

Research Article

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Uniqueness of solutions for a ψ -Hilfer fractional integral boundary value problem with the p -Laplacian operator

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Abstract: In this article, we discuss the existence of a unique solution to a ψ -Hilfer fractional differential equation involving the p -Laplacian operator subject to nonlocal ψ -Riemann-Liouville fractional integral boundary conditions. Banach's fixed point theorem is the main tool of our study. Examples are given for illustrating the obtained results.

Keywords: ψ -Hilfer fractional derivative, the p -Laplacian operator, existence, fixed point

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1 Introduction

Fractional differential operators are found to be of great utility in the mathematical modeling of natural and engineering phenomena, for example, see [1–6] and the references cited therein. In contrast to the integer-order differential operators, these operators are nonlocal in nature and account for the history of the physical phenomena under consideration. In the literature, there do exist several kinds of fractional integrals and derivatives, for instance, see [7–10]. Hilfer in [11] proposed a generalized fractional derivative of order α and type β , which is known as the Hilfer fractional derivative, and it interpolates between the Riemann-Liouville and Caputo derivatives. This two-parameter fractional derivative operator appeared in the modeling of diffusion models, dielectric relaxation in glass-forming materials, etc. [12,13]. For some recent works on Hilfer fractional differential equations, we refer the reader to the articles [14–18]. In [19], Sousa and Capelas de Oliveira generalized the Hilfer fractional derivative by introducing the concept of ψ -Hilfer fractional derivative. One of the advantages of this derivative is that it covers a wide class of fractional derivatives, which can be fixed by choosing the function ψ appropriately. Later, this derivative gained much attention, and many researchers turned to investigate it, for example, see [20–26].

On the other hand, differential equations with the p -Laplacian operator appeared for the first time when Leibenson [27] was attempting to derive an accurate formula to model turbulent flow in the porous medium. Keeping in mind the application of differential equations involving the p -Laplacian operator in

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the areas of mechanics, nonlinear dynamics, glaciology, nonlinear elasticity, flow through porous media, and so on, many authors focused on this topic. For details and examples, one can see the article [28–32]. As far as we know, there is no work dealing with ψ -Hilfer fractional differential equations with the p -Laplacian operator.

The objective this study is to investigate a ψ -Hilfer fractional boundary value problem involving the p -Laplacian operator $\phi_p(\cdot)$ and ψ -Riemann-Liouville fractional integral boundary conditions given as follows:

$$\begin{cases} (\phi_p({}^H D_{0+}^{\alpha,\beta,\psi} y(t)))' + f(t, y(t)) = 0, & 0 < t < 1, \\ y(0) = 0, {}^H D_{0+}^{\alpha,\beta} y(0) = 0, & y(1) = \sum_{i=1}^m a_i I_{0+}^{\sigma_i,\psi} y(\mu_i), \quad 0 < \mu_i < 1, \quad a_i \in \mathbb{R}, \end{cases} \quad (1)$$

where $\phi_p(t) = t|t|^{p-2}$, $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$, ${}^H D_{0+}^{\alpha,\beta,\psi}$ denotes the ψ -Hilfer fractional derivative operator of order $\alpha \in (1, 2]$ and type $\beta \in [0, 1]$, $I_{0+}^{\sigma_i,\psi}$ is the ψ -Riemann-Liouville fractional integral operator of order $\sigma_i > 0$, and $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

The rest of the article is organized as follows: Section 2 presents the background material related to our work, Section 3 contains the main results for problem (1), and Section 4 presents examples to illustrate the obtained results.

2 Preliminaries

Let $C([a, b], \mathbb{R})$ denote the Banach space of all continuous functions from $[a, b]$ into $\mathbb{R}$ endowed with the supremum norm $\|y\| = \sup_{t \in [a,b]} |y(t)|$. Let $C^n([a, b], \mathbb{R})$ be the class all n -times continuously differential functions from $[a, b]$ into $\mathbb{R}$.

In the forthcoming analysis, it is assumed that ψ is an increasing and positive monotone function on $(0, 1]$ possessing a continuous derivative $\psi'(t) \neq 0$ on $(0, 1)$.

Definition 1. [7] For $\alpha > 0$ and $-\infty \leq a < b \leq \infty$, the left-sided ψ -Riemann-Liouville fractional integral of an integrable function f on $[a, b]$ is defined as follows:

$$(I_{a+}^{\alpha,\psi} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} f(s) ds. \quad (2)$$

Definition 2. [7] Let $n - 1 \leq \alpha < n$ ($n = [\alpha] + 1$), the left-sided ψ -Riemann-Liouville fractional derivative of a function $f \in C([a, b], \mathbb{R})$ is given as follows:

$$(D_{a+}^{\alpha,\psi} f)(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_a^t \psi'(s) (\psi(t) - \psi(s))^{n-\alpha-1} f(s) ds. \quad (3)$$

Definition 3. [33] Let $n - 1 \leq \alpha < n$ ($n = [\alpha] + 1$), the left-sided ψ -Caputo fractional derivative of a function $f \in C^n([a, b], \mathbb{R})$ is defined as follows:

$$({}^C D_{a+}^{\alpha,\psi} f)(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{\psi'(s)}{[\psi(t) - \psi(s)]^{1-n+\alpha}} \left(\frac{1}{\psi'(s)} \frac{d}{ds} \right)^n f(s) ds. \quad (4)$$

Definition 4. [19] For $\alpha > 0$, $n = [\alpha] + 1$, $\beta \in [0, 1]$, and $-\infty \leq a < x < b \leq \infty$, the left-sided ψ -Hilfer fractional derivative for a function $f \in C^n([a, b], \mathbb{R})$ is defined as follows:

$$({}^H D_{a+}^{\alpha, \beta, \psi} f)(t) = I_{a+}^{\beta(n-\alpha), \psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{a+}^{(1-\beta)(n-\alpha), \psi} f(t), \quad (5)$$

which can alternatively be written as

$$({}^H D_{a+}^{\alpha, \beta, \psi} f)(t) = I_{a+}^{\gamma-\alpha, \psi} D_{a+}^{\gamma, \psi} f(t), \quad (6)$$

where $\gamma = \alpha + \beta(n - \alpha)$.

Lemma 1. [19] Let $n - 1 < \alpha \leq n$, $\beta \in [0, 1]$ and $\gamma = \alpha + \beta(n - \alpha)$.

(1) If $f \in C^n([a, b], \mathbb{R})$, then

$$I_{a+}^{\alpha, \psi} ({}^H D_{a+}^{\alpha, \beta, \psi} f)(t) = f(t) - \sum_{i=1}^n \frac{(\psi(t) - \psi(a))^{\gamma-i}}{\Gamma(\gamma-i+1)} d_{\psi}^{[n-i]} I_{a+}^{(1-\beta)(n-\alpha), \psi} f(a), \quad (7)$$

$$\text{where } d_{\psi}^{[n-i]} f(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^{n-i}.$$

(2) If $f \in C([a, b], \mathbb{R})$, then

$${}^H D_{a+}^{\alpha, \beta, \psi} I_{a+}^{\alpha, \psi} f(t) = f(t). \quad (8)$$

In the following lemma, we solve the linear variant of problem (1).

Lemma 2. For $1 < \alpha < 2$, $\beta \in [0, 1]$, $\gamma = \alpha + \beta(2 - \alpha)$, and $h \in C([0, 1], \mathbb{R})$, the integral representation of the solution for the following linear ψ -Hilfer p -Laplacian fractional integral boundary value problem:

$$\begin{cases} (\phi_p({}^H D_{0+}^{\alpha, \beta, \psi} y(t)))' + h(t) = 0, \\ y(0) = 0, {}^H D_{0+}^{\alpha, \beta, \psi} y(0) = 0, \quad y(1) = \sum_{i=1}^m a_i I_{0+}^{\sigma_i, \psi} y(\mu_i), \end{cases} \quad (9)$$

is as follows:

$$\begin{aligned} y(t) = & -\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\tau) (\psi(t) - \psi(\tau))^{\alpha-1} \left(\phi_q \left(\int_0^{\tau} h(s) ds \right) \right) d\tau + \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Lambda \Gamma(\gamma)} \\ & \times \left\{ \frac{1}{\Gamma(\alpha)} \int_0^1 \psi'(\tau) (\psi(1) - \psi(\tau))^{\alpha-1} \left(\phi_q \left(\int_0^{\tau} h(s) ds \right) \right) d\tau \right. \\ & \left. - \sum_{i=1}^m \frac{a_i}{\Gamma(\alpha + \sigma_i)} \int_0^{\mu_i} \psi'(\tau) (\psi(\mu_i) - \psi(\tau))^{\alpha+\sigma_i-1} \left(\phi_q \left(\int_0^{\tau} h(s) ds \right) \right) d\tau \right\}, \end{aligned} \quad (10)$$

where it is assumed that

$$\Lambda = \frac{(\psi(1) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} - \sum_{i=1}^m \frac{a_i (\psi(\mu_i) - \psi(0))^{\gamma+\sigma_i-1}}{\Gamma(\gamma + \sigma_i)} \neq 0.$$

Proof. Setting $\phi_p({}^H D_{0+}^{\alpha, \beta, \psi} y(t)) = \theta(t)$, problem (9) can be split into two problems:

$$\begin{cases} \theta(t)' + h(t) = 0, \\ \theta(0) = 0 \end{cases} \quad (11)$$

and

$$\begin{cases} \phi_p({}^H D_{0^+}^{\alpha,\beta,\psi} y(t)) = \theta(t), \\ y(0) = 0, \quad y(1) = \sum_{i=1}^n a_i I_{0^+}^{\sigma_i,\psi} y(\mu_i). \end{cases} \quad (12)$$

Solving (11), we obtain $\theta(t) = -\int_0^t h(s)ds$. Applying $I_{0^+}^{\alpha,\psi}$ to the ψ -Hilfer p -Laplacian fractional differential equation in (12), we obtain

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \phi_q(\theta(s))ds + c_0(\psi(t) - \psi(0))^{\gamma-2} + c_1(\psi(t) - \psi(0))^{\gamma-1}, \quad (13)$$

where c_0 and c_1 are arbitrary constants. Using the condition $y(0) = 0$ in (13) yields $c_0 = 0$. Then, from (13), we have

$$y(1) = \frac{1}{\Gamma(\alpha)} \int_0^1 \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \phi_q(\theta(s))ds + c_1(\psi(1) - \psi(0))^{\gamma-1} \quad (14)$$

and

$$I_{0^+}^{\sigma_i,\psi} y(\mu_i) = \frac{1}{\Gamma(\alpha + \sigma_i)} \int_0^{\mu_i} \psi'(s)(\psi(t) - \psi(s))^{\alpha+\sigma_i-1} \phi_q(\theta(s))ds + c_1 \frac{\Gamma(\gamma)}{\Gamma(\gamma + \sigma_i)} (\psi(\mu_i) - \psi(0))^{\gamma+\sigma_i-1}. \quad (15)$$

Inserting (14) and (15) in the condition: $y(1) = \sum_{i=1}^m a_i I_{0^+}^{\sigma_i,\psi} y(\mu_i)$, we find that

$$\begin{aligned} c_1 = \frac{1}{\Lambda \Gamma(\gamma)} & \left\{ \sum_{i=1}^m \frac{a_i}{\Gamma(\alpha + \sigma_i)} \int_0^{\mu_i} \psi'(s)(\psi(t) - \psi(s))^{\alpha+\sigma_i-1} \phi_q(\theta(s))ds \right. \\ & \left. - \frac{1}{\Gamma(\alpha)} \int_0^1 \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \phi_q(\theta(s))ds \right\}. \end{aligned} \quad (16)$$

Substituting the values of c_0 , c_1 , and $\theta(t) = -\int_0^t h(s)ds$ in (13), we obtain the solution (10). This completes the proof. $\square$

The following lemma provides bounds for the p -Laplace operator, which can easily be proved by using the mean value theorem when the function $\phi_p(k) = |k|^{p-2}k$ is differentiable for all values of k except $k = 0$. Moreover, $\left| \frac{\partial \phi(k)}{\partial k} \right|$ is bounded by $(p-1) \max |k|^{p-2}$ in the case of $p > 2$ and bounded by $(p-1) \min |k|^{p-2}$ when $1 < p < 2$.

Lemma 3. (See (2.1) and (2.2) on page 3268 in [28])

(i) For $1 < p \leq 2$, $|k_1|, |k_2| \geq \Delta_1 > 0$, and $k_1 k_2 > 0$, we have

$$|\phi_p(k_2) - \phi_p(k_1)| \leq (p-1) \Delta_1^{p-2} |k_2 - k_1|;$$

(ii) For $p > 2$, $|k_1|, |k_2| \leq \Delta_2$, and $k_1 k_2 > 0$, we have

$$|\phi_p(k_2) - \phi_p(k_1)| \leq (p-1) \Delta_2^{p-2} |k_2 - k_1|.$$

3 Main results

This section is devoted to the uniqueness results for problem (1) for different values of p . Consider the following composition operator:

$$\mathcal{G} = \mathcal{G}_2 \circ \mathcal{G}_1, \quad (17)$$

where $\mathcal{G}_1 : C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$ is defined as follows:

$$(\mathcal{G}_1 y)(t) = \phi_q \left(\int_0^t f(s, y(s)) ds \right), \quad (18)$$

and $\mathcal{G}_2 : C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$ is given as follows:

$$\begin{aligned} (\mathcal{G}_2 y)(t) = & -\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\tau) (\psi(t) - \psi(\tau))^{\alpha-1} y(\tau) d\tau + \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Lambda\Gamma(\gamma)} \\ & \times \left\{ \frac{1}{\Gamma(\alpha)} \int_0^1 \psi'(\tau) (\psi(1) - \psi(\tau))^{\alpha-1} y(\tau) d\tau \right. \\ & \left. - \sum_{i=1}^{m-1} a_i \frac{1}{\Gamma(\alpha + \sigma_i)} \int_0^{\mu_i} \psi'(\tau) (\psi(\mu_i) - \psi(\tau))^{\alpha+\sigma_i-1} y(\tau) d\tau \right\}. \end{aligned} \quad (19)$$

Clearly $\mathcal{G} : C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$ is a continuous operator. For computational convenience, we set

$$\Omega = \frac{(\psi(1) - \psi(0))^\alpha}{\Gamma(\alpha + 1)} + \frac{(\psi(1) - \psi(0))^{\gamma+\alpha-1}}{\Lambda\Gamma(\gamma)\Gamma(\alpha + 1)} + \sum_{i=1}^m a_i \frac{(\psi(1) - \psi(0))^{\gamma-1} (\psi(\mu_i) - \psi(0))^{\alpha+\sigma_i}}{\Lambda\Gamma(\gamma)\Gamma(\alpha + \sigma_i + 1)}. \quad (20)$$

Now, we present our main results, which will be proved with the aid of the Banach contraction mapping principle and Lemma 2.

Theorem 1. Assume that $1 < p \leq 2$ and the following conditions hold:

(A₁) There exist a nonnegative integrable function g on $[0, 1]$ and a positive constant M such that

$$0 < \int_0^1 g(t) dt \leq M \quad \text{and} \quad |f(t, y)| \leq g(t) \quad \text{for } (t, y) \in [0, 1] \times \mathbb{R};$$

(A₂) There exists a positive constant k such that

$$|f(t, y_1) - f(t, y_2)| \leq k|y_2 - y_1|, \quad (t, y_i) \in [0, 1] \times \mathbb{R}, \quad i = 1, 2.$$

Then, the boundary value problem (1) has a unique solution provided that

$$0 < k < \frac{1}{(q-1)M^{q-2}\Omega}, \quad (21)$$

where Ω is given by (20).

Proof. By the assumption (A₁), we have

$$\left| \int_0^t f(s, y(s)) ds \right| = \int_0^t |f(s, y(s))| ds \leq \int_0^t g(s) ds \leq M.$$

Setting $r \geq M^{q-1}\Omega$, we define $B_r = \{y \in C([0, 1], \mathbb{R}) : \|y\| \leq r\}$ and show that $\mathcal{G}B_r \subset B_r$, where the operator $\mathcal{G}$ is defined by (17). For $y \in B_r$, it follows by the definition of $\phi_p(\cdot)$ that

$$|(\mathcal{G}_1 y)(t)| = \left| \phi_q \left(\int_0^t f(s, y(s)) ds \right) \right| = \phi_q \left(\left| \int_0^t f(s, y(s)) ds \right| \right) \leq M^{q-1}.$$

Consequently, we obtain

$$\begin{aligned}
|(\mathcal{G}y)(t)| &= |(\mathcal{G}_2 \circ \mathcal{G}_1y)(t)| \\
&= \left| \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1}((\mathcal{G}_1y)(\tau))d\tau + \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Lambda\Gamma(\gamma)} \right. \\
&\quad \times \left\{ \frac{1}{\Gamma(\alpha)} \int_0^1 \psi'(\tau)(\psi(1) - \psi(\tau))^{\alpha-1}((\mathcal{G}_1y)(\tau))d\tau \right. \\
&\quad \left. \left. - \sum_{i=1}^{m-1} a_i \frac{1}{\Gamma(\alpha + \sigma_i)} \int_0^{\mu_i} \psi'(\tau)(\psi(\mu_i) - \psi(\tau))^{\alpha+\sigma_i-1}((\mathcal{G}_1y)(\tau))d\tau \right\} \right| \\
&\leq M^{q-1} \left(\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1}d\tau + \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Lambda\Gamma(\gamma)} \right. \\
&\quad \times \left. \left\{ \frac{1}{\Gamma(\alpha)} \int_0^1 \psi'(\tau)(\psi(1) - \psi(\tau))^{\alpha-1}d\tau + \sum_{i=1}^{m-1} a_i \frac{1}{\Gamma(\alpha + \sigma_i)} \int_0^{\mu_i} \psi'(\tau)(\psi(\mu_i) - \psi(\tau))^{\alpha+\sigma_i-1}d\tau \right\} \right) \\
&\leq M^{q-1} \left(\frac{(\psi(1) - \psi(0))^\alpha}{\Gamma(\alpha + 1)} + \frac{(\psi(1) - \psi(0))^{\gamma+\alpha-1}}{\Lambda\Gamma(\gamma)\Gamma(\alpha + 1)} + \sum_{i=1}^m a_i \frac{(\psi(1) - \psi(0))^{-1}(\psi(\mu_i) - \psi(0))^{\alpha+\sigma_i}}{\Lambda\Gamma(\gamma)\Gamma(\alpha + \sigma_i + 1)} \right) = M^{q-1}\Omega,
\end{aligned}$$

which, on taking the norm for $t \in [0, 1]$, yields $\|\mathcal{G}y\| \leq r$. Because $y \in B_r$ is an arbitrary element, we obtain $\mathcal{G}B_r \subset B_r$.

In order to show that $\mathcal{G}$ is a contraction, let $y, z \in C([0, 1])$. Then, using Lemma 3 and the fact that $1 < p \leq 2 \Rightarrow q > 2$, we obtain

$$\begin{aligned}
|(\mathcal{G}_1y)(t) - (\mathcal{G}_1z)(t)| &= \left| \phi_q \left(\int_0^t f(s, y(s))ds \right) - \phi_q \left(\int_0^t f(s, z(s))ds \right) \right| \\
&\leq (q-1)M^{q-2} \left| \int_0^t f(s, y(s))ds - \int_0^t f(s, z(s))ds \right| \\
&\leq k(q-1)M^{q-2}t \|y - z\|.
\end{aligned}$$

In consequence, we obtain

$$\begin{aligned}
|(\mathcal{G}y)(t) - (\mathcal{G}z)(t)| &= |(\mathcal{G}_2 \circ \mathcal{G}_1y)(t) - (\mathcal{G}_2 \circ \mathcal{G}_1z)(t)| \\
&= \left| \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1}((\mathcal{G}_1y)(\tau) - (\mathcal{G}_1z)(\tau))d\tau + \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Lambda\Gamma(\gamma)} \right. \\
&\quad \times \left\{ \frac{1}{\Gamma(\alpha)} \int_0^1 \psi'(\tau)(\psi(1) - \psi(\tau))^{\alpha-1}((\mathcal{G}_1y)(\tau) - (\mathcal{G}_1z)(\tau))d\tau \right. \\
&\quad \left. \left. - \sum_{i=1}^{m-1} a_i \frac{1}{\Gamma(\alpha + \sigma_i)} \int_0^{\mu_i} \psi'(\tau)(\psi(\mu_i) - \psi(\tau))^{\alpha+\sigma_i-1}((\mathcal{G}_1y)(\tau) - (\mathcal{G}_1z)(\tau))d\tau \right\} \right| \\
&\leq k(q-1)M^{q-2}\|y - z\| \left(\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1}\tau d\tau + \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Lambda\Gamma(\gamma)} \right. \\
&\quad \times \left. \left\{ \frac{1}{\Gamma(\alpha)} \int_0^1 \psi'(\tau)(\psi(1) - \psi(\tau))^{\alpha-1}\tau d\tau + \sum_{i=1}^{m-1} a_i \frac{1}{\Gamma(\alpha + \sigma_i)} \int_0^{\mu_i} \psi'(\tau)(\psi(\mu_i) - \psi(\tau))^{\alpha+\sigma_i-1}\tau d\tau \right\} \right) \\
&\leq k(q-1)M^{q-2} \left(\frac{(\psi(1) - \psi(0))^\alpha}{\Gamma(\alpha + 1)} + \frac{(\psi(1) - \psi(0))^{\gamma+\alpha-1}}{\Lambda\Gamma(\gamma)\Gamma(\alpha + 1)} \right. \\
&\quad \left. + \sum_{i=1}^m a_i \frac{(\psi(1) - \psi(0))^{\gamma-1}(\psi(\mu_i) - \psi(0))^{\alpha+\sigma_i}}{\Lambda\Gamma(\gamma)\Gamma(\alpha + \sigma_i + 1)} \right) \|y - z\| = k(q-1)M^{q-2}\Omega\|y - z\|.
\end{aligned}$$

Taking the norm of the above inequality for $t \in [0, 1]$, we obtain

$$\|(\mathcal{G}y)(t) - (\mathcal{G}z)(t)\| \leq k(q-1)M^{q-2}\Omega,$$

which together with the condition (21) implies that the operator $\mathcal{G}$ is a contraction. Hence, by the Banach's contraction mapping principle, the operator $\mathcal{G}$ has a unique fixed point. Hence, problem (1) has a unique solution on $[0, 1]$. This completes the proof. $\square$

Theorem 2. Assume that $p > 2$ and the following conditions hold:

(A₃) There exist constants $m > 0$ and δ with $0 < \delta \leq \frac{1}{2-q}$ such that

$$f(t, y) \geq m\delta t^{\delta-1}, \quad \text{for } (t, y) \in [0, 1] \times \mathbb{R};$$

(A₄) There exists a constant $k > 0$ such that

$$|f(t, y) - f(t, z)| \leq k|y - z|, \quad \text{for } (t, y_i) \in [0, 1] \times \mathbb{R}, \quad i = 1, 2.$$

If

$$0 < k < \frac{1}{(q-1)m^{q-2}\Omega}, \quad (22)$$

where Ω is given by (20), then there exists a unique solution to the boundary value problem (1) on $[0, 1]$.

Proof. By (A₃), we have

$$\int_0^t f(s, y(s))ds \geq \int_0^t m\delta s^{\delta-1}ds = mt^{\delta}.$$

Choosing $\bar{r} \geq \max\left\{\frac{1-f_0}{k}, \frac{f_0\Omega}{1-k\Omega}\right\}$, where $f_0 = \max_{t \in [0,1]} |f(t, 0)|$, we define $B_{\bar{r}} = \{y \in C([0, 1], \mathbb{R}) : \|y\| \leq \bar{r}\}$ and show that $\mathcal{G}B_{\bar{r}} \subset B_{\bar{r}}$, where the operator $\mathcal{G}$ is defined by (17). Using the definition of $\phi_q(\cdot)$ and the values of $\bar{r}$ and q , we have for $y \in B_{\bar{r}}$ that

$$\begin{aligned} |(\mathcal{G}y)(t)| &= \left| \phi_q \left(\int_0^t f(s, y(s))ds \right) \right| = \phi_q \left(\left| \int_0^t f(s, y(s))ds \right| \right) \\ &\leq \phi_q \left(\int_0^t [|f(s, y(s)) - f(s, 0)| + |f(s, 0)|]ds \right) \\ &\leq \phi_q \left(\int_0^1 [k\|y\| + f_0]ds \right) \leq (k\bar{r} + f_0)^{q-1} \leq (k\bar{r} + f_0). \end{aligned}$$

As in the previous theorem, one can obtain

$$\|(\mathcal{G}y)(t)\| \leq (k\bar{r} + f_0)\Omega \leq \bar{r},$$

which means that $\|\mathcal{G}y\| \leq \bar{r}$. Thus, $\mathcal{G}y \in B_{\bar{r}}$. Because $y \in B_{\bar{r}}$ is an arbitrary element, we obtain $\mathcal{G}B_{\bar{r}} \subset B_{\bar{r}}$.

Next, it will be shown that $\mathcal{G}$ is a contraction. Let $y, z \in C([0, 1])$. Because $p > 2$ implies $1 < q < 2$, by Lemma (3), we obtain

$$\begin{aligned} |(\mathcal{G}y)(t) - (\mathcal{G}z)(t)| &= \left| \phi_q \left(\int_0^t f(s, y(s))ds \right) - \phi_q \left(\int_0^t f(s, z(s))ds \right) \right| \\ &\leq (q-1)(mt^{\delta})^{q-2} \left| \int_0^t f(s, y(s))ds - \int_0^t f(s, z(s))ds \right| \\ &\leq k(q-1)m^{q-2}t^{\delta(q-2)+1}\|y - z\| \leq k(q-1)m^{q-2}\|y - z\|. \end{aligned}$$

Consequently, we have

$$\begin{aligned}
 |(\mathcal{G}y)(t) - (\mathcal{G}z)(t)| &= |(\mathcal{G}_2 \circ \mathcal{G}_1 y)(t) - (\mathcal{G}_2 \circ \mathcal{G}_1 z)(t)| \\
 &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\tau) (\psi(t) - \psi(\tau))^{\alpha-1} ((\mathcal{G}_1 y)(\tau) - (\mathcal{G}_1 z)(\tau)) d\tau + \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Lambda\Gamma(\gamma)} \right. \\
 &\quad \times \left\{ \frac{1}{\Gamma(\alpha)} \int_0^1 \psi'(\tau) (\psi(1) - \psi(\tau))^{\alpha-1} ((\mathcal{G}_1 y)(\tau) - (\mathcal{G}_1 z)(\tau)) d\tau \right. \\
 &\quad \left. - \sum_{i=1}^{m-1} a_i \frac{1}{\Gamma(\alpha + \sigma_i)} \int_0^{\mu_i} \psi'(\tau) (\psi(\mu_i) - \psi(\tau))^{\alpha+\sigma_i-1} ((\mathcal{G}_1 y)(\tau) - (\mathcal{G}_1 z)(\tau)) d\tau \right\} \Bigg| \\
 &\leq k(q-1)m^{q-2}\|y - z\| \left(\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\tau) (\psi(t) - \psi(\tau))^{\alpha-1} \tau^{\delta(q-2)+1} d\tau + \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Lambda\Gamma(\gamma)} \right. \\
 &\quad \times \left\{ \frac{1}{\Gamma(\alpha)} \int_0^1 \psi'(\tau) (\psi(1) - \psi(\tau))^{\alpha-1} \tau^{\delta(q-2)+1} d\tau \right. \\
 &\quad \left. + \sum_{i=1}^{m-1} a_i \frac{1}{\Gamma(\alpha + \sigma_i)} \int_0^{\mu_i} \psi'(\tau) (\psi(\mu_i) - \psi(\tau))^{\alpha+\sigma_i-1} \tau^{\delta(q-2)+1} d\tau \right\} \Bigg) \\
 &\leq k(q-1)m^{q-2} \left(\frac{(\psi(1) - \psi(0))^\alpha}{\Gamma(\alpha + 1)} + \frac{(\psi(1) - \psi(0))^{\gamma+\alpha-1}}{\Lambda\Gamma(\gamma)\Gamma(\alpha + 1)} \right. \\
 &\quad \left. + \sum_{i=1}^m a_i \frac{(\psi(1) - \psi(0))^{\gamma-1} (\psi(\mu_i) - \psi(0))^{\alpha+\sigma}}{\Lambda\Gamma(\gamma)\Gamma(\alpha + \sigma_i + 1)} \right) \|y - z\| \\
 &= k(q-1)m^{q-2}\Omega\|y - z\|,
 \end{aligned}$$

which, on taking norm for $t \in [0, 1]$, yields $\|(\mathcal{G}_1 y) - (\mathcal{G}_1 z)\| \leq k(q-1)m^{q-2}\Omega\|y - z\|$. Thus, it follows from condition (22) that the operator $\mathcal{G}$ is a contraction. Hence, by Banach's contraction principle, the operator $\mathcal{G}$ has a unique fixed point, which is indeed a unique solution to problem (1) on $[0, 1]$. This finishes the proof. $\square$

Theorem 3. Let $p > 2$. Assume that (A_4) and the following condition are satisfied:

(A_5) There exist a number $m > 0$ and $0 < \delta \leq \frac{1}{2-q}$ such that

$$f(t, y) \leq -m\delta t^{\delta-1}, \quad \text{for } (t, y) \in [0, 1] \times \mathbb{R}.$$

Then, the boundary value problem (1) has a unique solution on $[0, 1]$ if k satisfies

$$0 < k < \frac{1}{(q-1)m^{q-2}\Omega}.$$

Proof. Since the proof is similar to that of the last theorem, we omit it. $\square$

4 Examples

Consider the following problem:

$$\begin{cases} (\phi_p({}^H D_{0^+}^{3/2, 1/2, \psi} y(x)))' + f(t, y(t)) = 0, \\ y(0) = {}^H D_{0^+}^{3/2, 1/2, \psi} y(0) = 0, \quad y(1) = 1/4 I_{0^+}^{5/3} y(1/2) + 3 I_{0^+}^{4/5} y(3/4), \end{cases} \quad (23)$$

where $\alpha = 3/2$, $\beta = 1/2$, $\sigma_1 = 5/3$, $\sigma_2 = 4/5$, $\mu_1 = 1/2$, $\mu_2 = 3/4$, $a_1 = 1/4$, $a_2 = 3$, and $\psi(t) = t^2 + t$. We will fix p and $f(t, y(t))$ later.

For illustrating Theorem 3.1, let us take

$$f(t, y) = \frac{1}{\sqrt{400 + t}} \left[e^t \left(\sin y + \frac{|y|}{|y| + 1} \right) + \cos t \right], \quad (24)$$

and $p = 4/3$, which implies that $q = 4$. From the given data, we find that $\gamma = 1.75$ and $\Omega = 7.222698841$. Condition (A_1) is satisfied with $g(t) = \frac{2e^t + \cos t}{\sqrt{t + 400}}$, and so $M = 0.2137530673$. In addition, (A_2) and (21) are satisfied with $k = 0.2718281828$. Thus, all the conditions of Theorem 3.1 are satisfied, and hence, its conclusion implies that problem (23) with $p = 4/3$ and $f(t, y)$ given by (24) has a unique solution on $[0, 1]$.

Next, we illustrate Theorem 3.2 by choosing

$$f(t, y) = \frac{2e^t}{3} \left(\frac{|y|}{|y| + 12} + 1 \right), \quad (25)$$

and $p = 7/2$, which means that $q = 7/5$. Let us take $\delta = 4/3 < 1/(2 - q)$ and $m = 1/2$. Then, $f(t, y) \geq 2/3 t^{1/3} = \delta m t^{\delta-1}$, that is, (A_3) is satisfied. In addition, (A_4) and (22) hold true with $k = 0.1510156571$. Thus, it follows by the conclusion of Theorem 3.2 that problem (23) with $p = 7/2$ and $f(t, y)$ given by (25) has a unique solution on $[0, 1]$.

5 Conclusion

We have presented the uniqueness results for a ψ -Hilfer fractional differential equation involving the p -Laplacian operator complemented with nonlocal ψ -Riemann-Liouville fractional integral boundary conditions with the aid of Banach's contraction mapping principle. Our first result deals with the case $1 < p \leq 2$, while the second and third results are obtained under different conditions on the nonlinear function $f(t, y)$ involved in the given problem when $p > 2$. Our results are new in the given configuration and enrich the literature on the p -Laplacian fractional order boundary value problems.

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