Research Article

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A novel conservative numerical approximation scheme for the Rosenau-Kawahara equation

https://doi.org/10.1515/dema-2022-0204
received October 27, 2021; accepted January 20, 2023

Abstract: In this article, a numerical solution for the Rosenau-Kawahara equation is considered. A new conservative numerical approximation scheme is presented to solve the initial boundary value problem of the Rosenau-Kawahara equation, which preserves the original conservative properties. The proposed scheme is based on the finite difference method. The existence of the numerical solutions for the scheme has been shown by Browder fixed point theorem. The priori bound and error estimates, as well as the conservation of discrete mass and discrete energy for the finite difference solutions, are discussed. The discrepancies of discrete mass and energy are computed and shown by the curves of these quantities over time. Unconditional stability, second-order convergence, and uniqueness of the scheme are proved based on the discrete energy method. Numerical examples are given to show the effectiveness of the proposed scheme and confirm the theoretical analysis.

Keywords: Rosenau-Kawahara equation, conservative difference scheme, existence, convergence

MSC 2020: 65N06, 65N12

1 Introduction

The well-known Korteweg-de Vries (KdV) equation is a classical paradigm of integrable nonlinear evolution equations, which plays an important role in the development of soliton theory [1,2]. However, in the study of the dynamics of dense discrete systems, the case of wave-wave and wave-wall interactions cannot be treated by the well-known KdV equation. Furthermore, the slope and behavior of high-amplitude waves may not be well predicted by the KdV equation because it was modeled under the assumption of weak anharmonicity. In order to overcome the shortcoming of the KdV equation, Rosenau [3] proposed the KdV-like Rosenau equation

$$u_t + u_{xxxxt} + u_x + uu_x = 0. \tag{1.1}$$

The theoretical results on the existence, uniqueness, and regularity of the solution to (1.1) have been investigated by Park [4]. Since then, various numerical techniques have been proposed for (1.1) [5–11] and also the references therein; especially in [11], operator time-splitting techniques combined with the quintic B-spline collocation method were developed for the generalized Rosenau-KdV equation. For wide, interesting, and related topics covered, we should also recall the numerical study done on the equations in [12–21].

For further consideration of the nonlinear wave, Zuo [22] added the viscous terms u_{xxx} and $-u_{xxxxx}$ to the Rosenau equation and proposed the Rosenau-Kawahara equation

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$$u_t + u_{xxxxt} + u_{xxx} - u_{xxxxx} + u_x + uu_x = 0. \tag{1.2}$$

Recently, Zuo [22] obtained solitons and periodic solutions for the Rosenau-Kawahara equation. Hu et al. [23] proposed two conservative schemes for the Rosenau-Kawahara equation. The numerical results are interesting. However, both conservative laws of the Rosenau-Kawahara equation were not proved in [23]. But, both conservative laws play very important roles in the following numerical analysis of the proposed scheme. Furthermore, as far as the computational studies are concerned, the handling of the nonlinear term uu_x also has a different discretization method. Therefore, in this article, we give a modified proof of the conservative properties, and an attempt has been made to propose a new difference scheme for the Rosenau-Kawahara equation (1.2) with the initial condition

$$u(x,0) = u_0(x), \quad x \in [a,b], \tag{1.3}$$

and the following boundary conditions

$$u(a,t) = u(b,t) = u_x(a,t) = u_x(b,t) = u_{xx}(a,t) = u_{xx}(b,t) = 0, \quad t \in [0,T], \tag{1.4}$$

where $u_0(x)$ is a known smooth function.

In [24], Biswas et al. studied the solitary solution of the Rosenau-Kawahara equation; by the solitary wave assumptions, the solitary solution and its derivatives have the following asymptotic values: $u \to 0$ as $x \to \pm\infty$ and $\frac{\partial^n u}{\partial x^n} \to 0$ as $x \to \pm\infty$ for $n \geq 1$. Thus, the problem can be set up in a compact subset $[a,b]$. This implies that the initial boundary value problems (1.2)–(1.4) is consistent with the initial value problems (1.2) and (1.3) for $-a \gg 0$ and $b \gg 0$. Hence, we can prove that the system (1.2)–(1.4) possesses the following conservative laws [23]:

$$Q(t) = \int_a^b u(x,t)\mathrm{d}x = \int_a^b u(x,0)\mathrm{d}x = Q(0), \tag{1.5}$$

$$E(t) = ||u||_{L_2}^2 + ||u_{xx}||_{L_2}^2 = E(0). \tag{1.6}$$

Proof. Integrating over the interval $[a,b]$ in (1.2) yields

$$\int_a^b (u_t + u_{xxxxt} + u_{xxx} - u_{xxxxx} + u_x + uu_x)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}u_{xxx}|_a^b + u_{xx}|_a^b - u_{xxxx}|_a^b + u|_a^b + \frac{1}{2}u^2|_a^b = 0. \tag{1.7}$$

According to the asymptotic values of $u \to 0$ and $\frac{\partial^n u}{\partial x^n} \to 0$ with $n \geq 1$ in the aforementioned compact subset $[a,b]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u\mathrm{d}x = 0. \tag{1.8}$$

Let $Q(t) = \int_a^b u(x,t)\mathrm{d}x$. (1.5) holds.

Next, we shall prove (1.6). Consider $u_t + u_{xxxxt} = -u_{xxx} + u_{xxxxx} - u_x - uu_x$. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t) &= 2\int_a^b uu_t\mathrm{d}x + 2\int_a^b u_{xx}u_{xxt}\mathrm{d}x \\
&= 2\int_a^b uu_t\mathrm{d}x + 2u_xu_{xxt}|_a^b - 2\int_a^b u_xu_{xxxt}\mathrm{d}x \\
&= 2\int_a^b uu_t\mathrm{d}x - 2uu_{xxxt}|_a^b + 2\int_a^b uu_{xxxxt}\mathrm{d}x \\
&= 2\int_a^b u(u_t + u_{xxxxt})\mathrm{d}x \\
&= 2\int_a^b u(-u_{xxx} + u_{xxxxx} - u_x - uu_x)\mathrm{d}x \\
&= u_x^2|_a^b - u_{xx}^2|_a^b - u^2|_a^b - \frac{2}{3}u^3|_a^b \\
&= 0.
\end{aligned} \tag{1.9}$$

By the definition of $E(t)$, (1.6) holds. □

It is well known that the conservative difference schemes perform better than the non-conservative ones, and the non-conservative difference schemes may easily show nonlinear blow-up [25]. Some conservative schemes have been proposed in the literature [26–34]. Recently, efficient and conservative numerical techniques were used to solve the general nonlinear wave equation [35] and the nonlinear fractional Schrödinger equations [36], with references therein. The numerical results of all the schemes are very encouraging. For the wide topics of structure-preserving schemes covered, the efficient numerical methods proposed in [37–40] are efficient. The main purpose of this article is to construct a new numerical scheme that has the following advantages: coupling with the Richardson extrapolation, the proposed scheme is uniquely solvable, unconditionally stable, and of second-order accuracy; the new scheme preserves the original conservative property; the coefficient matrices of the scheme are symmetric and seven-diagonal, and the Thomas algorithm can be employed to solve it effectively.

The remainder of this article is organized as follows. In Section 2, an energy conservative C–N difference scheme for the Rosenau-Kawahara equation is described and the discrete conservative laws of the difference scheme are discussed. In Section 3, we prove the existence of the scheme. In Section 4, convergence, stability, and uniqueness of the scheme are proved. In Section 5, numerical experiments are reported.

2 A conservative scheme and its discrete conservative law

In this section, we describe a new conservative difference scheme for the problems (1.2)–(1.4). Let h and τ be the uniform step size in the spatial and temporal directions, respectively. Denote $x_j = a + jh$ $(0 \le j \le J)$, $t_n = n\tau$ $(0 \le n \le N)$, $u_j^n \approx u(x_j, t_n)$, and $Z_h^0 = \{u = (u_j) | u_{-2} = u_{-1} = u_0 = u_J = u_{J+1} = u_{J+2} = 0, j = -2, -1, 0, 1, 2, \ldots, J, J+1, J+2\}$. Define the difference operators:

$$(\omega_j^n)_x = \frac{\omega_{j+1}^n - \omega_j^n}{h}, \quad (\omega_j^n)_{\bar{x}} = \frac{\omega_j^n - \omega_{j-1}^n}{h}, \quad (\omega_j^n)_{\hat{x}} = \frac{\omega_{j+1}^n - \omega_{j-1}^n}{2h},$$

$$(\omega_j^n)_t = \frac{\omega_j^{n+1} - \omega_j^n}{\tau}, \quad (\omega_j^n)_{x\bar{x}} = \frac{\omega_{j+1}^n - 2\omega_j^n + \omega_{j-1}^n}{h^2}, \quad \omega_j^{n+\frac{1}{2}} = \frac{\omega_j^{n+1} + u_j^n}{2},$$

$$(\omega^n, v^n) = h\sum_{j=1}^{J-1} \omega_j^n v_j^n, \quad ||\omega^n||^2 = (\omega^n, \omega^n), \quad ||\omega^n||_\infty = \max_{1\le j\le J-1} |\omega_j^n|.$$

In the article, C denotes a positive constant independent of mesh steps h and τ, which may have different values in different occurrences.

The following conservative C–N difference scheme for the IBV problems (1.2)–(1.4) is considered:

$$(u_j^n)_t + (u_j^n)_{xx\bar{x}\bar{x}t} + \left(u_j^{n+\frac{1}{2}}\right)_{x\bar{x}\hat{x}} - \left(u_j^{n+\frac{1}{2}}\right)_{xx\bar{x}\bar{x}\hat{x}} + \left(u_j^{n+\frac{1}{2}}\right)_{\hat{x}} + \frac{1}{3}\left(u_{j+1}^{n+\frac{1}{2}} + u_j^{n+\frac{1}{2}} + u_{j-1}^{n+\frac{1}{2}}\right)\left(u_j^{n+\frac{1}{2}}\right)_{\hat{x}} = 0, \tag{2.1}$$

$$u_j^0 = u_0(x_j), \quad 1 \le j \le J-1, \tag{2.2}$$

$$u_0^n = u_J^n = (u_0^n)_{\hat{x}} = (u_J^n)_{\hat{x}} = (u_0^n)_{x\bar{x}} = (u_J^n)_{x\bar{x}} = 0. \tag{2.3}$$

For convenience, the last term of (2.1) is defined as follows:

$$\kappa\left(u^{n+\frac{1}{2}}, \left(u^{n+\frac{1}{2}}\right)_{\hat{x}}\right) = \frac{1}{3}\left(u_{j+1}^{n+\frac{1}{2}} + u_j^{n+\frac{1}{2}} + u_{j-1}^{n+\frac{1}{2}}\right)\left(u_j^{n+\frac{1}{2}}\right)_{\hat{x}}.$$

To obtain conservative laws, we introduce the following Lemma. It can be proved without difficulty [8].

Lemma 2.1. *For any two mesh functions: $u, v \in Z_h^0$, we have*

$$(u_x, v) = -\left(u, v_{\bar{x}}\right), \quad \left(v u_{x\bar{x}}\right) = -(v_x, (u_x)),$$

and

$$\left(u, u_{x\bar{x}}\right) = -(u_x, u_x) = -||u_x||^2.$$

Furthermore, if $(u_0^n)_{x\bar{x}} = (u_J^n)_{x\bar{x}} = 0$, *then*

$$\left(u, u_{xx\bar{x}\bar{x}}\right) = ||u_{xx}||^2.$$

Theorem 2.1. *Suppose* $u_0 \in H_0^2[a, b]$, *then the scheme* (2.1)–(2.3) *admits the following invariants*:

$$Q^n = h\sum_{j=1}^{J-1} u_j^n = Q^{n-1} = \cdots = Q^0. \tag{2.4}$$

$$E^n = ||u^n||^2 + ||u_{xx}^n||^2 = E^{n-1} = \cdots = E^0. \tag{2.5}$$

Proof. Multiplying (2.1) with h, according to the boundary conditions (2.3), then summing up for j from 1 to $J-1$, we obtain

$$h\sum_{j=1}^{J-1}(u_j^{n+1} - u_j^n) = 0. \tag{2.6}$$

Let

$$Q^n = h\sum_{j=1}^{J-1} u_j^n, \tag{2.7}$$

then (2.4) is obtained from (2.6).

Taking the inner product of (2.1) with $2u^{n+\frac{1}{2}}$, according to boundary condition (2.3) and Lemma 2.1, we have

$$\begin{aligned}&\frac{1}{\tau}(||u^{n+1}||^2 - ||u^n||^2) + \frac{1}{\tau}(||u_{xx}^{n+1}||^2 - ||u_{xx}^n||^2) + \left(\left(u^{n+\frac{1}{2}}\right)_{x\bar{x}\hat{x}}, 2u^{n+\frac{1}{2}}\right) + \left(\left(u^{n+\frac{1}{2}}\right)_{\hat{x}}, 2u^{n+\frac{1}{2}}\right)\\ &\quad + \left(\kappa(u^{n+\frac{1}{2}}, \left(u^{n+\frac{1}{2}}\right)_{\hat{x}}), 2u^{n+\frac{1}{2}}\right) = 0.\end{aligned} \tag{2.8}$$

Note that

$$\left(\left(u^{n+\frac{1}{2}}\right)_{x\bar{x}\hat{x}}, 2u^{n+\frac{1}{2}}\right) = 0, \quad \left(\left(u^{n+\frac{1}{2}}\right)_{xx\bar{x}\bar{x}\hat{x}}, 2u^{n+\frac{1}{2}}\right) = 0, \quad \left(\left(u^{n+\frac{1}{2}}\right)_{\hat{x}}, 2u^{n+\frac{1}{2}}\right) = 0, \tag{2.9}$$

and

$$\begin{aligned}\left(\kappa\left(u^{n+\frac{1}{2}}, \left(u^{n+\frac{1}{2}}\right)_{\hat{x}}\right), 2u^{n+\frac{1}{2}}\right) &= \frac{2}{3}h\sum_{j=1}^{J-1}\left[\left(u_{j+1}^{n+\frac{1}{2}} + u_j^{n+\frac{1}{2}} + u_{j-1}^{n+\frac{1}{2}}\right)\left(u_j^{n+\frac{1}{2}}\right)_{\hat{x}}\right]u_j^{n+\frac{1}{2}}\\ &= \frac{1}{3}\sum_{j=1}^{J-1}\left[\left(u_{j+1}^{n+\frac{1}{2}} + u_j^{n+\frac{1}{2}} + u_{j-1}^{n+\frac{1}{2}}\right)\left(u_{j+1}^{n+\frac{1}{2}} - u_{j-1}^{n+\frac{1}{2}}\right)\right]u_j^{n+\frac{1}{2}}\\ &= \frac{1}{3}\sum_{j=1}^{J-1}\left(u_{j+1}^{n+\frac{1}{2}} + u_j^{n+\frac{1}{2}}\right)u_{j+1}^{n+\frac{1}{2}}u_j^{n+\frac{1}{2}} - \frac{1}{3}\sum_{j=1}^{J-1}\left(u_j^{n+\frac{1}{2}} + u_{j-1}^{n+\frac{1}{2}}\right)u_{j-1}^{n+\frac{1}{2}}u_j^{n+\frac{1}{2}}\\ &= 0.\end{aligned} \tag{2.10}$$

It follows from (2.8)–(2.10) that

$$(||u^{n+1}||^2 - ||u^n||^2) + (||u_{xx}^{n+1}||^2 - ||u_{xx}^n||^2) = 0. \tag{2.11}$$

By the definition of E^n, (2.5) holds. □

3 Existence

To prove the existence of solution for scheme (2.1)–(2.3), the following Browder fixed point theorem should be introduced. For the proof, see [41].

Lemma 3.1. (Browder fixed point Theorem). *Let H be a finite dimensional inner product space. Suppose that $g : H \to H$ is continuous and there exists an $\alpha > 0$ such that $(g(x), x) \geq 0$ for all $x \in H$ with $||x|| = \alpha$. Then, there exists $x^* \in H$ such that $g(x^*) = 0$ and $||x^*|| \leq \alpha$.*

Theorem 3.1. *There exists $u^n \in Z_h^0$, which satisfies the difference scheme* (2.1)–(2.3).

Proof. It follows from the original problems (1.2)–(1.4) that u^0 satisfies the scheme (2.1)–(2.3). For $n \leq N-1$, assume that $u^1, u^2, \dots, u^n$ satisfy (2.1)–(2.3). Next, we prove that there exists u^{n+1}, which satisfies (2.1).

Define an operator g on Z_h^0 as follows:

$$g(v) = 2v - 2u^n + 2v_{xx\bar{x}\bar{x}} - 2u^n_{xx\bar{x}\bar{x}} + \tau v_{x\bar{x}\hat{x}} - \tau v_{xx\bar{x}\bar{x}\hat{x}} + \tau v_{\hat{x}} + \frac{\tau}{3}(v_{j+1} + v_j + v_{j-1})v_{\hat{x}}. \tag{3.1}$$

Taking the inner product of (3.1) with v and using

$$\left(v_{x\bar{x}\hat{x}}, v\right) = 0, \ \left(v_{xx\bar{x}\bar{x}\hat{x}}, v\right) = 0, \ \left(v_{\hat{x}}, v\right) = 0, \quad \left((v_{j+1} + v_j + v_{j-1})v_{\hat{x}}, v\right) = 0,$$

we obtain that

$$\begin{aligned}(g(v), v) &= 2||v||^2 - 2(u^n, v) + 2||v_{xx}||^2 - 2(u^n_{xx}, v_{xx}) \\ &\geq 2||v||^2 - 2||u^n||\cdot||v|| + 2||v_{xx}||^2 - 2||u^n_{xx}||\cdot||v_{xx}|| \\ &\geq 2||v||^2 - (||u||^2 + ||v||^2) + 2||v_{xx}||^2 - (||u_{xx}||^2 + ||v_{xx}||^2) \\ &\geq ||v||^2 - (||u^n||^2 + ||u_{xx}||^2) + ||v_{xx}||^2 \\ &\geq ||v||^2 - (||u^n||^2 + ||u^n_{xx}||^2).\end{aligned} \tag{3.2}$$

Obviously, for $\forall v \in Z_h^0$, $(g(v), v) \geq 0$ with $||v||^2 = ||u^n||^2 + ||u^n_{xx}||^2 + 1$. It follows from Lemma 3.1 that there exists $v^* \in Z_h^0$ that satisfies $g(v^*) = 0$. Let $u^{n+1} = 2v^* - u^n$, it can be proved that u^{n+1} is the solution of the scheme (2.1)–(2.3). This completes the proof of Theorem 3.1. □

4 Convergence, stability, and uniqueness of the scheme

Next, we discuss the convergence stability and uniqueness of the scheme (2.1)–(2.3). Let $v(x, t)$ be the solution to problems (1.2)–(1.4), $v_j^n = v(x_j, t_n)$, then we define the truncation error of the scheme (2.1) as follows:

$$Er_j^n = (v_j^n)_t + (v_j^n)_{xx\bar{x}\bar{x}t} + \left(v_j^{n+\frac{1}{2}}\right)_{x\bar{x}\hat{x}} - \left(v_j^{n+\frac{1}{2}}\right)_{xx\bar{x}\bar{x}\hat{x}} + \left(v_j^{n+\frac{1}{2}}\right)_{\hat{x}} + \frac{1}{3}\left(v_{j+1}^{n+\frac{1}{2}} + v_j^{n+\frac{1}{2}} + v_{j-1}^{n+\frac{1}{2}}\right)\left(v_j^{n+\frac{1}{2}}\right)_{\hat{x}}. \tag{4.1}$$

According to the Taylor expansion, we know that $Er_j^n = O(\tau^2 + h^2)$ holds if $\tau, h \to 0$.

Lemma 4.1. (Discrete Sobolev's inequality [42]). *There exist two constants C_1 and C_2 such that*

$$||u^n||_\infty \leq C_1||u^n|| + C_2||u_x^n||.$$

Lemma 4.2. *Suppose $u_0 \in H_0^2[a, b]$, then the estimate of the solution of the initial boundary value problems* (1.2)–(1.4) *satisfies*

$$||u||_{L_2} \leq C, \quad ||u_x||_{L_2} \leq C, \quad ||u||_{L_\infty} \leq C.$$

Proof. It follows from (1.6) that

$$||u||_{L_2} \le C, \quad ||u_{xx}||_{L_2} \le C. \tag{4.2}$$

An application of Hölder inequality and Schwartz inequality yields

$$||u_x||_{L_2}^2 \le ||u||_{L_2} \cdot ||u_{xx}||_{L_2} \le \frac{1}{2}\Big(||u||_{L_2}^2 + ||u_{xx}||_{L_2}^2\Big). \tag{4.3}$$

Hence, $||u_x||_{L_2} \le C$. By Sobolev inequality, we have $||u||_{L_\infty} \le C$. □

Lemma 4.3. (Discrete Gronwall inequality [42]). *Suppose $w(k)$ and $\rho(k)$ are nonnegative mesh functions and $\rho(k)$ is nondecreasing. If $C > 0$ and*

$$w(k) \le \rho(k) + C\tau \sum_{l=0}^{k-1} w(l) \quad \forall k,$$

then

$$w(k) \le \rho(k) e^{C\tau k} \quad \forall k.$$

Lemma 4.4. *Suppose $u_0 \in H_0^2[a, b]$, then there is the estimate of the solution u^n of* (2.1): $||u^n|| \le C$, $||u_x^n|| \le C$, *which yields $||u^n||_\infty \le C$.*

Proof. It follows from (2.5) that

$$||u^n|| \le C, \quad ||u_{xx}^n|| \le C. \tag{4.4}$$

Using Lemma 2.1 and Schwartz inequality, we obtain

$$||u_x^n||^2 \le ||u^n|| ||u_{xx}^n|| \le \frac{1}{2}(||u^n||^2 + ||u_{xx}^n||^2) \le C. \tag{4.5}$$

According to Lemma 4.1, we have $||u^n||_\infty \le C$. □

Remark 4.1. Lemma 4.4 implies that scheme (2.1)–(2.3) is unconditionally stable.

Theorem 4.1. *Suppose $u_0 \in H_0^2[a, b]$ and $u(x, t) \in C^{7,3}$, then the solution u^n of the scheme* (2.1)–(2.3) *converges to the solution of problems* (1.2)–(1.4) *in the sense of $||\cdot||_\infty$ norm and the rate of convergence is $O(\tau^2 + h^2)$.*

Proof. Subtracting (4.1) from (2.1) and letting $e_j^n = v_j^n - u_j^n$, we have

$$Er_j^n = (e_j^n)_t + (e_j^n)_{xx\bar{x}\bar{x}t} + \left(e_j^{n+\frac{1}{2}}\right)_{x\bar{x}\hat{x}} - \left(e_j^{n+\frac{1}{2}}\right)_{xx\bar{x}\bar{x}\hat{x}} + \left(e_j^{n+\frac{1}{2}}\right)_{\hat{x}} + P + Q, \tag{4.6}$$

where

$$P = \frac{1}{3}\left(v_{j+1}^{n+\frac{1}{2}} + v_j^{n+\frac{1}{2}} + v_{j-1}^{n+\frac{1}{2}}\right)\left(v_j^{n+\frac{1}{2}}\right)_{\hat{x}}, \quad Q = -\frac{1}{3}\left(u_{j+1}^{n+\frac{1}{2}} + u_j^{n+\frac{1}{2}} + u_{j-1}^{n+\frac{1}{2}}\right)\left(u_j^{n+\frac{1}{2}}\right)_{\hat{x}}.$$

Computing the inner product of (4.6) with $2e^{n+\frac{1}{2}}$, and noting that

$$\left(\left(e^{n+\frac{1}{2}}\right)_{x\bar{x}\hat{x}}, 2e^{n+\frac{1}{2}}\right) = 0, \quad \left(\left(e^{n+\frac{1}{2}}\right)_{xx\bar{x}\bar{x}\hat{x}}, 2e^{n+\frac{1}{2}}\right) = 0, \quad \left(\left(e^{n+\frac{1}{2}}\right)_{\hat{x}}, 2e^{n+\frac{1}{2}}\right) = 0,$$

we obtain

$$\left(Er^n, 2e^{n+\frac{1}{2}}\right) = \frac{1}{\tau}(||e^{n+1}||^2 - ||e^n||^2) + \frac{1}{\tau}(||e_{xx}^{n+1}||^2 - ||e_{xx}^n||^2) + \left(P + Q, 2e^{n+\frac{1}{2}}\right). \tag{4.7}$$

According to Lemma 4.2, 4.4 and Cauchy-Schwartz inequality, we have

$$\begin{aligned}\left(P+Q,2e^{n+\frac{1}{2}}\right)&=\frac{2h}{3}\sum_{j=1}^{J-1}\left[\left(v_{j+1}^{n+\frac{1}{2}}+v_j^{n+\frac{1}{2}}+v_{j-1}^{n+\frac{1}{2}}\right)\left(v_j^{n+\frac{1}{2}}\right)_{\hat{x}}-\left(u_{j+1}^{n+\frac{1}{2}}+u_j^{n+\frac{1}{2}}+u_{j-1}^{n+\frac{1}{2}}\right)\left(u_j^{n+\frac{1}{2}}\right)_{\hat{x}}\right]e_j^{n+\frac{1}{2}}\\&=\frac{2h}{3}\sum_{j=1}^{J-1}\left[\left(e_{j+1}^{n+\frac{1}{2}}+e_j^{n+\frac{1}{2}}+e_{j-1}^{n+\frac{1}{2}}\right)\left(u_j^{n+\frac{1}{2}}\right)_{\hat{x}}+\left(u_{j+1}^{n+\frac{1}{2}}+u_j^{n+\frac{1}{2}}+u_{j-1}^{n+\frac{1}{2}}\right)\left(e_j^{n+\frac{1}{2}}\right)_{\hat{x}}\right]e_j^{n+\frac{1}{2}}\\&\le C(\|e_x^{n+1}\|^2+\|e_x^n\|^2+\|e^{n+1}\|^2+\|e^n\|^2).\end{aligned}\tag{4.8}$$

In addition, there exists obviously that

$$\left(Er^n,2e^{n+\frac{1}{2}}\right)\le\|Er^n\|^2+\frac{1}{2}(\|e^{n+1}\|^2+\|e^n\|^2).\tag{4.9}$$

Substituting (4.8)–(4.9) into (4.7), we obtain

$$(\|e^{n+1}\|^2-\|e^n\|^2)+(\|e_{xx}^{n+1}\|^2-\|e_{xx}^n\|^2)\le C\tau(\|e^{n+1}\|^2+\|e^n\|^2+\|e_x^{n+1}\|^2+\|e_x^n\|^2)+\tau\|Er^n\|^2.\tag{4.10}$$

Similarly to the proof of (4.5), we have

$$\|e_x^{n+1}\|^2\le\frac{1}{2}(\|e^{n+1}\|^2+\|e_{xx}^{n+1}\|^2),\quad \|e_x^n\|^2\le\frac{1}{2}(\|e^n\|^2+\|e_{xx}^n\|^2).\tag{4.11}$$

Thus, (4.10) yields

$$(\|e^{n+1}\|^2-\|e^n\|^2)+(\|e_{xx}^{n+1}\|^2-\|e_{xx}^n\|^2)\le C\tau(\|e^{n+1}\|^2+\|e^n\|^2+\|e_{xx}^{n+1}\|^2+\|e_{xx}^n\|^2)+\tau\|Er^n\|^2.\tag{4.12}$$

Let $B^n=\|e^n\|^2+\|e_{xx}^n\|^2$, then (4.12) can be rewritten as follows:

$$(1-C\tau)(B^{n+1}-B^n)\le 2C\tau B^n+\tau\|Er^n\|^2.$$

If τ is sufficiently small, which satisfies $1-C\tau>0$, then

$$B^{n+1}-B^n\le C\tau B^n+C\tau\|Er^n\|^2.\tag{4.13}$$

Summing up (4.13) from 0 to $n-1$, we have

$$B^n\le B^0+C\tau\sum_{l=0}^{n-1}\|Er^l\|^2+C\tau\sum_{l=0}^{n-1}B^l.$$

Noting that

$$\tau\sum_{l=0}^{n-1}\|Er^l\|^2\le n\tau\max_{0\le l\le n-1}\|Er^l\|^2\le T\cdot[O(\tau^2+h^2)]^2,$$

and $e^0=0$, we have $B^0=[O(\tau^2+h^2)]^2$. Hence,

$$B^n\le[O(\tau^2+h^2)]^2+C\tau\sum_{l=0}^{n-1}B^l.\tag{4.14}$$

According to Lemma 4.3, we obtain $B^n\le[O(\tau^2+h^2)]^2$, that is

$$\|e^n\|\le O(\tau^2+h^2),\quad \|e_{xx}^n\|\le O(\tau^2+h^2).\tag{4.15}$$

This together with (4.11) and Lemma 4.1 gives

$$\|e^n\|_\infty\le O(\tau^2+h^2).\tag{4.16}$$

This completes the proof of Theorem 4.1. □

Similarly, we can prove the stability of the difference solution. The details are omitted.

Theorem 4.2. *Under the conditions of Theorem* 4.1, *the solution of scheme* (2.1)–(2.3) *is stable by the* $||\cdot||_\infty$ *norm.*

Theorem 4.3. *The scheme* (2.1)–(2.3) *is uniquely solvable.*

Proof. Suppose that u^n and U^n both satisfy the scheme (2.1)–(2.3). Let $w^n = u^n - U^n$, then we have

$$(w_j^n)_t + (w_j^n)_{xx\bar{x}\bar{x}t} + (w_j^{n+\frac{1}{2}})_{x\bar{x}\hat{x}} - (w_j^{n+\frac{1}{2}})_{xx\bar{x}\bar{x}\hat{x}} + (w_j^{n+\frac{1}{2}})_{\hat{x}} + I + II = 0, \tag{4.17}$$

$$w_j^0 = 0, \tag{4.18}$$

$$w_0^n = w_J^n = (w_0^n)_{\hat{x}} = (w_J^n)_{\hat{x}} = (w_0^n)_{x\bar{x}} = (w_J^n)_{x\bar{x}} = 0, \tag{4.19}$$

where

$$I = \frac{1}{3}\left(u_{j+1}^{n+\frac{1}{2}} + u_j^{n+\frac{1}{2}} + u_{j-1}^{n+\frac{1}{2}}\right)\left(u_j^{n+\frac{1}{2}}\right)_{\hat{x}}, \quad II = -\frac{1}{3}(U_{j+1}^{n+\frac{1}{2}} + U_j^{n+\frac{1}{2}} + U_{j-1}^{n+\frac{1}{2}})(U_j^{n+\frac{1}{2}})_{\hat{x}}.$$

Similarly to the proof of Theorem 4.1, we obtain that

$$||w^n||^2 + ||w_{xx}^n||^2 = 0. \tag{4.20}$$

This completes the proof of Theorem 4.3. □

5 Numerical experiments

In this section, we shall compute some numerical experiments to verify the correction of our analysis in the above sections.

Consider the initial-boundary value problem for the Rosenau-Kawahara equation (1.2)–(1.4). The exact solution of the system (1.2)–(1.4) has the following form:

$$u(x,t) = \left(-\frac{35}{12} + \frac{35}{156}\sqrt{205}\right)\operatorname{sech}^4\left[\frac{1}{12}\sqrt{-13+\sqrt{205}}\left(x - \frac{1}{13}\sqrt{205}\,t\right)\right]. \tag{5.1}$$

The initial condition of the studied model is obtained from (5.1):

$$u(x,0) = \left(-\frac{35}{12} + \frac{35}{156}\sqrt{205}\right)\operatorname{sech}^4\left(\frac{1}{12}\sqrt{-13+\sqrt{205}}\,x\right). \tag{5.2}$$

Table 1: The errors of numerical solutions at $t = 6$ with various h and τ for the scheme (2.1)

h	τ	$\|\|v^n - u^n\|\|$	$\|\|v^n - u^n\|\|_\infty$	$\frac{\|\| v^{\frac{n}{4}} - u^{\frac{n}{4}} \|\|}{\|\| v^n - u^n \|\|}$	$\frac{\|\| v^{\frac{n}{4}} - u^{\frac{n}{4}} \|\|_\infty}{\|\| v^n - u^n \|\|_\infty}$
0.4	0.4	2.08567×10^{-3}	7.23237×10^{-4}	—	—
0.2	0.2	5.23337×10^{-4}	1.81471×10^{-4}	3.96965	3.96596
0.1	0.1	1.30957×10^{-4}	4.54316×10^{-5}	3.99242	3.99037
0.05	0.05	3.30439×10^{-5}	1.13603×10^{-5}	3.99809	3.99761
0.4	0.2	1.47546×10^{-3}	5.08196×10^{-4}	—	—
0.2	0.1	3.69368×10^{-4}	1.27183×10^{-4}	3.98720	3.98563
0.1	0.05	9.23770×10^{-5}	3.18208×10^{-5}	3.99682	3.99561

Table 2: The errors of numerical solutions at $t = 10$ with various h and τ for the scheme (2.1)

h	τ	$\|v^n - u^n\|$	$\|v^n - u^n\|_\infty$	$\frac{\|v_4^n - u_4^n\|}{\|v^n - u^n\|}$	$\frac{\|v_4^n - u_4^n\|_\infty}{\|v^n - u^n\|_\infty}$	CPU time
0.4	0.4	3.43837×10^{-3}	1.19758×10^{-3}	—	—	29.047 s
0.2	0.2	8.63021×10^{-4}	3.00546×10^{-4}	3.96007	3.95519	287.656 s
0.1	0.1	2.15981×10^{-4}	7.52081×10^{-5}	3.99010	3.98661	4.23×10^3 s
0.05	0.05	5.42160×10^{-5}	1.88069×10^{-5}	3.99747	3.99692	6.367×10^4 s
0.4	0.2	2.43357×10^{-3}	8.43732×10^{-4}	—	—	56.625 s
0.2	0.1	6.09377×10^{-4}	2.11201×10^{-4}	3.97956	3.97587	569.047 s
0.1	0.05	1.52477×10^{-4}	5.28166×10^{-5}	3.99496	3.99193	8.068×10^3 s

Table 3: The errors in the sense of L_∞-norm of numerical solutions u^n of the scheme (2.1) at different time t with various h and τ

t	$h = \tau = 0.4$	$h = \tau = 0.2$	$h = \tau = 0.1$	$h = \tau = 0.05$
2	2.36285×10^{-4}	5.94147×10^{-5}	1.48662×10^{-5}	3.71732×10^{-6}
4	4.78784×10^{-4}	1.20237×10^{-4}	3.00929×10^{-5}	7.52510×10^{-6}
6	7.23237×10^{-4}	1.81471×10^{-4}	4.54316×10^{-5}	1.13603×10^{-5}
8	9.63714×10^{-4}	2.41833×10^{-4}	6.05150×10^{-5}	1.51333×10^{-5}
10	1.19758×10^{-3}	3.00546×10^{-4}	7.52081×10^{-5}	1.88069×10^{-5}

Table 4: The errors in the sense of L_2-norm of numerical solutions u^n of the scheme (2.1) at different time t with various h and τ

t	$h = \tau = 0.4$	$h = \tau = 0.2$	$h = \tau = 0.1$	$h = \tau = 0.05$
2	6.99455×10^{-4}	1.75479×10^{-4}	4.39154×10^{-5}	1.10141×10^{-5}
4	1.39565×10^{-3}	3.50160×10^{-4}	8.76227×10^{-5}	2.19159×10^{-5}
6	2.08567×10^{-3}	5.23337×10^{-4}	1.30957×10^{-4}	3.30439×10^{-5}
8	2.76715×10^{-3}	6.94432×10^{-4}	1.73777×10^{-4}	4.54320×10^{-5}
10	3.43837×10^{-3}	8.63021×10^{-4}	2.15981×10^{-4}	5.42160×10^{-5}

Table 5: Discrete mass and discrete energy of the scheme (2.1) when $h = \tau = 0.05$

t	Q^n	E^n
$t = 0$	4.12089321499933	0.83620118172401
$t = 2$	4.12089302402904	0.83620118171116
$t = 4$	4.12089342992713	0.83620118172960
$t = 6$	4.12089416312627	0.83620118175778
$t = 8$	4.12089522384561	0.83620118242616
$t = 10$	4.12089327111637	0.83620118182357

It follows from (5.1) that the initial boundary value problems (1.2)–(1.4) is consistent to the initial value problems (1.2)–(1.3) for $-a \gg 0, b \gg 0$. In the numerical experiments, we take $a = -40$, $b = 120$, and $T = 10$. The errors in the sense of L_∞ and L_2-norm of the numerical solutions are listed on Tables 1–4 under different mesh steps h and τ. Tables 1 and 2 verify that the scheme (2.1) has an accuracy of $O(\tau^2 + h^2)$. CPU time is given to show that the proposed scheme is easily solved and stored. Tables 3 and 4 are presented to show the good stability of the numerical solutions. Table 5 and Figure 1 of the curves of discrepancies of the discrete mass and discrete energy are given to show that the scheme (2.1) preserves the discrete

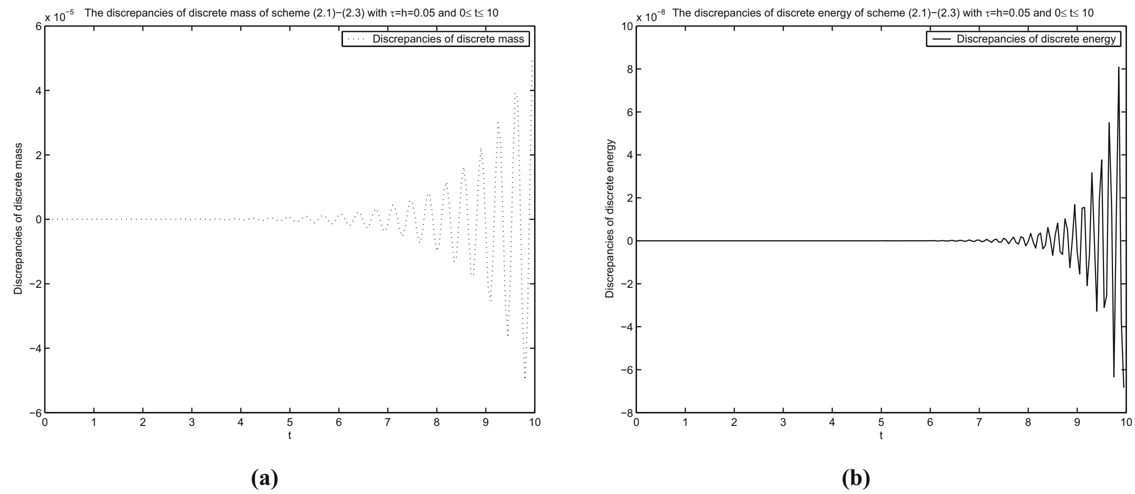

Figure 1: The discrepancies of conservative laws of the Scheme (2.1) with $h = \tau = 0.05$ and $0 \le t \le 10$.

Table 6: L_∞-norm errors comparison of Schemes I–III at different time t with $h = \tau = 0.2$

t	2	4	6	8	10
I	5.63242×10^{-5}	1.137091×10^{-4}	1.72392×10^{-4}	2.30840×10^{-4}	2.88211×10^{-4}
II	1.15741×10^{-4}	2.33753×10^{-4}	3.53667×10^{-4}	4.72674×10^{-4}	5.89342×10^{-4}
III	5.94147×10^{-5}	1.20237×10^{-4}	1.81471×10^{-4}	2.41833×10^{-4}	3.00546×10^{-4}

Table 7: L_2-norm errors comparison of Schemes I–III at different time t with $h = \tau = 0.2$

t	2	4	6	8	10
I	1.70833×10^{-4}	3.41048×10^{-4}	5.10109×10^{-4}	6.77538×10^{-4}	8.42946×10^{-4}
II	3.48375×10^{-4}	6.95306×10^{-4}	1.03963×10^{-3}	1.38033×10^{-3}	1.71663×10^{-3}
III	1.75479×10^{-4}	3.50160×10^{-4}	5.23337×10^{-4}	6.94432×10^{-4}	8.63021×10^{-4}

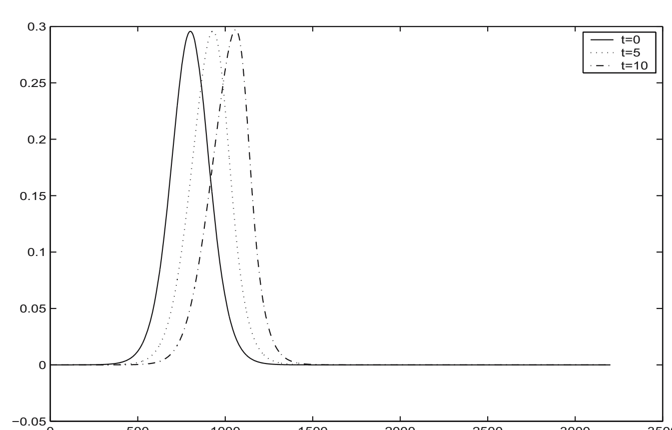

Figure 2: Exact solutions of $u(x, t)$ at $t = 0$ and numerical solutions computed by the scheme (2.1) with $h = \tau = 0.05$ at $t = 5, 10$.

conservative laws very well. We denote the presented scheme as Scheme III, the nonlinear scheme as Scheme I and the linear scheme as II in [23]. Error comparisons have been made between Schemes I–III in Tables 6 and 7. From Tables 6 and 7, it is shown that the errors obtained by our method are much better or in good agreement with the others in [23].

The curves of the solitary wave with time computed by the scheme (2.1) with the mesh steps $h = \tau = 0.05$ are given in Figure 2, the waves at $t = 5$ and 10 agree with the ones at $t = 0$ quite well, which also shows the efficiency and accuracy of the scheme in the present article.

6 Conclusion

In this article, an attempt has been made to construct a new numerical scheme to solve the initial-boundary problem of the Rosenau-Kawahara equation. The presented scheme has the following advantages: the new scheme is conservative and preserves the original conservative properties; the algebraic system obtained from the presented scheme is easy to store and solve by the software systems of nowadays. The existence of the numerical solutions for the scheme has been shown by Browder fixed point theorem. A detailed numerical analysis of the scheme is presented including a convergence analysis result. Numerical examples are reported to show the efficiency and accuracy of the scheme.

Acknowledgements: The authors are very grateful to the editor and the reviewers for their valuable suggestions to improve this work.

Funding information: This work is supported by the Natural Science Foundation of China (No. 11201343), the Natural Science Foundation of Shandong Province (No. ZR2013FL032), a Project of Shandong Province Higher Educational Science and Technology Program (No. J15LI56), and the Doctoral Research Foundation of WFU (No. 2019BS01).

Author contributions: The article was carried out in collaboration between all authors. Both authors have contributed to, reads and approved the manuscript.

Conflict of interest: The authors state that there is no conflict of interest.

References

[1] S. Zhu and J. Zhao, *The alternating segment explicit-implicit scheme for the dispersive equation*, Appl. Math. Lett. **14** (2001), no. 6, 657–662, DOI: https://doi.org/10.1016/S0893-9659(01)80022-7.
[2] A. R. Bahadir, *Exponential finite-difference method applied to Korteweg-de Vries equation for small times*, Appl. Math. Comput. **160** (2005), no. 3, 675–682, DOI: https://doi.org/10.1016/j.amc.2003.11.025.
[3] P. Rosenau, *Dynamics of dense discrete systems*, Progr. Theor. Phys. **79** (1988), no. 5, 1028–1042, DOI: https://doi.org/10.1143/PTP.79.1028.
[4] M. A. Park, *On the Rosenau equation*, Math. Appl. Comput. **9** (1990), 145–152.
[5] S. M. Choo, S. K. Chung, and K. I. Kim, *A discontinuous Galerkin method for the Rosenau equation*, Appl. Numer. Math. **58** (2008), no. 6, 783–799, DOI: https://doi.org/10.1016/j.apnum.2007.02.008.
[6] S. K. Chung and A. K. Pani, *Numerical methods for the Rosenau equation*, Appl. Anal. **77** (2001), 351–369, DOI: https://doi.org/10.1080/00036810108840914.
[7] K. Omrani, F. Abidi, T. Achouri, and N. Khiari, *A new conservative finite difference scheme for the Rosenau equation*, Appl. Math. Comput. **201** (2008), no. 1–2, 35–43, DOI: https://doi.org/10.1016/j.amc.2007.11.039.
[8] J. Hu and K. Zheng, *Two conservative difference schemes for the generalized Rosenau equation*, Boundary Value Problems, **2010** (2010), Article ID 543503, 18 pages, DOI: https://doi.org/10.1155/2010/543503.

[9] J. Hu, Y. Xu, and B. Hu, *Conservative linear difference scheme for Rosenau-KdV equation*, Adv. Math. Phys. **2013** (2013), Article ID 423718, 7 pages.

[10] S. A. Manickam, A. K. Pani, and S. K. Chung, *A second-order splitting combined with orthogonal cubic spline collocation method for the Rosenau equation*, Numer. Methods Partial Differential Equations **14** (1998), no. 6, 695–716, DOI: https://doi.org/10.1002/(SICI)1098-2426(199811)14:6.

[11] S. Kutluay, M. Karta, and N. M. Yağmurlu, *Operator time-splitting techniques combined with quintic B-spline collocation method for the generalized Rosenau-KdV equation*, Numer. Methods Partial Differential Equations **35** (2019), no. 6, 2221–2235, DOI: https://doi.org/10.1002/num.22409.

[12] S. B. G. Karakoc, *A new numerical application of the generalized Rosenau-RLW equation*, Scientia Iranica B **27** (2020), no. 2, 772–783, DOI: https://doi.org/10.24200/sci.2018.50490.172.

[13] S. B. G. Karakoc, S. K. Bhowmik, and F. Gao, *A numerical study using finite element method for generalized Rosenau-Kawahara-RLW equation*, Comput. Methods Differ. Equ. **7** (2019), no. 3, 1–15.

[14] S. B. G. Karakoc, *A detailed numerical study on generalized Rosenau-KdV equation with finite element method*, J. Sci. Art **4** (2018), no. 45, 837–852.

[15] S. B. G. Karakoc, F. Gao, and S. K. Bhowmik, *Solitons and shock waves solutions for the Rosenau-KdV-RLW equation*, J. Sci. Art **3** (2018), no. 44, 1073–1088.

[16] T. Ak and S. B. G. Karakoc, *A numerical technique based on collocation method for solving modified Kawahara equation*, J. Ocean Eng. Sci. **3** (2018), no. 1, 67–75, DOI: https://doi.org/10.1016/j.joes.2017.12.004.

[17] T. Ak, S. B. G. Karakoc, and H. Ttiki, *Numerical simulation for treatment of dispersive shallow water waves with Rosenau KdV equation*, Eur. Phys. J. Plus **131** (2016), 1–15, DOI: https://doi.org/10.1140/epjp/i2016-16356-3.

[18] S. B. G. Karakoc and T. Ak, *Numerical solution of Rosenau KdV equation using subdomain finite element method*, New Trends Math. Sci. **4** (2016), no. 1, 223–235, DOI: http://dx.doi.org/10.20852/ntmsci.2016115857.

[19] S. B. G. Karakoc and T. Ak, *Numerical simulation of dispersive shallow water waves with Rosenau-KdV equation*, Int. J. Adv. Appl. Math. Mech. **3** (2016), 32–40.

[20] S. B. G. Karakoc, A. Saha, and D. Sucu, *A novel implementation of Petrov-Galerkin method to shallow water solitary wave pattern and superperiodic traveling wave and its multistability generalized Korteweg-de Vries equation*, Chinese J. Phys. **68** (2020), 605–617, DOI: https://doi.org/10.1016/j.cjph.2020.10.010.

[21] G. M. Coclite and L. diRuvo, *On the classical solutions for a Rosenau-Korteweg-deVries-Kawahara type equation*, Asymptot. Anal. **129** (2022), no. 1, 51–73, DOI: https://doi.org/10.3233/ASY-211721.

[22] J. Zuo, *Solitons and periodic solutions for the Rosenau-Kdv and Rosenau-Kawahara equations*, Appl. Math. Comput. **215** (2009), no. 2, 835–840, DOI: https://doi.org/10.1016/j.amc.2009.06.011.

[23] J. Hu, Y. Xu, B. Hu, and X. Xie, *Two conservative difference schemes for Rosenau-Kawahara equation*, Adv. Math. Phys. **2014** (2014), Article ID 217393, 11 pages, DOI: https://doi.org/10.1155/2014/217393.

[24] A. Biswas, H. Triki, and M. Labidi, *Bright and dark solutions of Rosenau-Kawahara equation with power law nonlinearty*, Phys. Wave Phen. **19** (2011), 24–29, DOI: https://doi.org/10.3103/S1541308X11010067.

[25] Z. Fei, V. M. Pérez-García, and L. Vázquez, *Numerical simulation of nonlinear Schrödinger system: a new conservative scheme*, Appl. Math. Comput. **71** (1995), no. 2–3, 165–177, DOI: https://doi.org/10.1016/0096-3003(94)00152-T.

[26] Z. Fei and L. Vázquez, *Two energy conserving numerical schemes for the Sine-Gordon equation*, Appl. Math. Comput. **45** (1991), no. 1, 17–30, DOI: https://doi.org/10.1016/0096-3003(91)90087-4.

[27] S. M. Choo, S. K. Chung, and K. I. Kim, *Conservative nonlinear difference scheme for the Cahn-Hilliard equation II*, Comput. Math. Appl. **39** (2000), no. 1–2, 229–243, DOI: https://doi.org/10.1016/S0898-1221(99)00326-0.

[28] Q. Chang, E. Jia, and W. Sun, *Difference schemes for solving the generalized nonlinear Schrödinger equation*, J. Comput. Phys. **148** (1999), no. 2, 397–415, DOI: https://doi.org/10.1006/jcph.1998.6120.

[29] Q. Chang, B. Guo, and H. Jiang, *Finite difference method for generalized Zakharov equations*, Math. Comput. **64** (1995), 537–553.

[30] Q. Chang and H. Jiang, *A conservative scheme for the Zakharov equations*, J. Comput. Phys. **113** (1994), no. 2, 309–319, DOI: https://doi.org/10.1006/jcph.1994.1138.

[31] J. Zuo, Y. Zhang, T. Zhang, and F. Chang, *A new conservative difference scheme for the General Rosenau-RLW equation*, Boundary Value Problems **2010** (2010), Article ID 516260, 13 pages, DOI: https://doi.org/10.1155/2010/516260.

[32] X. Pan and L. Zhang, *On the convergence of a conservative numerical scheme for the usual Rosenau-RLW equation*, Appl. Math. Model. **36** (2012), no. 8, 3371–3378, DOI: https://doi.org/10.1016/j.apm.2011.08.022.

[33] L. Zhang, *A finite difference scheme for the generalized regularized long-wave equation*, Appl. Math. Comput. **168** (2005), no. 2, 962–972, DOI: https://doi.org/10.1016/j.amc.2004.09.027.

[34] L. Zhang, *Convergence of a conservative difference scheme for a class of Klein-Gordon-Schrödinger equations in one space dimension*, Appl. Math. Comput. **163** (2005), no. 1, 343–355, DOI: https://doi.org/10.1016/j.amc.2004.02.010.

[35] D. Li and W. Sun, *Linearly implicit and high-order energy-conserving schemes for nonlinear wave equations*, J. Sci. Comput. **83** (2020), 65, DOI: https://doi.org/10.1007/s10915-020-01245-6.

[36] X. Li, J. Wen, and D. Li, *Mass- and energy-conserving difference schemes for nonlinear fractional Schrödinger equations*, Appl. Math. Lett. **111** (2021), 106686, DOI: https://doi.org/10.1016/j.aml.2020.106686.

[37] T. Wang, J. Wang, and B. Guo, *Two completely explicit and unconditionally convergent Fourier pseudo-spectral methods for solving the nonlinear Schrödinger equation*, J. Comput. Phys. **404** (2020), 109116, DOI: https://doi.org/10.1016/j.jcp.2019.109116.
[38] Y. Cheng, T. Wang, and B. Guo, *An efficient and unconditionally convergent Galerkin finite-element method for the nonlinear Schrödinger equation in high dimensions*, Adv. Appl. Math. Mech. **13** (2021), no. 4, 735–760, DOI: https://doi.org/10.4208/aamm.OA-2020-0033.
[39] S. Li, T. Wang, J. Wang, and B. Guo, *An efficient and accurate Fourier pseudo-spectral method for the nonlinear Schrödinger equation with wave operator*, Int. J. Comput. Math. **98** (2020), no. 2, 340–356, DOI: https://doi.org/10.1080/00207160.2020.1745785.
[40] X. Li, Y. Gong, and L. Zhang, *Two novel classes of linear high-order structure-preserving schemes for the generalized nonlinear Schrödinger equation*, Appl. Math. Lett. **104** (2020), 106273, DOI: https://doi.org/10.1016/j.aml.2020.106273.
[41] F. E. Browder, *Existence and uniqueness theorems for solutions of nonlinear boundary value problems*, Proc. Symp. Appl. Math. **17** (1965), 24–49, DOI: http://dx.doi.org/10.1090/psapm/017/0197933.
[42] Y. Zhou, *Application of Discrete Functional Analysis to the Finite Difference Method*, Inter. Acad. Publishers, Beijing, 1990.