Research Article

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The asymptotic behaviors of solutions for higher-order (m_1, m_2)-coupled Kirchhoff models with nonlinear strong damping

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Abstract: The Kirchhoff model is derived from the vibration problem of stretchable strings. This article focuses on the long-time dynamics of a class of higher-order coupled Kirchhoff systems with nonlinear strong damping. The existence and uniqueness of the solutions of these equations in different spaces are proved by prior estimation and the Faedo-Galerkin method. Subsequently, the family of global attractors of these problems is proved using the compactness theorem. In this article, we systematically propose the definition and proof process of the family of global attractors and enrich the related conclusions of higher-order coupled Kirchhoff models. The conclusions lay a theoretical foundation for future practical applications.

Keywords: higher-order coupled Kirchhoff system, nonlinear strong damping, global attractor family

MSC 2020: 35B41, 35G31

1 Introduction

In this study, we consider the dynamic behavior of the following higher-order coupled Kirchhoff models in a bounded smooth domain $\Omega \subset R^n$:

$$\begin{cases} u_{tt} + N_1\left(\|\nabla^{m_1}u\|^2\right)(-\Delta)^{m_1}u_t + M\left(\|\nabla^{m_1}u\|^2 + \|\nabla^{2m_2}v\|^2\right)(-\Delta)^{m_1}u + g_1(u,v) = f_1(x), \\ v_{tt} + N_2\left(\|\nabla^{2m_2}v\|^2\right)(-\Delta)^{2m_2}v_t + M\left(\|\nabla^{m_1}u\|^2 + \|\nabla^{2m_2}v\|^2\right)(-\Delta)^{2m_2}v + g_2(u,v) = f_2(x), \end{cases} \tag{1}$$

under the following boundary conditions:

$$\begin{aligned} u(x) = 0, \quad \frac{\partial^i u}{\partial \mathbf{n}^i} = 0,\ i = 1, \ldots, m_1 - 1,\ m_1 > 1 \\ v(x) = 0, \quad \frac{\partial^j v}{\partial \mathbf{n}^j} = 0,\ j = 1, \ldots, 2m_2 - 1,\ m_2 > 1, \end{aligned} \tag{2}$$

and the following initial conditions:

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \Omega, \tag{3}$$

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where Δ is the Laplace operator, N_1, N_2 and M_1, M_2 are scalar functions specified later, g_1, g_2 are the given source terms, and f_1, f_2 are the given functions.

(1) is a set of important generalized higher-order quasi-linear wave equations. The proposed equation in this article originated from Kirchhoff's vibration problem of stretchable strings in 1883:

$$\rho h \frac{\partial^2 u}{\partial t^2} = \left\{ p_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 \mathrm{d}x \right\} \frac{\partial^2 u}{\partial t^2}, \tag{4}$$

where $0 < x < L, t \geq 0, u = u(x, t)$ is the lateral displacement at space coordinate x and time coordinate t, E represents the Young's modulus, ρ represents the mass density, h represents the cross-sectional area, L represents the length, and p_0 represents the axial tension of the accident. The long-time behavior of various forms of Kirchhoff equations have attracted the attention of many scholars in recent decades, and abundant research results have been produced [1–13].

Chueshov [1] studied the well-posedness and long-time dynamical behavior of the following Kirchhoff equation with a nonlinear strong damping term:

$$u_{tt} + \sigma(\|\nabla u\|^2)\Delta u_t - \phi(\|\nabla u\|^2)\Delta u + f(u) = h(x). \tag{5}$$

Lin et al. [2] studied the global dynamics of the following generalized nonlinear Kirchhoff-Boussinesq equations with a strong damping:

$$u_{tt} + \alpha u_t - \beta \Delta u_t + \Delta^2 u = \operatorname{div}(g(|\nabla u|^2)\nabla u) + \Delta h(u) + f(x). \tag{6}$$

This article proved that the semi-group conformed to the squeezing property and demonstrated the existence of the exponential attractor of the system. Then, the spectral interval theory was used to prove that the system had an inertial manifold.

Ghisi and Gobbino [3] studied the global and local existence of solutions to the following Kirchhoff model with strong damping:

$$u_{tt}(t) + 2\delta A^{\sigma} u_t(t) + M(|A^{1/2}u(t)|^2)Au(t) = 0. \tag{7}$$

Nakao [4] proved the initial-boundary value problem of the quasi-linear Kirchhoff-type wave equation with standard dissipation u_t:

$$u_{tt} - (1 + \|\nabla u(t)\|_2^2)\Delta u + u_t + g(x, u) = f(x). \tag{8}$$

With the advance of research, scholars began to turn their attention to the dynamics of the higher-order Kirchhoff equations. Ye and Tao [14] studied the initial-boundary value problem of the following kind of higher-order Kirchhoff-type equation with a nonlinear dissipation term:

$$u_{tt} + \Phi(\|D^m u\|^2)(-\Delta)^m u + a|u_t|^{q-2}u_t = b|u|^{r-2}u. \tag{9}$$

Lin and Zhu [15] studied the initial and boundary value problems of the following nonlinear nonlocal higher-order Kirchhoff-type equations:

$$u_{tt} + M(\|D^m u\|^2)(-\Delta)^m u + \beta(-\Delta)^m u_t + g(x, u_t) = f(x). \tag{10}$$

This study demonstrated the existence and uniqueness of the solution, proved the existence of a global attractor family of the problem through the compact method, and obtained the finite Hausdorff and Fractal dimensions.

Originated from physics, system coupling is a measure where two entities depend on each other. With suitable conditions or parameters, a connected system is coupled, and the potential energy of the system can enable the combination of structural functions of different systems and generate new functions. As a mathematical equation derived from physics, the Kirchhoff model is favorable for considering coupled system. Scholars gradually considered the dynamics of coupled Kirchhoff equations. For example, Wang and Zhang [16] studied the long-time dynamics problem of a class of coupled beam equations with strong damping under nonlinear boundary conditions. Lin and Zhang [17] studied the initial-boundary value problem of the following Kirchhoff coupling group with strong damping and source terms:

$$\begin{cases} u_{tt} - \beta\Delta u_t - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta u + g_1(u,v) = f_1(x), \\ v_{tt} - \beta\Delta v_t - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta v + g_2(u,v) = f_2(x). \end{cases} \tag{11}$$

The finite Hausdorff dimension of the global attractor can be obtained in [17].

In recent years, Lin et al. [18–20] focused on the dynamics of a class of higher-order coupled Kirchhoff equations and obtained a series of ideal results.

At present, few articles focus on the higher-order coupled Kirchhoff problems, and the problem of higher-order beam-plate coupled with nonlinear strong damping has not been studied. The main difficulty lies in the estimation and processing of the harmonic term and the nonlinear damping term and the nonlinear damping when proving the uniqueness. Therefore, under reasonable assumptions, this article overcomes these difficulties by using Holder's inequality, Young's inequality, Poincare inequality, and Gagliardo-Nirenberg inequality and obtains the global solution of the problem and the family of global attractors. This study could refine the definition and existence theorem of the family of global attractors. The conclusions could fill the gap of the family of global attractors of higher-order coupled models (regardless of whether m_1 is equal to m_2) and lay the foundation for subsequent engineering applications.

This article is organized as follows. Section 2 presents the fundamentals for this work. Section 3 states and proves the main results. Finally, conclusions of this article are presented in Section 4.

2 Preparatory knowledge

In this article, $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and the inner product in $H = L^2(\Omega)$. Let $H_0^1 = D((-\Delta)^{\frac{1}{2}})$ be the scale of the Hilbert space generated by the Laplacian with Dirichlet boundary condition on H and endowed with standard inner product and norm, respectively, $(\cdot,\cdot)_{H_0^1} = ((-\Delta)^{\frac{1}{2}}\cdot, (-\Delta)^{\frac{1}{2}}\cdot)$ and $\|\cdot\|_{H_0^1} = \|(-\Delta)^{\frac{1}{2}}\cdot\|$. The main goal is to study the well-posedness and long-time dynamics of problems (1) to (3) under the following set of hypotheses:

(A1). $M(s)$ is a continuous function on intervals $[0, +\infty)$, $M(s) \in C^1(R^+)$, and

(1) $M'(s) \geq 0$,

(2) $M(0) \equiv M_0 > 0$.

(A2). For any $u, v \in H$, let

$$J(u,v) = \int_\Omega [G_1(u,v) + G_2(u,v)]\mathrm{d}x,$$

where $G_1(u,v) = \int_0^u g_1(s,v)\mathrm{d}s$, $G_2(u,v) = \int_0^v g_2(u,s)\mathrm{d}s$, then for any $\mu \geq 0$, there exists $C_1 \geq 0$, $C_\mu \geq 0$, $C'_\mu \geq 0$ that

$$\begin{aligned} &G_1(u,v) + G_2(u,v) - C_1 J(u,v) + \mu\left(\|\nabla^{m_1}u\|^2 + \|\nabla^{2m_2}v\|^2\right) \geq -C_\mu, \\ &J(u,v) + \mu\left(\|\nabla^{m_1}u\|^2 + \|\nabla^{2m_2}v\|^2\right) \geq -C'_\mu. \end{aligned}$$

(A3). $g_j(u,v)$ $(j = 1,2) \in C^1(R)$, and

$$\begin{aligned} &|g_j(u,v)| \leq C_2\left(1 + |u|^{p_j} + |v|^{q_j}\right); \\ &|g_{ju}(u,v)| \leq C_3\left(1 + |u|^{p_j-1} + |v|^{q_j}\right); \\ &|g_{jv}(u,v)| \leq C_4\left(1 + |u|^{p_j} + |v|^{q_j-1}\right). \end{aligned}$$

Specifically, when $n = 1, 2$, $1 \leq p_j(q_j)$; when $3 \leq n \leq 4m$, $1 \leq p_j(q_j) \leq \frac{2m}{n-2m}$, where $m = \min\{m_1, m_2\}$.

(A4). $N_j(s_j) \geq N_{j0}$ and N_{j0} $(j = 1, 2)$ are positive constants, and $\rho > 0$. Thus, $M(s_1 + s_2) - \rho N_1(s_1) - \rho N_2(s_2) > 0$.

Then, the research phase space of this study is obtained:

$$V_0 = H, \quad V_k = H^k(\Omega) \cap H_0^1, X_{0\times 0} = V_{m_1}(\Omega) \times H \times V_{2m_2} \times H,$$
$$X_{k_1\times k_2} = V_{m_1+k_1}(\Omega) \times V_{k_1}(\Omega) \times V_{2m_2+2k_2} \times V_{2k_2}(\Omega), \quad k_1 = 0, 1, 2, \ldots, m_1, k_2 = 0, 1, 2, \ldots, m_2,$$

and the norms of the corresponding spaces are as follows:

$$\|(u, y_1, v, y_2)\|^2_{X_{k_1\times k_2}} = \|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{k_1}y_1\|^2 + \|\nabla^{2m_2+2k_2}v\|^2 + \|\nabla^{2k_2}y_2\|^2.$$

Meanwhile, the general form of the Poincare inequality is as follows: $\lambda_1\|\nabla^r u\|^2 \leq \|\nabla^{r+1}u\|^2$, where λ_1 is the first eigenvalue of $-\Delta$ with a homogeneous Dirichlet boundary on Ω. In this article, C_i is a constant, and $C(\cdot)$ is a constant depending on the parameters in parentheses.

Lemma 1. [21] *Let $y : R^+ \to R^+$ be an absolutely continuous positive function on* $[0, +\infty)$, *which satisfies the following differential inequality for some $\delta > 0$:*

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 2\delta y(t) \leq g(t)y(t) + K, \quad t > 0,$$

where $K \geq 0$, and $a \geq 0$ if $t \geq s \geq 0$ so that $\int_s^t g(\tau)\mathrm{d}\tau \leq \delta(t - s) + a$. Then,

$$y(t) \leq \mathrm{e}^a y(0)\mathrm{e}^{-\delta t} + \frac{K\mathrm{e}^a}{\delta}, \quad t \geq 0.$$

Lemma 2. [15] *Let X be a Banach space, and the continuous operator semi-group $\{S(t)\}_{t\geq 0}$ satisfies the following*:

(1) *The semi-group $\{S(t)\}_{t\geq 0}$ is uniformly bounded in X, i.e., for all $R_0 > 0$, and there exists a positive constant $C_0(R_0)$ that when $\|u\|_X \leq R_0$,*

$$\|S(t)u\|_X \leq C_0(R_0), (\textit{for all } t \in [0, +\infty));$$

(2) *there exists a bounded absorbing set B_0 in X, and for any bounded set $B \subset X$, there exists a moment t_0 that*

$$S(t)B \subset B_0(t \geq t_0);$$

(3) *if $t > 0$, and $S(t)$ is a fully continuous operator, then the semi-group $\{S(t)\}_{t\geq 0}$ has a global attractor A in X, and*

$$A = \omega(B_0) = \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau} S(t)B_0}.$$

3 Main conclusion

Let $\varepsilon > 0$ be small enough and $\lambda_1^m N_{10} - 2 - 4\varepsilon - 2\varepsilon^2 \geq 0, \lambda_1^{2m} N_{20} - 2 - 4\varepsilon - 2\varepsilon^2 \geq 0$.

Lemma 3. *Assume that assumptions* (A1)–(A4) *hold, if $f_j \in H$ $(j = 1, 2)$ and initial data $(u_0, u_1, v_0, v_1) \in X_{0\times 0}$, then for $R_0 > 0$, there exist positive constants $C(R_0)$ and t_0, so that when $t \geq t_0$, (u, y_1, v, y_2) determined by problems* (1)–(3) *satisfies*

$$\|(u, y_1, v, y_2)\|^2_{X_{0\times 0}} = \|\nabla^{m_1}u\|^2 + \|y_1\|^2 + \|\nabla^{2m_2}v\|^2 + \|y_2\|^2 \leq C(R_0), \tag{12}$$

where $y_1 = u_t + \varepsilon u$, $y_2 = v_t + \varepsilon v$.

Proof. Multiplying the first equation of (1) by y_1 in H and the second one by y_2 in H, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|y_1\|^2+\|y_2\|^2+\int\limits_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2}M(\tau)\mathrm{d}\tau+2J(u,v)\right]+\varepsilon M\left(\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2\right)\left(\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2\right)\\
&\quad-\varepsilon(\|y_1\|^2+\|y_2\|^2)+\varepsilon^2((u,y_1)+(v,y_2))+N_1\left(\|\nabla^{m_1}u\|^2\right)\|\nabla^{m_1}y_1\|^2+N_2\left(\|\nabla^{2m_2}v\|^2\right)\|\nabla^{2m_2}y_2\|^2\\
&\quad-\varepsilon N_1\left(\|\nabla^{m_1}u\|^2\right)\left(\nabla^{m_1}y_1,\nabla^{m_1}u\right)-\varepsilon N_2\left(\|\nabla^{2m_2}v\|^2\right)\left(\nabla^{2m_2}y_2,\nabla^{2m_2}v\right)+\varepsilon(g_1(u,v),u)+\varepsilon(g_2(u,v),v)\\
&=(f_1,y_1)+(f_2,y_2).
\end{aligned}\tag{13}$$

By Holder's inequality, Young's inequality, and Poincare inequality, we have

$$\begin{aligned}
-\varepsilon(\|y_1\|^2+\|y_2\|^2)+\varepsilon^2((u,y_1)+(v,y_2))&\geq\left(-\varepsilon-\frac{\varepsilon^2}{2}\right)(\|y_1\|^2+\|y_2\|^2)-\frac{\varepsilon^2}{2}(\|u\|^2+\|v\|^2)\\
&\geq\left(-\varepsilon-\frac{\varepsilon^2}{2}\right)(\|y_1\|^2+\|y_2\|^2)-\frac{\varepsilon^2}{2}\lambda_1^{-m_1}\|\nabla^{m_1}u\|^2-\frac{\varepsilon^2}{2}\lambda_1^{-2m_2}\|\nabla^{2m_2}v\|^2,
\end{aligned}\tag{14}$$

$$\begin{aligned}
&N_1\left(\|\nabla^{m_1}u\|^2\right)\|\nabla^{m_1}y_1\|^2+N_2\left(\|\nabla^{2m_2}v\|^2\right)\|\nabla^{2m_2}y_2\|^2-\varepsilon N_1\left(\|\nabla^{m_1}u\|^2\right)\left(\nabla^{m_1}y_1,\nabla^{m_1}u\right)\\
&\quad-\varepsilon N_2\left(\|\nabla^{2m_2}v\|^2\right)\left(\nabla^{2m_2}y_2,\nabla^{2m_2}v\right)\\
&\geq\frac{1}{2}N_1\left(\|\nabla^{m_1}u\|^2\right)\|\nabla^{m_1}y_1\|^2+\frac{1}{2}N_2\left(\|\nabla^{2m_2}v\|^2\right)\|\nabla^{2m_2}y_2\|^2-\frac{\varepsilon^2}{2}N_1\left(\|\nabla^{m_1}u\|^2\right)\|\nabla^{m_1}u\|^2\\
&\quad-\frac{\varepsilon^2}{2}N_2\left(\|\nabla^{2m_2}v\|^2\right)\|\nabla^{2m_2}v\|^2\\
&\geq\frac{1}{2}\lambda_1^{m_1}N_1\left(\|\nabla^{m_1}u\|^2\right)\|y_1\|^2+\frac{1}{2}\lambda_1^{2m_2}N_2\left(\|\nabla^{2m_2}v\|^2\right)\|y_2\|^2-\frac{\varepsilon^2}{2}N_1\left(\|\nabla^{m_1}u\|^2\right)\|\nabla^{m_1}u\|^2\\
&\quad-\frac{\varepsilon^2}{2}N_2\left(\|\nabla^{2m_2}v\|^2\right)\|\nabla^{2m_2}v\|^2,
\end{aligned}\tag{15}$$

$$(f_1,y_1)+(f_2,y_2)\leq\|f_1\|\|y_1\|+\|f_2\|\|y_2\|\leq\frac{1}{2}\|y_1\|^2+\frac{1}{2}\|y_2\|^2+\frac{1}{2}\|f_1\|^2+\frac{1}{2}\|f_2\|^2.\tag{16}$$

Inserting the aforementioned estimates into (13) gives

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|y_1\|^2+\|y_2\|^2+\int\limits_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2}M(\tau)\mathrm{d}\tau+2J(u,v)\right]+\left(\frac{1}{2}\lambda_1^{m_1}N_1\left(\|\nabla^{m_1}u\|^2\right)-\frac{1}{2}-\varepsilon-\frac{\varepsilon^2}{2}\right)\|y_1\|^2\\
&\quad+\left(\frac{1}{2}\lambda_1^{2m_2}N_2\left(\|\nabla^{2m_2}v\|^2\right)-\frac{1}{2}-\varepsilon-\frac{\varepsilon^2}{2}\right)\|y_2\|^2+\varepsilon M\left(\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2\right)\left(\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2\right)\\
&\quad-\left(\frac{\varepsilon^2}{2}N_1\left(\|\nabla^{m_1}u\|^2\right)+\frac{\varepsilon^2}{2}\lambda_1^{-m_1}\right)\|\nabla^{m_1}u\|^2-\left(\frac{\varepsilon^2}{2}N_2\left(\|\nabla^{2m_2}v\|^2\right)+\frac{\varepsilon^2}{2}\lambda_1^{-2m_2}\right)\|\nabla^{2m_2}v\|^2\\
&\leq-\varepsilon(g_1(u,v),u)-\varepsilon(g_2(u,v),v)+\frac{1}{2}\|f_1\|^2+\frac{1}{2}\|f_2\|^2.
\end{aligned}\tag{17}$$

According to (A_1), we have

$$\begin{aligned}
&\varepsilon M\left(\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2\right)\left(\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2\right)\\
&\geq\frac{\varepsilon}{4}\int\limits_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2}M(\tau)\mathrm{d}\tau+\frac{3\varepsilon}{4}M\left(\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2\right)\left(\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2\right),
\end{aligned}\tag{18}$$

and according to (A_2), we have

$$-\varepsilon(g_1(u,v),u)-\varepsilon(g_2(u,v),v)\leq-\varepsilon C_1J(u,v)+\varepsilon\mu\left(\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2\right)+\varepsilon C_\mu.\tag{19}$$

Inserting (18) and (19) into (17) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|y_1\|^2+\|y_2\|^2+\int\limits_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2}M(\tau)\mathrm{d}\tau+2J(u,v)\right]+\left(\lambda_1^{m_1}N_1\left(\|\nabla^{m_1}u\|^2\right)-1-2\varepsilon-\varepsilon^2\right)\|y_1\|^2$$
$$+\left(\lambda_1^{2m_2}N_2\left(\|\nabla^{2m_2}v\|^2\right)-1-2\varepsilon-\varepsilon^2\right)\|y_2\|^2+\frac{\varepsilon}{2}\int\limits_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2}M(\tau)\mathrm{d}\tau+2\varepsilon C_1J(u,v)$$
$$+\left(\frac{3\varepsilon}{2}M\left(\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2\right)-2\varepsilon\mu-\varepsilon^2N_1\left(\|\nabla^{m_1}u\|^2\right)-\varepsilon^2\lambda_1^{-m_1}\right)\|\nabla^{m_1}u\|^2$$
$$+\left(\frac{3\varepsilon}{2}M\left(\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2\right)-2\varepsilon\mu-\varepsilon^2N_2\left(\|\nabla^{2m_2}v\|^2\right)-\varepsilon^2\lambda_1^{-2m_2}\right)\|\nabla^{2m_2}v\|^2\le 2\varepsilon C_\mu+\|f_1\|^2+\|f_2\|^2. \tag{20}$$

Let $H_1(t)=\|y_1\|^2+\|y_2\|^2+\int_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2}M(\tau)\mathrm{d}\tau+2J(u,v)$, and $\sigma_1=\min\{\lambda_1^{m_1}N_{10}-1-2\varepsilon-\varepsilon^2,\lambda_1^{2m_2}N_{20}-1-2\varepsilon-\varepsilon^2,\frac{\varepsilon}{2},\varepsilon C_1\}$, we can infer from (20) that

$$\frac{\mathrm{d}}{\mathrm{d}t}H_1(t)+\sigma_1H_1(t)\le 2\varepsilon C_\mu+\|f_1\|^2+\|f_2\|^2. \tag{21}$$

According to Gronwall's inequality, we have

$$H_1(t)\le H_1(0)\mathrm{e}^{-\sigma_1t}+\frac{2\varepsilon C_\mu+\|f_1\|^2+\|f_2\|^2}{\sigma_1}, \tag{22}$$

and according to $(A_1)(A_2)$, we have

$$\begin{aligned}H_1(t)&\ge\|y_1\|^2+\|y_2\|^2+M_0\left(\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2\right)+2J(u,v)\\&\ge\|y_1\|^2+\|y_2\|^2+\frac{M_0}{2}\left(\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2\right)-2C_\mu'\\&\ge C_5\left(\|y_1\|^2+\|y_2\|^2+\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v|^2\right)-2C_\mu',\end{aligned} \tag{23}$$

where $\mu=\frac{M_0}{4}$, $C_5=\min\left\{1,\frac{M_0}{2}\right\}$, then

$$\begin{aligned}\|(u,y_1,v,y_2)\|^2_{X_{0\times0}}&=\|\nabla^{m_1}u\|^2+\|y_1\|^2+\|\nabla^{2m_2}v\|^2+\|y_2\|^2\\&\le\frac{(H_1(t)+2C_\mu')}{C_5}\\&\le\frac{H_1(0)\mathrm{e}^{-\sigma_1t}+2C_\mu'}{C_5}+\frac{2\varepsilon C_\mu+\|f_1\|^2+\|f_2\|^2}{\sigma_1C_5},\end{aligned} \tag{24}$$

i.e.,

$$\overline{\lim_{t\to\infty}}\|(u,y_1,v,y_2)\|^2_{X_{0\times0}}\le\frac{2C_\mu'}{C_5}+\frac{2\varepsilon C_\mu+\|f_1\|^2+\|f_2\|^2}{\sigma_1C_5}=R_0. \tag{25}$$

Therefore, there exist positive constants $C(R_0)$ and t_0 that when $t\ge t_0$, we have

$$\|(u,y_1,v,y_2)\|^2_{X_{0\times0}}=\|\nabla^{m_1}u\|^2+\|y_1\|^2+\|\nabla^{2m_2}v\|^2+\|y_2\|^2\le C(R_0). \tag{26}$$

Thus, Lemma 3 is proved. □

Lemma 4. *Assume that assumptions (A1)–(A4) hold, if $f_1\in V_{k_1}$, $f_2\in V_{2k_2}$, $k_1=1,2,\ldots,m_1$, $k_2=1,2,\ldots,m_2$, and initial data $(u_0,u_1,v_0,v_1)\in X_{k_1\times k_2}$. Then for $R_{k_1\times k_2}>0$, there exist positive constants $C(R_{k_1\times k_2})$ and $t_{k_1\times k_2}$ that when $t\ge t_{k_1\times k_2}$, (u,y_1,v,y_2) determined by problems (1)–(3) satisfies*

$$\|(u,y_1,v,y_2)\|^2_{X_{k_1\times k_2}}=\|\nabla^{m_1+k_1}u\|^2+\|\nabla^{k_1}y_1\|^2+\|\nabla^{2m_2+2k_2}v\|^2+\|\nabla^{2k_2}y_2\|^2\le C\left(R_{k_1\times k_2}\right), \tag{27}$$

where $y_1 = u_t + \varepsilon u$, $y_2 = v_t + \varepsilon v$.

Proof. Multiplying the first equation of (1) by $(-\Delta)^{k_1} y_1$, $k_1 = 1, 2, \ldots, m_1$ in H, the second one by $(-\Delta)^{2k_2} y_2$, $k_2 = 1, 2, \ldots, m_2$ in H, and then integrating over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\|\nabla^{k_1}y_1\|^2 + \|\nabla^{2k_2}y_2\|^2 + M\left(\|\nabla^{m_1}u\|^2 + \|\nabla^{2m_2}v\|^2\right)\left(\|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{2m_2+2k_2}v\|^2\right)]\\
&+ \varepsilon M\left(\|\nabla^{m_1}u\|^2 + \|\nabla^{2m_2}v\|^2\right)\left(\|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{2m_2+2k_2}v\|^2\right) - \varepsilon\left(\|\nabla^{k_1}y_1\|^2 + \|\nabla^{2k_2}y_2\|^2\right)\\
&+ \varepsilon^2\left(\left(\nabla^{k_1}u, \nabla^{k_1}y_1\right) + \left(\nabla^{2k_2}v, \nabla^{2k_2}y_2\right)\right) + N_1\left(\|\nabla^{m_1}u\|^2\right)\|\nabla^{m_1+k_1}y_1\|^2\\
&+ N_2\left(\|\nabla^{2m_2}v\|^2\right)\|\nabla^{2m_2+2k_2}y_2\|^2 - \varepsilon N_1\left(\|\nabla^{m_1}u\|^2\right)\left(\nabla^{m_1+k_1}y_1, \nabla^{m_1+k_1}u\right)\\
&- \varepsilon N_2\left(\|\nabla^{2m_2}v\|^2\right)\left(\nabla^{2m_2+2k_2}y_2, \nabla^{2m_2+2k_2}v\right) + \left(g_1(u,v), (-\Delta)^{k_1}y_1\right) + \left(g_2(u,v), (-\Delta)^{2k_2}y_2\right)\\
=\ &\frac{\|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{2m_2+2k_2}v\|^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}M\left(\|\nabla^{m_1}u\|^2 + \|\nabla^{2m_2}v\|^2\right) + \left(f_1, (-\Delta)^{k_1}y_1\right) + \left(f_2, (-\Delta)^{2k_2}y_2\right).
\end{aligned} \tag{28}$$

According to Holder's inequality, Young's inequality, and Poincare inequality, we have

$$\begin{aligned}
&-\varepsilon\left(\|\nabla^{k_1}y_1\|^2 + \|\nabla^{2k_2}y_2\|^2\right) + \varepsilon^2\left(\left(\nabla^{k_1}u, \nabla^{k_1}y_1\right) + \left(\nabla^{2k_2}v, \nabla^{2k_2}y_2\right)\right)\\
&\geq \left(-\varepsilon - \frac{\varepsilon^2}{2}\right)\left(\|\nabla^{k_1}y_1\|^2 + \|\nabla^{2k_2}y_2\|^2\right) - \frac{\varepsilon^2}{2}\left(\|\nabla^{k_1}u\|^2 + \|\nabla^{2k_2}v\|^2\right)\\
&\geq \left(-\varepsilon - \frac{\varepsilon^2}{2}\right)\left(\|\nabla^{k_1}y_1\|^2 + \|\nabla^{2k_2}y_2\|^2\right) - \frac{\varepsilon^2}{2}\lambda_1^{-m_1}\|\nabla^{m_1+k_1}u\|^2 - \frac{\varepsilon^2}{2}\lambda_1^{-2m_2}\|\nabla^{2m_2+2k_2}v\|^2,
\end{aligned} \tag{29}$$

$$\begin{aligned}
&N_1\left(\|\nabla^{m_1}u\|^2\right)\|\nabla^{m_1+k_1}y_1\|^2 + N_2\left(\|\nabla^{2m_2}v\|^2\right)\|\nabla^{2m_2+2k_2}y_2\|^2 - \varepsilon N_1\left(\|\nabla^{m_1}u\|^2\right)\left(\nabla^{m_1+k_1}y_1, \nabla^{m_1+k_1}u\right)\\
&\quad - \varepsilon N_2\left(\|\nabla^{2m_2}v\|^2\right)\left(\nabla^{2m_2+2k_2}y_2, \nabla^{2m_2+2k_2}v\right)\\
&\geq \frac{1}{2}N_1\left(\|\nabla^{m_1}u\|^2\right)\|\nabla^{m_1+k_1}y_1\|^2 + \frac{1}{2}N_2\left(\|\nabla^{2m_2}v\|^2\right)\|\nabla^{2m_2+2k_2}y_2\|^2 - \frac{\varepsilon^2}{2}N_1\left(\|\nabla^{m_1}u\|^2\right)\|\nabla^{m_1+k_1}u\|^2\\
&\quad - \frac{\varepsilon^2}{2}N_2\left(\|\nabla^{2m_2}v\|^2\right)\|\nabla^{2m_2+2k_2}v\|^2,
\end{aligned} \tag{30}$$

$$\begin{aligned}
&\left(g_1(u,v), (-\Delta)^{k_1}y_1\right) + \left(g_2(u,v), (-\Delta)^{2k_2}y_2\right)\\
&\leq \|g_1(u,v)\|\|\nabla^{2k_2}y_1\| + \|g_2(u,v)\|\|\nabla^{4k_2}y_2\|\\
&\leq \frac{N_{10}}{4}\|\nabla^{m_1+k_1}y_1\|^2 + \frac{\lambda_1^{k_1-m_1}}{N_{10}}\|g_1(u,v)\|^2 + \frac{N_{20}}{4}\|\nabla^{2m_2+2k_2}y_2\|^2 + \frac{\lambda_1^{2k_2-2m_2}}{N_{20}}\|g_2(u,v)\|^2,
\end{aligned} \tag{31}$$

$$\begin{aligned}
\left(f_1, (-\Delta)^{k_1}y_1\right) + \left(f_2, (-\Delta)^{2k_2}y_2\right) &\leq \|\nabla^{k_1}f_1\|\|\nabla^{k_1}y_1\| + \|\nabla^{2k_2}f_2\|\|\nabla^{2k_2}y_2\|\\
&\leq \frac{1}{2}\|\nabla^{k_1}y_1\|^2 + \frac{1}{2}\|\nabla^{2k_2}y_2\|^2 + \frac{1}{2}\|\nabla^{k_1}f_1\|^2 + \frac{1}{2}\|\nabla^{2k_2}f_2\|^2,
\end{aligned} \tag{32}$$

and according to (A_3), we have

$$\begin{aligned}
\|g_1(u,v)\|^2 &= \int_\Omega |g_1(u,v)|^2\mathrm{d}x\\
&\leq \int_\Omega \left|C_2\left(1 + |u|^{p_1} + |v|^{q_1}\right)\right|^2\mathrm{d}x\\
&\leq C_6\int_\Omega \left(1 + |u|^{2p_1} + |v|^{2q_1}\right)\mathrm{d}x\\
&\leq C_7\left(1 + \|u\|_{2p_1}^{2p_1} + \|v\|_{2q_1}^{2q_1}\right),\\
\|g_2(u,v)\|^2 &\leq C_8\left(1 + \|u\|_{2p_2}^{2p_2} + \|v\|_{2q_2}^{2q_2}\right).
\end{aligned} \tag{33}$$

Furthermore, on the basis of the Gagliardo-Nirenberg inequality, we can conclude that

$$\begin{cases} \|u\|_{2p_j}^{2p_j} \le C_{9j}\|\nabla^{m_1}u\|^{\frac{n(p_j-1)}{m_1}}\|u\|^{\frac{2m_1p_j-n(p_j-1)}{m_1}}, \\ \|v\|_{2q_j}^{2q_j} \le C_{10j}\|\nabla^{2m_2}v\|^{\frac{n(q_j-1)}{2m_2}}\|v\|^{\frac{4m_2q_j-n(q_j-1)}{2m_2}}. \end{cases}$$

Thus, we have

$$\|g_1(u,v)\|^2 + \|g_2(u,v)\|^2 \le C(R_0). \tag{34}$$

By inserting (29)–(32) and (34) into (28), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\|\nabla^{k_1}y_1\|^2 + \|\nabla^{2k_2}y_2\|^2 + M\left(\|\nabla^{m_1}u\|^2 + \|\nabla^{2m_2}v\|^2\right)\left(\|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{2m_2+2k_2}v\|^2\right)] \\ &\quad + \frac{(2N_1(\|\nabla^{m_1}u\|^2) - N_{10})\lambda_1^{m_1} - 2 - 4\varepsilon - 2\varepsilon^2}{4}\|\nabla^{k_1}y_1\|^2 + \frac{(2N_2(\|\nabla^{2m_2}v\|^2) - N_{20})\lambda_1^{2m_2} - 2 - 4\varepsilon - 2\varepsilon^2}{4} \\ &\quad \times \|\nabla^{2k_2}y_2\|^2 + (\varepsilon M\left(\|\nabla^{m_1}u\|^2 + \|\nabla^{2m_2}v\|^2\right) - \frac{\varepsilon^2}{2}N_1\left(\|\nabla^{m_1}u\|^2\right) - \frac{\varepsilon^2}{2}\lambda_1^{-m_1})\|\nabla^{m_1+k_1}u\|^2 \\ &\quad + (\varepsilon M\left(\|\nabla^{m_1}u\|^2 + \|\nabla^{2m_2}v\|^2\right) - \frac{\varepsilon^2}{2}N_2\left(\|\nabla^{2m_2}v\|^2\right) - \frac{\varepsilon^2}{2}\lambda_1^{-2m_2})\|\nabla^{2m_2+2k_2}v\|^2 \\ &\le \frac{\|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{2m_2+2k_2}v\|^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}M\left(\|\nabla^{m_1}u\|^2 + \|\nabla^{2m_2}v\|^2\right) + \frac{1}{2}\|\nabla^{k_1}f_1\|^2 + \frac{1}{2}\|\nabla^{2k_2}f_2\|^2 + C(R_0,\lambda_1) \\ &\le \left(\|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{2m_2+2k_2}v\|^2\right)M'\left(\|\nabla^{m_1}u\|^2 + \|\nabla^{2m_2}v\|^2\right) \\ &\quad \times \left(\left(\nabla^{m_1}u, \nabla^{m_1}u_t\right) + \left(\nabla^{2m_2}v, \nabla^{2m_2}v_t\right)\right) + \frac{1}{2}\|\nabla^{k_1}f_1\|^2 + \frac{1}{2}\|\nabla^{2k_2}f_2\|^2 + C(R_0,\lambda_1) \\ &\le C_8\left(\|\nabla^{m_1}u_t\| + \|\nabla^{2m_2}v_t\|\right)\left(\|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{2m_2+2k_2}v\|^2\right) + \frac{1}{2}\|\nabla^{k_1}f_1\|^2 + \frac{1}{2}\|\nabla^{2k_2}f_2\|^2 + C(R_0,\lambda_1). \end{aligned} \tag{35}$$

Let $H_2(t) = \|\nabla^{k_1}y_1\|^2 + \|\nabla^{2k_2}y_2\|^2 + M(\|\nabla^{m_1}u\|^2 + \|\nabla^{2m_2}v\|^2)(\|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{2m_2+2k_2}v\|^2)$, and $\sigma_2 = \min\left\{\frac{\lambda_1^{m_1}N_{10} - 2 - 4\varepsilon - 2\varepsilon^2}{2}, \frac{\lambda_1^{2m_2}N_{20} - 2 - 4\varepsilon - 2\varepsilon^2}{2}, \frac{\varepsilon}{2}\right\}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}H_2(t) + \sigma_2H_2(t) \le C_9\left(\|\nabla^{m_1}u_t\| + \|\nabla^{2m_2}v_t\|\right)H_2(t) + \|\nabla^{k_1}f_1\|^2 + \|\nabla^{2k_2}f_2\|^2 + C(R_0,\lambda_1). \tag{36}$$

By taking the scalar product in H of (1) with u_t, v_t, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|u_t\|^2 + \|v_t\|^2 + \int_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2} M(\tau)\mathrm{d}\tau + 2J(u,v) - 2(f_1,u) - 2(f_2,v)\right] + N_1\left(\|\nabla^{m_1}u\|^2\right)\|\nabla^{m_1}u_t\|^2 \\ &\quad + N_2\left(\|\nabla^{2m_2}v\|^2\right)\|\nabla^{2m_2}v_t\|^2 = 0, \end{aligned} \tag{37}$$

and integrating (37) in $\mathrm{d}t$ on $(0,t)$ derives

$$\begin{aligned} &\int_0^t \left(\|\nabla^{m_1}u_t(\tau)\|^2 + \|\nabla^{2m_2}v_t(\tau)\|^2\right)\mathrm{d}\tau \\ &\le \frac{1}{\min\{N_{10},N_{20}\}}\int_0^t \left(N_1\left(\|\nabla^{m_1}u(\tau)\|^2\right)\|\nabla^{m_1}u_t(\tau)\|^2 + N_2\left(\|\nabla^{2m_2}v(\tau)\|^2\right)\|\nabla^{2m_2}v_t(\tau)\|^2\right)\mathrm{d}\tau \\ &\le \frac{1}{\min\{N_{10},N_{20}\}}\left(\|u_1\|^2 + \|v_1\|^2 + \int_0^{\|\nabla^{m_1}u_0\|^2+\|\nabla^{2m_2}v_0\|^2} M(\tau)\mathrm{d}\tau + 2J(u_0,v_0) - 2(f_1,u_0) - 2(f_2,v_0)\right) \le C_{10}, \end{aligned} \tag{38}$$

then, we have

$$C_9\int_s^t\Big(\big(\|\nabla^{m_1}u_t(\tau)\|+\|\nabla^{2m_2}v_t(\tau)\|\big)\Big)\mathrm{d}\tau\le\frac{\sigma_2}{2}(t-s)+a,\tag{39}$$

for $t>s\ge 0$ and some $a>0$. Together with (36), (39), and Lemma 1, we can obtain that

$$H_2(t)\le C_{11}H_2(0)\mathrm{e}^{-\frac{\sigma_2}{2}t}+C_{12}.\tag{40}$$

According to (A_1), we have

$$\begin{aligned}H_2(t)&\ge\|\nabla^{k_1}y_1\|^2+\|\nabla^{2k_2}y_2\|^2+M\Big(\|\nabla^{m_1}u\|^2+\|\nabla^{2m_2}v\|^2\Big)\Big(\|\nabla^{m_1+k_1}u\|^2+\|\nabla^{2m_2+2k_2}v\|^2\Big)\\&\ge C_{13}\Big(\|\nabla^{k_1}y_1\|^2+\|\nabla^{2k_2}y_2\|^2+\|\nabla^{m_1+k_1}u\|^2+\|\nabla^{2m_2+2k_2}v\|^2\Big),\end{aligned}\tag{41}$$

then,

$$\begin{aligned}\|(u,y_1,v,y_2)\|^2_{X_{k_1\times k_2}}&=\|\nabla^{k_1}y_1\|^2+\|\nabla^{2k_2}y_2\|^2+\|\nabla^{m_1+k_1}u\|^2+\|\nabla^{2m_2+2k_2}v\|^2\\&\le\frac{C_{11}H_2(0)\mathrm{e}^{-\frac{\sigma_2}{2}t}+C_{12}}{C_{13}},\end{aligned}\tag{42}$$

i.e.,

$$\overline{\lim_{t\to\infty}}\|(u,y_1,v,y_2)\|^2_{X_{k_1\times k_2}}\le R_{k_1\times k_2}.\tag{43}$$

Therefore, there exist positive constants $C(R_{k_1\times k_2})$ and $t_{k_1\times k_2}$ that when $t\ge t_{k_1\times k_2}$, (u,y_1,v,y_2) satisfies

$$\begin{aligned}&\|(u,y_1,v,y_2)\|^2_{X_{k_1\times k_2}}=\|\nabla^{k_1}y_1\|^2+\|\nabla^{2k_2}y_2\|^2+\|\nabla^{m_1+k_1}u\|^2+\|\nabla^{2m_2+2k_2}v\|^2\le C\Big(R_{k_1\times k_2}\Big),\\&k_1=1,2,\ldots,m_1,\ \ k_2=1,2,\ldots,m_2.\end{aligned}\tag{44}$$

Thus, Lemma 4 is proved. □

Theorem 1. *Assume that assumptions* (*A*1)–(*A*4) *hold, if* $f_1\in V_{k_1}$, $f_2\in V_{2k_2}$ *and initial data* $(u_0,u_1,v_0,v_1)\in X_{k_1\times k_2}$, $k_1=0,1,2,\ldots,m_1$, $k_2=0,1,2,\ldots,m_2$, *then problems* (1)–(3) *admit a unique solution* (u,v) *satisfying*

$$\begin{aligned}&u\in L^\infty\Big(0,\infty;V_{m_1+k_1}\Big);\\&u_t\in L^\infty(0,\infty;H)\cap L^2\Big(0,T;\ V_{k_1}\Big);\\&v\in L^\infty\Big(0,\infty;V_{2m_2+2k_2}\Big);\\&v_t\in L^\infty(0,\infty;H)\cap L^2\Big(0,T;\ V_{2k_2}\Big).\end{aligned}$$

Proof. According to [15] and the Faedo-Galerkin method, (1)–(3) have global solutions combining with Lemmas 3 and 4.

Let u^1,v^1 and u^2,v^2 be two solutions of problems (1)–(3) corresponding to the same initial data, respectively, $w=u^1-u^2$, $z=v^1-v^2$. Then, (w,z) solves

$$\begin{cases}w_{tt}+\dfrac{1}{2}\sigma_{12}(t)(-\Delta)^{m_1}w_t+\dfrac{1}{2}\Phi_{12}(t)(-\Delta)^{m_1}w+G_1(u^1,u^2,v^1,v^2;\ t)=0,\\ z_{tt}+\dfrac{1}{2}\sigma_{34}(t)(-\Delta)^{2m_2}z_t+\dfrac{1}{2}\Phi_{12}(t)(-\Delta)^{2m_2}z+G_2(u^1,u^2,v^1,v^2;\ t)=0,\end{cases}\tag{45}$$

where $\sigma_{12}=\sigma_1(t)+\sigma_2(t)$, $\Phi_{12}(t)=\Phi_1(t)+\Phi_2(t)$, $\sigma_i(t)=N_1(\|\nabla^{m_1}u^i\|^2)$, $\Phi_i(t)=M(\|\nabla^{m_1}u^i\|^2+\|\nabla^{2m_2}v^i\|^2)$, $i=1,2$, $\sigma_{34}=\sigma_3(t)+\sigma_4(t)$, $\sigma_j(t)=N_2(\|\nabla^{2m_2}v^j\|^2)$, $j=3,4$, $G_1(u^1,u^2,v^1,v^2;\ t)=\frac{1}{2}\{[\sigma_1(t)-\sigma_2(t)](-\Delta)^{m_1}(u_t^1+u_t^2)+[\Phi_1(t)-\Phi_2(t)](-\Delta)^{m_1}(u^1+u^2)\}+g_1(u_1,v_1)-g_1(u_2,v_2)$, $G_2(u^1,u^2,v^1,v^2;\ t)=\frac{1}{2}\{[\sigma_3(t)-\sigma_4(t)](-\Delta)^{2m_2}(v_t^1+v_t^2)+[\Phi_1(t)-\Phi_2(t)](-\Delta)^{2m_2}(v^1+v^2)\}+g_2(u_1,v_1)-g_2(u_2,v_2)$.

According to Lemma 3, $\sigma_{12}' \le C(R_0)(\|\nabla^{m_1}u_t^1\| + \|\nabla^{m_1}u_t^2\|)$, $\sigma_{34}' \le C(R_0)(\|\nabla^{2m_2}v_t^1\| + \|\nabla^{2m_2}v_t^2\|)$.

By taking the scalar product in H of (45) with w_t, z_t, we can obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|w_t\|^2 + \|z_t\|^2 + \frac{1}{4}\Phi_0\cdot\left(\|\nabla^{m_1}w\|^2 + \|\nabla^{2m_2}z\|^2\right)\right] + \frac{1}{2}\sigma_{12}(t)\|\nabla^{m_1}w_t\|^2 + \frac{1}{2}\sigma_{34}(t)\|\nabla^{2m_2}z_t\|^2 + (G_1(u^1, u^2, v^1, v^2; t), w_t) + (G_2(u^1, u^2, v^1, v^2; t), z_t) = 0. \tag{46}$$

According to Lemma 3 and $(A1)$, $M_0 \le M \le C(R_0, H_1(0)) \equiv M_1$. When $\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^{m_1}w\|^2 + \|\nabla^{2m_2}z\|^2) \ge 0$, $\Phi_0 = 2M_0$; otherwise $\Phi_0 = 2M_1$.

Let $(G_1(u^1, u^2, v^1, v^2; t), w_t) = G_{11} + G_{12} + G_{13}$, $(G_2(u^1, u^2, v^1, v^2; t), z_t) = G_{21} + G_{22} + G_{23}$, we have

$$\begin{aligned} G_{11} &= \frac{1}{2}(\sigma_1(t) - \sigma_2(t))\left(\nabla^{m_1}(u_t^1 + u_t^2), \nabla^{m_1}w_t\right) \\ &\le C(R_0)\left(\|\nabla^{m_1}u_t^1\| + \|\nabla^{m_1}u_t^2\|\right)\|\nabla^{m_1}w\|\|\nabla^{m_1}w_t\| \\ &\le \frac{\sigma_{120}}{8}\|\nabla^{m_1}w_t\|^2 + \frac{2C(R_0)}{\sigma_{120}}\left(\|\nabla^{m_1}u_t^1\|^2 + \|\nabla^{m_1}u_t^2\|^2\right)\|\nabla^{m_1}w\|^2, \end{aligned} \tag{47}$$

$$\begin{aligned} G_{12} &= \frac{1}{2}(\Phi_1(t) - \Phi_2(t))\left(\nabla^{m_1}(u^1 + u^2), \nabla^{m_1}w_t\right) \\ &\le C(R_0)\left(\|\nabla^{m_1}w\| + \|\nabla^{2m_2}z\|\right)\|\nabla^{m_1}w_t\| \\ &\le \frac{\sigma_{120}}{8}\|\nabla^{m_1}w_t\|^2 + \frac{2C(R_0)}{\sigma_{120}}\left(\|\nabla^{m_1}w\|^2 + \|\nabla^{2m_2}z\|^2\right), \end{aligned} \tag{48}$$

$$G_{13} = (g_1(u_1, v_1) - g_1(u_2, v_2), w_t) \le C(R_0)\left(\|w_t\|^2 + \|\nabla^{m_1}w\|^2 + \|\nabla^{2m_2}z\|^2\right), \tag{49}$$

$$\begin{aligned} G_{21} &= \frac{1}{2}(\sigma_3(t) - \sigma_4(t))\left(\nabla^{2m_2}(v_t^1 + v_t^2), \nabla^{2m_2}z_t\right) \\ &\le C(R_0)\left(\|\nabla^{2m_2}v_t^1\| + \|\nabla^{2m_2}v_t^2\|\right)\|\nabla^{2m_2}z\|\|\nabla^{2m_2}z_t\| \\ &\le \frac{\sigma_{340}}{8}\|\nabla^{2m_2}z_t\|^2 + \frac{2C(R_0)}{\sigma_{340}}\left(\|\nabla^{2m_2}v_t^1\|^2 + \|\nabla^{2m_2}v_t^2\|^2\right)\|\nabla^{2m_2}z\|^2, \end{aligned} \tag{50}$$

$$\begin{aligned} G_{22} &= \frac{1}{2}(\Phi_1(t) - \Phi_2(t))\left(\nabla^{2m_2}(v^1 + v^2), \nabla^{2m_2}z_t\right) \\ &\le C(R_0)\left(\|\nabla^{m_1}w\| + \|\nabla^{2m_2}z\|\right)\|\nabla^{2m_2}z_t\| \\ &\le \frac{\sigma_{340}}{8}\|\nabla^{2m_2}z_t\|^2 + \frac{2C(R_0)}{\sigma_{340}}\left(\|\nabla^{m_1}w\|^2 + \|\nabla^{2m_2}z\|^2\right), \end{aligned} \tag{51}$$

$$G_{23} = (g_2(u_1, v_1) - g_2(u_2, v_2), z_t) \le C(R_0)\left(\|z_t\|^2 + \|\nabla^{m_1}w\|^2 + \|\nabla^{2m_2}z\|^2\right), \tag{52}$$

where $\sigma_{120} = 2N_{10}$, $\sigma_{340} = 2N_{20}$.

By inserting (46)–(52) into (45), we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|w_t\|^2 + \|z_t\|^2 + \frac{1}{4}\Phi_0\cdot\left(\|\nabla^{m_1}w\|^2 + \|\nabla^{2m_2}z\|^2\right)\right] \\ &\le C_{14}\left(1 + \|\nabla^{2m_2}v_t^1\|^2 + \|\nabla^{2m_2}v_t^2\|^2\right)\left[\|w_t\|^2 + \|z_t\|^2 + \frac{1}{4}\Phi_0\cdot\left(\|\nabla^{m_1}w\|^2 + \|\nabla^{2m_2}z\|^2\right)\right]. \end{aligned} \tag{53}$$

Solving this differential inequality yields

$$\begin{aligned} &\left[\|w_t\|^2 + \|z_t\|^2 + \frac{1}{4}\Phi_0\cdot\left(\|\nabla^{m_1}w\|^2 + \|\nabla^{2m_2}z\|^2\right)\right] \\ &\le \left[\|w_1\|^2 + \|z_1\|^2 + \frac{1}{4}\Phi_0\cdot\left(\|\nabla^{m_1}w_0\|^2 + \|\nabla^{2m_2}z_0\|^2\right)\right]\exp\left(\int_0^t C_{14}\left(1 + \|\nabla^{2m_2}v_t^1\|^2 + \|\nabla^{2m_2}v_t^2\|^2\right)\mathrm{d}s\right). \end{aligned} \tag{54}$$

Thus, the uniqueness of the solution is proved. □

Therefore, problems (1)–(3) possess a unique solution u, v. Theorem 1 is proved.

According to Theorem 1, we define a nonlinear operator $\{S(t)\}_{t\geq 0}$ on space $X_{0\times 0}$: $S(t)(u_0, u_1, v_0, v_1) = (u, u_t, v, v_t)$, for all $t \geq 0$. Theorem 1 shows that $\{S(t)\}_{t\geq 0}$ compose a continuous semi-group in $X_{0\times 0}$. Before proving the family of global attractors, we first give their definition.

Definition 1. Let X_0 be a Banach space, and $\{S(t)\}_{t\geq 0}$ be a continuous operator semi-group, if there exist compact sets $A_{k_1\times k_2}$ that satisfies

(i) Invariance: all $A_{k_1\times k_2}$ are invariant sets under the action of the semi-group $\{S(t)\}_{t\geq 0}$,

$$S(t)A_{k_1\times k_2} = A_{k_1\times k_2}; \quad \text{for all } t \geq 0.$$

(ii) Attractiveness: all $A_{k_1\times k_2}$ attract all bounded sets in X_0, i.e., for any bounded $B \subset X_0$,

$$\operatorname{dist}\Big(S(t)B, A_{k_1\times k_2}\Big) = \sup_{x\in B}\inf_{y\in A_{k_1\times k_2}} \|S(t)x - y\|_{X_0} \to 0,\ t \to \infty.$$

In particular, when $t \to \infty$, all trajectories $S(t)u_0$ from u_0 converge to $A_{k_1\times k_2}$, i.e.,

$$\operatorname{dist}\Big(S(t)u_0, A_{k_1\times k_2}\Big) \to 0, \quad t \to \infty.$$

Then, compact sets A_k are all global attractors of the semi-group $\{S(t)\}_{t\geq 0}$. Let $\mathcal{A} = \{A_{k_1\times k_2} \subset X_0 : k_1 = 1, 2, \ldots, m_1, k_2 = 1, 2, \ldots, m_2\}$ be a family of subsets in X_0. Finally, the family $\mathcal{A}$ is the family of global attractors in X_0.

Theorem 2. *Assume that assumptions (A1)–(A4) hold, if $f_1 \in V_{m_1}$, $f_2 \in V_{2m_2}$ and initial data $(u_0, u_1, v_0, v_1) \in X_{m_1\times m_2}$, then, problems (1)–(3) have a family of global attractors $\mathcal{A}$ in $X_{0\times 0}$*

$$\mathcal{A} = \Big\{A_{k_1\times k_2}\Big\}, A_{k_1\times k_2} = \omega\Big(B_{k_1\times k_2,0}\Big) = \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau} S(t)B_{k_1\times k_2,0}}, \quad k_1 = 1, 2, \ldots, m_1, k_2 = 1, 2, \ldots, m_2,$$

where $B_{k_1\times k_2,0} = \{(u, u_t, v, v_t) \in X_{k_1\times k_2} : \|(u, u_t, v, v_t)\|^2_{X_{k_1\times k_2}} = \|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{k_1}u_t\|^2 + \|\nabla^{2m_2+2k_2}v\|^2 + \|\nabla^{2k_2}v_t\|^2 \leq C(R_0) + C(R_{k_1\times k_2})\}$ are bounded absorbing sets in $X_{0\times 0}$, $B_{k_1\times k_2,0}$ are compact in $X_{0\times 0}$, $A_{k_1\times k_2} \subset X_{0\times 0}$.

(1) $S(t)A_{k_1\times k_2} = A_{k_1\times k_2}$, *(for all $t \geq 0$),*

(2) $A_{k_1\times k_2}$ *attract all bounded sets in $X_{0\times 0}$, i.e., for all $B_{k_1\times k_2} \subset X_{0\times 0}$ are bounded sets in $X_{0\times 0}$, and*

$$\operatorname{dist}\Big(S(t)B_{k_1\times k_2}, A_{k_1\times k_2}\Big) = \sup_{x\in B_{k_1\times k_2}}\inf_{y\in A_{k_1\times k_2}} \|S(t)x - y\|_{X_{0\times 0}} \to 0 (t \to \infty),$$

where $\{S(t)\}_{t\geq 0}$ is the solution semi-group generated by problems (1)–(3).

Proof. According to Lemma 3, for all $R_0 > 0$, $\|(u_0, u_1, v_0, v_1)\|_{X_{0\times 0}} \leq R_0$. Thus,

$$\|S(t)(u_0, u_1, v_0, v_1)\|^2_{X_{0\times 0}} = \|u\|^2_{V_{m_1}} + \|u_t\|^2_{V_0} + \|v\|^2_{V_{2m_2}} + \|v_t\|^2_{V_0} \leq C(R_0)$$

show that $\{S(t)\}_{t\geq 0}$ are uniformly bounded in $X_{0\times 0}$; further, $B_{k_1\times k_2,0} = \{(u, u_t, v, v_t) \in X_{k_1\times k_2} : \|(u, u_t, v, v_t)\|^2_{X_{k_1\times k_2}} = \|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{k_1}u_t\|^2 + \|\nabla^{2m_2+2k_2}v\|^2 + \|\nabla^{2k_2}v_t\|^2 \leq C(R_0) + C(R_{k_1\times k_2})\}$ are bounded absorbing sets of the semi-group $\{S(t)\}_{t\geq 0}$ in $X_{0\times 0}$; because $X_{k_1\times k_2} \hookrightarrow X_{0\times 0}$ are compactly embedding, i.e., the bounded sets in $X_{k_1\times k_2}$ are compact sets in $X_{0\times 0}$, the solution semi-group $\{S(t)\}_{t\geq 0}$ is a fully continuous operator.

To sum up, we obtained the family of global attractors $\mathcal{A} = \{A_{k_1\times k_2}\}$ of the solution semi-group $\{S(t)\}_{t\geq 0}$ in $X_{0\times 0}$, and

$$A_{k_1\times k_2} = \omega\Big(B_{k_1\times k_2,0}\Big) = \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau} S(t)B_{k_1\times k_2,0}}, A_{k_1\times k_2} \subset X_{0\times 0}, \quad k_1 = 1, 2, \ldots, m_1, k_2 = 1, 2, \ldots, m_2.$$

Theorem 2 is proved. □

Note 1. Lemma 4 and Theorem 2 show that the bounded absorbing sets $B_{k_1\times k_2,0} = \{(u, u_t, v, v_t) \in X_{k_1\times k_2} : \|(u, u_t, v, v_t)\|^2_{X_{k_1\times k_2}} = \|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{k_1}u_t\|^2 + \|\nabla^{2m_2+2k_2}v\|^2 + \|\nabla^{2k_2}v_t\|^2 \le C(R_0) + C(R_{k_1\times k_2})\}$ are compact bounded absorbing sets in $X_{0\times 0}$. Therefore, based on condition 3 in Lemma 2, the operator semi-group $\{S(t)\}_{t\ge 0}$ only needs to be a continuous operator. According to Theorem 1, the semi-group $\{S(t)\}_{t\ge 0}$ is already a continuous semi-group. Thus, the family of global attractors $\mathcal{A} = \{A_{k_1\times k_2}\}$ of problems (1)–(3) in $X_{0\times 0}$ can also be obtained.

4 Conclusion

In this article, we studied a class of higher-order coupled Kirchhoff systems with nonlinear strong damping. For the first time, we systematically defined the family of global attractors and proved the existence of the family of global attractors of problems (1)–(3). The results enriched the related conclusions of higher-order coupled Kirchhoff models and laid a theoretical foundation for future practical applications.

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