Research Article

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Semi-Hyers-Ulam-Rassias stability for an integro-differential equation of order n

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Abstract: The Laplace transform method is applied in this article to study the semi-Hyers-Ulam-Rassias stability of a Volterra integro-differential equation of order n, with convolution-type kernel. This kind of stability extends the original Hyers-Ulam stability whose study originated in 1940. A general integral equation is formulated first, and then some particular cases (polynomial function and exponential function) for the function from the kernel are considered.

Keywords: Volterra integro-differential equation, Laplace transform, semi-Hyers-Ulam-Rassias stability

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1 Introduction

An important aspect from the qualitative theory of differential and integral equations is the stability of solutions. It is well known that this concept has various meanings in the literature, one of them being the Hyers-Ulam stability. This notion appeared first in connection with homomorphisms, in an open problem formulated by Ulam (see the book [1]), and answered by Hyers in [2]. A vast field of research developed afterward, with many authors generalizing the initial definition of stability (Hyers-Ulam-Rassias stability, generalized Hyers-Ulam-Rassias stability, semi-Hyers-Ulam-Rassias stability, and Mittag-Leffler-Hyers-Ulam stability) and obtaining results for various classes of functional equations (see, for instance, [3–7] and the references therein). Systematic approaches of the subject are given in [8,9].

The study of Hyers-Ulam stability for ordinary differential equations started with the papers of Obloza [10], and Alsina and Ger [11]. Further, many interesting results were obtained for linear differential equations or systems of equations, in papers like [12–17], or for some special classes of differential equations in [18,19]. Among the papers about partial differential equations, we mention [20–24]. An overall view of the subject is provided in the book [25].

The present work is about a class of Volterra integro-differential equations. The study of Hyers-Ulam stability for integral or integro-differential equations is not so extended as in the case of differential equations, but we can mention several papers approaching this problem, by various methods: [26–30].

In this article, we use the Laplace transform method. In the context of Hyers-Ulam stability, this method appeared first for linear differential equations, in the article by Rezaei et al. [31]. Then, the Laplace transform was used to obtain stability results in several other articles: [32] for linear differential equations, [33] for Laguerre differential equation and Bessel differential equation, [34] for the Mittag-Leffler-Hyers-Ulam

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stability of a linear differential equation of first order, [35] for fractional differential equations, and [24] for the convection partial differential equation.

In the following, inspired by [31] and continuing the research from [36], we will study a Volterra integro-differential equation of order n with a convolution type kernel:

$$x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + \cdots + a_0 x(t) + \int_0^t g(t-u)x(u)\mathrm{d}u - f(t) = 0, \tag{1.1}$$

$x \in C^n(0,\infty)$, $a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$, $n \in \mathbb{N}$, $n > 1$.

In the previous article [36], we obtained stability results for the integro-differential equation (1.1) of order I. Those results will be extended here for a class of equations of order n. The outline of this article is the following: In Section 2, we present the stability notion, properties of the Laplace transform, and some auxiliary results. The main results are given in Section 3 and concern the semi-stability of the integro-differential equation (1.1), for some particular cases of the function g.

2 Preliminary notions and results

In the rest of the article, we denote by $\Re(s)$ the real part of a complex number s and by $\mathbb{F}$ the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Throughout the work, we assume that the functions $f, g, x : (0,\infty) \to \mathbb{F}$ are continuous and of the exponential order.

The Laplace transform of the function f is denoted by $\mathcal{L}(f)$ and is defined by

$$\mathcal{L}(f)(s) = F(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t,$$

on the open half plane $\{s \in \mathbb{C} | \Re(s) > \sigma_f\}$, where σ_f is the abscissa of convergence of the function f. The inverse Laplace transform of a function F is denoted by $\mathcal{L}^{-1}(F)$. In the next section of the article, we will use the following auxiliary results, proved in [31].

Lemma 2.1. [31] *Let $P(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ and $Q(s) = b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0$ where m, n are nonnegative integers with $m < n$ and a_i, b_i are scalars, $i \in \{0, 1, \ldots, n\}$. Then there exists an infinitely differentiable function $g : (0,\infty) \to \mathbb{F}$ such that*

$$\mathcal{L}(g) = \frac{Q(s)}{P(s)}, \quad \Re(s) > \sigma_P$$

and

$$g^{(k)}(0) = \begin{cases} 0, & k \in \{0, 1, \ldots, n-m-2\} \\ \dfrac{b_m}{a_n}, & k = n-m-1, \end{cases}$$

where $\sigma_P = \max\{\Re(s) : P(s) = 0\}$.

Lemma 2.2. [31] *Given an integer $n > 1$, let $f : (0,\infty) \to \mathbb{F}$ be a continuous function and let $P(s)$ be a complex polynomial of degree n. Then there exists an n times differentiable function $h : (0,\infty) \to \mathbb{F}$ such that*

$$\mathcal{L}(h) = \frac{\mathcal{L}(f)}{P(s)}, \quad \Re(s) > \max\{\sigma_P, \sigma_f\},$$

where $\sigma_P = \max\{\Re(s) : P(s) = 0\}$ and σ_f is the abscissa of convergence for f. It holds that $h^{(k)}(0) = 0$ for every $k \in \{0, 1, \ldots, n-1\}$.

In the rest of the article, we write $x(0), x'(0), \dots, x^{(n)}(0)$ instead of the lateral limits $x(0^+), x'(0^+), \dots, x^{(n)}(0^+)$, respectively.

Let $\varepsilon > 0$ and consider the inequality

$$|x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + \dots + a_0 x(t) + \int_0^t g(t-u)x(u)\mathrm{d}u - f(t)| \le \varepsilon, \tag{2.1}$$

$t \in (0, \infty)$.

We say, as in [27], that equation (1.1) is *semi-Hyers-Ulam-Rassias stable*, if there exists a real number $c > 0$ and a function $k : (0, \infty) \to (0, \infty)$ such that for each x that verifies the inequality (2.1), there exists a solution x_0 of the equation (1.1) with

$$|x(t) - x_0(t)| \le c \cdot k(t), \quad \forall t \in (0, \infty). \tag{2.2}$$

3 Stability results

We prove the stability results for the solution of equation (1.1) in some particular cases for the function g.

Theorem 3.1. *Let $g : (0, \infty) \to \mathbb{F}$, $g(t) = t^m$, $m \in \mathbb{N}$. Then, for every function $x : (0, \infty) \to \mathbb{F}$ satisfying the inequality* (2.1), *for all $t \in (0, \infty)$ and some $\varepsilon > 0$, there exists a solution $x_0 : (0, \infty) \to \mathbb{F}$ of equation* (1.1) *such that*

$$|x(t) - x_0(t)| \le \frac{\varepsilon c}{\alpha}(e^{\alpha t} - 1), \quad \forall t \in (0, \infty),$$

for all $\alpha > \max\{0, \sigma, \sigma_f\}$, where

$$c = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{(\alpha + \beta i)^{m+1}}{(\alpha + \beta i)^{n+m+1} + \dots + a_0(\alpha + \beta i)^{m+1} + m!}\right|\mathrm{d}\beta,$$

and σ is defined in (3.2).

Proof. Let $p : (0, \infty) \to \mathbb{F}$,

$$p(t) = x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + \dots + a_0 x(t) + \int_0^t g(t-u)x(u)\mathrm{d}u - f(t), \quad t \in (0, \infty). \tag{3.1}$$

The Laplace transform of the function p is

$$\begin{aligned}\mathcal{L}(p) = {} & (s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0)\mathcal{L}(x) - (s^{n-1} + a_{n-1}s^{n-2} + \dots + a_1)x(0) - (s^{n-2} + a_{n-1}s^{n-3} + \dots + a_2)x'(0) \\ & - \dots - (s + a_{n-1})x^{(n-2)}(0) - x^{(n-1)}(0) + \mathcal{L}(x)\cdot \mathcal{L}(g) - \mathcal{L}(f).\end{aligned}$$

Denoting

$$P_{n,i} = s^{n-i} + a_{n-1}s^{n-i-1} + \dots + a_{i+1}s + a_i, \quad \text{for } i \in \{0, 1, \dots, n\},$$

we can write

$$\mathcal{L}(p) = P_{n,0}(s)\mathcal{L}(x) - \sum_{i=1}^{n} P_{n,i}(s)x^{(i-1)}(0) + \mathcal{L}(x)\cdot \mathcal{L}(g) - \mathcal{L}(f).$$

We obtain

$$\mathcal{L}(x) = \frac{1}{P_{n,0}(s) + \mathcal{L}(g)}\sum_{i=1}^{n} P_{n,i}(s)x^{(i-1)}(0) + \frac{1}{P_{n,0}(s) + \mathcal{L}(g)}\mathcal{L}(f) + \frac{1}{P_{n,0}(s) + \mathcal{L}(g)}\mathcal{L}(p).$$

If $g(t) = t^m$, then the Laplace image is $\mathcal{L}(g) = \frac{m!}{s^{m+1}}$. Hence,

$$P_{n,0}(s) + \mathcal{L}(g) = \frac{s^{n+m+1} + a_{n-1}s^{n+m} + \cdots + a_0 s^{m+1} + m!}{s^{m+1}},$$

so we obtain

$$\begin{aligned}\mathcal{L}(x) = {} & \frac{s^{m+1}}{s^{n+m+1} + a_{n-1}s^{n+m} + \cdots + a_0 s^{m+1} + m!} \sum_{i=1}^{n} P_{n,i}(s) x^{(i-1)}(0) \\ & + \frac{s^{m+1}}{s^{n+m+1} + a_{n-1}s^{n+m} + \cdots + a_0 s^{m+1} + m!} \mathcal{L}(f) \\ & + \frac{s^{m+1}}{s^{n+m+1} + a_{n-1}s^{n+m} + \cdots + a_0 s^{m+1} + m!} \mathcal{L}(p).\end{aligned}$$

For each $i \in \{0, 1, \ldots, n\}$, we have

$$\begin{aligned}\frac{s^{m+1} P_{n,i}(s)}{s^{n+m+1} + a_{n-1}s^{n+m} + \cdots + a_0 s^{m+1} + m!} &= \frac{s^{n+m+1-i} + a_{n-1}s^{n+m-i} + \cdots + a_i s^{m+1}}{s^{n+m+1} + a_{n-1}s^{n+m} + \cdots + a_0 s^{m+1} + m!} \\ &= \frac{1}{s^i} - \frac{a_{i-1}s^{m+i} + \cdots + a_0 s^{m+1} + m!}{s^i(s^{n+m+1} + a_{n-1}s^{n+m} + \cdots + a_0 s^{m+1} + m!)}.\end{aligned}$$

Let $s_1, s_2, \ldots, s_{n+m+1}$ be the roots of the polynomial $s^{n+m+1} + a_{n-1}s^{n+m} + \cdots + a_0 s^{m+1} + m!$, and let

$$\sigma = \max\{\Re(s_j) | j \in \{1, 2, \ldots, n + m + 1\}\}. \tag{3.2}$$

Let σ_f be the abscissa of convergence for f.

By using Lemma 2.1, it follows that there exists an infinitely differentiable function h_i such that

$$\mathcal{L}(h_i) = \frac{a_{i-1}s^{m+i} + \cdots + a_0 s^{m+1} + m!}{s^i(s^{n+m+1} + a_{n-1}s^{n+m} + \cdots + a_0 s^{m+1} + m!)},$$

for every s with $\Re(s) > \max\{0, \sigma\}$, and

$$h_i(0) = h_i'(0) = \cdots = h_i^{(n-1)}(0),$$

for any $i \in \{0, 1, \ldots, n\}$. We also have

$$\mathcal{L}^{-1}\left(\frac{1}{s^i}\right) = \frac{t^{i-1}}{(i-1)!}.$$

Let us define

$$f_i(t) = \frac{t^{i-1}}{(i-1)!} - h_i(t), \quad i \in \{0, 1, \ldots, n\}.$$

We notice that

$$f_i^{(k)}(t) = \begin{cases} 0, k \neq i - 1 \\ 1, k = i - 1 \end{cases}, \quad k \in \{0, 1, \ldots, n - 1\}.$$

By Lemma 2.1, there exists an infinitely differentiable function z such that

$$\mathcal{L}(z) = \frac{s^{m+1}}{s^{n+m+1} + a_{n-1}s^{n+m} + \cdots + a_0 s^{m+1} + m!},$$

for every s with $\Re(s) > \max\{0, \sigma\}$, and $z(0) = z'(0) = \cdots = z^{(n-2)}(0)$.

If we define f_0 as a convolution product, $f_0 = f * z$, then

$$\mathcal{L}(f_0) = \mathcal{L}(f * z) = \mathcal{L}(f)\mathcal{L}(z),$$

$$f_0(t) = \int_0^t f(u)\cdot z(t-u)\mathrm{d}u,$$

and from here $f_0(0) = 0$. Also

$$f_0'(t) = \underbrace{z(0)}_{0} f(t) + \int_0^t f(u)\cdot z'(t-u)\mathrm{d}u = \int_0^t f(u)\cdot z'(t-u)\mathrm{d}u,$$

and from here, $f_0'(0) = 0$. Analog, $f_0''(0) = \cdots = f_0^{(n-2)}(0) = 0$. But

$$f_0^{(n-2)}(t) = \int_0^t f(u)\cdot z^{(n-2)}(t-u)\mathrm{d}u,$$

and from here,

$$f_0^{(n-1)}(t) = \underbrace{z^{(n-2)}(0)}_{0} f(t) + \int_0^t f(u)\cdot z^{(n-1)}(t-u)\mathrm{d}u,$$

hence, $f_0^{(n-1)}(0) = 0$. In conclusion,

$$f_0^{(k)}(0) = 0, \quad \forall k \in \{0, 1, \ldots, n-1\}.$$

Let us define

$$x_0(t) = \sum_{i=1}^{n} x^{(i-1)}(0) f_i(t) + f_0(t).$$

We have $x_0^{(k)}(0) = x^{(k)}(0), \quad \forall k \in \{0, 1, \ldots, n-1\}$ and

$$\mathcal{L}(x_0) = \frac{s^{m+1}}{s^{n+m+1} + a_{n-1}s^{n+m} + \cdots + a_0 s^{m+1} + m!} \sum_{i=1}^{n} P_{n,i}(s) x_0^{(i-1)}(0)$$
$$+ \frac{s^{m+1}}{s^{n+m+1} + a_{n-1}s^{n+m} + \cdots + a_0 s^{m+1} + m!} \mathcal{L}(f),$$

for every s with $\Re(s) > \max\{0, \sigma, \sigma_f\}$.

We obtain

$$\mathcal{L}(x) - \mathcal{L}(x_0) = \frac{s^{m+1}}{s^{n+m+1} + a_{n-1}s^{n+m} + \cdots + a_0 s^{m+1} + m!} \mathcal{L}(p).$$

Since the Laplace transform of the function $z : (0, \infty) \to \mathbb{F}$ is

$$\mathcal{L}(z) = \frac{s^{m+1}}{s^{n+m+1} + a_{n-1}s^{n+m} + \cdots + a_0 s^{m+1} + m!},$$

we obtain

$$\mathcal{L}(x) - \mathcal{L}(x_0) = \mathcal{L}(z)\mathcal{L}(p) = \mathcal{L}(z * p).$$

Hence,

$$|x(t) - x_0(t)| = |z * p(t)| \le \int_0^t |z(t-u)|\cdot|p(u)|\mathrm{d}u \le \varepsilon \int_0^t |z(t-u)|\mathrm{d}u.$$

Further on, from the formula for the inverse Laplace transform follows

$$
\begin{aligned}
|z(t-u)| &= \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(\alpha+\beta i)(t-u)}(\alpha+\beta i)^{m+1}}{(\alpha+\beta i)^{n+m+1} + \cdots + a_0(\alpha+\beta i)^{m+1} + m!} \right| \mathrm{d}\beta \\
&\le \frac{1}{2\pi} e^{\alpha(t-u)} \int_{-\infty}^{\infty} \left| \frac{(\alpha+\beta i)^{m+1}}{(\alpha+i\beta)^{n+m+1} + \cdots + a_0(\alpha+\beta i)^{m+1} + m!} \right| \mathrm{d}\beta \\
&\le c e^{\alpha(t-u)},
\end{aligned}
$$

for all $\alpha > \max\{0, \sigma, \sigma_f\}$, where

$$
c = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{(\alpha+i\beta)^{m+1}}{(\alpha+\beta i)^{n+m+1} + \cdots + a_0(\alpha+\beta i)^{m+1} + m!} \right| \mathrm{d}\beta < \infty,
$$

since $n \in \mathbb{N}$, $n > 1$. It follows that

$$
|x(t) - x_0(t)| \le \varepsilon c \int_0^t e^{\alpha(t-u)} \mathrm{d}u = \frac{\varepsilon c e^{\alpha t}}{-\alpha}[e^{-\alpha t} - 1] = \frac{\varepsilon c}{\alpha}(e^{\alpha t} - 1).
$$

□

Example 3.1. Let us consider $m = 0$, $n = 2$ and the following equation:

$$
x''(t) + 3x'(t) + 3x(t) + \int_0^t x(u)\mathrm{d}u = t. \tag{3.3}
$$

Consider also $\varepsilon > 0$ and the inequality

$$
\left| x''(t) + 3x'(t) + 3x(t) + \int_0^t x(u)\mathrm{d}u - t \right| \le \varepsilon. \tag{3.4}
$$

By applying Theorem 3.1, we obtain that for every function $x : (0, \infty) \to \mathbb{F}$ satisfying the inequality (3.4), there exists a solution $x_0 : (0, \infty) \to \mathbb{F}$ of equation (3.3) such that

$$
|x(t) - x_0(t)| \le \frac{\varepsilon}{2\alpha}(e^{\alpha t} - 1), \quad \forall t \in (0, \infty), \quad \forall \alpha > 0.
$$

Indeed, we have

$$
\frac{s^{m+1}}{a_n s^{n+m+1} + a_{n-1}s^{n+m} + \cdots + a_0 s^{m+1} + m!} = \frac{s}{s^3 + 3s^2 + 3s + 1} = \frac{s}{(s+1)^3}
$$

and

$$
\begin{aligned}
c &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\alpha+\beta i}{(\alpha+1+\beta i)^3} \right| \mathrm{d}\beta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{\alpha^2+\beta^2}}{(\sqrt{(\alpha+1)^2+\beta^2})^3} \mathrm{d}\beta \le \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(\alpha+1)^2+\beta^2} \mathrm{d}\beta \\
&= \frac{1}{2\pi} \frac{1}{\alpha+1} \pi < \frac{1}{2}, \quad \text{for all } \alpha > \max\{0, \sigma\} = 0,
\end{aligned}
$$

since

$$
\sigma = \max\{\Re(s_j) | j \in \{1, 2, 3\}\} = -1,
$$

s_1, s_2, s_3 being the roots of the polynomial $s^3 + 3s^2 + 3s + 1$.

Hence, by using Theorem 3.1, we obtain

$$
|x(t) - x_0(t)| \le \frac{\varepsilon c}{\alpha}(e^{\alpha t} - 1) \le \frac{\varepsilon}{2\alpha}(e^{\alpha t} - 1), \quad \forall t \in (0, \infty), \quad \forall \alpha > 0.
$$

As in [31], a Corollary of Theorem 3.1 can be formulated.

Corollary 3.1. *Let $g(t) = t^m$, for every $t \in (0, \infty)$. There exists $c > 0$ such that, for every $x : (0, \infty) \to \mathbb{F}$ verifying inequality* (2.1), *for all $t \in (0, \infty)$ and some $\varepsilon > 0$, there exists a solution $x_0 : (0, \infty) \to \mathbb{F}$ of the equation* (1.1) *such that*

$$|x(t) - x_0(t)| \le \varepsilon cet, \quad \begin{cases} \text{for all} \quad t \in (0, \infty), \quad \text{if} \ \max\{0, \sigma, \sigma_f\} = 0 \\ \text{or} \\ \text{for all } t \in \left(0, \dfrac{1}{\max\{0, \sigma, \sigma_f\}}\right) \quad \text{if} \ \max\{0, \sigma, \sigma_f\} > 0. \end{cases}$$

Proof. Theorem 3.1 yields

$$|x(t) - x_0(t)| \le \frac{\varepsilon c}{\alpha}(e^{\alpha t} - 1) \le \frac{\varepsilon c}{\alpha} e^{\alpha t},$$

for all $t > 0$ and $\alpha > \max\{0, \sigma, \sigma_f\}$.

As a function of α, the expression $\frac{e^{\alpha t}}{\alpha}$ attains its minimum for $\alpha = \frac{1}{t}$. Replacing α by $\frac{1}{t}$ in the aforementioned inequality, we obtain

$$|x(t) - x_0(t)| \le \varepsilon cte.$$

If $\max\{0, \sigma, \sigma_f\} = 0$, then the inequality holds for any $t > 0$. If $\max\{0, \sigma, \sigma_f\} > 0$, since $\frac{1}{t} = \alpha > \max\{0, \sigma, \sigma_f\}$, the inequality holds only for $t < \frac{1}{\max\{0, \sigma, \sigma_f\}}$. □

When the function g is an exponential function, we obtain the following result.

Theorem 3.2. *Let $g : (0, \infty) \to \mathbb{F}$, $g(t) = e^{\omega t}$, with $\omega \in \mathbb{R}$. Then, for every function $x : (0, \infty) \to \mathbb{F}$ satisfying the inequality* (2.1), *for all $t \in (0, \infty)$ and some $\varepsilon > 0$, there exists a solution $x_0 : (0, \infty) \to \mathbb{F}$ of the equation* (1.1) *such that*

$$|x(t) - x_0(t)| \le \frac{\varepsilon c}{\alpha}(e^{\alpha t} - 1), \quad \forall t \in (0, \infty),$$

for all $\alpha > \max\{0, \sigma, \sigma_f\}$, where

$$c = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\alpha + \beta i - \omega}{(\alpha + \beta i - \omega)((\alpha + \beta i)^n + \dots + a_1(\alpha + \beta i) + a_0) + 1} \right| \mathrm{d}\beta,$$

and σ is defined in (3.5).

Proof. In the same way and with the same notation as in Theorem 3.1, we obtain

$$\mathcal{L}(x) = \frac{1}{P_{n,0}(s) + \mathcal{L}(g)} \sum_{i=1}^{n} P_{n,i}(s) x^{(i-1)}(0) + \frac{1}{P_{n,0}(s) + \mathcal{L}(g)} (\mathcal{L}(f) + \mathcal{L}(p)).$$

For $g(t) = e^{\omega t}$, the Laplace image is $\mathcal{L}(g) = \frac{1}{s - \omega}$. Hence,

$$P_{n,0}(s) + \mathcal{L}(g) = \frac{(s - \omega)P_{n,0}(s) + 1}{s - \omega},$$

so we obtain

$$\mathcal{L}(x) = \frac{s - \omega}{(s - \omega)P_{n,0}(s) + 1} \sum_{i=1}^{n} P_{n,i}(s) x^{(i-1)}(0) + \frac{s - \omega}{(s - \omega)P_{n,0}(s) + 1} \mathcal{L}(f) + \frac{s - \omega}{(s - \omega)P_{n,0}(s) + 1} \mathcal{L}(p).$$

For each $i \in \{0, 1, \dots, n\}$, we have

$$\frac{(s - \omega)P_{n,i}(s)}{(s - \omega)P_{n,0}(s) + 1} = \frac{1}{s^i} + \frac{(s - \omega)[s^i P_{n,i}(s) - P_{n,0}(s)] - 1}{s^i[(s - \omega)P_{n,0}(s) + 1]} = \frac{1}{s^i} - \frac{(s - \omega)(a_0 + a_1 s + \dots + a_{i-1} s^{i-1}) + 1}{s^i[(s - \omega)P_{n,0}(s) + 1]}.$$

Let $s_1, s_2, \dots, s_{n+1}$ be the roots of the polynomial $(s-\omega)P_{n,0}(s)+1$, and let

$$\sigma = \max\{\Re(s_j) | j \in \{1,2,\dots,n+1\}\}. \tag{3.5}$$

According to Lemma 2.1, there exists an infinitely differentiable function h_i such that

$$\mathcal{L}(h_i) = \frac{(s-\omega)(\alpha_0+\alpha_1 s + \dots + \alpha_{i-1}s^{i-1})+1}{s^i[(s-\omega)P_{n,0}(s)+1]},$$

for every s with $\Re(s) > \max\{0,\sigma\}$, and

$$h_i(0) = h_i'(0) = \dots = h_i^{(n-1)}(0),$$

for any $i \in \{0,1,\dots,n\}$. We also have

$$\mathcal{L}^{-1}\left(\frac{1}{s^i}\right) = \frac{t^{i-1}}{(i-1)!},$$

so we define

$$f_i(t) = \frac{t^{i-1}}{(i-1)!} - h_i(t), \quad i \in \{0,1,\dots,n\}.$$

We notice that

$$f_i^{(k)}(t) = \begin{cases} 0, & k \neq i-1 \\ 1, & k = i-1 \end{cases}, \quad k \in \{0,1,\dots,n-1\}.$$

Using again Lemma 2.1, there exists an infinitely differentiable function z such that

$$\mathcal{L}(z) = \frac{s-\omega}{(s-\omega)P_{n,0}(s)+1},$$

for every s with $\Re(s) > \max\{0,\sigma\}$, and $z(0) = z'(0) = \dots = z^{(n-2)}(0)$.

Let

$$f_0(t) = \int_0^t f(u)\cdot z(t-u).\,\mathrm{d}u,$$

In the same way as in Theorem 3.1, we have that $\mathcal{L}(f_0) = \mathcal{L}(f)\mathcal{L}(z)$, and

$$f_0(0) = f_0'(0) = \dots = f_0^{(n-1)}(0) = 0.$$

Let us define

$$x_0(t) = \sum_{i=1}^{n} x^{(i-1)}(0) f_i(t) + f_0(t).$$

We have $x_0^{(i)}(0) = x^{(i)}(0), \forall i \in \{0,1,\dots,n-1\}$ and

$$\mathcal{L}(x_0) = \frac{s-\omega}{(s-\omega)P_{n,0}(s)+1}\sum_{i=1}^{n} P_{n,i}(s)x^{(i-1)}(0) + \frac{s-\omega}{(s-\omega)P_{n,0}(s)+1}\mathcal{L}(f)$$

for every s with $\Re(s) > \max\{0,\sigma,\sigma_f\}$.

We obtain

$$\mathcal{L}(x) - \mathcal{L}(x_0) = \frac{s-\omega}{(s-\omega)P_{n,0}(s)+1}\mathcal{L}(p) = \mathcal{L}(z)\mathcal{L}(p) = \mathcal{L}(z^* p).$$

Hence,

$$|x(t) - x_0(t)| = |z * p(t)| \le \varepsilon \int_0^t |z(t-u)|\mathrm{d}u.$$

But

$$
\begin{aligned}
|z(t-u)| &= \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(\alpha+\beta i)(t-u)}(\alpha+\beta i-\omega)}{(\alpha+\beta i-\omega)((\alpha+\beta i)^n + \dots + a_1(\alpha+\beta i)+a_0)+1} \right| d\beta \\
&\leq \frac{1}{2\pi} e^{\alpha(t-u)} \int_{-\infty}^{\infty} \left| \frac{\alpha+\beta i-\omega}{(\alpha+\beta i-\omega)((\alpha+\beta i)^n + \dots + a_0)+1} \right| d\beta \\
&\leq c e^{\alpha(t-u)},
\end{aligned}
$$

for all $\alpha > \max\{0, \sigma, \sigma_f\}$, where

$$
c = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\alpha+\beta i-\omega}{(\alpha+\beta i-\omega)((\alpha+\beta i)^n + \dots + a_1(\alpha+\beta i)+a_0)+1} \right| d\beta < \infty,
$$

since $n \in \mathbb{N}$, $n > 1$. It follows that

$$
|x(t) - x_0(t)| \leq \varepsilon c \int_0^t e^{\alpha(t-u)} du = \frac{\varepsilon c e^{\alpha t}}{-\alpha}[e^{-\alpha t} - 1] = \frac{\varepsilon c}{\alpha}(e^{\alpha t} - 1). \qquad \square
$$

A consequence similar to Corollary 3.1 can be also formulated in this case.

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