Research Article

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A class of strongly convergent subgradient extragradient methods for solving quasimonotone variational inequalities

https://doi.org/10.1515/dema-2022-0202
received June 29, 2021; accepted January 20, 2023

Abstract: The primary goal of this research is to investigate the approximate numerical solution of variational inequalities using quasimonotone operators in infinite-dimensional real Hilbert spaces. In this study, the sequence obtained by the proposed iterative technique for solving quasimonotone variational inequalities converges strongly toward a solution due to the viscosity-type iterative scheme. Furthermore, a new technique is proposed that uses an inertial mechanism to obtain strong convergence iteratively without the requirement for a hybrid version. The fundamental benefit of the suggested iterative strategy is that it substitutes a monotone and non-monotone step size rule based on mapping (operator) information for its Lipschitz constant or another line search method. This article also provides a numerical example to demonstrate how each method works.

Keywords: variational inequality problem, subgradient extragradient method, strong convergence results, quasimonotone operator, Lipschitz continuity

MSC 2020: 65Y05, 65K15, 68W10, 47H05, 47H10

1 Introduction

The primary goal of this article is to investigate the iterative methods used to approximate the solution of the *variational inequality problem* (VIP) using quasimonotone operators in any real Hilbert space. Let $\mathcal{E}$ be a real Hilbert space and $\mathcal{M}$ be a nonempty convex, closed subset of $\mathcal{E}$. Let $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ be an operator. A problem (VIP) for $\mathcal{K}$ on $\mathcal{M}$ is described as follows [28]:

$$\text{Find } r^* \in \mathcal{M} \quad \text{such that } \langle \mathcal{K}(r^*), y - r^* \rangle \geq 0, \quad \forall y \in \mathcal{M}. \tag{VIP}$$

To validate the strong convergence, it is assumed that the following requirements are met:

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($\mathcal{K}$1) The solution set for problem (VIP) is denoted by $\mathrm{VI}(\mathcal{M}, \mathcal{K})$ and it is nonempty.
($\mathcal{K}$2) An operator $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ is said to be quasimonotone such that

$$\langle \mathcal{K}(y_1), y_2 - y_1 \rangle > 0 \Rightarrow \langle \mathcal{K}(y_2), y_2 - y_1 \rangle \geq 0, \ \forall y_1, y_2 \in \mathcal{M}.$$

($\mathcal{K}$3) An operator $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ is said to be *Lipschitz continuous* with constant $L > 0$ such that

$$\|\mathcal{K}(y_1) - \mathcal{K}(y_2)\| \leq L\|y_1 - y_2\|, \ \forall y_1, y_2 \in \mathcal{M}.$$

($\mathcal{K}$4) An operator $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ is said to be *sequentially weakly continuous*, i.e., $\{\mathcal{K}(u_n)\}$ weakly converges to $\mathcal{K}(u)$ because each sequence $\{u_n\}$ weakly converges to u.

It is well recognized that the problem (VIP) is a crucial problem in non-linear analysis. It is a key mathematical model that integrates a number of important concepts in applied mathematics, such as a non-linear system of equations, optimization conditions for problems with the optimization process, complementarity problems, network equilibrium problems, and finance (see for more details [15,18–21, 26,30]). As a result, this concept has several applications in mathematical programming, engineering, transportation analysis, network economics, game theory, and computer science.

The regularized approach and the projection method are two popular and generic methods for solving variational inequalities. It should also be mentioned that the first technique is most usually used to deal with variational inequalities accompanied by the monotone operator class. In this method, the regularized subproblem is strongly monotone, and its unique solution is obtained more conveniently than the initial problem. In this article, we will look into projection methods that are well known for their ease of numerical computation.

Furthermore, projection techniques can be used to find a numerical solution to variational inequalities. Many researchers have created original projection methods to solve various types of variational inequalities (for more details, see [5–8,12,14,17,22,23,25,27,29,31,38] and others in [3,4,9–11,16,32–36]). All techniques for resolving the (VIP) problem are focused on computing a projection on the appropriate set $\mathcal{M}$. Korpelevich [22] and Antipin [1] introduced the equivalent extragradient method. Their method is as follows:

$$\begin{cases} u_1 \in \mathcal{M}, \\ y_n = P_{\mathcal{M}}[u_n - \rho\mathcal{K}(u_n)], \\ u_{n+1} = P_{\mathcal{M}}[u_n - \rho\mathcal{K}(y_n)], \end{cases} \tag{1}$$

where $0 < \rho < \frac{1}{L}$. In keeping with the previous technique, we used two projections on the underlying set $\mathcal{M}$ for each iteration. Indeed, if the feasible set $\mathcal{M}$ has a complicated structure, the method's computing efficiency may decrease. This section will go through various methods to obtain through this limitation. In the study by Censor et al. [12], the subgradient extragradient technique was first used. The following strategy is used in this technique:

$$\begin{cases} u_1 \in \mathcal{M}, \\ y_n = P_{\mathcal{M}}[u_n - \rho\mathcal{K}(u_n)], \\ u_{n+1} = P_{\mathcal{E}_n}[u_n - \rho\mathcal{K}(y_n)], \end{cases} \tag{2}$$

where $0 < \rho < \frac{1}{L}$ through

$$\mathcal{E}_n = \{z \in \mathcal{E} : \langle u_n - \rho\mathcal{K}(u_n) - y_n, z - y_n \rangle \leq 0\}.$$

Tseng's extragradient technique [31], which uses only one projection every iteration, is another notable method that does not require two projections. The following strategy is used in this technique:

$$\begin{cases} u_1 \in \mathcal{M}, \\ y_n = P_{\mathcal{M}}[u_n - \rho\mathcal{K}(u_n)], \\ u_{n+1} = y_n + \rho[\mathcal{K}(u_n) - \mathcal{K}(y_n)], \end{cases} \tag{3}$$

where $0 < \rho < \frac{1}{L}$. It is important to note that the previous techniques have two main flaws: a fixed step size rule that is reliant on the Lipschitz modulus of the cost operator and a weakly converging iterative procedure. The Lipschitz modulus is frequently uncertain or difficult to calculate. A fixed step size limitation that impacts the method's efficacy and speed of convergence may be difficult to explain theoretically. Additionally, in the situation of an infinite-dimensional Hilbert space, the investigation of a strongly convergent iterative sequence is crucial.

The gradient projection technique was the first well-established projection method for determining variational inequalities, and it was followed by numerous additional projection methods, including the well-known extragradient approach [22], the subgradient extragradient methods [12,13], and others [14,17,25, 31,39]. The aforementioned methods are used to solve variational inequalities using monotone, strongly monotone, or inverse monotone. Furthermore, while generating approximation solutions and determining their convergence, fixed or variable step sizes frequently depend on the Lipschitz constants of the operators. This can limit implementations since, in some cases, some parameters are unknown or impossible to estimate.

The purpose of this research is to look at variational inequalities using quasimonotone operators in infinite-dimensional Hilbert spaces. Furthermore, this study shows that the iterative sequences generated by all four subgradient extragradient algorithms strongly converge to a solution. Subgradient extragradient methods use both monotone and non-monotone variable step size rules. The study of inertial algorithms is also presented, which typically enhances the efficiency of the iterative sequence. The article's main contribution is that it investigates explicit monotone and non-monotone step size rules using inertial schemes and achieves strong convergence.

This article is written as follows. Section 2 provides preliminary results. Section 3 describes four novel methods and their convergence analysis. Finally, Section 4 provides some numerical findings to explain the practical efficiency of the proposed methods.

2 Preliminaries

This section contains various important identities as well as significant lemmas. Let us define the following set:

$$\mathrm{VI}(\mathcal{M}, \mathcal{K})_+ = \{r^* \in \mathcal{M} : \langle \mathcal{M}(r^*), y - r^* \rangle > 0, \quad \forall y \in \mathcal{M}\}.$$

For any $u, y \in \mathcal{E}$, we have

$$\|u + y\|^2 = \|u\|^2 + 2\langle u, y \rangle + \|y\|^2.$$

A *metric projection* $P_{\mathcal{M}}(y_1)$ of $y_1 \in \mathcal{E}$ is described by:

$$P_{\mathcal{M}}(y_1) = \arg\min\{\|y_1 - y_2\| : y_2 \in \mathcal{M}\}.$$

Lemma 2.1. [2] *Suppose that* $P_{\mathcal{M}} : \mathcal{E} \to \mathcal{M}$ *is a metric projection. Then, the following conditions are satisfied*:
(1) $e_3 = P_{\mathcal{M}}(e_1)$ *if and only if*

$$\langle e_1 - e_3, e_2 - e_3 \rangle \le 0, \ \forall e_2 \in \mathcal{M},$$

(2)

$$\|e_1 - P_{\mathcal{M}}(e_2)\|^2 + \|P_{\mathcal{M}}(e_2) - e_2\|^2 \le \|e_1 - e_2\|^2, \quad e_1 \in \mathcal{M}, e_2 \in \Sigma,$$

(3)

$$\|e_1 - P_{\mathcal{M}}(e_1)\| \le \|e_1 - e_2\|, \quad e_2 \in \mathcal{M}, e_1 \in \Sigma.$$

Lemma 2.2. [37] *Let* $\{p_n\} \subset [0, +\infty)$ *be a sequence such that*

$$p_{n+1} \leq (1 - q_n)p_n + q_n r_n, \quad \forall n \in \mathbb{N}.$$

Moreover, two sequences $\{q_n\} \subset (0, 1)$ *and* $\{r_n\} \subset \mathbb{R}$ *such that*

$$\lim_{n\to+\infty} q_n = 0, \quad \sum_{n=1}^{+\infty} q_n = +\infty \quad \textit{and} \quad \limsup_{n\to+\infty} r_n \leq 0.$$

Then, $\lim_{n\to+\infty} p_n = 0$.

Lemma 2.3. [24] *Let* $\{p_n\}$ *be a real sequence, and there exists a subsequence* $\{n_i\}$ of $\{n\}$ *such that*

$$p_{n_i} < p_{n_{i+1}} \quad \textit{for all } i \in \mathbb{N}.$$

Then, there exists a non-decreasing sequence $m_k \subset \mathbb{N}$ *such that* $m_k \to +\infty$ *as* $k \to +\infty$, *and satisfying the following inequality for* $k \in \mathbb{N}$:

$$p_{m_k} \leq p_{m_{k+1}} \quad \textit{and} \quad p_k \leq p_{m_{k+1}}.$$

Indeed, $m_k = \max\{j \leq k : p_j \leq p_{j+1}\}$.

3 Main results

In this section, we propose four new methods to solve quasimonotone variational inequalities in a real Hilbert space and prove strong convergence results for the proposed method. The first and second methods involve a monotonic self-adaptive step rule to make the algorithm independent of the Lipschitz constant. Let $g : \mathcal{E} \to \mathcal{E}$ be a strict contraction function through constant $\xi \in [0, 1)$. The main algorithm is as follows:

Algorithm 1 (Explicit monotonic viscosity-type subgradient extragradient method)

Step 0: Let $u_1 \in \mathcal{M}$, $\mu \in (0, 1)$ and $\rho_1 > 0$. Moreover, sequence $\{\gamma_n\} \subset (0, 1)$ such that $\lim_{n\to+\infty} \gamma_n = 0$ and $\sum_{n=1}^{+\infty} \gamma_n = +\infty$.

Step 1: Compute

$y_n = P_{\mathcal{M}}(u_n - \rho_n \mathcal{K}(u_n))$.

If $u_n = y_n$, then STOP. Otherwise, go to **Step 2**.

Step 2: Construct a set $\mathcal{E}_n = \{z \in \mathcal{E} : \langle u_n - \rho_n \mathcal{K}(u_n) - y_n, z - y_n\rangle \leq 0\}$ and evaluate

$t_n = P_{\mathcal{E}_n}(u_n - \rho_n \mathcal{K}(y_n))$.

Step 3: Calculate

$u_{n+1} = \gamma_n g(u_n) + (1 - \gamma_n)t_n$.

Step 4: Calculate

$$\rho_{n+1} = \begin{cases} \min\left\{\rho_n, \frac{\mu \| u_n - y_n\|}{\|\mathcal{K}(u_n) - \mathcal{K}(y_n)\|}\right\} & \text{if } \mathcal{K}(u_n) \neq \mathcal{K}(y_n), \\ \rho_n & \text{else.} \end{cases} \tag{4}$$

Set $n := n + 1$s and go back to **Step 1**.

Lemma 3.1. *A step size sequence* $\{\rho_n\}$ *generated in* (4) *is decreasing monotonically with a lower bound* $\min\{\frac{\mu}{L}, \rho_0\}$ *and converges to a fixed* $\rho > 0$.

Proof. It is obvious that $\{\rho_n\}$ is a monotone and non-increasing sequence. It is given that operator $\mathcal{K}$ is Lipschitz continuous with a constant $L > 0$ such that

$$\|\mathcal{K}(u_n) - \mathcal{K}(y_n)\| \le L\|u_n - y_n\|.$$

Let $\mathcal{K}(u_n) \neq \mathcal{K}(y_n)$ such that

$$\frac{\mu\|u_n - y_n\|}{\|\mathcal{K}(u_n) - \mathcal{K}(y_n)\|} \ge \frac{\mu\|u_n - y_n\|}{L\|u_n - y_n\|} \ge \frac{\mu}{L}. \tag{7}$$

As a result of the aforementioned expression, $\{\rho_n\}$ has a lower bound of $\min\{\frac{\mu}{L}, \rho_0\}$. Moreover, there exists $\rho > 0$ such that $\lim_{n\to\infty}\rho_n = \rho$. □

Algorithm 2 (Inertial monotonic explicit subgradient extragradient method)

Step 0: Let $u_0, u_1 \in \mathcal{M}$, $\mu \in (0, 1)$ and $\rho_1 > 0$. Moreover, $\{\gamma_n\} \subset (0, 1)$ such that
$\lim_{n\to+\infty} \gamma_n = 0$ and $\sum_{n=1}^{+\infty}\gamma_n = +\infty$.
Step 1: Evaluate $s_n = u_n + \chi_n(u_n - u_{n-1}) - \gamma_n[u_n + \chi_n(u_n - u_{n-1})]$, where χ_n such that

$$0 \le \chi_n \le \hat{\chi}_n \quad \text{and} \quad \hat{\chi}_n = \begin{cases} \min\left\{\frac{\chi}{2}, \frac{\varepsilon_n}{\|u_n - u_{n-1}\|}\right\} & \text{if} \quad u_n \neq u_{n-1}, \\ \frac{\chi}{2} & \text{else}, \end{cases} \tag{5}$$

with positive sequence $\varepsilon_n = \circ(\gamma_n)$ such that $\lim_{n\to+\infty}\frac{\varepsilon_n}{\gamma_n} = 0$.
Step 2: Compute
$y_n = P_{\mathcal{M}}(s_n - \rho_n\mathcal{K}(s_n))$.
If $u_n = y_n$, then STOP. Otherwise, go to **Step 3**.
Step 3: Construct a set $\mathcal{E}_n = \{z \in \mathcal{E} : \langle s_n - \rho_n\mathcal{K}(s_n) - y_n, z - y_n\rangle \le 0\}$ and evaluate
$u_{n+1} = P_{\mathcal{E}_n}(s_n - \rho_n\mathcal{K}(y_n))$.
Step 4: Calculate

$$\rho_{n+1} = \begin{cases} \min\left\{\rho_n, \frac{\mu\,\|\,s_n - y_n\|}{\|\mathcal{K}(s_n) - \mathcal{K}(y_n)\|}\right\} & \text{if } \mathcal{K}(s_n) \neq \mathcal{K}(y_n), \\ \rho_n & \text{else}. \end{cases} \tag{6}$$

Set $n := n + 1$ and go back to **Step 1**.

Algorithm 3 (Non-monotonic explicit viscosity-type subgradient extragradient method)

Step 0: Let $u_1 \in \mathcal{M}$, $\rho_1 > 0$, $\mu \in (0, 1)$ and choose a non-negative real sequence $\{\varphi_n\}$ such that $\sum_n^{\infty}\varphi_n < +\infty$.
Moreover, $\{\gamma_n\} \subset (0, 1)$ such that
$\lim_{n\to+\infty} \gamma_n = 0$ and $\sum_{n=1}^{+\infty}\gamma_n = +\infty$.
Step 1: Compute
$y_n = P_{\mathcal{M}}(u_n - \rho_n\mathcal{K}(u_n))$.
If $u_n = y_n$, then STOP. Otherwise, go to **Step 2**.
Step 2: Construct a set $\mathcal{E}_n = \{z \in \mathcal{E} : \langle u_n - \rho_n\mathcal{K}(u_n) - y_n, z - y_n\rangle \le 0\}$ and evaluate
$t_n = P_{\mathcal{E}_n}(u_n - \rho_n\mathcal{K}(y_n))$.
Step 3: Calculate $u_{n+1} = \gamma_n g(u_n) + (1 - \gamma_n)t_n$.
Step 4: Calculate

$$\rho_{n+1} = \begin{cases} \min\left\{\rho_n + \varphi_n, \frac{\mu\,\|\,u_n - y_n\|}{\|\mathcal{K}(u_n) - \mathcal{K}(y_n)\|}\right\} & \text{if } \mathcal{K}(u_n) \neq \mathcal{K}(y_n), \\ \rho_n + \varphi_n & \text{else}. \end{cases} \tag{8}$$

Set $n := n + 1$ and go back to **Step 1**.

Algorithm 4 (Inertial non-monotonic explicit subgradient extragradient method)

Step 0: Let $u_0, u_1 \in \mathcal{M}$, $\mu \in (0,1)$, $\rho_1 > 0$ and choose a non-negative real sequence $\{\varphi_n\}$ such that $\sum_n^{\infty}\varphi_n < +\infty$. Moreover, $\{\gamma_n\} \subset (0,1)$ such that
$\lim_{n\to+\infty} \gamma_n = 0$ and $\sum_{n=1}^{+\infty}\gamma_n = +\infty$.
Step 1: Evaluate $s_n = u_n + \chi_n(u_n - u_{n-1}) - \gamma_n[u_n + \chi_n(u_n - u_{n-1})]$, where χ_n such that

$$0 \le \chi_n \le \hat{\chi}_n \quad \text{and} \quad \hat{\chi}_n = \begin{cases} \min\left\{\frac{\chi}{2}, \frac{\varepsilon_n}{\|u_n - u_{n-1}\|}\right\} & \text{if } u_n \neq u_{n-1}, \\ \frac{\chi}{2} & \text{else}, \end{cases} \tag{9}$$

with positive sequence $\varepsilon_n = \circ(\gamma_n)$ such that $\lim_{n\to+\infty}\frac{\varepsilon_n}{\gamma_n} = 0$.
Step 2: Compute
$y_n = P_{\mathcal{M}}(s_n - \rho_n\mathcal{K}(s_n))$.
If $u_n = y_n$, then STOP. Otherwise, go to **Step 3**.
Step 3: Construct a set $\mathcal{E}_n = \{z \in \mathcal{E} : \langle s_n - \rho_n\mathcal{K}(s_n) - y_n, z - y_n\rangle \le 0\}$ and evaluate
$u_{n+1} = P_{\mathcal{E}_n}(s_n - \rho_n\mathcal{K}(y_n))$.
Step 4: Calculate

$$\rho_{n+1} = \begin{cases} \min\left\{\rho_n + \varphi_n, \frac{\mu\|s_n - y_n\|}{\|\mathcal{K}(s_n) - \mathcal{K}(y_n)\|}\right\} & \text{if } \mathcal{K}(s_n) \neq \mathcal{K}(y_n), \\ \rho_n + \varphi_n & \text{else}. \end{cases} \tag{10}$$

Set $n := n+1$ and go back to **Step 1**.

Lemma 3.2. *A sequence $\{\rho_n\}$ generated by expression* (8) *is convergent to ρ and satisfying the following inequality*:

$$\min\left\{\frac{\mu}{L}, \rho_1\right\} \le \rho \le \rho_1 + P \quad \text{where } P = \sum_{n=1}^{+\infty}\varphi_n.$$

Proof. Due to the Lipschitz continuity of a mapping $\mathcal{K}$, there exists a fixed number $L > 0$. Let $\mathcal{K}(u_n) \neq \mathcal{K}(y_n)$ such that

$$\frac{\mu\|u_n - y_n\|}{\|\mathcal{K}(u_n) - \mathcal{K}(y_n)\|} \ge \frac{\mu\|u_n - y_n\|}{L\|u_n - y_n\|} \ge \frac{\mu}{L}. \tag{11}$$

By using mathematical induction on the definition of ρ_{n+1}, we have

$$\min\left\{\frac{\mu}{L}, \rho_1\right\} \le \rho_n \le \rho_1 + P.$$

Let

$$[\rho_{n+1} - \rho_n]^+ = \max\{0, \rho_{n+1} - \rho_n\}$$

and

$$[\rho_{n+1} - \rho_n]^- = \max\{0, -(\rho_{n+1} - \rho_n)\}.$$

From the definition of $\{\rho_n\}$, we have

$$\sum_{n=1}^{+\infty}(\rho_{n+1} - \rho_n)^+ = \sum_{n=1}^{+\infty}\max\{0, \rho_{n+1} - \rho_n\} \le P < +\infty. \tag{12}$$

That is, the series $\sum_{n=1}^{+\infty}(\rho_{n+1}-\rho_n)^+$ is convergent. Next, we need to prove the convergence of $\sum_{n=1}^{+\infty}(\rho_{n+1}-\rho_n)^-$. Let $\sum_{n=1}^{+\infty}(\rho_{n+1}-\rho_n)^- = +\infty$, due to the reason that

$$\rho_{n+1}-\rho_n = (\rho_{n+1}-\rho_n)^+ - (\rho_{n+1}-\rho_n)^-.$$

Thus, we have

$$\rho_{k+1}-\rho_1 = \sum_{n=0}^{k}(\rho_{n+1}-\rho_n) = \sum_{n=0}^{k}(\rho_{n+1}-\rho_n)^+ - \sum_{n=0}^{k}(\rho_{n+1}-\rho_n)^-. \tag{13}$$

By allowing $k \to +\infty$ in expression (13), we have $\rho_k \to -\infty$ as $k \to \infty$. This is a contradiction. Due to the convergence of $\sum_{n=0}^{k}(\rho_{n+1}-\rho_n)^+$ and $\sum_{n=0}^{k}(\rho_{n+1}-\rho_n)^-$ as $k \to +\infty$ in expression (13), we obtain $\lim_{n\to\infty}\rho_n = \rho$. This completes the proof. ☐

Lemma 3.3. *Let the mapping $\mathcal{K}:\mathcal{E}\to\mathcal{E}$ satisfy conditions ($\mathcal{K}$1)–($\mathcal{K}$4). For any $r^*\in \mathrm{VI}(\mathcal{M},\mathcal{K})_+$, we have*

$$\|t_n-r^*\|^2 \le \|u_n-r^*\|^2 - \left(1-\frac{\mu\rho_n}{\rho_{n+1}}\right)\|u_n-y_n\|^2 - \left(1-\frac{\mu\rho_n}{\rho_{n+1}}\right)\|t_n-y_n\|^2.$$

Proof. Let us consider that

$$\begin{aligned}\|t_n-r^*\|^2 &= \|P_{\mathcal{E}_n}[u_n-\rho_n\mathcal{K}(y_n)]-r^*\|^2\\ &= \|P_{\mathcal{E}_n}[u_n-\rho_n\mathcal{K}(y_n)]+[u_n-\rho_n\mathcal{K}(y_n)]-[u_n-\rho_n\mathcal{K}(y_n)]-r^*\|^2\\ &= \|[u_n-\rho_n\mathcal{K}(y_n)]-r^*\|^2+\|P_{\mathcal{E}_n}[u_n-\rho_n\mathcal{K}(y_n)]-[u_n-\rho_n\mathcal{K}(y_n)]\|^2\\ &\quad+2\langle P_{\mathcal{E}_n}[u_n-\rho_n\mathcal{K}(y_n)]-[u_n-\rho_n\mathcal{K}(y_n)],[u_n-\rho_n\mathcal{K}(y_n)]-r^*\rangle.\end{aligned} \tag{14}$$

By using $r^*\in \mathrm{VI}(\mathcal{M},\mathcal{K})_+\subset\mathcal{E}_n$, we obtain

$$\begin{aligned}&\|P_{\mathcal{E}_n}[u_n-\rho_n\mathcal{K}(y_n)]-[u_n-\rho_n\mathcal{K}(y_n)]\|^2+\langle P_{\mathcal{E}_n}[u_n-\rho_n\mathcal{K}(y_n)]-[u_n-\rho_n\mathcal{K}(y_n)],[u_n-\rho_n\mathcal{K}(y_n)]-r^*\rangle\\ &\quad=\langle [u_n-\rho_n\mathcal{K}(y_n)]-P_{\mathcal{E}_n}[u_n-\rho_n\mathcal{K}(y_n)],r^*-P_{\mathcal{E}_n}[u_n-\rho_n\mathcal{K}(y_n)]\rangle\le 0,\end{aligned} \tag{15}$$

which implies that

$$\langle P_{\mathcal{E}_n}[u_n-\rho_n\mathcal{K}(y_n)]-[u_n-\rho_n\mathcal{K}(y_n)],[u_n-\rho_n\mathcal{K}(y_n)]-r^*\rangle\le-\|P_{\mathcal{E}_n}[u_n-\rho_n\mathcal{K}(y_n)]-[u_n-\rho_n\mathcal{K}(y_n)]\|^2. \tag{16}$$

Combining (14) and (16), we obtain

$$\begin{aligned}\|t_n-r^*\|^2&\le\|u_n-\rho_n\mathcal{K}(y_n)-r^*\|^2-\|P_{\mathcal{E}_n}[u_n-\rho_n\mathcal{K}(y_n)]-[u_n-\rho_n\mathcal{K}(y_n)]\|^2\\ &\le\|u_n-r^*\|^2-\|u_n-t_n\|^2+2\rho_n\langle\mathcal{K}(y_n),r^*-t_n\rangle.\end{aligned} \tag{17}$$

Since $r^*\in \mathrm{VI}(\mathcal{M},\mathcal{K})_+$, we have

$$\langle\mathcal{K}(r^*),y-r^*\rangle>0,\quad \text{for all } y\in\mathcal{M}.$$

Thus, the aforementioned expression implies that

$$\langle\mathcal{K}(y),y-r^*\rangle\ge 0,\quad \text{for all } y\in\mathcal{M}.$$

By using $y=y_n\in\mathcal{M}$, we obtain

$$\langle\mathcal{K}(y_n),y_n-r^*\rangle\ge 0.$$

Thus, we have

$$\langle\mathcal{K}(y_n),r^*-t_n\rangle=\langle\mathcal{K}(y_n),r^*-y_n\rangle+\langle\mathcal{K}(y_n),y_n-t_n\rangle\le\langle\mathcal{K}(y_n),y_n-t_n\rangle. \tag{18}$$

Combining expressions (17) and (18), we obtain

$$\begin{aligned}\|t_n - r^*\|^2 &\le \|u_n - r^*\|^2 - \|u_n - t_n\|^2 + 2\rho_n\langle \mathcal{K}(y_n), y_n - t_n\rangle \\ &\le \|u_n - r^*\|^2 - \|u_n - y_n + y_n - t_n\|^2 + 2\rho_n\langle \mathcal{K}(y_n), y_n - t_n\rangle \\ &\le \|u_n - r^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle u_n - \rho_n\mathcal{K}(y_n) - y_n, t_n - y_n\rangle.\end{aligned} \tag{19}$$

Note that $t_n = P_{\mathcal{E}_n}[u_n - \rho_n\mathcal{K}(y_n)]$ implies that

$$\begin{aligned}2\langle u_n - \rho_n\mathcal{K}(y_n) - y_n, t_n - y_n\rangle &= 2\langle u_n - \rho_n\mathcal{K}(u_n) - y_n, t_n - y_n\rangle + 2\rho_n\langle \mathcal{K}(u_n) - \mathcal{K}(y_n), t_n - y_n\rangle \\ &\le \frac{\rho_n}{\rho_{n+1}}2\rho_{n+1}\|\mathcal{K}(u_n) - \mathcal{K}(y_n)\|\|t_n - y_n\| \\ &\le \frac{\mu\rho_n}{\rho_{n+1}}\|u_n - y_n\|^2 + \frac{\mu\rho_n}{\rho_{n+1}}\|t_n - y_n\|^2.\end{aligned} \tag{20}$$

Combining expressions (19) and (20), we obtain

$$\begin{aligned}\|t_n - r^*\|^2 &\le \|u_n - r^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + \frac{\rho_n}{\rho_{n+1}}[\mu\|u_n - y_n\|^2 + \mu\|t_n - y_n\|^2] \\ &\le \|u_n - r^*\|^2 - \left(1 - \frac{\mu\rho_n}{\rho_{n+1}}\right)\|u_n - y_n\|^2 - \left(1 - \frac{\mu\rho_n}{\rho_{n+1}}\right)\|t_n - y_n\|^2.\end{aligned} \tag{21}$$

□

Lemma 3.4. *Let the mapping* $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ *satisfy conditions* $(\mathcal{K}1)$–$(\mathcal{K}4)$. *If there exists a subsequence* $\{u_{n_k}\}$ *weakly convergent to* $\hat{u}$ *and* $\lim_{k\to\infty}\|u_{n_k} - y_{n_k}\| = 0$, *then* $\hat{u} \in \mathrm{VI}(\mathcal{M}, \mathcal{K})$.

Proof. Since $\{u_{n_k}\}$ is weakly convergent to $\hat{u}$ and $\lim_{k\to\infty}\|u_{n_k} - y_{n_k}\| = 0$, the sequence $\{y_{n_k}\}$ is weakly convergent to $\hat{u}$. Next, we need to show that $\hat{u} \in \mathrm{VI}(\mathcal{M}, \mathcal{K})$. Thus, we have

$$y_{n_k} = P_{\mathcal{M}}\left[u_{n_k} - \rho_{n_k}\mathcal{K}\left(u_{n_k}\right)\right],$$

which is equivalent to

$$\langle u_{n_k} - \rho_{n_k}\mathcal{K}\left(u_{n_k}\right) - y_{n_k}, y - y_{n_k}\rangle \le 0, \quad \forall\, y \in \mathcal{M}. \tag{22}$$

The aforementioned inequality implies that

$$\langle u_{n_k} - y_{n_k}, y - y_{n_k}\rangle \le \rho_{n_k}\langle \mathcal{K}\left(u_{n_k}\right), y - y_{n_k}\rangle, \quad \forall\, y \in \mathcal{M}. \tag{23}$$

Thus, we obtain

$$\frac{1}{\rho_{n_k}}\langle u_{n_k} - y_{n_k}, y - y_{n_k}\rangle + \langle \mathcal{K}\left(u_{n_k}\right), y_{n_k} - u_{n_k}\rangle \le \langle \mathcal{K}\left(u_{n_k}\right), y - u_{n_k}\rangle, \quad \forall y \in \mathcal{M}. \tag{24}$$

By the use of $\min\{\frac{\mu}{L}, \rho_1\} \le \rho \le \rho_1$ and $\{u_{n_k}\}$ is a bounded sequence. It is given that $\lim_{k\to\infty}\|u_{n_k} - y_{n_k}\| = 0$ and $k \to \infty$ in (24), we obtain

$$\liminf_{k\to\infty}\langle \mathcal{K}\left(u_{n_k}\right), y - u_{n_k}\rangle \ge 0, \quad \forall y \in \mathcal{M}. \tag{25}$$

Moreover, we have

$$\langle \mathcal{K}(y_{n_k}), y - y_{n_k}\rangle = \langle \mathcal{K}(y_{n_k}) - \mathcal{K}\left(u_{n_k}\right), y - u_{n_k}\rangle + \langle \mathcal{K}\left(u_{n_k}\right), y - u_{n_k}\rangle + \langle \mathcal{K}(y_{n_k}), u_{n_k} - y_{n_k}\rangle. \tag{26}$$

Since $\lim_{k\to\infty}\|u_{n_k} - y_{n_k}\| = 0$ and $\mathcal{K}$ is L-Lipschitz continuity on $\mathcal{E}$, we obtain

$$\lim_{k\to\infty}\|\mathcal{K}\left(u_{n_k}\right) - \mathcal{K}(y_{n_k})\| = 0. \tag{27}$$

From expressions (26) and (27), we obtain

$$\liminf_{k\to\infty}\langle \mathcal{K}(y_{n_k}), y - y_{n_k}\rangle \ge 0, \quad \forall y \in \mathcal{M}. \tag{28}$$

For the continuity of demonstration, let us take a positive sequence $\{\varepsilon_k\}$ that is decreasing to zero. For every $\{\varepsilon_k\}$, we denote by m_k the positive smallest integer in order that

$$\langle \mathcal{K}\big(u_{n_i}\big), y - u_{n_i}\rangle + \varepsilon_k > 0, \quad \forall i \geq m_k. \tag{29}$$

Since $\{\varepsilon_k\}$ is decreasing, it is easy to observe that the sequence $\{m_k\}$ is increasing.

Case I: If there exists subsequence $\{u_{n_{m_{k_j}}}\}$ of $\{u_{n_{m_k}}\}$ such that $\mathcal{K}(u_{n_{m_{k_j}}}) = 0$ $(\forall j)$. Let $j \to \infty$, we obtain

$$\langle \mathcal{K}(\hat{u}), y - \hat{u}\rangle = \lim_{j\to\infty} \langle \mathcal{K}\big(u_{n_{m_{k_j}}}\big), y - \hat{u}\rangle = 0. \tag{30}$$

Thus, $\hat{u} \in \mathcal{M}$ and imply that $\hat{u} \in \mathrm{VI}(\mathcal{M}, \mathcal{K})$.

Case II: If there exists a fixed number $N_0 \in \mathbb{N}$ such that for all $n_{m_k} \geq N_0$, $\mathcal{K}(u_{n_{m_k}}) \neq 0$. Consider that

$$\Upsilon_{n_{m_k}} = \frac{\mathcal{K}(u_{n_{m_k}})}{\|\mathcal{K}(u_{n_{m_k}})\|^2}, \ \forall n_{m_k} \geq N_0. \tag{31}$$

Due to the aforementioned definition, we obtain

$$\langle \mathcal{K}\big(u_{n_{m_k}}\big), \Upsilon_{n_{m_k}}\rangle = 1, \ \forall n_{m_k} \geq N_0. \tag{32}$$

Moreover, using expressions (29) and (32), for all $n_{m_k} \geq N_0$, we have

$$\langle \mathcal{K}\big(u_{n_{m_k}}\big), \quad y + \varepsilon_k \Upsilon_{n_{m_k}} - u_{n_{m_k}}\rangle > 0. \tag{33}$$

Since $\mathcal{K}$ is quasimonotone, then

$$\langle \mathcal{K}\big(y + \varepsilon_k \Upsilon_{n_{m_k}}\big), \quad y + \varepsilon_k \Upsilon_{n_{m_k}} - u_{n_{m_k}}\rangle > 0. \tag{34}$$

For all $n_{m_k} \geq N_0$, we have

$$\langle \mathcal{K}(y), y - u_{n_{m_k}}\rangle \geq \langle \mathcal{K}(y) - \mathcal{K}\big(y + \varepsilon_k \Upsilon_{n_{m_k}}\big), \quad y + \varepsilon_k \Upsilon_{n_{m_k}} - u_{n_{m_k}}\rangle - \varepsilon_k \langle \mathcal{K}(y), \Upsilon_{n_{m_k}}\rangle. \tag{35}$$

Since $\{u_{n_k}\}$ converges weakly to $\hat{u} \in \mathcal{M}$ through $\mathcal{K}$ is weakly continuous on the set $\mathcal{M}$, we obtain $\{\mathcal{K}(u_{n_k})\}$ converges weakly to $\mathcal{K}(\hat{u})$. Let $\mathcal{K}(\hat{u}) \neq 0$, we have

$$\|\mathcal{K}(\hat{u})\| \leq \liminf_{k\to\infty} \|\mathcal{K}\big(u_{n_k}\big)\|. \tag{36}$$

Since $\{u_{n_{m_k}}\} \subset \{u_{n_k}\}$ and $\lim_{k\to\infty}\varepsilon_k = 0$, we have

$$0 \leq \lim_{k\to\infty} \|\varepsilon_k \Upsilon_{n_{m_k}}\| = \lim_{k\to\infty} \frac{\varepsilon_k}{\|\mathcal{K}(u_{n_{m_k}})\|} \leq \frac{0}{\|\mathcal{K}(\hat{u})\|} = 0. \tag{37}$$

Next, letting $k \to \infty$ in expression (35), we obtain

$$\langle \mathcal{K}(y), y - \hat{u}\rangle \geq 0, \ \forall y \in \mathcal{M}. \tag{38}$$

Consider the case when $u \in \mathcal{M}$ is an arbitrary element and $0 < \rho \leq 1$. Thus, we have

$$\hat{u}_\rho = \rho u + (1 - \rho)\hat{u}. \tag{39}$$

Then, $\hat{u}_\rho \in \mathcal{M}$, and from expression (38), we have

$$\rho\langle \mathcal{K}(\hat{u}_\rho), u - \hat{u}\rangle \geq 0. \tag{40}$$

Hence, we have

$$\langle \mathcal{K}(\hat{u}_\rho), u - \hat{u}\rangle \geq 0. \tag{41}$$

It is clear from equation (41) that

$$\langle \mathcal{K}(\hat{u}), u - \hat{u}\rangle \geq 0. \tag{42}$$

Hence, $\hat{u} \in \mathrm{VI}(\mathcal{M}, \mathcal{K})$. This completes the proof of lemma. □

Theorem 3.5. *Let the mapping $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ satisfy conditions $(\mathcal{K}1)$–$(\mathcal{K}4)$. Then, $\{u_n\}$ generated by the* Algorithm 1 *strongly converges to $r^* = P_{\mathrm{VI}(\mathcal{M},\mathcal{K})} \circ \mathrm{g}(r^*)$.*

Proof. Since $\rho_n \to \rho$, there exists a fixed number $\varepsilon \in (0, 1 - \mu)$ such that

$$\lim_{n\to\infty}\left(1 - \frac{\mu\rho_n}{\rho_{n+1}}\right) = 1 - \mu > \varepsilon > 0.$$

Therefore, there exists a fixed number $M_1 \in \mathbb{N}$ such that

$$\left(1 - \frac{\mu\rho_n}{\rho_{n+1}}\right) > \varepsilon > 0, \ \ \forall n \geq M_1. \tag{43}$$

From expression (21), we obtain

$$\|t_n - r^*\|^2 \leq \|u_n - r^*\|^2, \ \ \forall n \geq M_1. \tag{44}$$

It is given that $r^* \in \Omega$ and due to the fact that g is a contraction with $\xi \in [0, 1)$, we have

$$\begin{aligned}\|u_{n+1} - r^*\| &= \|\gamma_n \mathrm{g}(u_n) + (1 - \gamma_n)t_n - r^*\| \\ &= \|\gamma_n[\mathrm{g}(u_n) - r^*] + (1 - \gamma_n)[t_n - r^*]\| \\ &= \|\gamma_n[\mathrm{g}(u_n) + \mathrm{g}(r^*) - \mathrm{g}(r^*) - r^*] + (1 - \gamma_n)[t_n - r^*]\| \\ &\leq \gamma_n\|\mathrm{g}(u_n) - \mathrm{g}(r^*)\| + \gamma_n\|\mathrm{g}(r^*) - r^*\| + (1 - \gamma_n)\|t_n - r^*\| \\ &\leq \gamma_n\xi\|u_n - r^*\| + \gamma_n\|\mathrm{g}(r^*) - r^*\| + (1 - \gamma_n)\|t_n - r^*\|.\end{aligned} \tag{45}$$

Combining expressions (44) and (45) and $\gamma_n \subset (0, 1)$, we obtain

$$\begin{aligned}\|u_{n+1} - r^*\| &\leq \gamma_n\xi\|u_n - r^*\| + \gamma_n\|\mathrm{g}(r^*) - r^*\| + (1 - \gamma_n)\|u_n - r^*\| \\ &= [1 - \gamma_n + \xi\gamma_n]\|u_n - r^*\| + \gamma_n(1 - \xi)\frac{\|\mathrm{g}(r^*) - r^*\|}{(1 - \xi)} \\ &\leq \max\left\{\|u_n - r^*\|, \frac{\|\mathrm{g}(r^*) - r^*\|}{(1 - \xi)}\right\} \\ &\leq \max\left\{\|u_{M_1} - r^*\|, \frac{\|\mathrm{g}(r^*) - r^*\|}{(1 - \xi)}\right\}.\end{aligned} \tag{46}$$

Therefore, we conclude that $\{u_n\}$ is a bounded sequence. By using expression (21), we have

$$\begin{aligned}\|u_{n+1} - r^*\|^2 &= \|\gamma_n \mathrm{g}(u_n) + (1 - \gamma_n)t_n - r^*\|^2 \\ &= \|\gamma_n[\mathrm{g}(u_n) - r^*] + (1 - \gamma_n)[t_n - r^*]\|^2 \\ &= \gamma_n\|\mathrm{g}(u_n) - r^*\|^2 + (1 - \gamma_n)\|t_n - r^*\|^2 - \gamma_n(1 - \gamma_n)\|\mathrm{g}(u_n) - t_n\|^2 \\ &\leq \gamma_n\|\mathrm{g}(u_n) - r^*\|^2 + (1 - \gamma_n)\left[\|u_n - r^*\|^2 - \left(1 - \frac{\mu\rho_n}{\rho_{n+1}}\right)\|u_n - y_n\|^2 - \left(1 - \frac{\mu\rho_n}{\rho_{n+1}}\right)\|t_n - y_n\|^2\right] \\ &\quad - \gamma_n(1 - \gamma_n)\|\mathrm{g}(u_n) - t_n\|^2 \\ &\leq \gamma_n\|\mathrm{g}(u_n) - r^*\|^2 + \|u_n - r^*\|^2 - (1 - \gamma_n)\left(1 - \frac{\mu\rho_n}{\rho_{n+1}}\right)\|u_n - y_n\|^2 - (1 - \gamma_n)\left(1 - \frac{\mu\rho_n}{\rho_{n+1}}\right)\|t_n - y_n\|^2.\end{aligned} \tag{47}$$

The aforementioned expression implies that

$$(1 - \gamma_n)\left(1 - \frac{\mu\rho_n}{\rho_{n+1}}\right)\|u_n - y_n\|^2 + (1 - \gamma_n)\left(1 - \frac{\mu\rho_n}{\rho_{n+1}}\right)\|t_n - y_n\|^2 \leq \gamma_n\|\mathrm{g}(u_n) - r^*\|^2 + \|u_n - r^*\|^2 - \|u_{n+1} - r^*\|^2. \tag{48}$$

By using expression (44), we obtain

$$
\begin{aligned}
\|u_{n+1} - r^*\|^2 &= \|\gamma_n g(u_n) + (1-\gamma_n)t_n - r^*\|^2 \\
&= \|\gamma_n[g(u_n) - r^*] + (1-\gamma_n)[t_n - r^*]\|^2 \\
&\le (1-\gamma_n)^2\|t_n - r^*\|^2 + 2\gamma_n\langle g(u_n) - r^*, (1-\gamma_n)[t_n - r^*] + \gamma_n[g(u_n) - r^*]\rangle \\
&= (1-\gamma_n)^2\|t_n - r^*\|^2 + 2\gamma_n\langle g(u_n) - g(r^*) + g(r^*) - r^*, u_{n+1} - r^*\rangle \\
&= (1-\gamma_n)^2\|t_n - r^*\|^2 + 2\gamma_n\langle g(u_n) - g(r^*), u_{n+1} - r^*\rangle + 2\gamma_n\langle g(r^*) - r^*, u_{n+1} - r^*\rangle \\
&\le (1-\gamma_n)^2\|t_n - r^*\|^2 + 2\gamma_n\xi\|u_n - r^*\|\|u_{n+1} - r^*\| + 2\gamma_n\langle g(r^*) - r^*, u_{n+1} - r^*\rangle \\
&\le (1+\gamma_n^2 - 2\gamma_n)\|u_n - r^*\|^2 + 2\gamma_n\xi\|u_n - r^*\|^2 + 2\gamma_n\langle g(r^*) - r^*, u_{n+1} - r^*\rangle \\
&= (1-2\gamma_n)\|u_n - r^*\|^2 + \gamma_n^2\|u_n - r^*\|^2 + 2\gamma_n\xi\|u_n - r^*\|^2 + 2\gamma_n\langle g(r^*) - r^*, u_{n+1} - r^*\rangle \\
&= [1 - 2\gamma_n(1-\xi)]\|u_n - r^*\|^2 + 2\gamma_n(1-\xi)\left[\frac{\gamma_n\|u_n - r^*\|^2}{2(1-\xi)} + \frac{\langle g(r^*) - r^*, u_{n+1} - r^*\rangle}{1-\xi}\right].
\end{aligned}
\tag{49}
$$

Case 1: Let us consider $M_2 \in \mathbb{N}$ ($M_2 \ge M_1$) such that

$$\|u_{n+1} - r^*\| \le \|u_n - r^*\|, \quad \forall n \ge M_2. \tag{50}$$

Then, $\lim_{n\to\infty}\|u_n - r^*\|$ exists. Let $\lim_{n\to\infty}\|u_n - r^*\| = l$. By the use of expression (48), we obtain

$$(1-\gamma_n)\left(1 - \frac{\mu\rho_n}{\rho_{n+1}}\right)\|u_n - y_n\|^2 + (1-\gamma_n)\left(1 - \frac{\mu\rho_n}{\rho_{n+1}}\right)\|t_n - y_n\|^2 \le \gamma_n\|g(u_n) - r^*\|^2 + \|u_n - r^*\|^2 - \|u_{n+1} - r^*\|^2. \tag{51}$$

Since $\lim_{n\to\infty}\|u_n - r^*\|$ exists and $\gamma_n \to 0$, we obtain

$$\lim_{n\to\infty}\|u_n - y_n\| = \lim_{n\to\infty}\|t_n - y_n\| = 0. \tag{52}$$

By using the aforementioned results, we obtain

$$\lim_{n\to\infty}\|u_n - t_n\| \le \lim_{n\to\infty}\|u_n - y_n\| + \lim_{n\to\infty}\|y_n - t_n\| = 0. \tag{53}$$

It further implies that

$$
\begin{aligned}
\|u_{n+1} - u_n\| &= \|\gamma_n g(u_n) + (1-\gamma_n)t_n - u_n\| \\
&= \|\gamma_n[g(u_n) - u_n] + (1-\gamma_n)[t_n - u_n]\| \le \gamma_n\|g(u_n) - u_n\| + (1-\gamma_n)\|t_n - u_n\| \longrightarrow 0.
\end{aligned}
\tag{54}
$$

The aforementioned term specifically includes that

$$\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0. \tag{55}$$

By using Lemma 3.4, we obtain

$$\limsup_{n\to\infty}\langle g(r^*) - r^*, u_n - r^*\rangle = \limsup_{k\to\infty}\langle g(r^*) - r^*, u_{n_k} - r^*\rangle = \langle g(r^*) - r^*, \hat{u} - r^*\rangle \le 0. \tag{56}$$

By the use of $\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0$, we may deduce that

$$\limsup_{n\to\infty}\langle g(r^*) - r^*, u_{n+1} - r^*\rangle \le \limsup_{n\to\infty}\langle g(r^*) - r^*, u_{n+1} - u_n\rangle + \limsup_{n\to\infty}\langle g(r^*) - r^*, u_n - r^*\rangle \le 0. \tag{57}$$

It is evident from expressions (49) and (57) that we are going to have it

$$\limsup_{n\to\infty}\left[\frac{\gamma_n\|u_n - r^*\|^2}{2(1-\xi)} + \frac{\langle g(r^*) - r^*, u_{n+1} - r^*\rangle}{1-\xi}\right] \le 0. \tag{58}$$

By choosing $n \ge M_3 \in \mathbb{N}$ ($M_3 \ge M_2$) large enough such that $2\gamma_n(1-\xi) < 1$, and by the use of (49), (58), and through Lemma 2.2, we conclude that $\|u_n - r^*\| \to 0$ as $n \to \infty$.

Case 2: Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|u_{n_i} - r^*\| \le \|u_{n_{i+1}} - r^*\|, \quad \forall i \in \mathbb{N}.$$

From Lemma 2.3, there exists a sequence $\{m_k\} \subset \mathbb{N}$ as $\{m_k\} \to \infty$, such that

$$\|u_{m_k} - r^*\| \le \|u_{m_{k+1}} - r^*\| \quad \text{and} \quad \|u_k - r^*\| \le \|u_{m_{k+1}} - r^*\|, \ \forall k \in \mathbb{N}. \tag{59}$$

As in case 1, expression (47) implies that:

$$\begin{aligned}&\left(1-\gamma_{m_k}\right)\left(1-\frac{\mu\rho_{m_k}}{\rho_{m_k+1}}\right)\|u_{m_k}-y_{m_k}\|^2+\left(1-\gamma_{m_k}\right)\left(1-\frac{\mu\rho_{m_k}}{\rho_{m_k+1}}\right)\|t_{m_k}-y_{m_k}\|^2\\&\le \gamma_{m_k}\|g\left(u_{m_k}\right)-r^*\|^2+\|u_{m_k}-r^*\|^2-\|u_{m_k+1}-r^*\|^2.\end{aligned}\tag{60}$$

By using $\gamma_{m_k} \to 0$, we obtain the following result:

$$\lim_{k\to\infty}\|u_{m_k}-y_{m_k}\| = \lim_{k\to\infty}\|t_{m_k}-y_{m_k}\| = 0. \tag{61}$$

Next, we are going to obtain the following:

$$\begin{aligned}\|u_{m_k+1}-u_{m_k}\| &= \|\gamma_{m_k} g\left(u_{m_k}\right)+\left(1-\gamma_{m_k}\right)t_{m_k}-u_{m_k}\|\\&=\|\gamma_{m_k}\left[g\left(u_{m_k}\right)-u_{m_k}\right]+\left(1-\gamma_{m_k}\right)\left[t_{m_k}-u_{m_k}\right]\|\\&\le \gamma_{m_k}\|g\left(u_{m_k}\right)-u_{m_k}\|+\left(1-\gamma_{m_k}\right)\|t_{m_k}-u_{m_k}\| \longrightarrow 0.\end{aligned}\tag{62}$$

We use the same argument as in Case 1 such that

$$\limsup_{k\to\infty}\langle g(r^*)-r^*, u_{m_k+1}-r^*\rangle \le 0. \tag{63}$$

By using expressions (49) and (59), we have

$$\begin{aligned}\|u_{m_k+1}-r^*\|^2 &\le [1-2\gamma_{m_k}(1-\xi)]\|u_{m_k}-r^*\|^2+2\gamma_{m_k}(1-\xi)\left[\frac{\gamma_{m_k}\|u_{m_k}-r^*\|^2}{2(1-\xi)}+\frac{\langle g(r^*)-r^*, u_{m_k+1}-r^*\rangle}{1-\xi}\right]\\&\le [1-2\gamma_{m_k}(1-\xi)]\|u_{m_k+1}-r^*\|^2+2\gamma_{m_k}(1-\xi)\left[\frac{\gamma_{m_k}\|u_{m_k}-r^*\|^2}{2(1-\xi)}+\frac{\langle g(r^*)-r^*, u_{m_k+1}-r^*\rangle}{1-\xi}\right].\end{aligned}\tag{64}$$

It continues on from that

$$\|u_{m_k+1}-r^*\|^2 \le \frac{\gamma_{m_k}\|u_{m_k}-r^*\|^2}{2(1-\xi)}+\frac{\langle g(r^*)-r^*, u_{m_k+1}-r^*\rangle}{1-\xi}. \tag{65}$$

Since $\gamma_{m_k} \to 0$ and $\|u_{m_k}-r^*\|$ is a bounded sequence, then, expressions (63) and (65) indicate that

$$\|u_{m_k+1}-r^*\|^2 \to 0, \quad \text{as } k \to \infty. \tag{66}$$

The aforementioned equation means that

$$\lim_{k\to\infty}\|u_k-r^*\|^2 \le \lim_{k\to\infty}\|u_{m_k+1}-r^*\|^2 \le 0. \tag{67}$$

Consequently, $u_n \to r^*$. This completes the proof of the theorem. □

Theorem 3.6. *Let the mapping $\mathcal{K}: \mathcal{E} \to \mathcal{E}$ satisfy conditions $(\mathcal{K}1)$–$(\mathcal{K}4)$. Then, $\{u_n\}$ generated by the Algorithm 2 converges strongly to a solution $r^* = P_{VI(\mathcal{M},\mathcal{K})}(0)$.*

Proof. By using the definition of $\{s_n\}$, we obtain

$$\begin{aligned}\|s_n-r^*\| &= \|u_n+\chi_n(u_n-u_{n-1})-\gamma_n u_n-\chi_n\gamma_n(u_n-u_{n-1})-r^*\|\\&=\|(1-\gamma_n)(u_n-r^*)+(1-\gamma_n)\chi_n(u_n-u_{n-1})-\gamma_n r^*\|\end{aligned}\tag{68}$$

$$\begin{aligned}&\le (1-\gamma_n)\|u_n-r^*\|+(1-\gamma_n)\chi_n\|u_n-u_{n-1}\|+\gamma_n\|r^*\|\\&\le (1-\gamma_n)\|u_n-r^*\|+\gamma_n K_1,\end{aligned}\tag{69}$$

where

$$(1-\gamma_n)\frac{\chi_n}{\gamma_n}\|u_n-u_{n-1}\|+\|r^*\|\le K_1.$$

It is given that $\rho_n\to\rho$ such that there exists a finite number $\Im\in(0,1-\mu)$ such that

$$\lim_{n\to\infty}\left(1-\frac{\mu\rho_n}{\rho_{n+1}}\right)=1-\mu>\Im>0.$$

Thus, there is $N_1\in\mathbb{N}$ such that

$$\left(1-\frac{\mu\rho_n}{\rho_{n+1}}\right)>\Im>0,\ \ \forall n\ge N_1. \tag{70}$$

From Lemma 3.3, we may rewrite

$$\|u_{n+1}-r^*\|^2\le\|s_n-r^*\|^2,\ \ \forall n\ge N_1. \tag{71}$$

By the use of expressions (69) and (71), we obtain

$$\|u_{n+1}-r^*\|\le(1-\gamma_n)\|u_n-r^*\|+\gamma_nK_1\le\max\{\|u_n-r^*\|,K_1\}\cdots\le\max\{\|u_{N_1}-r^*\|,K_1\}. \tag{72}$$

As a result, we can conclude that $\{u_n\}$ is a bounded sequence. Indeed, by expression (69), we have

$$\begin{aligned}\|s_n-r^*\|^2&\le(1-\gamma_n)^2\|u_n-r^*\|^2+\gamma_n^2K_1^2+2K_1\gamma_n(1-\gamma_n)\|u_n-r^*\|\\&\le\|u_n-r^*\|^2+\gamma_n[\gamma_nK_1^2+2K_1(1-\gamma_n)\|u_n-r^*\|]\\&\le\|u_n-r^*\|^2+\gamma_nK_2,\end{aligned} \tag{73}$$

for some $K_2>0$. Both expressions (21) and (73) imply that

$$\|u_{n+1}-r^*\|^2\le\|u_n-r^*\|^2+\gamma_nK_2-\left(1-\frac{\mu\rho_n}{\rho_{n+1}}\right)\|s_n-y_n\|^2-\left(1-\frac{\mu\rho_n}{\rho_{n+1}}\right)\|u_{n+1}-y_n\|^2. \tag{74}$$

From expression (68), we can write

$$\begin{aligned}\|s_n-r^*\|^2&=\|u_n+\chi_n(u_n-u_{n-1})-\gamma_nu_n-\chi_n\gamma_n(u_n-u_{n-1})-r^*\|^2\\&=\|(1-\gamma_n)(u_n-r^*)+(1-\gamma_n)\chi_n(u_n-u_{n-1})-\gamma_nr^*\|^2\\&\le\|(1-\gamma_n)(u_n-r^*)+(1-\gamma_n)\chi_n(u_n-u_{n-1})\|^2+2\gamma_n\langle -r^*,s_n-r^*\rangle\\&=(1-\gamma_n)^2\|u_n-r^*\|^2+(1-\gamma_n)^2\chi_n^2\|u_n-u_{n-1}\|^2+2\chi_n(1-\gamma_n)^2\|u_n-r^*\|\|u_n\\&\quad-u_{n-1}\|+2\gamma_n\langle -r^*,s_n-u_{n+1}\rangle+2\gamma_n\langle -r^*,u_{n+1}-r^*\rangle\\&\le(1-\gamma_n)\|u_n-r^*\|^2+\chi_n^2\|u_n-u_{n-1}\|^2+2\chi_n(1-\gamma_n)\|u_n-r^*\|\|u_n-u_{n-1}\|+2\gamma_n\|r^*\|\|s_n\\&\quad-u_{n+1}\|+2\gamma_n\langle -r^*,u_{n+1}-r^*\rangle\\&=(1-\gamma_n)\|u_n-r^*\|^2+\gamma_n\left[\chi_n\|u_n-u_{n-1}\|\frac{\chi_n}{\gamma_n}\|u_n-u_{n-1}\|+2(1-\gamma_n)\|u_n-r^*\|\frac{\chi_n}{\gamma_n}\|u_n\right.\\&\quad\left.-u_{n-1}\|+2\|r^*\|\|s_n-u_{n+1}\|+2\langle r^*,r^*-u_{n+1}\rangle\right].\end{aligned} \tag{75}$$

From expressions (71) and (75), we obtain

$$\begin{aligned}\|u_{n+1}-r^*\|^2&\le(1-\gamma_n)\|u_n-r^*\|^2+\gamma_n\left[\chi_n\|u_n-u_{n-1}\|\frac{\chi_n}{\gamma_n}\|u_n-u_{n-1}\|+2(1-\gamma_n)\|u_n-r^*\|\frac{\chi_n}{\gamma_n}\|u_n\right.\\&\quad\left.-u_{n-1}\|+2\|r^*\|\|s_n-u_{n+1}\|+2\langle r^*,r^*-u_{n+1}\rangle\right].\end{aligned} \tag{76}$$

Case 1: Consider that there exists a finite number $N_2\in\mathbb{N}$ $(N_2\ge N_1)$ such that

$$\|u_{n+1} - r^*\| \le \|u_n - r^*\|, \ \forall n \ge N_2. \tag{77}$$

Thus, the aforementioned relation implies that $\lim_{n\to\infty}\|u_n - r^*\|$ exists and let $\lim_{n\to\infty}\|u_n - r^*\| = l$, for $l \ge 0$. By using expression (74), we can rewrite

$$\left(1 - \frac{\mu\rho_n}{\rho_{n+1}}\right)\|s_n - y_n\|^2 + \left(1 - \frac{\mu\rho_n}{\rho_{n+1}}\right)\|u_{n+1} - y_n\|^2 \le \|u_n - r^*\|^2 + \gamma_n K_2 - \|u_{n+1} - r^*\|^2. \tag{78}$$

Since limit of $\|u_n - r^*\|$ exists and $\gamma_n \to 0$, we can deduce that

$$\|s_n - y_n\| \to 0 \quad \text{and} \quad \|u_{n+1} - y_n\| \to 0 \quad \text{as } n \to \infty. \tag{79}$$

It continues from expression (79) that

$$\lim_{n\to\infty} \|s_n - u_{n+1}\| \le \lim_{n\to\infty} \|s_n - y_n\| + \lim_{n\to\infty} \|y_n - u_{n+1}\| = 0. \tag{80}$$

Next, we need to evaluate

$$\begin{aligned}\|s_n - u_n\| &= \|u_n + \chi_n(u_n - u_{n-1}) - \gamma_n[u_n + \chi_n(u_n - u_{n-1})] - u_n\| \\ &\le \chi_n\|u_n - u_{n-1}\| + \gamma_n\|u_n\| + \chi_n\gamma_n\|u_n - u_{n-1}\| \\ &= \gamma_n\frac{\chi_n}{\gamma_n}\|u_n - u_{n-1}\| + \gamma_n\|u_n\| + \gamma_n^2\frac{\chi_n}{\gamma_n}\|u_n - u_{n-1}\| \longrightarrow 0.\end{aligned} \tag{81}$$

Thus, the aforementioned expression implies that

$$\lim_{n\to\infty} \|u_n - u_{n+1}\| \le \lim_{n\to\infty} \|u_n - s_n\| + \lim_{n\to\infty} \|s_n - u_{n+1}\| = 0. \tag{82}$$

From given $r^* = P_{\mathrm{VI}(\mathcal{M},\mathcal{K})}(0)$, we have

$$\langle 0 - r^*, y - r^*\rangle \le 0, \ \forall y \in \mathrm{VI}(\mathcal{M}, \mathcal{K}). \tag{83}$$

Moreover, it is considered that

$$\limsup_{n\to\infty}\langle r^*, r^* - u_n\rangle = \lim_{k\to\infty} \langle r^*, r^* - u_{n_k}\rangle = \langle r^*, r^* - \hat{u}\rangle \le 0. \tag{84}$$

By using the fact, $\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0$. Thus, using expression (84), we can deduce that

$$\limsup_{n\to\infty}\langle r^*, r^* - u_{n+1}\rangle \le \limsup_{n\to\infty}\langle r^*, r^* - u_n\rangle + \limsup_{n\to\infty}\langle r^*, u_n - u_{n+1}\rangle \le 0. \tag{85}$$

By using expressions (76) and (85) and taking Lemma 2.2 imply that $\|u_n - r^*\| \to 0$ as $n \to \infty$.

Case 2: Let there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|u_{n_i} - r^*\| \le \|u_{n_{i+1}} - r^*\|, \ \forall i \in \mathbb{N}.$$

Thus, by using Lemma 2.3, there exists a sequence $\{m_k\} \subset \mathbb{N}$ as $\{m_k\} \to \infty$ such that

$$\|u_{m_k} - r^*\| \le \|u_{m_{k+1}} - r^*\| \quad \text{and} \quad \|u_k - r^*\| \le \|u_{m_{k+1}} - r^*\|, \quad \text{for all } k \in \mathbb{N}. \tag{86}$$

Expression (78) implies that

$$\left(1 - \frac{\mu\rho_{m_k}}{\rho_{m_k+1}}\right)\|s_{m_k} - y_{m_k}\|^2 + \left(1 - \frac{\mu\rho_{m_k}}{\rho_{m_k+1}}\right)\|u_{m_k+1} - y_{m_k}\|^2 \le \|u_{m_k} - r^*\|^2 + \gamma_{m_k} K_2 - \|u_{m_k+1} - r^*\|^2. \tag{87}$$

Due to sequence $\gamma_{m_k} \to 0$, we deduce the following:

$$\lim_{k\to\infty} \|s_{m_k} - y_{m_k}\| = \lim_{k\to\infty} \|u_{m_k+1} - y_{m_k}\| = 0. \tag{88}$$

It follows that

$$\lim_{k\to\infty} \|u_{m_{k+1}} - s_{m_k}\| \le \lim_{k\to\infty} \|u_{m_{k+1}} - y_{m_k}\| + \lim_{k\to\infty} \|y_{m_k} - s_{m_k}\| = 0. \tag{89}$$

Next, we have to evaluate

$$\begin{aligned}\|s_{m_k} - u_{m_k}\| &= \|u_{m_k} + \alpha_{m_k}\big(u_{m_k} - u_{m_k-1}\big) - \gamma_{m_k}[u_{m_k} + \alpha_{m_k}\big(u_{m_k} - u_{m_k-1}\big)] - u_{m_k}\| \\ &\leq \alpha_{m_k}\|u_{m_k} - u_{m_k-1}\| + \gamma_{m_k}\|u_{m_k}\| + \alpha_{m_k}\gamma_{m_k}\|u_{m_k} - u_{m_k-1}\| \\ &= \gamma_{m_k}\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k} - u_{m_k-1}\| + \gamma_{m_k}\|u_{m_k}\| + \gamma_{m_k}^2\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k} - u_{m_k-1}\| \longrightarrow 0.\end{aligned} \tag{90}$$

It follows that

$$\lim_{k\to\infty}\|u_{m_k} - u_{m_k+1}\| \leq \lim_{k\to\infty}\|u_{m_k} - s_{m_k}\| + \lim_{k\to\infty}\|s_{m_k} - u_{m_k+1}\| = 0. \tag{91}$$

By using the same argument as in Case 1, such that

$$\limsup_{k\to\infty}\langle r^*, r^* - u_{m_k+1}\rangle \leq 0. \tag{92}$$

Now, using expressions (76) and (86), we have

$$\begin{aligned}\|u_{m_k+1} - r^*\|^2 &\leq \big(1 - \gamma_{m_k}\big)\|u_{m_k} - r^*\|^2 + \gamma_{m_k}\Bigg[\alpha_{m_k}\|u_{m_k} - u_{m_k-1}\|\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k} - u_{m_k-1}\| + 2\big(1 - \gamma_{m_k}\big)\|u_{m_k} \\ &\quad - r^*\|\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k} - u_{m_k-1}\| + 2\|r^*\|\|s_{m_k} - u_{m_k+1}\| + 2\langle r^*, r^* - u_{m_k+1}\rangle\Bigg] \\ &\leq \big(1 - \gamma_{m_k}\big)\|u_{m_{k+1}} - r^*\|^2 + \gamma_{m_k}\Bigg[\alpha_{m_k}\|u_{m_k} - u_{m_k-1}\|\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k} - u_{m_k-1}\| + 2\big(1 - \gamma_{m_k}\big)\|u_{m_k} \\ &\quad - r^*\|\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k} - u_{m_k-1}\| + 2\|r^*\|\|s_{m_k} - u_{m_k+1}\| + 2\langle r^*, r^* - u_{m_k+1}\rangle\Bigg].\end{aligned} \tag{93}$$

Thus, we obtain

$$\begin{aligned}\|u_{m_k+1} - r^*\|^2 &\leq \Bigg[\alpha_{m_k}\|u_{m_k} - u_{m_k-1}\|\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k} - u_{m_k-1}\| \\ &\quad + 2\big(1 - \gamma_{m_k}\big)\|u_{m_k} - r^*\|\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k} - u_{m_k-1}\| + 2\|r^*\|\|s_{m_k} - u_{m_k+1}\| + 2\langle r^*, r^* - u_{m_k+1}\rangle\Bigg].\end{aligned} \tag{94}$$

Since $\gamma_{m_k} \to 0$ and $\|u_{m_k} - r^*\|$ is a bounded sequence, then expressions (92) and (94) imply that

$$\|u_{m_k+1} - r^*\|^2 \to 0 \text{ , as } k \to \infty. \tag{95}$$

It implies that

$$\lim_{n\to\infty}\|u_k - r^*\|^2 \leq \lim_{n\to\infty}\|u_{m_k+1} - r^*\|^2 \leq 0. \tag{96}$$

As a consequence, $u_n \to r^*$. This will complete the proof of the theorem. □

Theorem 3.7. *Let the mapping $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ satisfy conditions $(\mathcal{K}1)$–$(\mathcal{K}4)$. Then, $\{u_n\}$ generated by* Algorithm 3 *converges strongly to a solution of the problem* VIP.

Proof. The proof is the same as the proof of Theorem 3.5. □

Theorem 3.8. *Let the mapping $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ meet conditions $(\mathcal{K}1)$–$(\mathcal{K}4)$. Then, $\{u_n\}$ generated by* Algorithm 4 *converges strongly to a solution of the problem* VIP.

Proof. The proof is the same as the proof of Theorem 3.6. □

4 Numerical illustrations

The numerical results of the proposed iterative schemes are given in this section, in contrast to some related work in the literature and also in the analysis of how variations in control parameters affect the numerical effectiveness of the proposed algorithms. All computations are done in MATLAB R2018b and run on an HP i5-6200 8.00 GB (7.78 GB usable) RAM laptop.

Example 4.1. Let $\mathbb{H} = l_2$ be a real Hilbert space with sequences of real numbers satisfying the following condition:

$$\|u_1\|^2 + \|u_2\|^2 + \cdots + \|u_n\|^2 + \cdots < +\infty. \tag{97}$$

Assume that $\mathcal{K} : \mathcal{M} \to \mathcal{M}$ is defined by:

$$G(u) = (5 - \|u\|)u, \quad \forall u \in \mathbb{H},$$

where $C = \{u \in \mathbb{H} : \|u\| \leq 3\}$. It is to note that $\mathcal{K}$ is sequentially weakly continuous on $\mathcal{E}$ and $\mathrm{VI}(\mathcal{M}, \mathcal{K}) = \{0\}$. For each $u, y \in \mathcal{E}$, we have

$$\begin{aligned} \|\mathcal{K}(u) - \mathcal{K}(y)\| &= \|(5 - \|u\|)u - (5 - \|y\|)y\| \\ &= \|5(u - y) - \|u\|(u - y) - (\|u\| - \|y\|)y\| \\ &\leq 5\|u - y\| + \|u\|\|u - y\| + |\|u\| - \|y\||\|y\| \\ &\leq 5\|u - y\| + 3\|u - y\| + 3\|u - y\| \\ &\leq 11\|u - y\|. \end{aligned} \tag{98}$$

Hence, $\mathcal{K}$ is L-Lipschitz continuous with $L = 11$. For any $u, y \in \mathcal{E}$, let $\langle \mathcal{K}(u), y - u \rangle > 0$ such that

$$(5 - \|u\|)\langle u, y - u \rangle > 0.$$

Since $\|u\| \leq 3$, it implies that

$$\langle u, y - u \rangle > 0.$$

Thus, it implies that

$$\begin{aligned} \langle \mathcal{K}(y), y - u \rangle &= (5 - \|y\|)\langle y, y - u \rangle \\ &\geq (5 - \|y\|)\langle y, y - u \rangle - (5 - \|y\|)\langle u, y - u \rangle \\ &\geq 2\|u - y\|^2 \geq 0. \end{aligned} \tag{99}$$

Thus, we show that $\mathcal{K}$ is quasimonotone on $\mathcal{M}$. Let $u = (\frac{5}{2}, 0, 0, \ldots, 0, \ldots)$ and $y = (3, 0, 0, \ldots, 0, \ldots)$ such that

$$\langle \mathcal{K}(u) - \mathcal{K}(y), u - y \rangle = (2.5 - 3)^2 < 0.$$

The formula for a projection on C is given in the following manner:

Table 1: Numerical values for Example 4.1

	Number of iterations		Execution time in seconds	
u_1	Alg1	Alg3	Alg1	Alg3
$(2, 2, \ldots, 2_{5,000}, 0, 0, \ldots)$	41	36	3.93847480000000	3.33763350000000
$(1, 2, \ldots, 5{,}000, 0, 0, \ldots)$	57	48	4.87583390000000	4.24728740000000
$(5, 5, \ldots, 5_{10,000}, 0, 0, \ldots)$	48	39	4.37341940000000	3.82418350000000
$(50, 50, \ldots, 50_{10,000}, 0, 0, \ldots)$	61	49	5.84570940000000	4.57478350000000
$(500, 500, \ldots, 500_{10,000}, 0, 0, \ldots)$	89	67	8.34746348000000	6.92528350000000

Table 2: Numerical values for Example 4.1

	Number of iterations		Execution time in seconds	
u_1	Alg2	Alg4	Alg2	Alg4
$(2, 2, \ldots, 2_{5,000}, 0, 0, \ldots)$	29	21	2.14638330000000	2.00018330000000
$(1, 2, \ldots, 5,000, 0, 0, \ldots)$	33	27	2.87463850000000	2.14639210000000
$(5, 5, \ldots, 5_{10,000}, 0, 0, \ldots)$	30	21	2.72444340000000	1.92738850000000
$(50, 50, \ldots, 50_{10,000}, 0, 0, \ldots)$	41	34	4.56044940000000	3.93847350000000
$(500, 500, \ldots, 500_{10,000}, 0, 0, \ldots)$	47	38	7.46292840000000	5.46293350000000

$$P_C(u) = \begin{cases} u & \text{if } \|u\| \le 3, \\ \dfrac{3u}{\|u\|}, & \text{otherwise}. \end{cases}$$

The following conditions have been taken for numerical study. (i) Algorithm 1 (**Alg1**): $\rho_1 = 0.22$, $\mu = 0.44$, $\gamma_n = \frac{1}{(n+2)}$; (ii) Algorithm 2 (**Alg2**): $\rho_1 = 0.22$, $\mu = 0.44$, $\gamma_n = \frac{1}{(n+2)}$, $p_n = \frac{100}{(n+1)^2}$; (iii) Algorithm 3 (**Alg3**): $\rho_1 = 0.22$, $\mu = 0.44$, $\chi = 0.50$, $\gamma_n = \frac{1}{(n+2)}$, $\varepsilon_n = \frac{1}{(n+1)^2}$; (iv) Algorithm 4 (**Alg4**): $\rho_1 = 0.22$, $\mu = 0.44$, $\chi = 0.50$, $\gamma_n = \frac{1}{(n+2)}$, $\varepsilon_n = \frac{1}{(n+1)^2}$, $p_n = \frac{100}{(n+1)^2}$ (Tables 1 and 2).

5 Conclusion

We developed various types of explicit extragradient-type methods for finding a numerical solution to quasimonotone variational inequalities in a real Hilbert space. This approach is seen as a variant of the two-step extragradient method. Two strongly convergent findings are well proven and correspond to the suggested methods. The numerical findings were analyzed to demonstrate the numerical performance of the suggested methods. These computational results show that the non-monotone variable step size rule continues to improve the effectiveness of the iterative sequence in this scenario.

Funding information: This research was supported by The Science, Research and Innovation Promotion Funding (TSRI) (Grant No. FRB660012/0168). This research block grants were managed under Rajamangala University of Technology Thanyaburi (FRB66E0653S.2).

Author contributions: All the authors contributed equally to this manuscript.

Conflict of interest: No potential conflict of interest was reported by the authors.

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