Research Article

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Bernstein-type operators on elliptic domain and their interpolation properties

https://doi.org/10.1515/dema-2022-0199
received January 21, 2022; accepted January 20, 2023

Abstract: The aim of this article is to construct univariate Bernstein-type operators $(\mathcal{B}_m^x G)(x, z)$ and $(\mathcal{B}_n^z G)(x, z)$, their products $(\mathcal{P}_{mn} G)(x, z)$, $(\mathcal{Q}_{nm} G)(x, z)$, and their Boolean sums $(\mathcal{S}_{mn} G)(x, z)$, $(\mathcal{T}_{nm} G)(x, z)$ on elliptic region, which interpolate the given real valued function G defined on elliptic region on its boundary. The bound of the remainders of each approximation formula of corresponding operators are computed with the help of Peano's theorem and modulus of continuity, and the rate of convergence for functions of Lipschitz class is computed.

Keywords: Bernstein-type operators, elliptic domain, interpolation, product operators, Boolean sum operators, modulus of continuity, Peano's theorem

MSC 2020: 41A35, 41A36, 41A80

1 Introduction

Approximation of functions by simpler class of functions, especially polynomials and positive linear operators, has attracted lot of researchers to construct some other simpler class of operators in last decades. Approximating functions, some data, and a member of a given set are some of the examples of the approximation calculations. It was initiated basically in 1885, when great mathematician Weierstrass proposed a fundamental theorem known as the Weierstrass approximation theorem, which guarantees to construct polynomials to approximate continuous function on compact interval in $\mathbb{R}$. Weirstrass itself proposed a proof of theorem. A new era in the approximation theory started in 1912, when great Russian mathematician, Bernstein [1] constructed the sequence of operators (polynomials) $\mathcal{B}_n: C[0, 1] \to C[0, 1]$ for any bounded function G defined on $[0, 1]$ to provide constructive proof of Weierstrass approximation theorem for all $x \in [0, 1]$, $n \in \mathbb{N}$ as follows:

$$\mathcal{B}_n(G;\, x) = \sum_{r=0}^{n} \binom{n}{r} x^r (1 - x)^{n-r} G\left(\frac{r}{n}\right). \tag{1.1}$$

This proof is based on probabilistic approach and is simpler, elegant, and constructive. The advantage of using Bernstein polynomial is that it is compatible with computers and easy to implement for simulation purposes. Space $C[a, b]$ and $C[0, 1]$ are identical as normed space with sup-norm. This Bernstein operator can be extended to any arbitrary compact interval $[a, b]$ of $\mathbb{R}$ with the help of the map $\sigma : [a, b] \to [0, 1]$ defined by $\sigma(x) = \frac{x-a}{b-a}$. Another development started in the approximation theory when Korovkin in 1953

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discovered a simple criteria whether the sequence of positive linear operator converges uniformly to continuous function on [0, 1] by simply checking the uniform convergence of the Chebychev like test functions 1, x, and x^2 in the space $C[0,1]$ of all continuous functions on the real interval [0, 1]. As space $C[0,1]$ is not strictly convex with respect to sup-norm, best approximation may not be unique. This idea motivates to create some other positive linear operators on [0, 1].

In the finite element method for differential equations with given boundary conditions approximation operators on polygonal domains are required. Thus, many researchers generalized Bernstein-type operators on different domains and constructed some other operators for improved approximation. In this sequel, In 1973, Barnhill et al. [2–4] initiated and investigated smooth interpolation in triangles. Stancu studied polynomial interpolation on boundary data on triangles and error bound for smooth interpolation [5,6]. Cătinaş extended some interpolation operators to triangle with one curved side [7]. Cai et al. constructed λ-Bernstein operators and studied its approximation properties in [8,9]. Braha et al. studied λ-Bernstein operators via power series summability methods in [10]. Mursaleen et al. studied approximation properties by q-Bernstein shifted operators and q-Bernstein Schurer operators in [11,12]. Recently Khan et al. generalized Bernstein-type operators and studied applications of its basis in computer aided geometric design (CAGD) [13,14]. For other applications of Bernstein-type operators related to construction of Bezier curves and surfaces, one can see [15–19].

In 2009, Blaga and Coman [20] constructed Bernstein-type operators $(\mathcal{B}_m^x G)(x,z)$ and $(\mathcal{B}_n^z G)(x,z)$, their products $(\mathcal{P}_{mn}G)(x,z)$, $(\mathcal{Q}_{nm}G)(x,z)$, and their Boolean sums $(\mathcal{S}_{mn}G)(x,z)$, $(\mathcal{T}_{nm}G)(x,z)$ to approximate any real valued function G defined on triangular domain. For other similar kind of works one can see [21–25].

Inspired by the idea of [20] and the recent work [15,26], first, we construct Bernstein-type operators $(\mathcal{B}_m^x G)(x,z)$ and $(\mathcal{B}_n^z G)(x,z)$ in Section 2. In Section 3, we calculate some moments of the operators $(\mathcal{B}_m^x G)(x,z)$ and $(\mathcal{B}_n^z G)(x,z)$. In Section 4, we define the approximation formula and present the estimate for error bound. In Section 5, we discuss the rate of convergence for functions of Lipschitz class. Section 6 deals with product operators $(\mathcal{P}_{mn}G)(x,z)$, $(\mathcal{Q}_{nm}G)(x,z)$ and their remainders bound. In Section 7, Boolean sum operators $(\mathcal{S}_{mn}G)(x,z)$, $(\mathcal{T}_{nm}G)(x,z)$ and their remainders are computed on elliptic domain. Finally, graphical analysis is presented to demonstrate theoretical findings in Section 8. These operators interpolate the real valued function defined on the elliptic domain on its boundary.

2 Construction of univariate operators on elliptic domain

Let us consider the standard ellipse and elliptic region in the two-dimensional space $\mathbb{R}^2$ defined as follows:

$$E = \left\{(x,z) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{z^2}{b^2} = 1\right\},$$

$$E^* = \left\{(x,z) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{z^2}{b^2} \le 1\right\}.$$

Consider,

$$\Gamma_1 = \{(x,z) \in E : x \le 0\}, \quad \Gamma_2 = \{(x,z) \in E : x \ge 0\},$$
$$\Gamma_3 = \{(x,z) \in E : z \le 0\} \quad \text{and} \quad \Gamma_4 = \{(x,z) \in E : z \ge 0\}.$$

Notice that $A_z = \left(-a\sqrt{1-\frac{z^2}{b^2}}, z\right)$ and $B_z = \left(a\sqrt{1-\frac{z^2}{b^2}}, z\right)$ be the end points of line segment A_zB_z from Γ_1 to Γ_2 parallel to axis Ox. Similarly, $C_x = \left(x, -b\sqrt{1-\frac{x^2}{a^2}}\right)$ and $D_x = \left(x, b\sqrt{1-\frac{x^2}{a^2}}\right)$ be the endpoints of line segment C_xD_x from Γ_3 to Γ_4 parallel to axis Oz. Line segment A_zB_z and C_xD_x intersect at the point $(x,z) \in E^*$ as shown in Figures 1 and 2. Let $\circ_m^x = \{-a\sqrt{1-\frac{z^2}{b^2}} + 2a\frac{i}{m}\sqrt{1-\frac{z^2}{b^2}}, i = 0, 1, \cdots, m\}$ and $\circ_n^z =$

$\{-b\sqrt{1-\frac{x^2}{a^2}} + 2b\frac{j}{n}\sqrt{1-\frac{x^2}{a^2}},\quad j=0,1,\ldots,n\}$ be uniform partitions of the intervals $\left[-a\sqrt{1-\frac{z^2}{b^2}}, a\sqrt{1-\frac{z^2}{b^2}}\right]$ and $\left[-b\sqrt{1-\frac{x^2}{a^2}}, b\sqrt{1-\frac{x^2}{a^2}}\right]$, respectively. We denote $g(z) = a\sqrt{1-\frac{z^2}{b^2}}$, $z \in [-b, b]$ and $h(x) = b\sqrt{1-\frac{x^2}{a^2}}$, $x \in [-a, a]$ throughout the article. Line segment A_zB_z represents the interval $[-g(z), g(z)]$, and its uniform partition is given by $\circ_m^x = \{-g(z) + 2\frac{i}{m}g(z),\quad i = 0, 1, \ldots, m\}$. Similarly, line segment C_xD_x represents the interval $[-h(x), h(x)]$, and its uniform partition is given by $\circ_m^z = \{-h(x) + 2\frac{j}{n}h(x),\quad j = 0, 1, \ldots, n\}$ as shown in Figures 1 and 2.

Now we introduce Bernstein-type operators $(\mathcal{B}_m^x G)$ and $(\mathcal{B}_n^z G)$ for the function $G : E^* \to \mathbb{R}$ as follows:

$$(\mathcal{B}_m^x G)(x,z) = \begin{cases} \sum_{i=0}^{m} \tilde{p}_{m,i}(x,z)G\left(-g(z) + 2\frac{i}{m}g(z), z\right), & (x,z) \in E^*\backslash\{(0,-b),(0,b)\} \\ G(0,-b), & (0,-b) \in E \\ G(0,b), & (0,b) \in E \end{cases} \tag{2.1}$$

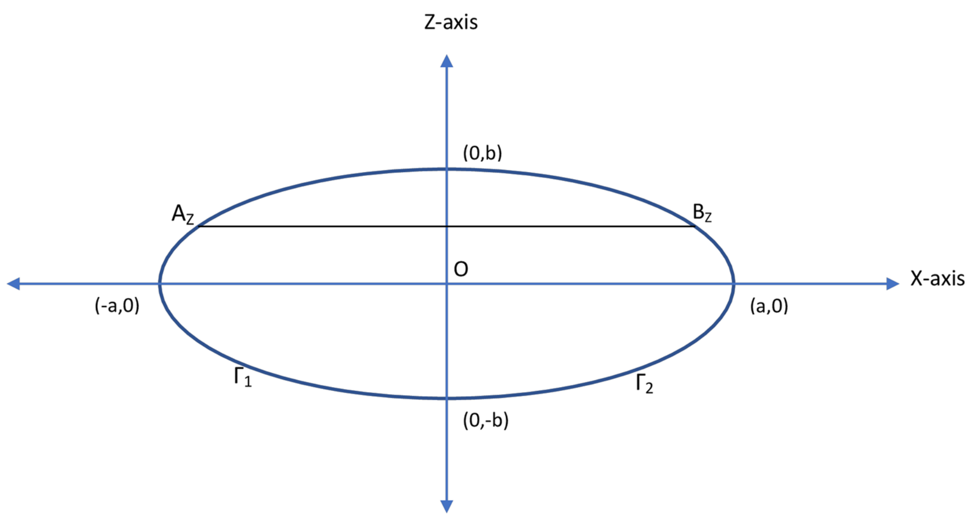

Figure 1: Elliptic domain.

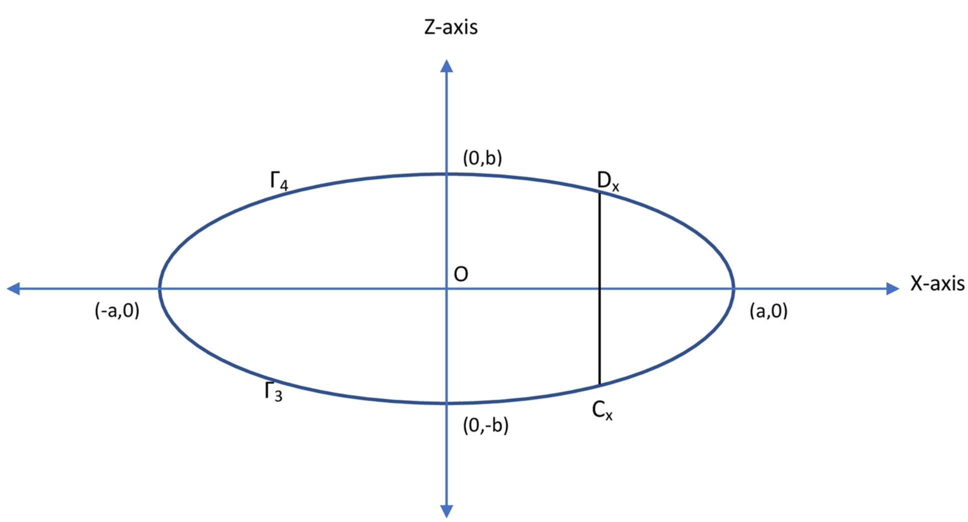

Figure 2: Elliptic domain.

and

$$(\mathcal{B}_n^z G)(x,z) = \begin{cases} \sum_{j=0}^{n} \tilde{q}_{n,j}(x,z) G\left(x, -h(x) + 2\frac{j}{n}h(x)\right), & (x,z) \in E^* \backslash \{(-a,0),(a,0)\} \\ G(a,0) & (a,0) \in E \\ G(-a,0) & (-a,0) \in E, \end{cases} \tag{2.2}$$

where

$$\tilde{p}_{m,i}(x,z) = \frac{\binom{m}{i}(x+g(z))^i (g(z)-x)^{m-i}}{(2g(z))^m}, \quad (x,z) \in E^* \backslash \{(0,-b),(0,b)\} \tag{2.3}$$

$$\tilde{q}_{n,j}(x,z) = \frac{\binom{n}{j}(z+h(x))^j (h(x)-z)^{n-j}}{(2h(x))^n}, \quad (x,z) \in E^* \backslash \{(0,-a),(0,a)\}. \tag{2.4}$$

We emphasize on the interpolation properties, the order of accuracy, and the remainder of the approximation formulas for the constructed operators.

3 Some preliminary results

Definition 3.1. If the operator $\mathcal{B}_m^x$ preserve the monomial of highest degree say k, i.e., $(\mathcal{B}_m^x e_{k0})(x,z) = x^k$, then we say that operator $\mathcal{B}_m^x$ has degree of exactness k. Then we write $\operatorname{dex}(\mathcal{B}_m^x) = k$.

Theorem 3.2. *For any function $G : E^* \to \mathbb{R}$, we have*

$$(i) \quad \mathcal{B}_m^x G = G \quad \text{on } E,$$

$$(ii) \quad (\mathcal{B}_m^x e_{i0})(x,z) = x^i, \quad i = 0, 1 (\operatorname{dex}(\mathcal{B}_m^x) = 1),$$

$$(\mathcal{B}_m^x e_{20})(x,z) = x^2 + \frac{(g(z)^2 - x^2)}{m}$$

$$(\mathcal{B}_m^x e_{ij})(x,z) = \begin{cases} z^j x^i, & i = 0,1, j \in \mathbb{N} \\ z^j\left(x^2 + \frac{(g(z)^2 - x^2)}{m}\right), & i = 2, \ j \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $e_{ij}(x,z) = x^i z^j$ and $\operatorname{dex}(\mathcal{B}_m^x)$ denote the degree of exactness of the operator $\mathcal{B}_m^x$.

Proof. (i) One can easily notice from the basis functions (2.3)

$$\tilde{p}_{m,i}(-g(z),z) = \begin{cases} 1, & \text{for } i = 0, \\ 0, & \text{for } i > 0, \end{cases}$$

and

$$\tilde{p}_{m,i}(g(z),z) = \begin{cases} 1, & \text{for } i = m, \\ 0, & \text{for } i < m. \end{cases}$$

Then one can easily verify with the help of definition of $\mathcal{B}_m^x G$ and aforementioned property of basis function that $\mathcal{B}_m^x G = G$ on $\Gamma_1 \cup \Gamma_2 = E$.

(ii)

$$(\mathcal{B}^x_m e_{00})(x,z) = \sum_{i=0}^m \tilde{p}_{m,i}(x,z) = \sum_{i=0}^m \frac{\binom{m}{i}(x+g(z))^i(g(z)-x)^{m-i}}{(2g(z))^m} = \frac{(x+g(z)+g(z)-x)^m}{(2g(z))^m} = 1.$$

$$\begin{aligned}(\mathcal{B}^x_m e_{10})(x,z) &= \sum_{i=0}^m \tilde{p}_{m,i}(x,z)\Big(-g(z)+\frac{i}{m}2g(z)\Big)\\ &= -g(z)\sum_{i=0}^m \tilde{p}_{m,i}(x,z) + 2g(z)\sum_{i=0}^m \frac{i}{m}\tilde{p}_{m,i}(x,z)\\ &= -g(z) + 2g(z)\sum_{i=0}^m \frac{\frac{i}{m}\binom{m}{i}(x+g(z))^i(g(z)-x)^{m-i}}{(2g(z))^m}\\ &= -g(z) + \frac{2g(z)}{2g(z)}(x+g(z))\sum_{i=1}^m \frac{\binom{m-1}{i-1}(x+g(z))^{i-1}(g(z)-x)^{m-i}}{(2g(z))^{m-1}}\\ &= -g(z) + (x+g(z))\sum_{i=0}^{m-1} \frac{\binom{m-1}{i}(x+g(z))^{i}(g(z)-x)^{m-1-i}}{(2g(z))^{m-1}}\\ &= -g(z)+x+g(z) = x.\end{aligned}$$

$$\begin{aligned}(\mathcal{B}^x_m e_{20})(x,z) &= \sum_{i=0}^m \tilde{p}_{m,i}(x,z)\Big(-g(z)+\frac{i}{m}2g(z)\Big)^2\\ &= \sum_{i=0}^m \frac{\binom{m}{i}(x+g(z))^i(g(z)-x)^{m-i}}{(2g(z))^m}\Big(g(z)^2+4\frac{i^2}{m^2}g(z)^2-4\frac{i}{m}g(z)^2\Big).\end{aligned}$$

Putting $x = -g(z)+2g(z)t$, $t \in [0,1]$,

$$\begin{aligned}(\mathcal{B}^x_m e_{20})(x,z) &= \sum_{i=0}^m \binom{m}{i} t^i(1-t)^{m-i}\Big(g(z)^2+4\frac{i^2}{m^2}g(z)^2-4\frac{i}{m}g(z)^2\Big)\\ &= g(z)^2 + 4g(z)^2\sum_{i=0}^m \frac{i^2}{m^2}\binom{m}{i}t^i(1-t)^{m-i} - 4g(z)^2\sum_{i=0}^m \frac{i}{m}\binom{m}{i}t^i(1-t)^{m-i}\\ &= g(z)^2 + 4g(z)^2\Big((1-\frac{1}{m})t^2+\frac{t}{m}\Big) - 4tg(z)^2\\ &= g(z)^2 + 4g(z)^2\Big(1-\frac{1}{m}\Big)\frac{(x+g(z))^2}{(2g(z))^2} + 4g(z)^2\frac{1}{m}\frac{x+g(z)}{2g(z)} - 4\frac{x+g(z)}{2g(z)}g(z)^2\\ &= g(z)^2 + \Big(1-\frac{1}{m}\Big)(x+g(z))^2 + \frac{2g(z)}{m}(x+g(z)) - 2g(z)(x+g(z))\\ &= (-g(z)+x+g(z))^2 - \frac{1}{m}(x^2-g(z)^2)\\ &= x^2 + \frac{1}{m}(g(z)^2-x^2).\end{aligned}$$

For $i = 0, 1, 2$ and $j \in \mathbb{N}$, we have

$$\begin{aligned}(B^x_m e_{ij})(x,z) &= \sum_{i=0}^m \tilde{p}_{m,i}(x,z)\Big(-g(z)+2\frac{i}{m}g(z)\Big)^i z^j\\ &= z^j\sum_{i=0}^m \tilde{p}_{m,i}(x,z)\Big(-g(z)+2\frac{i}{m}g(z)\Big)^i\\ &= z^j(B^x_m e_{i0})(x,z).\end{aligned}$$

Hence, (3.1) follows immediately. □

In the similar way, the following theorem is easy to prove.

Theorem 3.3. *For any function $G : E^* \to \mathbb{R}$, then*

$$(i) \quad \mathcal{B}_n^z G = G \text{ on } E,$$

$$(ii) \quad (\mathcal{B}_n^z e_{0i})(x,z) = z^i, \quad i = 0, 1(\mathrm{dex}(\mathcal{B}_n^z) = 1),$$

$$(\mathcal{B}_n^z e_{02})(x,z) = z^2 + \frac{(h(x)^2 - z^2)}{n}$$

$$(\mathcal{B}_n^z e_{ij})(x,z) = \begin{cases} x^i z^j, & j = 0, 1, \quad i \in \mathbb{N} \\ x^i\left(z^2 + \dfrac{(h(x)^2 - z^2)}{n}\right), & j = 2, \quad i \in \mathbb{N}, \end{cases}$$

where $e_{ij}(x,z) = x^i z^j$ and $\mathrm{dex}(\mathcal{B}_n^z)$ denote the degree of exactness of the operator $\mathcal{B}_n^z$.

4 Approximation formulae and remainder

Now we consider the approximation formula

$$G = \mathcal{B}_m^x G + \mathcal{R}_m^x G,$$

where $\mathcal{R}_m^x G$ denote the remainder if a function G is approximated by the approximants $\mathcal{B}_m^x G$.

Theorem 4.1. *If $G(.,z) \in C[-g(z), g(z)]$, then*

$$|(\mathcal{R}_m^x G)(x,z)| \le \left(1 + \frac{a}{\delta\sqrt{m}}\right) w(G(.,z);\ \delta), \quad z \in [-b, b], \tag{4.1}$$

where $w(G(.,z);\ \delta)$ represents the modulus of continuity of the function G with respect to the variable x.

Moreover, if $\delta = \frac{1}{\sqrt{m}}$, then

$$|(\mathcal{R}_m^x G)(x,z)| \le (1 + a) w\left(G(.,z);\ \frac{1}{\sqrt{m}}\right), \quad z \in [-b, b]. \tag{4.2}$$

Proof. Since we have by definition of remainder,

$$|(\mathcal{R}_m^x G)(x,z)| \le \sum_{i=0}^{m} \tilde{p}_{m,i}(x,z) |G(x,z) - G\left(-g(z) + \frac{2i}{m} g(z), z\right)|.$$

By using the inequality,

$$\left| G(x,z) - G\left(-g(z) + \frac{2i}{m} g(z), z\right) \right| \le \left(1 + \frac{1}{\delta} \left| x + g(z) - \frac{i}{m} 2g(z) \right|\right) w(G(.,z);\ \delta)$$

$$|(\mathcal{R}_m^x G)(x,z)| \le \sum_{i=0}^{m} \tilde{p}_{m,i}(x,z) \left(1 + \frac{1}{\delta} \left| x + g(z) - \frac{i}{m} 2g(z) \right|\right) w(G(.,z);\ \delta)$$

$$\le \left[1 + \frac{1}{\delta}\left(\sum_{i=0}^{m} \tilde{p}_{m,i}(x,z)(x + g(z) - \frac{i}{m} 2g(z))^2\right)^{\frac{1}{2}}\right] w(f(.,z);\ \delta)$$

$$|(\mathcal{R}_m^x G)(x,z)| \le \left[1 + \frac{1}{\delta}\frac{1}{\sqrt{m}}\sqrt{(g(z)^2 - x^2)}\right] w(G(.,z);\ \delta).$$

Since

$$\max_{-g(z)\leq x\leq g(z)}[g^2(z)-x^2]=g^2(z)\quad \text{and}\quad \max_{-b\leq z\leq b}[g^2(z)]=a^2.$$

Hence,

$$|(\mathcal{R}_m^x G)(x,z)|\leq\left(1+\frac{a}{\delta\sqrt{m}}\right)w(G(.,z);\,\delta).$$

Now, for $\delta=\frac{1}{\sqrt{m}}$, we obtain (4.2). □

Theorem 4.2. *If* $G(.,z)\in C^2[-g(z),g(z)]$, *then*

$$(\mathcal{R}_m^x G)(x,z)=\frac{x^2-g^2(z)}{2m}G^{(2,0)}(\xi,z),\quad \xi\in[-g(z),g(z)],\tag{4.3}$$

for all $z\in[-b,b]$ *and*

$$|(\mathcal{R}_m^x G)(x,z)|\leq\frac{a^2}{2m}M_{20}G,\quad (x,z)\in E^*,\tag{4.4}$$

where

$$M_{ij}G=\max_{E^*}|G^{(i,j)}(x,z)|.$$

Proof. As $\mathrm{dex}(\mathcal{B}_m^x)=1$, by Peano's theorem, one obtains. For more details on remainder calculation, see [5].

$$(\mathcal{R}_m^x G)(x,z)=\int_{-g(z)}^{g(z)}K_{20}(x,z;\,t)G^{(2,0)}(t,z)\mathrm{d}t,$$

where $G^{(2,0)}(t,z)$ denotes the second-order partial derivative with respect to first variable, and kernel

$$K_{20}(x,z;\,t):=\mathcal{R}_m^x[(x-t)_+]=(x-t)_+-\sum_{i=0}^{m}\tilde{p}_{m,i}(x,z)\left(-g(z)+i\frac{2g(z)}{m}-t\right)_+$$

does not change the sign on the interval $[-g(z),g(z)]$, i.e.,

$$K_{20}(x,z;\,t)\leq 0,\quad x\in[-g(z),g(z)].$$

By using the mean value theorem, we obtain

$$(\mathcal{R}_m^x G)(x,z)=G^{(2,0)}(\xi,z)\int_{-g(z)}^{g(z)}K_{20}(x,z;\,t)\mathrm{d}t,\quad \xi\in[-g(z),g(z)].$$

After an easy calculation, we obtain

$$(\mathcal{R}_m^x G)(x,z)=-\left(\frac{g^2(z)-x^2}{2m}\right)G^{(2,0)}(\xi,z),$$

where $\xi\in[-g(z),g(z)]$.

Since

$$\max_{-g(z)\leq x\leq g(z)}[g^2(z)-x^2]\leq a^2.$$

Hence, (4.4) follows immediately. □

Remark 4.3. From Theorem (4.2), for all $x\in[-g(z),g(z)]$ and $z\in[-b,b]$, we have

- If $G(.,z)$ is a concave function, then $(\mathcal{R}_m^x G)(x,z)\geq 0$, i.e., $(\mathcal{B}_m^x G)(x,z)\leq G(x,z)$.

• If $G(.,z)$ is a convex function, then $(\mathcal{R}_m^x G)(x,z) \le 0$, i.e., $(\mathcal{B}_m^x G)(x,z) \ge G(x,z)$.

Remark 4.4. For the remainder $\mathcal{R}_n^z G$ of the approximation formula,

$$G = \mathcal{B}_n^z G + \mathcal{R}_n^z G,$$

A: If $G(x,\cdot) \in C[-h(x), h(x)]$, then

$$|(\mathcal{R}_n^z)(x,z)| \le \left(1 + \frac{b}{\delta\sqrt{n}}\right) w(G(x,.);\delta), \tag{4.5}$$

where $w(G(x,.);\delta)$ is the modulus of continuity of the function G with respect to the variable z.

B: If $G(x,.) \in C^2[-h(x), h(x)]$, then

$$(\mathcal{R}_n^z G)(x,z) = \frac{z^2 - h^2(x)}{2n} G^{(0,2)}(x,\eta), \quad \eta \in [-h(x), h(x)] \tag{4.6}$$

for all $x \in [-a,a]$ and

$$|(\mathcal{R}_n^z G)(x,z)| \le \frac{b^2}{2n} M_{02} G, \quad (x,z) \in E^*,$$

where

$$M_{ij} G = \max_{E^*} |G^{(i,j)}(x,z)|.$$

5 Rate of convergence

Now, we study the rate of convergence of the operators $(\mathcal{B}_m^x G)(x,z)$ with the help of functions of Lipschitz class $\mathrm{Lip}_M(\alpha)$ with respect to first variable x, where $M > 0$ and $0 < \alpha \le 1$.

A function $G(.,z)$ belongs to $\mathrm{Lip}_M(\alpha)$ if

$$|G(x_1,z) - G(x_2,z)| \le M|x_1 - x_2|^\alpha \quad (x_1, x_2, z \in \mathbb{R}). \tag{5.1}$$

We have the following theorem.

Theorem 5.1. *Let $G(.,z) \in \mathrm{Lip}_M(\alpha)$, then we have*

$$|(\mathcal{B}_m^x G)(x,z) - G(x,z)| \le M[g^2(z) - x^2]^{\alpha/2}, \tag{5.2}$$

for all $x \in [-g(z), g(z)]$ and $z \in [-b,b]$.

Proof. Since $(\mathcal{B}_m^x G)(.,z) : C[-g(z), g(z)] \to C[-g(z), g(z)]$ are linear positive operators and $G(.,z) \in \mathrm{Lip}_M(\alpha)$, we have,

$$\begin{aligned}
|(\mathcal{B}_m^x G)(x,z) - G(x,z)| &\le \mathcal{B}_m^x(|G(s,z) - G(x,z)|) \\
&= \sum_{i=0}^{m} \tilde{p}_{m,i}(x,z) \left| G\left(-g(z) + \frac{2i}{m} g(z), z\right) - G(x,z) \right| \\
&\le M \sum_{i=0}^{m} \tilde{p}_{m,i}(x,z) \left| -g(z) + \frac{2i}{m} g(z) - x \right|^\alpha \\
&\le M \sum_{i=0}^{m} \left[\tilde{p}_{m,i}(x,z) \left| -g(z) + \frac{2i}{m} g(z) - x \right|^2 \right]^{\frac{\alpha}{2}} [\tilde{p}_{m,i}(x,z)]^{\frac{2-\alpha}{2}}.
\end{aligned} \tag{5.3}$$

By applying Hölder's inequality for sums, we obtain

$$|(\mathcal{B}_m^x G)(x,z) - G(x,z)| \le M\left[\sum_{i=0}^{m} \tilde{p}_{m,i}(x,z)\left| -g(z) + \frac{2i}{m}g(z) - x\right|^2\right]^{\frac{\alpha}{2}}\left[\sum_{i=0}^{m}\tilde{p}_{m,i}(x,z)\right]^{\frac{2-\alpha}{2}} \tag{5.4}$$
$$= M[(\mathcal{B}_m^x(s-x)^2)(x,z)]^{\frac{\alpha}{2}}.$$

Since

$$(\mathcal{B}_m^x(s-x)^2)(x,z) = g^2(z) - x^2.$$

Hence, we obtain (5.2) and the theorem is proved. □

Remark 5.2. Let $G(x,.) \in \mathrm{Lip}_M(\alpha)$, then we have

$$|(\mathcal{B}_n^z G)(x,z) - G(x,z)| \le M[h^2(x) - z^2]^{\alpha/2},$$

for all $z \in [-h(x), h(x)]$ and $x \in [-a, a]$.

6 Product operators

Let $\mathcal{P}_{mn} = \mathcal{B}_m^x\mathcal{B}_n^z$ and $\mathcal{Q}_{nm} = \mathcal{B}_n^z\mathcal{B}_m^x$ be the products of operators $\mathcal{B}_m^x$ and $\mathcal{B}_n^z$.

We have

$$(\mathcal{P}_{mn}G)(x,z) = \sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(x,z)\tilde{q}_{n,j}(x_i,z)G\left(x_i, -h(x_i) + 2\frac{j}{n}h(x_i)\right), (x,z) \in E^*, \tag{6.1}$$

where $x_i = -g(z) + 2g(z)\frac{i}{m}$, $i = 0, 1, \dots, m$.

The product operator $\mathcal{Q}_{nm}$ is defined by

$$(\mathcal{Q}_{nm}G)(x,z) = \sum_{j=0}^{n}\sum_{j=0}^{n}\tilde{q}_{n,j}(x,z)\tilde{p}_{m,i}(x,z_j)G\left(-g(z_j) + 2\frac{i}{m}g(z_j), z_j\right), (x,z) \in E^*, \tag{6.2}$$

where $z_j = -h(x) + \frac{2j}{n}h(x)$, $j = 0, 1, \dots, n$.

Theorem 6.1. *The product operators $\mathcal{P}_{mn}$ and $\mathcal{Q}_{nm}$ interpolate the function G on boundary of elliptic domain, i.e.,*

$$(\mathcal{P}_{mn}G)(x,z) = G(x,z), \quad \textit{for all } (x,z) \in E.$$

The aforementioned proofs follow from some simple computation.

Let $\mathcal{R}_{mn}^{\mathcal{P}}G$ be the remainder of the approximation formula

$$G = \mathcal{P}_{mn}G + \mathcal{R}_{mn}^{\mathcal{P}}G.$$

One can see that remainders $\mathcal{R}_{mn}^{P}G$ for G on boundary of elliptic region is zero. Hence, we compute bounds for remainders $\mathcal{R}_{mn}^{P}G$ on $E^*\backslash E$.

Theorem 6.2. *If $G \in C(E^*)$, then*

$$|(\mathcal{R}_{mn}^{P}G)(x,z)| \le \left(\frac{1}{\delta_1}\sqrt{\frac{g^2(z) - x^2}{m}} + \frac{1}{\delta_2}\sqrt{\frac{h^2(x) - z^2}{n}} + 1\right)w(G;\, \delta_1, \delta_2). \tag{6.3}$$

Proof. We have

$$
\begin{aligned}
|(\mathcal{R}^{\mathcal{P}}_{mn}G)(x,z)| \leq \Bigg[& \frac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(x,z)\tilde{q}_{n,j}\left(-g(z)+2i\frac{g(z)}{m},z\right)\left| x+g(z)-2i\frac{g(z)}{m}\right| \\
& + \frac{1}{\delta_2}\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(x,z)\tilde{q}_{n,j}\left(-g(z)+2i\frac{g(z)}{m},z\right)\left| z+h(x)-2\frac{j}{n}h(x)\right| \\
& + \sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(x,z)\tilde{q}_{n,j}\left(-g(z)+2i\frac{g(z)}{m},z\right)\Bigg] w(G;\,\delta_1,\delta_2).
\end{aligned}
$$

$$
\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(x,z)\tilde{q}_{n,j}\left(-g(z)+2i\frac{g(z)}{m},z\right)\left| x+g(z)-2i\frac{g(z)}{m}\right| \leq \sqrt{(\mathcal{B}^x_m(s-x)^2)(x,z)},
$$

$$
\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(x,z)\tilde{q}_{n,j}\left(-g(z)+2i\frac{g(z)}{m},z\right)\left| z+h(x)-\frac{j}{n}h(x)\right| \leq \sqrt{(\mathcal{B}^z_n(t-z)^2)(x,z)},
$$

while

$$
\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(x,z)\tilde{q}_{n,j}\left(-g(z)+2i\frac{g(z)}{m},z\right)=1.
$$

It follows,

$$
|(\mathcal{R}^{\mathcal{P}}_{mn}G)(x,z)| \leq \left(\frac{1}{\delta_1}\sqrt{(\mathcal{B}^x_m(s-x)^2)(x,z)}+\frac{1}{\delta_2}\sqrt{(\mathcal{B}^z_n(t-z)^2)(x,z)}+1\right)w(G;\,\delta_1,\delta_2).
$$

Since

$$
(\mathcal{B}^x_m(s-x)^2)(x,z)=\frac{g^2(z)-x^2}{m}
$$

and

$$
(\mathcal{B}^z_n(t-z)^2)(x,z)=\frac{h^2(x)-z^2}{n},
$$

we have,

$$
|(\mathcal{R}^{\mathcal{P}}_{mn}G)(x,z)| \leq \left(\frac{1}{\delta_1}\sqrt{\frac{g^2(z)-x^2}{m}}+\frac{1}{\delta_2}\sqrt{\frac{h^2(x)-z^2}{n}}+1\right)w(G;\,\delta_1,\delta_2).
$$

□

7 Boolean sum operators

Let

$$
\mathcal{S}_{mn} := \mathcal{B}^x_m \oplus \mathcal{B}^z_n = \mathcal{B}^x_m + \mathcal{B}^z_n - \mathcal{B}^x_m\mathcal{B}^z_n,
$$
$$
\mathcal{T}_{nm} := \mathcal{B}^z_n \oplus \mathcal{B}^x_m = \mathcal{B}^z_n + \mathcal{B}^x_m - \mathcal{B}^z_n\mathcal{B}^x_m
$$

be the Boolean sums of the Bernstein-type operators $\mathcal{B}^x_m$ and $\mathcal{B}^z_n$.

Theorem 7.1. *For function $G : E^* \to \mathbb{R}$, we have*

$$
\mathcal{S}_{mn}G\,|_{\partial E^*} = G\,|_{\partial E^*}.
$$

Proof. We have

$$
\mathcal{S}_{mn}G = (\mathcal{B}^x_m + \mathcal{B}^z_n - \mathcal{B}^x_m\mathcal{B}^z_n)G.
$$

The interpolation properties of $\mathcal{B}^x_m$, $\mathcal{B}^z_n$, and $\mathcal{P}_{mn}$ imply that

For all $(x, z) \in E$, we have

$$(\mathcal{S}_{mn}G)(x, z) = (\mathcal{B}_m^x G)(x, z) + (\mathcal{B}_n^z)G(x, z) - \mathcal{B}_m^x(\mathcal{B}_n^z G)(x, z) = G(x, z).$$

Let $\mathcal{R}_{mn}^{S}G$ denote the remainder of the Boolean sum approximation formula,

$$G = \mathcal{S}_{mn}G + \mathcal{R}_{mn}^{S}G.$$

Similarly here remainder $\mathcal{R}_{mn}^{S}G$ on the boundary of elliptic region is zero. Hence, we compute bounds for remainders $\mathcal{R}_{mn}^{S}G$ on $E^*\backslash E$. □

Theorem 7.2. *If $G \in C(E^*)$, then*

$$\begin{aligned}|(\mathcal{R}_{mn}^{S}G)(x, z)| \le &\left(1 + \frac{1}{\delta_1}\sqrt{\frac{g^2(z) - x^2}{m}}\right)w(G(.,z);\delta_1) + \left(1 + \frac{1}{\delta_2}\sqrt{\frac{h^2(x) - z^2}{n}}\right)w(G(x, .);\delta_2)\\ &+ \left(\frac{1}{\delta_1}\sqrt{\frac{g^2(z) - x^2}{m}} + \frac{1}{\delta_2}\sqrt{\frac{h^2(x) - z^2}{n}} + 1\right)w(G;\ \delta_1, \delta_2),\end{aligned} \tag{7.1}$$

for all $(x, z) \in E^*$.

Proof. From the equality,

$$G - \mathcal{S}_{mn}G = G - \mathcal{B}_m^x G + G - \mathcal{B}_n^z - (G - \mathcal{P}_{mn}G),$$

we obtain,

$$|(\mathcal{R}_{mn}^{S}G)(x, z)| \le |(\mathcal{R}_m^x G)(x, z)| + |(\mathcal{R}_n^z G)(x, z)| + |(\mathcal{R}_{mn}^{P}G)(x, y)|.$$

Now, the proof easily follows from idea involved in proof of (4.2), (4.5), and inequality (6.3). □

Remark 7.3. Analoguous results for remainders of the product approximation formula can easily be obtained.

$$G = Q_{nm}G + R_{nm}^{Q}G = \mathcal{B}_n^z\mathcal{B}_m^x G + \mathcal{R}_{nm}^{Q}G$$

and for the Boolean sum formula

$$G = \mathcal{T}_{nm}G + \mathcal{R}_{nm}^{\mathcal{T}}G = (\mathcal{B}_n^z \oplus \mathcal{B}_m^x)G + \mathcal{R}_{nm}^{\mathcal{T}}G.$$

8 Graphical analysis

Let us consider the function $G(x, z) = x\exp(-x^2 - z^2)$ for graphical demonstration defined on elliptic domain $E^* = \{(x, z) \in \mathbb{R}^2 : \frac{x^2}{4} + \frac{z^2}{9} \le 1\}$. We present the graph of the function $f(x, z)$ in Figure 3(a). Figure 3(b)–(e) represents the Bernstein-type operators $\mathcal{B}_m^x G$, $\mathcal{B}_n^z G$, product operator $\mathcal{P}_{mn}G$, and Boolean sum operator $\mathcal{S}_{mn}G$ for $m = n = 5$. One can easily observe that approximation can be made better by increasing the value of m and n. Also notice that from each figure, each operators $\mathcal{B}_m^x G$, $\mathcal{B}_n^z G$, $\mathcal{P}_{mn}G$, and $\mathcal{S}_{mn}G$ is interpolating the given function $f(x, z) = x\exp(-x^2 - z^2)$ on the boundary of elliptic domain $E^* = \{(x, z) \in \mathbb{R}^2 : \frac{x^2}{4} + \frac{z^2}{9} \le 1\}$.

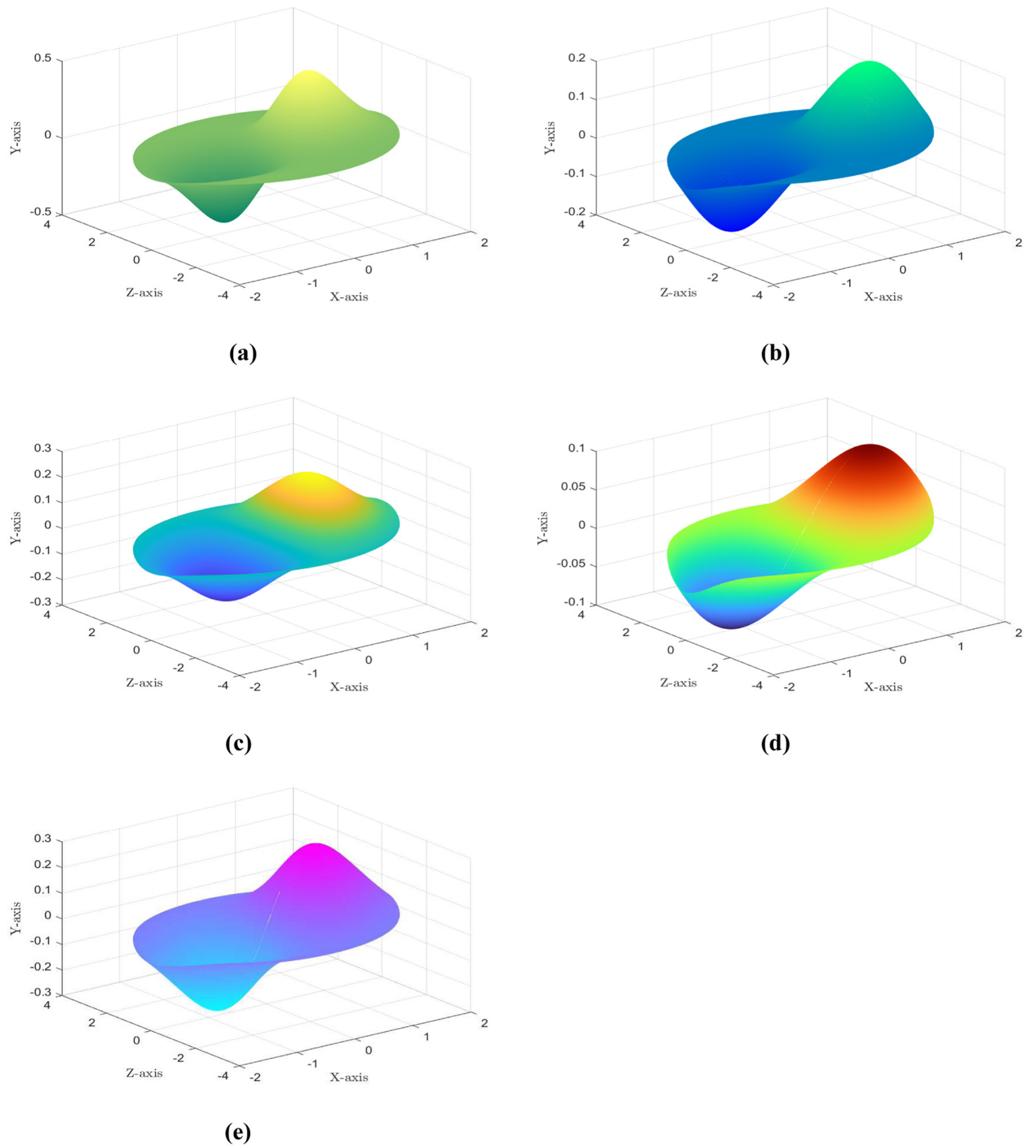

Figure 3: Bernstein operators, product operator and Boolean sum operator approximating function $f(x, z) = x \exp(-x^2 - z^2)$ on elliptic domain. (a) $f(x, z) = x \exp(-x^2 - z^2)$. (b) The operator $\mathcal{B}_m^x G$. (c) The operator $\mathcal{B}_n^z G$. (d) The operator $\mathcal{P}_{mn} G$. (e) The operator $\mathcal{S}_{mn} G$.

Acknowledgement: Not applicable.

Funding information: Not applicable.

Author contributions: All the authors contributed equally and significantly in writing this article.

Conflict of interest: The authors declare that they have no conflict of interest.

Data availability statement: Not applicable.

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