus, condition (47) is satisfied with $h(\varsigma) = \varsigma^2$ and $\gamma_{h_1} = \frac{2}{\Gamma_{0.25}\left(\frac{17}{4}\right)}$, $\gamma_{h_2} = \frac{2}{\Gamma_{0.25}\left(\frac{9}{2}\right)}$. It follows from Theorem 12 that problem (54) is Ulam-Hyers-Rassias stable with

$$|(\kappa_1(\varsigma), \nu_1(\varsigma)) - (\kappa(\varsigma), \nu(\varsigma))| \le 7.34\lambda h(\varsigma), \ \varsigma \in [0, 1].$$

Example 14. Consider the following hybrid fractional $\frac{\sqrt{e}}{5}$-differential system:

$$\begin{cases} {}^{C}D_{\frac{\sqrt{e}}{5}}^{\frac{\ln 3}{3}}\left({}^{C}D_{\frac{\sqrt{e}}{5}}^{\frac{\sqrt{5}}{4}}\left(\frac{\kappa(\varsigma)}{\frac{|\cos(\kappa(\varsigma)+\nu(\varsigma))|+1}{31}}\right)\right) = \frac{1+e^{-\varsigma}}{16\ln 2+\varsigma^2} \times \frac{e^{-2\varsigma}\cos\kappa(\varsigma)}{15(\varsigma+4)} + \frac{e^{-\varsigma}\nu(\varsigma)}{25\sqrt{80+\varsigma^2}}, \varsigma \in [0,1], \\ {}^{C}D_{\frac{\sqrt{e}}{5}}^{\frac{1}{3}}\left({}^{C}D_{\frac{\sqrt{e}}{5}}^{\frac{e}{5}}\left(\frac{\nu(\varsigma)}{\frac{|\sin\kappa(\varsigma)|+|\cos\nu(\varsigma)|+1}{37}}\right)\right) = \frac{\sin\varsigma}{\sqrt{32+\varsigma^2}} \times \frac{\kappa(\varsigma)}{5e(\varsigma+4)^2} + \frac{\sin\nu(\varsigma)}{35e^2\sqrt{1+\varsigma}}, \varsigma \in [0,1], \\ \kappa(0)=\kappa(1)=0, \quad \nu(0)=\nu(1)=0. \end{cases} \tag{55}$$

For this example, we have $\delta_1 = \frac{\ln 3}{3}$, $\theta_1 = \frac{\sqrt{5}}{4}$, $\delta_2 = \frac{1}{2}$, $\theta_2 = \frac{e}{5}$, $q = \frac{\sqrt{e}}{5}$, and

$$\psi_1(\varsigma,\kappa,\nu) = \frac{1+e^{-\varsigma}}{16\ln 2+\varsigma^2} + \frac{e^{-2\varsigma}\cos\kappa}{15(\varsigma+4)} + \frac{e^{-\varsigma}\nu}{3\sqrt{80+\varsigma^2}},$$
$$\psi_2(\varsigma,\kappa,\nu) = \frac{\sin\varsigma}{\sqrt{32+\varsigma^2}} + \frac{\kappa}{5e(\varsigma+4)^2} + \frac{\sin\nu}{35e^2\sqrt{1+\varsigma}}$$

It is easy to find that

$$|\psi_1(\varsigma,\kappa,\nu)| \le \frac{1}{8\ln 2} + \frac{1}{60}|\kappa| + \frac{1}{120}|\nu|,$$
$$|\psi_2(\varsigma,\kappa,\nu)| \le \frac{1}{16} + \frac{1}{80e}|\kappa| + \frac{1}{35e^2}|\nu|,$$
$$|\varphi_1(\varsigma,\kappa,\nu)| \le \frac{2}{31}, \quad |\varphi_2(\varsigma,\kappa,\nu)| \le \frac{3}{37},$$

which implies $\vartheta_0 = \frac{1}{8\ln 2}$, $\vartheta_1 = \frac{1}{60}$, $\vartheta_2 = \frac{1}{120}$, $\mu_0 = \frac{1}{16}$, $\mu_1 = \frac{1}{80e}$, $\mu_2 = \frac{1}{35e^2}$, $\Lambda_1 = \frac{2}{31}$, and $\Lambda_2 = \frac{3}{37}$. Therefore, we obtain

$$\frac{\Lambda_1\vartheta_1}{\Gamma_q(\delta_1+\theta_1+1)} + \frac{\Lambda_2\mu_1}{\Gamma_q(\delta_2+\theta_2+1)} = 1.4605 \times 10^{-3} < \frac{1}{2}$$

and

$$\frac{\Lambda_1\vartheta_2}{\Gamma_q(\delta_1+\theta_1+1)} + \frac{\Lambda_2\mu_2}{\Gamma_q(\delta_2+\theta_2+1)} = 8.5619 \times 10^{-4} < \frac{1}{2}.$$

Thus, all the conditions of Theorem 8 are satisfied, and problem (55) has at least one solution on $[0, 1]$.

5 Conclusion

In this study, we considered acoupled hybrid fractional q-differential systems involving two sequential Caputo fractional q-derivatives. The uniqueness, existence, and Ulam stability of the solutions have been discussed. The existence and uniqueness of solutions for the mentioned problem is established by applying contraction mapping principles. By the aid of the Leray-Schauder alternative the existence of at least one solution is established. Furthermore, the Ulam-Hyers stability and Ulam-Hyers-Rassias stability results are obtained. Finally, a simulative examples are proposed in order to enlighten the theoretical results. Since, in this field of interest, it is important to increase ability of scholars for investigating differential equations with fractional quantum calculus and trying to find applicability of the studied problems in real word phenomena, in this work we have discussed a coupled hybrid fractional q-differential systems with two sequential Caputo fractional q-derivatives. However, for future developments, we think that it will be more suitable to discuss the above q-fractional problem by considering n-sequential Caputo q-derivatives. For further consideration in the future, we will continue to study the Ulam-Hyers-Mittag-Leffler stability for the above proposed system by using Henry-Gronwall inequality.

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