Research Article

Ginkyu Choi and Soon-Mo Jung*

Hyers-Ulam stability of isometries on bounded domains-II

https://doi.org/10.1515/dema-2022-0196
received July 2, 2022; accepted January 20, 2023

Abstract: The question of whether there is a true isometry approximating the ε-isometry defined in the bounded subset of the n-dimensional Euclidean space has long been considered an interesting question. In 1982, Fickett published the first article on this topic, and in early 2000, Alestalo et al. and Väisälä improved Fickett's result significantly. Recently, the second author of this article published a paper improving the previous results. The main purpose of this article is to significantly improve all of the aforementioned results by applying a basic and intuitive method.

Keywords: isometry, ε-isometry, Hyers-Ulam stability, bounded domain

MSC 2020: Primary 46C99, Secondary 39B82, 39B62, 46B04

1 Introduction

We assume that $(E, \langle\cdot,\cdot\rangle)$ and $(F, \langle\cdot,\cdot\rangle)$ are real (or complex) Hilbert spaces and D is a nonempty subset of E. Given $\varepsilon > 0$, a function $f : D \to F$ is called an *ε-isometry* if f satisfies the inequality

$$|\|f(x) - f(y)\| - \|x - y\|| \le \varepsilon$$

for all $x, y \in D$. If there exists a positive constant K (independent of f and ε) such that for each ε-isometry $f : D \to F$, there exists an isometry $U : D \to F$ satisfying the inequality $\|f(x) - U(x)\| \le K\varepsilon$ for every $x \in D$, then the equation, $\|f(x) - f(y)\| = \|x - y\|$, is said to have the *Hyers-Ulam stability*.

Hyers and Ulam [1] were the first mathematicians to publish a article on the stability of isometries. Indeed, they were able to prove the Hyers-Ulam stability of surjective isometries defined on the entire space by using properties of the inner product of Hilbert space. Readers interested in more literature on similar subjects are referred to the papers [2–12] and the references cited therein.

The question of whether there is a true isometry approximating the ε-isometry defined in the bounded subset of the n-dimensional Euclidean space has long been considered an interesting question.

To our knowledge, Fickett [13] was the first mathematician who tried to study the Hyers-Ulam stability of isometries whose domains are bounded subsets of $\mathbb{R}^n$. After studying the stability of isometries in the bounded domain, Fickett used this result to prove Ulam's conjecture about the invariance of measures. Indeed, the research results of the Hyers-Ulam stability of isometries in the bounded domain can be used in various fields of application.

*** Corresponding author: Soon-Mo Jung,** Mathematics Section, College of Science and Technology, Hongik University, 30016 Sejong, Republic of Korea, e-mail: smjung747@gmail.com, smjung@hongik.ac.kr
Ginkyu Choi: Department of Electronic and Electrical Engineering, College of Science and Technology, Hongik University, 30016 Sejong, Republic of Korea, e-mail: gkchoi@hongik.ac.kr

Theorem 1.1. (Fickett) *Given an integer* $n \geq 2$, *let* D *be a bounded subset of* $\mathbb{R}^n$ *and let* $\varepsilon > 0$ *be given. If a function* $f : D \to \mathbb{R}^n$ *is an* ε*-isometry, then there exists an isometry* $U : D \to \mathbb{R}^n$ *such that*

$$\|f(x) - U(x)\| \leq 27\varepsilon^{1/2^n} \tag{1}$$

for any $x \in D$.

Comparing Theorem 1.1 with the definition of Hyers-Ulam stability mentioned earlier, although Fickett did not prove Hyers-Ulam stability of isometries in a strict sense, it is obvious that his goal was to prove Hyers-Ulam stability of isometries on the bounded domain.

For the same kind of inequalities as (1) concerning the stability of isometries, the speed of convergence of the upper bounds of the inequalities seems to be most important when ε goes to zero. From this point of view, an obvious disadvantage of Fickett's theorem is that the upper bound of inequality (1) decreases very slowly to 0 as ε approaches 0. Roughly speaking, the problem is that the speed of convergence is too slow.

Because Fickett's theorem has this shortcoming, it justifies that the purpose of this article is to further improve Fickett's theorem.

In the 40 years since Fickett published his result, several mathematicians have constantly tried to improve Fickett's theorem. Unfortunately, however, most attempts do not appear to have significantly improved Fickett's theorem, with the exception of the two cases by Alestalo et al. [14] and Väisälä [15]. Moreover, recently, the second author was stimulated by the two results just mentioned and succeeded in further improving them (see [16]).

In this article, we will further improve Fickett's theorem by applying the purely analytical method used in [16]. The purely analytic method to be used in this article is completely different from the methods used in [14,15]. Indeed, we prove the Hyers-Ulam stability of isometries defined on the bounded subsets of $\mathbb{R}^n$ for $n \geq 3$.

The advantage of this article is that the intermediate process is more precisely refined, and better results can be obtained compared to the previous paper [16], even though it assumes almost the same conditions as in the previous paper. Indeed, according to Theorem 4.1 or Corollary 4.2, we will find that the upper bound of the relevant inequality in this article is only less than half that of the previous paper [16].

2 Real version of QR decomposition

Throughout this article, we assume that n is a fixed integer not less than 3 and $\{e_1, e_2, \ldots, e_n\}$ is the standard basis for the n-dimensional Euclidean space $\mathbb{R}^n$. Furthermore, we assume that D, a subset of $\mathbb{R}^n$, contains $\{0, e_1, e_2, \ldots, e_n\}$. It does not matter at all whether D is a bounded set.

According to the real version of QR decomposition (see [16, Theorem 2.1]), for each function $f : D \to \mathbb{R}^n$, there exists an orthogonal matrix $\mathbf{Q}$ such that we may express

$$f(e_i) = (e'_{i1}, e'_{i2}, \ldots, e'_{ii}, 0, \ldots, 0)^{tr} = \sum_{j=1}^{i} e'_{ij}\mathbf{Q}e_j$$

with respect to the new basis $\{\mathbf{Q}e_1, \mathbf{Q}e_2, \ldots, \mathbf{Q}e_n\}$ for $\mathbb{R}^n$, where $e'_{ii} \geq 0$ for all $i \in \{1, 2, \ldots, n\}$. Therefore, we can assume that from now on we write $f(e_i) = (e'_{i1}, e'_{i2}, \ldots, e'_{ii}, 0, \ldots, 0)$ as row vectors for convenience, where each e'_{ii} is nonnegative.

3 A preliminary theorem

In the following theorem, we do not know in advance the exact values of c_{ij}. But we define σ as if we knew the values of c_{ij} in advance, because we know that there is nothing wrong with doing so. However, by (20) and (22), we note that

$$\sigma = \max_{2\le j\le n}\sum_{i=1}^{j-1} c_{ji}^2 = \sum_{i=1}^{n-1} c_{(i+1)i}^2 = \frac{6{,}481 + 3{,}504\sqrt{2}}{436}(n-1) = 26.2302852(n-1)$$

and

$$\min_{1\le i\le n}\frac{1}{2c_{ii}} = \frac{12+\sqrt{109}}{70} = 0.3205759.$$

In practice, σ defined in this article is the same as σ given in the previous paper [16]. Moreover, we note that $\min\left\{\frac{1}{\sigma}, \min_{1\le i\le n}\frac{1}{2c_{ii}}, \frac{1}{12}\right\} = \frac{1}{\sigma} = \frac{436}{6{,}481+3{,}504\sqrt{2}}\frac{1}{n-1} = 0.0381239\frac{1}{n-1}$.

The proof of the following theorem is similar to the proof of [16, Theorem 3.1], but in many places, they are quite different. Therefore, even if there is a overlap in a significant part of the proof, we consider it to be inefficient to point out them one by one and to cite the proof of the previous article. Hence, we will proceed with the proof of this theorem as far as possible without omission.

We remember that $\{e_1, e_2, \ldots, e_n\}$ is the standard basis for the n-dimensional Euclidean space $\mathbb{R}^n$ and 0 denotes the origin of $\mathbb{R}^n$.

Theorem 3.1. *Given an integer $n \ge 3$, let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and let D be a subset of $\mathbb{R}^n$ including the set $\{0, e_1, e_2, \ldots, e_n\}$. If a function $f : D \to \mathbb{R}^n$ satisfies $f(0) = 0$ and*

$$\big|\|f(x) - f(y)\| - \|x - y\|\big| \le \varepsilon \tag{2}$$

for all $x, y \in \{0, e_1, e_2, \ldots, e_n\}$ and for some constant ε with $0 < \varepsilon < \min\left\{\frac{1}{\sigma}, \min_{1\le i\le n}\frac{1}{2c_{ii}}, \frac{1}{12}\right\}$, where σ is defined as $\sigma = \max_{2\le j\le n}\sum_{i=1}^{j-1}c_{ji}^2$ and the c_{ii}'s and $c_{(i+1)i}$'s will be determined by formulas (20) and (22), respectively, then there exist positive real numbers c_{ij}'s, $i, j \in \{1, 2, \ldots, n\}$ with $j \le i$, such that

$$\begin{cases} -c_{ij}\varepsilon \le e'_{ij} \le c_{ij}\varepsilon & (\textit{for } i > j), \\ 1 - c_{ii}\varepsilon \le e'_{ii} \le 1 + \varepsilon & (\textit{for } i = j) \end{cases} \tag{3}$$

and such that the c_{ij}'s satisfy the relations in (19) for all $i, j \in \{1, 2, \ldots, n\}$ with $j \le i$.

Proof. (a) By using inequality (2) and assumption $f(0) = 0$, we have

$$\big|\|f(e_j)\| - 1\big| \le \varepsilon \quad \text{and} \quad \big|\|f(e_k) - f(e_\ell)\| - \sqrt{2}\big| \le \varepsilon$$

for any $j, k, \ell \in \{1, 2, \ldots, n\}$ with $k < \ell$. Since $f(e_j) = (e'_{j1}, \ldots, e'_{jj}, 0, \ldots, 0)$ for all $j \in \{1, 2, \ldots, n\}$ and $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$, from the last inequalities, we obtain the following two inequalities, which are equivalent to the inequality (2) for $x, y \in \{0, e_1, e_2, \ldots, e_n\}$:

$$(1-\varepsilon)^2 \le \sum_{i=1}^{j} e_{ji}'^2 \le (1+\varepsilon)^2 \tag{4}$$

for each $j \in \{1, 2, \ldots, n\}$ and

$$(\sqrt{2}-\varepsilon)^2 \le \sum_{i=1}^{k} e_{ki}'^2 - \sum_{i=1}^{k} 2e'_{ki}e'_{\ell i} + \sum_{i=1}^{\ell} e_{\ell i}'^2 \le (\sqrt{2}+\varepsilon)^2 \tag{5}$$

for every $k, \ell \in \{1, 2, \ldots, n\}$ with $k < \ell$. From now on, we will prove this theorem by using inequalities (4) and (5) instead of inequality (2).

(b) We will apply the "main" induction to prove the array of equations presented in (19). Proving the array in (19) is the most important and longest part of this proof.

(b.1) According to the QR decomposition, e'_{11} is a nonnegative real number, so setting $j = 1$ in (4) gives us the inequality, $1 - \varepsilon \le e'_{11} \le 1 + \varepsilon$, and we select $c_{11} = 1$ as the smallest positive real number that satisfies the following inequality:

$$1 - c_{11}\varepsilon \le 1 - \varepsilon \le e'_{11} \le 1 + \varepsilon. \tag{6}$$

This fact guarantees the existence of c_{11} satisfying the second condition of (3) for $i = j = 1$. If we set $j = 2$ in (4) and substitute $k = 1$ and $\ell = 2$ in (5) and then combine the resulting inequalities, then we obtain

$$\frac{-(2c_{11} + 2 + 2\sqrt{2})\varepsilon + c_{11}^2\varepsilon^2}{2(1 - c_{11}\varepsilon)} \le e'_{21} \le \frac{(4 + 2\sqrt{2})\varepsilon + \varepsilon^2}{2(1 - c_{11}\varepsilon)}.$$

By using the conditions $\varepsilon < \frac{1}{12}$ and $c_{11} = 1$, we choose $c_{21} = \frac{49 + 24\sqrt{2}}{24 - 2c_{11}} = 3.7700512$ as the smallest positive real number that satisfies the rightmost inequality of

$$-c_{21}\varepsilon \le \frac{-(2c_{11} + 2 + 2\sqrt{2})\varepsilon + c_{11}^2\varepsilon^2}{2(1 - c_{11}\varepsilon)} \le e'_{21} \le \frac{(4 + 2\sqrt{2})\varepsilon + \varepsilon^2}{2(1 - c_{11}\varepsilon)} \le c_{21}\varepsilon. \tag{7}$$

(Indeed, $c_{21} = 3.7700512$ is the "smallest" solution to the rightmost inequality in (7) when $\varepsilon = \frac{1}{12}$.) This fact confirms the existence of c_{21} satisfying the first condition of (3) for $i = 2$ and $j = 1$.

Furthermore, if we set $j = 3$ in (4) and substitute $k = 1$ and $\ell = 3$ into (5) and then combine the resulting inequalities, then we obtain the inner part of the following inequalities:

$$-c_{31}\varepsilon \le \frac{-(2c_{11} + 2 + 2\sqrt{2})\varepsilon + c_{11}^2\varepsilon^2}{2(1 - c_{11}\varepsilon)} \le e'_{31} \le \frac{(4 + 2\sqrt{2})\varepsilon + \varepsilon^2}{2(1 - c_{11}\varepsilon)} \le c_{31}\varepsilon, \tag{8}$$

and we select c_{31} as the smallest positive real number that satisfies the outermost inequalities of (8). By comparing inequalities (7) and (8), we choose c_{31} that satisfies $c_{31} = c_{21} = 3.7700512$. In this way, we can check the existence of c_{31} that satisfies the first condition in (3).

Moreover, the definition of σ and (4) with $j = 2$ imply the existence of a positive constant α that satisfies

$$(1 - \alpha\varepsilon)^2 \le (1 - \varepsilon)^2 - \varepsilon < (1 - \varepsilon)^2 - \sigma\varepsilon^2 \le (1 - \varepsilon)^2 - c_{21}^2\varepsilon^2 \le (1 - \varepsilon)^2 - e'^2_{21} \le e'^2_{22} \le (1 + \varepsilon)^2. \tag{9}$$

Now we want to select the smallest positive value of α that satisfies the first inequality of (9):

$$(1 - \alpha\varepsilon)^2 \le (1 - \varepsilon)^2 - \varepsilon.$$

The previous inequality is transformed into the quadratic inequality with respect to α:

$$\varepsilon\alpha^2 - 2\alpha + (3 - \varepsilon) \le 0,$$

whose solution is given by

$$\frac{3 - \varepsilon}{1 + \sqrt{1 - 3\varepsilon + \varepsilon^2}} \le \alpha \le \frac{3 - \varepsilon}{1 - \sqrt{1 - 3\varepsilon + \varepsilon^2}}.$$

Since we want the smallest possible value of α to satisfy the previous inequalities, we choose

$$\alpha = \alpha(\varepsilon) = \frac{3 - \varepsilon}{1 + \sqrt{1 - 3\varepsilon + \varepsilon^2}},$$

where $0 < \varepsilon < \min\left\{\frac{1}{\sigma}, \min_{1\le i\le n}\frac{1}{2c_{ii}}, \frac{1}{12}\right\}$. Furthermore, since $\frac{d}{d\varepsilon}\alpha(\varepsilon) > 0$ for any ε satisfying $0 < \varepsilon < \frac{1}{12}$, $\alpha(\varepsilon)$ increases as ε increases within such a small range of ε and

$$\alpha\left(\frac{1}{12}\right) = 12 - \sqrt{109} = 1.5596935.$$

In view of (9) and the previous argument, it holds that

$$1 - \alpha\left(\frac{1}{12}\right)\varepsilon < e'_{22} \le 1 + \varepsilon. \tag{10}$$

Thus, (10) assures the existence of $c_{22} = \alpha(\frac{1}{12}) = 12 - \sqrt{109} = 1.5596935$ satisfying the second condition of (3) for $i = j = 2$.

In a similar way, substitute $k = 2$ and $\ell = 3$ into (5) and using (4), a routine calculation shows the existence of c_{32} satisfying the first condition of (3). For example, since $-c_{21}^2\varepsilon^2 \le e'_{21}e'_{31} \le c_{21}^2\varepsilon^2$, it holds that

$$-c_{32}\varepsilon \le \frac{-(4 + 2\sqrt{2})\varepsilon + \varepsilon^2 - 2c_{21}^2\varepsilon^2}{2(1 - c_{22}\varepsilon)} \le e'_{32} \le \frac{(4 + 2\sqrt{2})\varepsilon + \varepsilon^2 + 2c_{21}^2\varepsilon^2}{2(1 - c_{22}\varepsilon)} \le c_{32}\varepsilon.$$

Since $\frac{1}{\varepsilon} > \sigma \ge c_{21}^2$, we have $2c_{21}^2\varepsilon^2 \le 2\varepsilon(c_{21}^2\varepsilon) < 2\varepsilon$. Moreover, since $\varepsilon < \frac{1}{12}$ and $c_{22} = 12 - \sqrt{109}$, we obtain $\frac{1}{2(1 - c_{22}\varepsilon)} < \frac{6}{\sqrt{109}}$. Hence, we obtain $c_{32} = \frac{73 + 24\sqrt{2}}{2\sqrt{109}} = 5.1215511$.

Analogously, the definition of σ and (4) with $j = 3$ implies the existence of a positive constant α that satisfies

$$(1 - \alpha\varepsilon)^2 \le (1 - \varepsilon)^2 - \varepsilon < (1 - \varepsilon)^2 - \sigma\varepsilon^2 \le (1 - \varepsilon)^2 - (c_{31}^2 + c_{32}^2)\varepsilon^2 \le (1 - \varepsilon)^2 - (e'^2_{31} + e'^2_{32}) \le e'^2_{33} \le (1 + \varepsilon)^2.$$

Repeating the same from (9) to (10), we obtain

$$1 - \alpha\left(\frac{1}{12}\right)\varepsilon < e'_{33} \le 1 + \varepsilon.$$

The last inequality assures the existence of $c_{33} = \alpha(\frac{1}{12}) = 12 - \sqrt{109} = 1.5596935$ satisfying the second condition of (3) for $i = j = 3$.

Therefore, all the constants c_{ij} considered in $(b.\ 1)$ satisfy the conditions in (3) and (19) for $n = 3$. By doing this, we start the induction (with $m = 3$).

(b.2) Induction hypothesis. Let m be some integer satisfying $3 \le m < n$. It is assumed that the smallest positive real numbers c_{ij}, $i, j \in \{1, 2, \ldots, m\}$ with $j \le i$, were found by the methods we did in the subsection $(b.\ 1)$, and that these numbers satisfy the following inequalities:

$$\begin{cases} -c_{ij}\varepsilon \le e'_{ij} \le c_{ij}\varepsilon & (\text{for } i > j), \\ 1 - c_{ii}\varepsilon \le e'_{ii} \le 1 + \varepsilon & (\text{for } i = j) \end{cases}$$

as well as the array of equations:

$$\begin{cases} c_{m1} = c_{(m-1)1} = c_{(m-2)1} = \ldots = c_{41} = c_{31} = c_{21}, \\ c_{m2} = c_{(m-1)2} = c_{(m-2)2} = \ldots = c_{42} = c_{32}, \\ c_{m3} = c_{(m-1)3} = c_{(m-2)3} = \ldots = c_{43}, \\ \vdots \qquad \vdots \qquad \vdots \\ c_{m(m-3)} = c_{(m-1)(m-3)} = c_{(m-2)(m-3)}, \\ c_{m(m-2)} = c_{(m-1)(m-2)}, \\ c_{m(m-1)}. \end{cases}$$

The last line in the aforementioned array consisting of only $c_{m(m-1)}$ means that there exists the smallest possible positive constant $c_{m(m-1)}$ that satisfies $-c_{m(m-1)}\varepsilon \le e'_{m(m-1)} \le c_{m(m-1)}\varepsilon$.

$(b.\ 3)$ We will now expand the equations in the direction of the arrows in the following equation.

$$\begin{cases} \leftarrow c_{m1} = c_{(m-1)1} = c_{(m-2)1} = \ldots = c_{41} = c_{31} = c_{21}, \\ \leftarrow c_{m2} = c_{(m-1)2} = c_{(m-2)2} = \ldots = c_{42} = c_{32}, \\ \leftarrow c_{m3} = c_{(m-1)3} = c_{(m-2)3} = \ldots = c_{43}, \\ \vdots \quad \vdots \qquad \vdots \qquad \vdots \\ \leftarrow c_{m(m-3)} = c_{(m-1)(m-3)} = c_{(m-2)(m-3)}, \\ \leftarrow c_{m(m-2)} = c_{(m-1)(m-2)}, \\ \leftarrow c_{m(m-1)}. \end{cases} \tag{11}$$

We let $j = m + 1$ in (4) and $\ell = m + 1$ in (5) to obtain

$$(1-\varepsilon)^2 \le \sum_{i=1}^{m+1} e_{(m+1)i}'^2 \le (1+\varepsilon)^2 \tag{12}$$

and

$$(\sqrt{2}-\varepsilon)^2 \le \sum_{i=1}^{k} e_{ki}'^2 - \sum_{i=1}^{k} 2e_{ki}'e_{(m+1)i}' + \sum_{i=1}^{m+1} e_{(m+1)i}'^2 \le (\sqrt{2}+\varepsilon)^2 \tag{13}$$

for every $k \in \{1, 2, \ldots, m\}$.

Similar to what we did to obtain (7), the inequalities (6), (12), and (13) with $k = 1$ yield the inner ones of the following inequalities:

$$-c_{(m+1)1}\varepsilon \le \frac{-(2c_{11} + 2 + 2\sqrt{2})\varepsilon + c_{11}^2\varepsilon^2}{2(1 - c_{11}\varepsilon)} \le e_{(m+1)1}' \le \frac{(4 + 2\sqrt{2})\varepsilon + \varepsilon^2}{2(1 - c_{11}\varepsilon)} \le c_{(m+1)1}\varepsilon, \tag{14}$$

and we find the smallest positive real number $c_{(m+1)1}$ satisfying the outermost inequalities of (14). By comparing both inequalities (7) and (14), we may conclude that $c_{(m+1)1} = c_{21}$, with which we initiate an "inner" induction that is subordinate to the main induction. (We start the induction in the direction of the arrow as shown in the following equation.)

$$\begin{cases} c_{(m+1)1} & = & c_{m1} & = & c_{(m-1)1} & = & c_{(m-2)1} & = \cdots = & c_{41} & = & c_{31} & = & c_{21}, \\ \downarrow & & c_{m2} & = & c_{(m-1)2} & = & c_{(m-2)2} & = \cdots = & c_{42} & = & c_{32}, \\ \downarrow & & c_{m3} & = & c_{(m-1)3} & = & c_{(m-2)3} & = \cdots = & c_{43}, \\ \vdots & & \vdots & & \vdots & & \vdots \\ \downarrow & & c_{m(m-3)} & = & c_{(m-1)(m-3)} & = & c_{(m-2)(m-3)}, \\ \downarrow & & c_{m(m-2)} & = & c_{(m-1)(m-2)}, \\ \downarrow & & c_{m(m-1)}. \end{cases}$$

(b.3.1) We choose some $k \in \{2, 3, \ldots, m\}$ and assume that $-c_{(m+1)i}\varepsilon \le e_{(m+1)i}' \le c_{(m+1)i}\varepsilon$ and $c_{(m+1)i} = c_{(i+1)i}$ for each $i \in \{1, 2, \ldots, k-1\}$. This is the hypothesis for our inner induction on i that operates inside the main induction on m. On the basis of hypothesis, we will prove that there exists a positive real number $c_{(m+1)k}$ that satisfies $-c_{(m+1)k}\varepsilon \le e_{(m+1)k}' \le c_{(m+1)k}\varepsilon$ as well as $c_{(m+1)k} = c_{(k+1)k}$. Roughly speaking, we apply this inner induction to horizontally expand each row to the left as shown in (11).

(b.3.2) It follows from (13) that

$$\begin{aligned} (\sqrt{2}-\varepsilon)^2 - \sum_{i=1}^{k} e_{ki}'^2 + \sum_{i=1}^{k-1} 2e_{ki}'e_{(m+1)i}' - \sum_{i=1}^{m+1} e_{(m+1)i}'^2 \\ \le -2e_{kk}'e_{(m+1)k}' \le \\ (\sqrt{2}+\varepsilon)^2 - \sum_{i=1}^{k} e_{ki}'^2 + \sum_{i=1}^{k-1} 2e_{ki}'e_{(m+1)i}' - \sum_{i=1}^{m+1} e_{(m+1)i}'^2 \end{aligned} \tag{15}$$

for any $k \in \{2, 3, \ldots, m\}$. On the other hand, by (4) and (12), we have

$$(1-\varepsilon)^2 \le \sum_{i=1}^{k} e_{ki}'^2 \le (1+\varepsilon)^2 \quad \text{and} \quad (1-\varepsilon)^2 \le \sum_{i=1}^{m+1} e_{(m+1)i}'^2 \le (1+\varepsilon)^2$$

for each $k \in \{2, 3, \ldots, m\}$. Moreover, it follows from the hypotheses $(b.\ 2)$ and $(b.\ 3.1)$ that

$$-\sum_{i=1}^{k-1} 2c_{ki}c_{(i+1)i}\varepsilon^2 = -\sum_{i=1}^{k-1} 2c_{ki}c_{(m+1)i}\varepsilon^2 \le \sum_{i=1}^{k-1} 2e_{ki}'e_{(m+1)i}' \le \sum_{i=1}^{k-1} 2c_{ki}c_{(m+1)i}\varepsilon^2 = \sum_{i=1}^{k-1} 2c_{ki}c_{(i+1)i}\varepsilon^2$$

for all $k \in \{2, 3, \ldots, m\}$.

Since $c_{kk}\varepsilon < \frac{1}{2}$ and $e'_{kk} > 0$ by (3), we use (15) and the last inequalities to obtain the inner ones of the following inequalities:

$$\begin{aligned} -c_{(m+1)k}\varepsilon &\leq \frac{1}{2e'_{kk}}\left(-(4+2\sqrt{2})\varepsilon + \varepsilon^2 - 2\sum_{i=1}^{k-1} c_{ki}c_{(i+1)i}\varepsilon^2\right) \\ &\leq e'_{(m+1)k} \leq \\ \frac{1}{2e'_{kk}}&\left((4+2\sqrt{2})\varepsilon + \varepsilon^2 + 2\sum_{i=1}^{k-1} c_{ki}c_{(i+1)i}\varepsilon^2\right) \leq c_{(m+1)k}\varepsilon \end{aligned} \tag{16}$$

for all $k \in \{2, 3, \ldots, m\}$, and we select the smallest positive constant $c_{(m+1)k}$ that satisfies the outermost inequalities of (16).

Similarly, by (4) and (5) with $\ell = k+1$, a routine calculation yields

$$\begin{aligned} -c_{(k+1)k}\varepsilon &\leq \frac{1}{2e'_{kk}}\left(-(4+2\sqrt{2})\varepsilon + \varepsilon^2 - 2\sum_{i=1}^{k-1} c_{ki}c_{(k+1)i}\varepsilon^2\right) \\ &\leq e'_{(k+1)k} \leq \\ \frac{1}{2e'_{kk}}&\left((4+2\sqrt{2})\varepsilon + \varepsilon^2 + 2\sum_{i=1}^{k-1} c_{ki}c_{(k+1)i}\varepsilon^2\right) \leq c_{(k+1)k}\varepsilon, \end{aligned} \tag{17}$$

where $c_{(k+1)k}$ is the smallest positive real number that satisfies the outmost conditions of (17). We note by $(b.\ 2)$ and $(b.\ 3.1)$ that $c_{(k+1)i} = c_{(i+1)i}$ for every integer i satisfying $0 < i < k$. Comparing (16) and (17), we conclude that $c_{(m+1)k} = c_{(k+1)k}$ for each $k \in \{2, 3, \ldots, m\}$. Furthermore, referring to the subsection $(b.\ 3)$, we see that $c_{(m+1)k} = c_{(k+1)k}$ holds for all $k \in \{1, 2, \ldots, m\}$, which proves the truth of the first column of the array of equations in subsection $(b.\ 3.3)$ below.

Moreover, inequality (4) with $j = m+1$ yields

$$(1-\varepsilon)^2 - \sum_{i=1}^{m} e'^2_{(m+1)i} \leq e'^2_{(m+1)(m+1)} \leq (1+\varepsilon)^2 - \sum_{i=1}^{m} e'^2_{(m+1)i}. \tag{18}$$

Since inequality (4) holds for all $j \in \{1, 2, \ldots, n\}$, the definition of σ and (4) imply that there exists a positive constant α such that

$$\begin{aligned} (1-\alpha\varepsilon)^2 &\leq (1-\varepsilon)^2 - \varepsilon < (1-\varepsilon)^2 - \sigma\varepsilon^2 \\ &\leq (1-\varepsilon)^2 - (c^2_{(m+1)1} + \cdots + c^2_{(m+1)m})\varepsilon^2 \\ &\leq (1-\varepsilon)^2 - (e'^2_{(m+1)1} + \cdots + e'^2_{(m+1)m}) \\ &\leq e'^2_{(m+1)(m+1)} \leq (1+\varepsilon)^2. \end{aligned}$$

By repeating the same from (9) to (10), we obtain

$$1 - \alpha\left(\frac{1}{12}\right)\varepsilon < e'_{(m+1)(m+1)} \leq 1 + \varepsilon.$$

The previous inequality assures the existence of $c_{(m+1)(m+1)} = \alpha(\frac{1}{12}) = 12 - \sqrt{109} = 1.5596935$ satisfying the second condition of (3) for $i = j = m+1$.

(b.3.3) We just proved in the subsections (b.2) to (b.3.2) that there exist positive real numbers c_{ij}, $i, j \in \{1, 2, \ldots, m+1\}$ with $j \leq i$, such that

$$\begin{cases} -c_{ij}\varepsilon \leq e'_{ij} \leq c_{ij}\varepsilon & (\text{for } i > j), \\ 1 - c_{ii}\varepsilon \leq e'_{ii} \leq 1 + \varepsilon & (\text{for } i = j), \end{cases}$$

and the c_{ij}'s satisfy the array:

$$\begin{cases} c_{(m+1)1} = c_{m1} = c_{(m-1)1} = \dots = c_{41} = c_{31} = c_{21}, \\ c_{(m+1)2} = c_{m2} = c_{(m-1)2} = \dots = c_{42} = c_{32}, \\ c_{(m+1)3} = c_{m3} = c_{(m-1)3} = \dots = c_{43}, \\ \quad\vdots \qquad\quad \vdots \qquad\quad \vdots \\ c_{(m+1)(m-2)} = c_{m(m-2)} = c_{(m-1)(m-2)}, \\ c_{(m+1)(m-1)} = c_{m(m-1)}, \\ c_{(m+1)m}. \end{cases}$$

The last row in the aforementioned array consisting of only $c_{(m+1)m}$ means that there exists the smallest positive real number $c_{(m+1)m}$ that satisfies $-c_{(m+1)m}\varepsilon \le e'_{(m+1)m} \le c_{(m+1)m}\varepsilon$.

(b.4) Altogether, by the main induction conclusion on m $(3 \le m < n)$, we may conclude that there exist positive constants c_{ij}, $i, j \in \{1, 2, \dots, n\}$ with $j \le i$, such that each inequality in (3) holds true and the c_{ij}'s satisfy

$$\begin{cases} c_{n1} = c_{(n-1)1} = c_{(n-2)1} = \dots = c_{41} = c_{31} = c_{21}, \\ c_{n2} = c_{(n-1)2} = c_{(n-2)2} = \dots = c_{42} = c_{32}, \\ c_{n3} = c_{(n-1)3} = c_{(n-2)3} = \dots = c_{43}, \\ \quad\vdots \qquad\quad \vdots \qquad\quad \vdots \\ c_{n(n-3)} = c_{(n-1)(n-3)} = c_{(n-2)(n-3)}, \\ c_{n(n-2)} = c_{(n-1)(n-2)}, \\ c_{n(n-1)}, \end{cases} \tag{19}$$

which completes the first part of our proof. We remark that the last row "$c_{n(n-1)}$" in the aforementioned array implies that there is a real number $c_{n(n-1)} > 0$ satisfying $-c_{n(n-1)}\varepsilon \le e'_{n(n-1)} \le c_{n(n-1)}\varepsilon$.

(c) As we showed at the end of (b.3.2), it holds that

$$c_{jj} = 12 - \sqrt{109} = 1.5596935 \tag{20}$$

for all $j \in \{1, 2, \dots, n\}$. Now, all that remains is to estimate the positive real numbers $c_{k(k-1)}$ more efficiently than the previous article [16] for $k \in \{2, 3, \dots, n\}$. From now on, we refine from (c) to the end of the proof of [16, Theorem 3.1] more precisely so that the positive constants $c_{k(k-1)}$ have smaller values.

We note that inequality (17) holds for every $k \in \{1, 2, \dots, n-1\}$. And then, we choose the smallest constant $c_{(k+1)k} > 0$ that satisfies the outermost inequalities of (17). Indeed, by inequalities (3) and (17), we can choose the smallest positive real number $c_{(k+1)k}$ as follows:

$$c_{(k+1)k} = \frac{1}{2(1 - c_{kk}\varepsilon)}\left(4 + 2\sqrt{2} + \varepsilon + 2\sum_{i=1}^{k-1} c_{ki}c_{(k+1)i}\varepsilon\right) \tag{21}$$

for $k \in \{1, 2, \dots, n-1\}$. Since $0 < \varepsilon < \frac{1}{2c_{kk}}$, we see that $\frac{1}{2(1-c_{kk}\varepsilon)} < 1$. Further, since $c_{ki} = c_{(k+1)i} = c_{(i+1)i}$ for any $i \in \{1, 2, \dots, k-1\}$ and $0 < \varepsilon < \frac{1}{\sigma}$, we know that $\sum_{i=1}^{k-1} c_{ki}c_{(k+1)i}\varepsilon \le \sigma\varepsilon < 1$. Thus, since $0 < \varepsilon < \frac{1}{12}$, we substitute $\varepsilon = \frac{1}{12}$ and $c_{kk} = 12 - \sqrt{109}$ in (21) to obtain

$$c_{(k+1)k} = \frac{73 + 24\sqrt{2}}{2\sqrt{109}} = 5.1215511 \tag{22}$$

for every $k \in \{1, 2, \dots, n-1\}$. □

Remark 3.1. The inequality (2) is a sufficient condition for inequalities in (3), and the inequalities in (3) are necessary conditions for the inequality (2).

4 Hyers-Ulam stability of isometries on bounded domains

As mentioned earlier, $\{e_1, e_2, \ldots, e_n\}$ denotes the standard basis for the n-dimensional Euclidean space $\mathbb{R}^n$ and 0 denotes the origin of $\mathbb{R}^n$. As already explained in Section 2, we can assume that $f(e_i) = (e'_{i1}, e'_{i2}, \ldots, e'_{ii}, 0, \ldots, 0)$ is written in row vector, where $e'_{ii} \geq 0$ for each $i \in \{1, 2, \ldots, n\}$. We denote by $B_d(0)$ the closed ball of radius d and centered at the origin of $\mathbb{R}^n$, *i.e.*, $B_d(0) = \{x \in \mathbb{R}^n : \|x\| \leq d\}$.

In the previous article [16], we restricted the values of c_{ij}s to positive integer values only. On the other hand, we were able to significantly reduce the sizes of the c_{ij}s in Theorem 3.1 by allowing them to have positive real numbers in this article.

The following theorem is practically identical to [16, Theorem 4.1], with only a few different assumptions. Slight differences between the two theorems are shown in the following table.

In [16]:	"integers"	$\sigma = \sum_{i=1}^{n-1} c_{(i+1)i}^2$
In this article:	"real numbers"	$\sigma = \max_{2\leq j\leq n} \sum_{i=1}^{j-1} c_{ji}^2$

These small differences do not affect the proof of the following theorem, so the proof proceeds the same as the proof of [16, Theorem 4.1]. Therefore, the proof of the following theorem is omitted.

Theorem 4.1. *Given a fixed integer $n \geq 3$, let D be a bounded subset of the n-dimensional Euclidean space $\mathbb{R}^n$ such that $\{0, e_1, e_2, \ldots, e_n\} \subset D \subset B_d(0)$ for some $d \geq 1$. If a function $f : D \to \mathbb{R}^n$ satisfies $f(0) = 0$ and the inequality* (2) *for all $x, y \in D$ and for some constant ε satisfying $0 < \varepsilon < \min\left\{\frac{1}{\sigma}, \min_{1\leq i\leq n} \frac{1}{2c_{ii}}, \frac{1}{12}\right\}$, where* $\sigma = \max_{2\leq j\leq n} \sum_{i=1}^{j-1} c_{ji}^2$ and the c_{ij}s *are positive real numbers estimated in Theorem* 3.1, *then there exists an isometry $U : D \to \mathbb{R}^n$ such that*

$$\|f(x) - U(x)\| \leq \left[\sum_{i=1}^{n} \left(\left(2 + \sum_{j=1}^{i} c_{ij}\right)(d+1) + 2\right)^2\right]^{1/2} \varepsilon \tag{23}$$

for all $x \in D$.

By using (19), (20), and (22), we obtain

$$3 + \sum_{j=1}^{i} c_{ij} = 3 + c_{21} + c_{32} + \cdots + c_{i(i-1)} + c_{ii} = \frac{(73 + 24\sqrt{2})\sqrt{109}}{218}(i-1) + 15 - \sqrt{109} = A(i-1) + B,$$

where we set

$$A = \frac{(73 + 24\sqrt{2})\sqrt{109}}{218} = 5.1215511 \quad \text{and} \quad B = 15 - \sqrt{109} = 4.5596935.$$

Furthermore, for $n \geq 3$, we have

$$\begin{aligned}
\sum_{i=1}^{n}\left(\left(2 + \sum_{j=1}^{i} c_{ij}\right)(d+1) + 2\right)^2 &\leq \sum_{i=1}^{n}\left(3 + \sum_{j=1}^{i} c_{ij}\right)^2 (d+1)^2 \\
&= \left(\frac{A^2}{3}n^3 + A\left(B - \frac{A}{2}\right)n^2 + \left(\frac{A^2}{6} - AB + B^2\right)n\right)(d+1)^2 \\
&\leq \left(\frac{A^2}{3}n^3 + \frac{A}{3}\left(B - \frac{A}{2}\right)n^3 + \frac{1}{9}\left(\frac{A^2}{6} - AB + B^2\right)n^3\right)(d+1)^2 \\
&= \frac{1}{27}(5A^2 + 6AB + 3B^2)(d+1)^2 n^3.
\end{aligned}$$

Then inequality (23) is transformed into

$$\|f(x) - U(x)\| \le \left[\sum_{i=1}^{n}\left(\left(2 + \sum_{j=1}^{i} c_{ij}\right)(d+1) + 2\right)^2\right]^{1/2} \varepsilon \le \left[\frac{1}{27}(5A^2 + 6AB + 3B^2)\right]^{1/2}(d+1)n\sqrt{n}\,\varepsilon$$

for all $x \in D$.

From Theorem 4.1 and the explanations described earlier, we obtain the following corollary.

Corollary 4.2. *Given a fixed integer $n \ge 3$, let D be a bounded subset of the n-dimensional Euclidean space $\mathbb{R}^n$ such that $\{0, e_1, e_2, \ldots, e_n\} \subset D \subset B_d(0)$ for some $d \ge 1$. If a function $f : D \to \mathbb{R}^n$ satisfies $f(0) = 0$ and the inequality (2) for all $x, y \in D$ and for some constant ε satisfying $0 < \varepsilon < \min\left\{\frac{1}{\sigma}, \min_{1\le i\le n}\frac{1}{2c_{ii}}, \frac{1}{12}\right\}$, where $\sigma = \max_{2\le j\le n}\sum_{i=1}^{j-1} c_{ji}^2$ and the c_{ij}s are positive real numbers estimated in Theorem 3.1, then there exists an isometry $U : D \to \mathbb{R}^n$ that satisfies*

$$\|f(x) - U(x)\| \le 3.5152581(d+1)n\sqrt{n}\ \varepsilon$$

for all $x \in D$.

5 Examples

Theorem 4.1 and Corollary 4.2 in this article can be applied only when $n \ge 3$. However, in a recent paper [17], the Hyers-Ulam stability of local isometries for $n \in \{2, 3\}$ was studied in detail. We introduce examples inferred from this result.

Example 5.1. Let $D = \{x \in \mathbb{R}^2 : \|x\| \le d\}$ for some $d \ge 1$ and let $f : D \to \mathbb{R}^2$ be a function satisfying $f(0) = 0$ and the inequality (2) for all $x, y \in D$ and some constant ε with $0 < \varepsilon < \frac{1}{13} \approx 0.0769231$. According to [17, Theorem 3.2], there exists an isometry $U : D \to \mathbb{R}^2$ that satisfies

$$\|f(x) - U(x)\| \le (8d + 4)\varepsilon$$

for all $x \in D$.

Example 5.2. Let $D = \{x \in \mathbb{R}^3 : \|x\| \le d\}$ for some $d \ge 1$ and let $f : D \to \mathbb{R}^3$ be a function satisfying $f(0) = 0$ and the inequality (2) for all $x, y \in D$ and some constant ε with $0 < \varepsilon < \frac{1}{13} \approx 0.0769231$. According to [17, Theorem 3.4], there exists an isometry $U : D \to \mathbb{R}^3$ that satisfies

$$\|f(x) - U(x)\| \le (16d + 5)\varepsilon$$

for all $x \in D$.

Now, we introduce some examples made by substituting $n \in \{4, 5\}$ in Corollary 4.2.

Example 5.3. Let $D = \{x \in \mathbb{R}^4 : \|x\| \le d\}$ for some $d \ge 1$ and let $f : D \to \mathbb{R}^4$ be a function satisfying $f(0) = 0$ and the inequality (2) for all $x, y \in D$ and some constant ε with $0 < \varepsilon < 0.0127079$. According to Corollary 4.2, there exists an isometry $U : D \to \mathbb{R}^4$ that satisfies

$$\|f(x) - U(x)\| \le 28.1220649(d+1)\varepsilon$$

for all $x \in D$.

Example 5.4. Let $D = \{x \in \mathbb{R}^5 : \|x\| \le d\}$ for some $d \ge 1$ and let $f : D \to \mathbb{R}^5$ be a function satisfying $f(0) = 0$ and the inequality (2) for all $x, y \in D$ and some constant ε with $0 < \varepsilon < 0.0095309$. According to Corollary 4.2, there exists an isometry $U : D \to \mathbb{R}^5$ that satisfies

$$\|f(x) - U(x)\| \le 39.3017804(d + 1)\varepsilon$$

for all $x \in D$.

6 Discussion and conclusion

We completed the following table by referring to the table presented in Section 6 of [16] and using the formula presented in Corollary 4.2.

i	1	2	3	4	5	⋯
$c^*(1, i)$	4	>79	>799	>7990	>79900	⋯
$8(d+1)i\sqrt{i}$	—	—	<84	128	<179	⋯
$3.5152581(d+1)i\sqrt{i}$	—	—	<37	<57	<79	⋯

The values in the first row of the aforementioned table were obtained by substituting $c = 1$ in the formula presented in the proof of [15, Theorem 4.1]. The values in the second row and the last row are due to the formulas given in [16, Corollary 4.2] and Corollary 4.2 of this article with $d = 1$, respectively. Comparing the values in the three rows of the aforementioned table, we see that our present result is more efficient than those of Väisälä and [16].

We emphasize that we greatly improved Fickett's theorem by using a purely intuitive method. It follows from Theorem 4.1 or Corollary 4.2 that if a function $f : D \to \mathbb{R}^n$ satisfies $f(0) = 0$ and inequality (2) for any $x, y \in D$ and for some small constant $\varepsilon > 0$, then there exists an isometry $U : D \to \mathbb{R}^n$ and a constant $K > 0$ such that the inequality $\|f(x) - U(x)\| \le K\varepsilon$ holds true for all $x \in D$. Unfortunately, however, it seems impossible to derive this useful conclusion using Fickett's theorem. From this point of view, we dare to say that we have significantly improved Fickett's theorem in this article.

The second author informed very recently that Vestfrid had obtained similar results [16]. It will be interesting to compare Vestfrid's results [18] with those of [16] and this article. Research on finding a smaller upper bound in the inequality of Corollary 4.2 seems necessary, and this kind of research will be an interesting task in the future.

Acknowledgments: The authors would like to thank the reviewers for their valuable suggestions and comments.

Funding information: This work was supported by 2022 Hongik University Research Fund. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2020R1F1A1A01049560).

Author contributions: Conceptualization: GC and SMJ; methodology: GC and SMJ; validation: GC and SMJ; investigation: GC and SMJ; writing – original draft preparation: GC and SMJ; writing – review and editing: GC and SMJ; project administration: SMJ; funding acquisition: GC and SMJ.

Conflict of interest: The authors state no conflict of interest.

References

[1] D. H. Hyers and S. M. Ulam, *On approximate isometries*, Bull. Amer. Math. Soc. **51** (1945), 288–292.
[2] D. G. Bourgin, *Approximate isometries*, Bull. Amer. Math. Soc. **52** (1946), 704–714.
[3] D. G. Bourgin, *Approximately isometric and multiplicative transformations on continuous function rings*, Duke Math. J. **16** (1949), 385–397.
[4] R. D. Bourgin, *Approximate isometries on finite dimensional Banach spaces*, Trans. Amer. Math. Soc. **207** (1975), 309–328.
[5] L. Cheng, Q. Cheng, K. Tu and J. Zhang, *A universal theorem for stability of ε-isometries of Banach spaces*, J. Funct. Anal. **269** (2015), no. 1, 199–214.
[6] L. Cheng and Y. Dong, *A note on the stability of nonsurjective ε-isometries of Banach spaces*, Proc. Amer. Math. Soc. **148** (2020), 4837–4844.
[7] J. Gevirtz, *Stability of isometries on Banach spaces*, Proc. Amer. Math. Soc. **89** (1983), 633–636.
[8] P. M. Gruber, *Stability of isometries*, Trans. Amer. Math. Soc. **245** (1978), 263–277.
[9] D. H. Hyers and S. M. Ulam, *Approximate isometries of the space of continuous functions*, Ann. Math. **48** (1947), 285–289.
[10] M. Omladicccc and P. SSSemrl, *On non linear perturbations of isometries*, Math. Ann. **303** (1995), 617–628.
[11] I. A. Vestfrid, *Stability of almost surjective ε-isometries of Banach spaces*, J. Funct. Anal. **269** (2015), no. 7, 2165–2170.
[12] I. A. Vestfrid, *Hyers-Ulam stability of isometries and non-expansive maps between spaces of continuous functions*, Proc. Amer. Math. Soc. **145** (2017), 2481–2494.
[13] J. W. Fickett, *Approximate isometries on bounded sets with an application to measure theory*, Studia Math. **72** (1982), 37–46.
[14] P. Alestalo, D. A. Trotsenko and J. Väisälä, *Isometric approximation*, Israel J. Math. **125** (2001), 61–82.
[15] J. Väisälä, *Isometric approximation property in Euclidean spaces*, Israel J. Math. **128** (2002), 1–27.
[16] S.-M. Jung, *Hyers-Ulam stability of isometries on bounded domains*, Open Math. **19** (2021), no. 1, 675–689.
[17] S.-M. Jung, J. Roh and D.-J. Yang, *On the improvement of Fickettas theorem on bounded sets*, J. Inequ. Appl. **2022** (2022), no. 17, 13.
[18] I. A. Vestfrid, *ε-Isometries in Euclidean spaces*, Nonlinear Anal. **63** (2005), 1191–1198.