

Research Article

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Certain aspects of Nörlund $\mathcal{I}$ -statistical convergence of sequences in neutrosophic normed spaces

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Abstract: The aim of this article is to investigate the neutrosophic Nörlund $\mathcal{I}$ -statistically convergent sequence space. We present some neutrosophic normed spaces (NNSs) in Nörlund convergent spaces. In addition, we also examine various topological and algebraic properties of these convergent sequence spaces. Theorems are proved in light of the NNS theory approach. Results are obtained via different perspectives and new examples are produced to justify the counterparts and show the existence of the introduced notions. The results established in this research work supply an exhaustive foundation in NNS and make a significant contribution to the theoretical development of NNS in the literature. The original aspect of this study is the first wholly up-to-date and thorough examination of the features and implementation of neutrosophic Nörlund $\mathcal{I}$ -statistically convergent sequences in NNS, based upon the standard definition.

Keywords: neutrosophic normed space, ideal convergence, Nörlund convergent space

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1 Introduction

Fuzzy theory has made a significant progress on the mathematical underpinnings of fuzzy set (FS) theory, which was pioneered by Zadeh [1] in 1965. Zadeh [1] mentioned that an FS assigns a membership value to each element of a given crisp universe set from $[0, 1]$. FSs cannot always overcome the absence of knowledge of membership degrees. Because of that, Atanassov [2] examined the intuitionistic FS (IFS), which is an extension of FS. Kramosil and Michalek [3] defined fuzzy metric space (FMS) by using the concepts of fuzzy and probabilistic metric spaces. For more information on FMSs and IF-normed spaces (IFNS), we refer the reader to [4–8]. Intuitionistic fuzzy fixed-point theory has become a subject of great interest for expert in fixed-point theory because this branch of mathematics has covered new possibilities for summability theory. In intuitionistic fuzzy metric space (IFMS), Mohamad [9] established the Banach fixed-point theorem. For more information on fixed point theory in FMS and IFMS, we refer the reader to [10–13]. The concept of neutrosophy implies impartial knowledge of thought, and then neutral describes the basic difference between neutral, fuzzy, intuitive FSs and logic. After the introduction of neutrosophic set (NS) by Smarandache [14], which is a generalization of the classical set, FS, and IFS, Maji [15] has introduced the combined concept of neutrosophic soft set (NSS). Taking everything into account, Smarandache applied the IFS theory by defining a new component, namely, the indeterminacy membership function. NS is determined as a set where every component of the universe has a degree of T , F , and I . In IFSSs, the “degree of

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non-belongingness” is not independent, but it is dependent on the “degree of belongingness.” FSs can be thought of as a remarkable case of an IFS where the “degree of non-belongingness” of an element is absolutely equal to “1-degree of belongingness.” Uncertainty is based on the belongingness degree in IFSs, whereas the uncertainty in NS is considered independently from T and F values. Since there are not any limitations among the degree of T , F , and I , NSs are actually more general than IFS. Consequently, several mathematicians have produced their research works in different mathematical structures, for example, Bera and Mahapatra [16–18]. The neutrosophic soft linear space was worked out by Bera and Mahapatra [16]. Afterward, in [17,18], the conception of neutrosophic soft normed linear set (NSNLS) was investigated, and various properties of NSNLS were proposed.

On the other hand, the notions of statistical convergent and $\mathcal{I}$ -convergent were further investigated from the sequence space point of view and linked with the summability theory by Fast [19] and Kostyrko et al. [20], respectively. Statistical convergence in the IFNS was presented by Karakuş et al. [21]. For extensive study in this topic, one may refer to the works of [22–44]. Kirişçi and Şimşek [45] investigated neutrosophic metric space (NMS) with continuous t -norms and continuous t -conorms. Kirişçi and Şimşek [46] proposed neutrosophic normed space (NNS) and statistical convergence in NNS. For more details on statistical convergence and ideal convergence, one may refer to [47–55].

The idea of convergence of sequence is important in the fundamental theory of mathematics. There are numerous convergence ideas in summability theory, such as classical measure theory, fuzzy theory, approximation theory, and probability theory, and the links between them are investigated. This study will do more research into the mathematical properties of Nörlund convergent spaces in light of this. Section 2 recalls some known definitions and theorems in neutrosophic and summability theory. In Section 3, we investigate the neutrosophic Nörlund $\mathcal{I}$ -statistically convergent sequence space. In addition, we present some NNS in Nörlund convergent spaces. Moreover, we also examine various topological and algebraic properties of these convergent sequence spaces.

2 Preliminaries

This section will serve to gather all the necessary results and techniques on which we will rely to accomplish our main results. First, we will go over some key terms. All along the article, let $\mathcal{I}$ be an admissible ideal, $\mathcal{H} = (\mathcal{P}, \mathcal{Q}, \mathcal{R})$ be a neutrosophic norm (NN), $\mathcal{N}^f$ be a Nörlund matrix, and $\mathcal{N}_p^f(\Theta)$ be $\mathcal{N}^f$ -transform of the sequence $\Theta = (\Theta_m) \in l_\infty$.

Triangular norms (t -norms) were investigated by Menger [56]. In the problem of computing the distance between two elements in space, Menger presented utilizing probability distributions instead of utilizing numbers for distance. T -norms are applied to generalize with the probability distribution of triangle inequality in metric space conditions. Triangular conorms (t -conorms) are identified as dual operations of t -norms.

Definition 1. [46] Let F be a vector space, $N = \{\langle \alpha, \mathcal{P}(\alpha), \mathcal{Q}(\alpha), \mathcal{R}(\alpha) \rangle : \alpha \in F\}$ be a normed space so that $N : F \times \mathbb{R}^+ \rightarrow [0, 1]$. Assume $\triangle$ and $\diamond$ demonstrate the continuous t -norm and continuous t -conorm, respectively. While following conditions supply, $V = (F, N, \triangle, \diamond)$ is named to be NNS. For all $\alpha, \beta \in F$ and $\kappa, \varpi > 0$ and for each $\rho \neq 0$,

- (a) $0 \leq \mathcal{P}(\alpha, \kappa) \leq 1, 0 \leq \mathcal{Q}(\alpha, \kappa) \leq 1, 0 \leq \mathcal{R}(\alpha, \kappa) \leq 1 \forall \kappa \in \mathbb{R}^+$,
- (b) $\mathcal{P}(\alpha, \kappa) + \mathcal{Q}(\alpha, \kappa) + \mathcal{R}(\alpha, \kappa) \leq 3$ (for $\kappa \in \mathbb{R}^+$),
- (c) $\mathcal{P}(\alpha, \kappa) = 1$ (for $\kappa > 0$) iff $\alpha = 0$,
- (d) $\mathcal{P}(\rho\alpha, \kappa) = \mathcal{P}\left(\alpha, \frac{\kappa}{|\rho|}\right)$,
- (e) $\mathcal{P}(\alpha, \varpi) \triangle \mathcal{P}(\beta, \kappa) \leq \mathcal{P}(\alpha + \beta, \varpi + \kappa)$,
- (f) $\mathcal{P}(\alpha, \cdot)$ is non-decreasing continuous function,
- (g) $\lim_{\kappa \rightarrow \infty} \mathcal{P}(\alpha, \kappa) = 1$,

- (h) $Q(\alpha, \kappa) = 0$ (for $\kappa > 0$) iff $\alpha = 0$,
- (i) $Q(\rho\alpha, \kappa) = Q\left(\alpha, \frac{\kappa}{|\rho|}\right)$,
- (j) $Q(\alpha, \varpi) \diamond Q(\beta, \kappa) \geq Q(\alpha + \beta, \varpi + \kappa)$,
- (k) $Q(\alpha, \cdot)$ is non-decreasing continuous function,
- (l) $\lim_{\kappa \rightarrow \infty} Q(\alpha, \kappa) = 0$,
- (m) $\mathcal{R}(\alpha, \kappa) = 0$ (for $\kappa > 0$) iff $\alpha = 0$,
- (n) $\mathcal{R}(\rho\alpha, \kappa) = \mathcal{R}\left(\alpha, \frac{\kappa}{|\rho|}\right)$,
- (o) $\mathcal{R}(\alpha, \varpi) \diamond \mathcal{R}(\beta, \kappa) \geq \mathcal{R}(\alpha + \beta, \varpi + \kappa)$,
- (p) $\mathcal{R}(\alpha, \cdot)$ is non-decreasing continuous function,
- (q) $\lim_{\kappa \rightarrow \infty} \mathcal{R}(\alpha, \kappa) = 0$,
- (r) If $\kappa \leq 0$, then $\mathcal{P}(\alpha, \kappa) = 0$, $Q(\alpha, \kappa) = 1$ and $\mathcal{R}(\alpha, \kappa) = 1$.

Then, $\mathcal{H} = (\mathcal{P}, Q, \mathcal{R})$ is an NN.

Statistical convergence and ideal convergence in NNS were proposed by Kirişçi and Şimşek [45] and Kişi [55], respectively, by using the concept of NN.

Nörlund sequence space was investigated by Wang [57] as follows:

$$\mathcal{N}^f = \left\{ \Theta = (\Theta_m) \in l_\infty : \sum_{k=0}^{\infty} \left| \frac{1}{A_k} \sum_{m=0}^k f_{k-m} \Theta_m \right|^p < \infty, \quad 1 \leq p < \infty \right\},$$

where $A_k = \sum_{m=0}^k f_m$. The spaces $l_\infty(\mathcal{N}^f)$ and $l_p(\mathcal{N}^f)$ consist of all sequences whose Nörlund transforms are in the spaces l_∞ and l_p , where $1 \leq p < \infty$.

Wang [57] used the Nörlund matrix $\mathcal{N}^f$ in the theory of sequence space for the first time. Recall that in [58], assume $f = (f_m)$ be a non-negative sequence of real numbers and $T_j = \sum_{m=0}^j f_m$ for each $j \in \mathbb{N}$ with $f_0 > 0$. At that time, the Nörlund matrix $\mathcal{N}^f = (a_{jm}^f)$ w.r.t. the sequence $f = (f_m)$ is determined as follows:

$$a_{jm}^f = \begin{cases} \frac{f_{j-m}}{T_j}, & \text{if } 0 \leq m \leq j \\ 0, & \text{if } m > j \end{cases}$$

for all $j, m \in \mathbb{N}$.

Wang [57] utilized the Nörlund matrix to determine the sequence space $l_\infty(\mathcal{N}^f)$ as the domain of Nörlund mean $\mathcal{N}^f$ -transform are in the space l_∞ . Tuğ and Başar [59] investigated the sequence spaces $c_0(\mathcal{N}^f)$ and $c(\mathcal{N}^f)$ as the set of all sequences with $\mathcal{N}^f$ in the spaces c_0 and c , respectively. In addition, Tuğ and Başar [59] identified the sequence $\mathcal{N}_p^f(\Theta)$ to indicate the $\mathcal{N}^f$ -transform of the sequence $(\Theta_m) \in w$, where the sequence $\mathcal{N}_p^f(\Theta)$ is determined as follows:

$$\mathcal{N}_p^f(\Theta) := \frac{1}{T_p} \sum_{m=0}^p f_{p-m} \Theta_m \quad (1)$$

for each $p \in \mathbb{N}$.

Recently, by using the concepts of the domain of Nörlund matrix $\mathcal{N}^f$ and $\mathcal{I}$ -convergence, Khan et al. [60] presented the space of Nörlund $\mathcal{I}$ -convergent sequences.

Ideal convergence in IFNS was determined with the help of membership and non-membership functions. Unlike prior works, this research considers the indeterminacy function while studying ideal convergence. The aim of this study is to put forward several recent advancements in NNS. For sequences, ideal convergence is known to be more general than statistical convergence. This has concentrated us to investigate the Nörlund $\mathcal{I}$ -statistical convergence of sequences in NNS. In a recent work, we proposed significant properties of this new type of convergence. In addition, it is denoted that Nörlund $\mathcal{I}$ -statistical convergence in NNS is generally dissimilar from $\mathcal{I}$ -statistical convergence in classical normed space (CNS), because

there is no “ $\mathcal{N}$ ” function in CNS. But, it is obvious that when particular conditions are met, all CNS can be NNS. When the NN is an additive positive integer, our conceptions and theorems yield the theoretical results of [20,23]. Since any crisp norm can generate an NN, the results found here are more general than the corresponding results for normed spaces. Several of the outcomes in this article either run parallel with classical ones or they are in the identical direction as the similar studies in this topic; however, in most conditions, the proofs use a different technique.

3 Main results

Throughout the article, we assume that the sequences $\Theta = (\Theta_m) \in l_\infty$ and $\mathcal{N}_p^f(\Theta_m)$ are connected as demonstrated in (1) and $\mathcal{I}$ is an admissible ideal of a subset of $\mathbb{N}$. In this section, by utilizing a domain of Nörlund matrix, which is used in [60] and $\mathcal{I}$ -convergence w.r.t. NN $\mathcal{H} = (\mathcal{P}, \mathcal{Q}, \mathcal{R})$ [55], we identify new Nörlund sequence spaces as follows.

Definition 2. A sequence $\Theta = (\Theta_m) \in l_\infty$ is said to be Nörlund $\mathcal{I}$ -statistically convergent to $\beta \in \mathbb{R}$, provided that, for each $\eta, \gamma > 0$.

$$A_1 = \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : |\mathcal{N}_p^f(\Theta_m) - \beta| \geq \eta\}| \geq \gamma \right\} \in \mathcal{I}.$$

Definition 3. A sequence $\Theta = (\Theta_m) \in l_\infty$ is said to be Nörlund $\mathcal{I}$ -statistically Cauchy, provided that, for each $\eta, \gamma > 0$, there is an $r \in \mathbb{N}$ such that

$$A_2 = \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : |\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_r^f(\Theta_m)| \geq \eta\}| \geq \gamma \right\} \in \mathcal{I}.$$

Definition 4. A sequence $\Theta = (\Theta_m) \in l_\infty$ is said to be neutrosophic Nörlund $\mathcal{I}$ -statistically convergent to $\beta \in \mathbb{R}$ w.r.t. NN $\mathcal{H} = (\mathcal{P}, \mathcal{Q}, \mathcal{R})$, provided that, for each $\sigma, \gamma > 0$ and $\eta \in (0, 1)$,

$$K_1 := \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \leq 1 - \eta \text{ or } \right. \\ \left. \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \geq \eta, \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \geq \eta\}| \geq \gamma \right\} \in \mathcal{I}.$$

Symbolically, we write $\mathcal{N}_p^f - \mathcal{I} - st \lim \Theta_m = \beta$ or $(\Theta_m) \xrightarrow{(\mathcal{P}, \mathcal{Q}, \mathcal{R})} \beta(S^{\mathcal{H}}(\mathcal{I}))$.

Definition 5. A sequence $\Theta = (\Theta_m) \in l_\infty$ is said to be neutrosophic Nörlund $\mathcal{I}$ -statistically Cauchy w.r.t. NN $\mathcal{H} = (\mathcal{P}, \mathcal{Q}, \mathcal{R})$, provided that, for each $\sigma, \gamma > 0$ and $\eta \in (0, 1)$, there is an $r \in \mathbb{N}$ such that the set K_2 belongs to $\mathcal{I}$, where

$$K_2 := \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_r^f(\Theta_m), \sigma) \leq 1 - \eta \text{ or } \right. \\ \left. \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_r^f(\Theta_m), \sigma) \geq \eta, \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_r^f(\Theta_m), \sigma) \geq \eta\}| \geq \gamma \right\} \in \mathcal{I}.$$

Now, we present the following sequence spaces:

$$\mathcal{N}_{S^{\mathcal{H}}(I_0)}^f := \left\{ \Theta = (\Theta_m) \in l_\infty : \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m), \sigma) \leq 1 - \eta \text{ or } \right. \right. \\ \left. \left. Q(\mathcal{N}_p^f(\Theta_m), \sigma) \geq \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m), \sigma) \geq \eta\} | \geq \gamma \right\} \in \mathcal{I} \right\},$$

$$\mathcal{N}_{S^{\mathcal{H}}(I)}^f := \left\{ \Theta = (\Theta_m) \in l_\infty : \left\{ p \in \mathbb{N} : \text{for some } \beta \in \mathbb{R}, \quad \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \leq 1 - \eta \text{ or } \right. \right. \\ \left. \left. Q(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \geq \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \geq \eta\} | \geq \gamma \right\} \in \mathcal{I} \right\},$$

$$\mathcal{N}_{S^{\mathcal{H}}(I^\infty)}^f := \left\{ \Theta = (\Theta_m) \in l_\infty : \left\{ p \in \mathbb{N} : \exists \zeta \in (0, 1), \quad \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m), \sigma) \leq 1 - \zeta \text{ or } \right. \right. \\ \left. \left. Q(\mathcal{N}_p^f(\Theta_m), \sigma) \geq \zeta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m), \sigma) \geq \zeta\} | \geq \gamma \right\} \in \mathcal{I} \right\}.$$

We identify an open ball and closed ball with center at Θ and radius $\sigma > 0$ w.r.t. the parameters of fuzziness $\eta \in (0, 1)$ and $\gamma > 0$ demonstrated by $\mathcal{B}(\Theta, \sigma, \eta, \gamma)$ and $\mathcal{B}[\Theta, \sigma, \eta, \gamma]$ as follows:

$$\mathcal{B}(\Theta, \sigma, \eta, \gamma) := \left\{ q = (q_m) \in l_\infty : \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_p^f(q), \sigma) \leq 1 - \eta \text{ or } \right. \right. \\ \left. \left. Q(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_p^f(q), \sigma) \geq \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_p^f(q), \sigma) \geq \eta\} | \geq \gamma \right\} \in \mathcal{I} \right\}$$

and

$$\mathcal{B}[\Theta, \sigma, \eta, \gamma] := \left\{ q = (q_m) \in l_\infty : \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_p^f(q), \sigma) < 1 - \eta \text{ or } \right. \right. \\ \left. \left. Q(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_p^f(q), \sigma) > \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_p^f(q), \sigma) > \eta\} | > \gamma \right\} \in \mathcal{I} \right\}.$$

When $(\Theta_m) \in \mathcal{N}_{S^{\mathcal{H}}(I)}^f$, then (Θ_m) $\mathcal{I}$ -statistically converges to some $\beta \in \mathbb{R}$, indicated by $(\Theta_m) \xrightarrow{S^{\mathcal{H}}(I)} \beta$, and in that case, we obtain $\mathcal{N}_p^f - \mathcal{I} - st \lim \Theta_m = \beta$.

Theorem 1. The inclusion relation $\mathcal{N}_{S^{\mathcal{H}}(I_0)}^f \subset \mathcal{N}_{S^{\mathcal{H}}(I)}^f \subset \mathcal{N}_{S^{\mathcal{H}}(I^\infty)}^f$ supplies.

Proof. It is obvious that $\mathcal{N}_{S^{\mathcal{H}}(I_0)}^f \subset \mathcal{N}_{S^{\mathcal{H}}(I)}^f$. Then, we have to denote that $\mathcal{N}_{S^{\mathcal{H}}(I)}^f \subset \mathcal{N}_{S^{\mathcal{H}}(I^\infty)}^f$. Contemplate $\Theta = (\Theta_m) \in \mathcal{N}_{S^{\mathcal{H}}(I)}^f$. At that time, there is an $\beta \in \mathbb{R}$ such that $\mathcal{N}_p^f - \mathcal{I} - st \lim \Theta_m = \beta$. So, for each $\eta \in (0, 1)$ and $\sigma, \gamma > 0$, the set

$$K := \left\{ p \in \mathbb{N} : \frac{1}{p} \left| \left\{ m \leq p : \mathcal{P}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) > 1 - \eta \text{ and } \right. \right. \right. \\ \left. \left. \left. Q\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < \eta, \quad \mathcal{R}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < \eta \right\} \right| < \gamma \right\} \in \mathcal{F}(\mathcal{I}).$$

Assume $\mathcal{P}\left(\beta, \frac{\sigma}{2}\right) = p$, $Q\left(\beta, \frac{\sigma}{2}\right) = q$, and $\mathcal{R}\left(\beta, \frac{\sigma}{2}\right) = r$ for all $\sigma > 0$. Since $p, q, r \in (0, 1)$ and $\eta \in (0, 1)$, there are $s_1, s_2, s_3 \in (0, 1)$ such that $(1 - \eta) \triangle p > 1 - s_1$, $\eta \diamond q < s_2$, and $\eta \diamond r < s_3$. So, we obtain

$$\begin{aligned}
\mathcal{P}(\mathcal{N}_p^f(\Theta_m), \sigma) &= \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta + \beta, \sigma) \\
&\geq \mathcal{P}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) \triangle \mathcal{P}\left(\beta, \frac{\sigma}{2}\right) \\
&\geq (1 - \eta) \triangle p \\
&> 1 - s_1,
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}(\mathcal{N}_p^f(\Theta_m), \sigma) &= \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \beta + \beta, \sigma) \\
&\leq \mathcal{Q}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) \diamond \mathcal{Q}\left(\beta, \frac{\sigma}{2}\right) \\
&< \eta \diamond q \\
&< s_2,
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}(\mathcal{N}_p^f(\Theta_m), \sigma) &= \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta + \beta, \sigma) \\
&\leq \mathcal{R}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) \diamond \mathcal{R}\left(\beta, \frac{\sigma}{2}\right) \\
&< \eta \diamond r \\
&< s_3.
\end{aligned}$$

When we have $s = \{s_1, s_2, s_3\}$, we obtain the set

$$\left\{ p \in \mathbb{N} : \exists s \in (0, 1), \frac{1}{p} \left| \{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta), \sigma) > 1 - s \text{ and } \mathcal{Q}(\mathcal{N}_p^f(\Theta), \sigma) < s, \mathcal{R}(\mathcal{N}_p^f(\Theta), \sigma) < \eta\} \right| < s \right\} \\
\in \mathcal{F}(I).$$

Hence, $\Theta = (\Theta_m) \in \mathcal{N}_{S^H(I^\infty)}^f$. This gives that $\mathcal{N}_{S^H(I)}^f \subset \mathcal{N}_{S^H(I^\infty)}^f$. $\square$

The converse of the inclusion relation does not supply. We establish the following example in support of our claim.

Example 1. Assume $(\mathbb{R}, \|\cdot\|)$ be a normed space such that $\|\Theta\| = \sup|\Theta_m|$, $u \triangle v = \min\{u, v\}$, and $u \diamond v = \max\{u, v\}$, $\forall u, v \in (0, 1)$. Now, we determine the norms $\mathcal{H} = (\mathcal{P}, \mathcal{Q}, \mathcal{R})$ on $\mathbb{R}^2 \times (0, \infty)$ as follows:

$$\mathcal{P}(\Theta, \sigma) = \frac{\sigma}{\sigma + \|\Theta\|}, \quad \mathcal{Q}(\Theta, \sigma) = \frac{\|\Theta\|}{\sigma + \|\Theta\|} \quad \text{and} \quad \mathcal{R}(\Theta, \sigma) = \frac{\|\Theta\|}{\sigma}.$$

Then, $(\mathbb{R}, \mathcal{H}, \triangle, \diamond)$ is an NNS. Contemplate the sequence $(\Psi_m) = \{1\}$. It can be easily examined that $(\Psi_m) \in \mathcal{N}_{S^H(I)}^f$ and $\mathcal{N}_p^f - I - st \lim \Theta_m = 1$ but $(\Theta_m) \notin \mathcal{N}_{S^H(I_0)}^f$.

Theorem 2. The spaces $\mathcal{N}_{S^H(I_0)}^f$ and $\mathcal{N}_{S^H(I)}^f$ are linear spaces.

Proof. It is obvious that $\mathcal{N}_{S^H(I_0)}^f \subset \mathcal{N}_{S^H(I)}^f$. At that time, we have to demonstrate the result for $\mathcal{N}_{S^H(I)}^f$. The proof of linearity of the space $\mathcal{N}_{S^H(I_0)}^f$ follows similarly. Assume sequences $\Theta = (\Theta_m)$ and $\Psi = (\Psi_m) \in \mathcal{N}_{S^H(I)}^f$. Then, there are $\beta_1, \beta_2 \in \mathbb{R}$ such that (Θ_m) and (Ψ_m) neutrosophic I -statistically converge to β_1 and β_2 , respectively.

$$\mathcal{N}_p^f - I - st \lim \Theta_m = \beta_1 \quad \text{and} \quad \mathcal{N}_p^f - I - st \lim \Psi_m = \beta_2.$$

We should denote that, for any scalars λ and ρ , the sequence $(\lambda\Theta_m + \rho\Psi_m)$ I -statistically converges to $\lambda\beta_1 + \rho\beta_2$. For $\sigma, \gamma > 0$ and $\eta \in (0, 1)$, take the following subsequent sets:

$$\begin{aligned}
K_1 &:= \left\{ p \in \mathbb{N} : \frac{1}{p} \left| \left\{ m \leq p : \mathcal{P}\left(\mathcal{N}_p^f(\Theta_m) - \beta_1, \frac{\sigma}{2|\lambda|}\right) \leq 1 - \eta \text{ or} \right. \right. \right. \\
&\quad \left. \left. \left. \mathcal{Q}\left(\mathcal{N}_p^f(\Theta_m) - \beta_1, \frac{\sigma}{2|\lambda|}\right) \geq \eta, \quad \mathcal{R}\left(\mathcal{N}_p^f(\Theta_m) - \beta_1, \frac{\sigma}{2|\lambda|}\right) \geq \eta \right\} \right| \geq \gamma \right\} \in I;
\end{aligned}$$

$$K_2 := \left\{ p \in \mathbb{N} : \frac{1}{p} \left| \left\{ m \leq p : \mathcal{P} \left(\mathcal{N}_p^f(\Psi_m) - \beta_2, \frac{\sigma}{2|\rho|} \right) \leq 1 - \eta \text{ or } \right. \right. \right. \\ \left. \left. \left. \mathcal{Q} \left(\mathcal{N}_p^f(\Psi_m) - \beta_2, \frac{\sigma}{2|\rho|} \right) \geq \eta, \quad \mathcal{R} \left(\mathcal{N}_p^f(\Psi_m) - \beta_2, \frac{\sigma}{2|\rho|} \right) \geq \eta \right\} \right| \geq \gamma \right\} \in \mathcal{I}.$$

So, we can write

$$K_1^c := \left\{ p \in \mathbb{N} : \frac{1}{p} \left| \left\{ m \leq p : \mathcal{P} \left(\mathcal{N}_p^f(\Theta_m) - \beta_1, \frac{\sigma}{2|\lambda|} \right) > 1 - \eta \text{ and } \right. \right. \right. \\ \left. \left. \left. \mathcal{Q} \left(\mathcal{N}_p^f(\Theta_m) - \beta_1, \frac{\sigma}{2|\lambda|} \right) < \eta, \quad \mathcal{R} \left(\mathcal{N}_p^f(\Theta_m) - \beta_1, \frac{\sigma}{2|\lambda|} \right) < \eta \right\} \right| < \gamma \right\} \in \mathcal{F}(\mathcal{I});$$

$$K_2^c := \left\{ p \in \mathbb{N} : \frac{1}{p} \left| \left\{ m \leq p : \mathcal{P} \left(\mathcal{N}_p^f(\Psi_m) - \beta_2, \frac{\sigma}{2|\rho|} \right) > 1 - \eta \text{ and } \right. \right. \right. \\ \left. \left. \left. \mathcal{Q} \left(\mathcal{N}_p^f(\Psi_m) - \beta_2, \frac{\sigma}{2|\rho|} \right) < \eta, \quad \mathcal{R} \left(\mathcal{N}_p^f(\Psi_m) - \beta_2, \frac{\sigma}{2|\rho|} \right) < \eta \right\} \right| < \gamma \right\} \in \mathcal{F}(\mathcal{I}).$$

Therefore, the set $K = K_1^c \cap K_2^c$ is non-empty and $K \in \mathcal{F}(\mathcal{I})$. Let $r \in K$, then

$$\begin{aligned} \mathcal{P}(\mathcal{N}_r^f(\lambda\Theta_m + \rho\Psi_m) - (\lambda\beta_1 + \rho\beta_2), \sigma) &\geq \mathcal{P} \left(\lambda\mathcal{N}_r^f(\Theta_m) - \lambda\beta_1, \frac{\sigma}{2} \right) \triangle \mathcal{P} \left(\rho\mathcal{N}_r^f(\Psi_m) - \rho\beta_2, \frac{\sigma}{2} \right) \\ &= \mathcal{P} \left(\mathcal{N}_r^f(\Theta_m) - \beta_1, \frac{\sigma}{2|\lambda|} \right) \triangle \mathcal{P} \left(\mathcal{N}_r^f(\Psi_m) - \beta_2, \frac{\sigma}{2|\rho|} \right) \\ &> (1 - \eta) \triangle (1 - \eta) \\ &> (1 - \eta). \end{aligned}$$

So, we obtain $\mathcal{P}(\mathcal{N}_r^f(\lambda\Theta_m + \rho\Psi_m) - (\lambda\beta_1 + \rho\beta_2), \sigma) > (1 - \eta)$. In addition,

$$\begin{aligned} \mathcal{Q}(\mathcal{N}_r^f(\lambda\Theta_m + \rho\Psi_m) - (\lambda\beta_1 + \rho\beta_2), \sigma) &\leq \mathcal{Q} \left(\lambda\mathcal{N}_r^f(\Theta_m) - \lambda\beta_1, \frac{\sigma}{2} \right) \diamond \mathcal{Q} \left(\rho\mathcal{N}_r^f(\Psi_m) - \rho\beta_2, \frac{\sigma}{2} \right) \\ &= \mathcal{Q} \left(\mathcal{N}_p^f(\Theta_m) - \beta_1, \frac{\sigma}{2|\lambda|} \right) \diamond \mathcal{Q} \left(\mathcal{N}_p^f(\Psi_m) - \beta_2, \frac{\sigma}{2|\rho|} \right) \\ &< \eta \diamond \eta \\ &< \eta. \end{aligned}$$

Then, we have $\mathcal{Q}(\mathcal{N}_r^f(\lambda\Theta_m + \rho\Psi_m) - (\lambda\beta_1 + \rho\beta_2), \sigma) < \eta$. Furthermore,

$$\begin{aligned} \mathcal{R}(\mathcal{N}_r^f(\lambda\Theta_m + \rho\Psi_m) - (\lambda\beta_1 + \rho\beta_2), \sigma) &\leq \mathcal{R} \left(\lambda\mathcal{N}_r^f(\Theta_m) - \lambda\beta_1, \frac{\sigma}{2} \right) \diamond \mathcal{R} \left(\rho\mathcal{N}_r^f(\Psi_m) - \rho\beta_2, \frac{\sigma}{2} \right) \\ &= \mathcal{R} \left(\mathcal{N}_r^f(\Theta_m) - \beta_1, \frac{\sigma}{2|\lambda|} \right) \diamond \mathcal{R} \left(\mathcal{N}_r^f(\Psi_m) - \beta_2, \frac{\sigma}{2|\rho|} \right) \\ &< \eta \diamond \eta \\ &< \eta. \end{aligned}$$

Therefore, we acquire $\mathcal{R}(\mathcal{N}_r^f(\lambda\Theta_m + \rho\Psi_m) - (\lambda\beta_1 + \rho\beta_2), \sigma) < \eta$. So,

$$r \in \left\{ p \in \mathbb{N} : \frac{1}{p} \left\{ m \leq p : \mathcal{P}(\mathcal{N}_p^f(\lambda\Theta_m + \rho\Psi_m) - (\lambda\beta_1 + \rho\beta_2), \sigma) > 1 - \eta \text{ and } \right. \right. \\ \left. \left. \mathcal{Q}(\mathcal{N}_p^f(\lambda\Theta_m + \rho\Psi_m) - (\lambda\beta_1 + \rho\beta_2), \sigma) < \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\lambda\Theta_m + \rho\Psi_m) - (\lambda\beta_1 + \rho\beta_2), \sigma) < \eta \right\} \right\}.$$

Hence,

$$K \subset \left\{ p \in \mathbb{N} : \frac{1}{p} \{ m \leq p : \mathcal{P}(\mathcal{N}_p^f(\lambda\Theta_m + \rho\Psi_m) - (\lambda\beta_1 + \rho\beta_2), \sigma) > 1 - \eta \text{ and } \right. \\ \left. \mathcal{Q}(\mathcal{N}_p^f(\lambda\Theta_m + \rho\Psi_m) - (\lambda\beta_1 + \rho\beta_2), \sigma) < \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\lambda\Theta_m + \rho\Psi_m) - (\lambda\beta_1 + \rho\beta_2), \sigma) < \eta \} \right\}.$$

Since $K \in \mathcal{F}(I)$, according to the definition of filter, we acquire

$$\left\{ p \in \mathbb{N} : \frac{1}{p} \{ m \leq p : \mathcal{P}(\mathcal{N}_p^f(\lambda\Theta_m + \rho\Psi_m) - (\lambda\beta_1 + \rho\beta_2), \sigma) > 1 - \eta \text{ and } \right. \\ \left. \mathcal{Q}(\mathcal{N}_p^f(\lambda\Theta_m + \rho\Psi_m) - (\lambda\beta_1 + \rho\beta_2), \sigma) < \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\lambda\Theta_m + \rho\Psi_m) - (\lambda\beta_1 + \rho\beta_2), \sigma) < \eta \} \right\} \in \mathcal{F}(I),$$

which means that the sequence $(\lambda\Theta_m + \rho\Psi_m)$ neutrosophic I -statistically converges to $\lambda\beta_1 + \rho\beta_2$. So, $(\lambda\Theta_m + \rho\Psi_m) \in \mathcal{N}_{S^H(I)}^f$. As a result, we obtain $\mathcal{N}_{S^H(I)}^f$ is a linear space. $\square$

Theorem 3. Every open ball with center at Θ and radius $\sigma > 0$ w.r.t. the parameters of fuzziness $\gamma > 0$, $\eta \in (0, 1)$, i.e., $\mathcal{B}(\Theta, \sigma, \eta, \gamma)$ is an open set in $\mathcal{N}_{S^H(I)}^f$ w.r.t. $\mathcal{NH} = (\mathcal{P}, \mathcal{Q}, \mathcal{R})$.

Proof. Assume $\mathcal{B}(\Theta, \sigma, \eta, \gamma)$ be an open ball with center at Θ and radius $\sigma > 0$ w.r.t. the parameters of fuzziness $\gamma > 0$, $\eta \in (0, 1)$,

$$\mathcal{B}(\Theta, \sigma, \eta, \gamma) := \left\{ q = (q_m) \in l_\infty : \left\{ p \in \mathbb{N} : \frac{1}{p} \{ m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_p^f(q), \sigma) \leq 1 - \eta \text{ or } \right. \right. \\ \left. \left. \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_p^f(q), \sigma) \geq \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_p^f(q), \sigma) \geq \eta \} \geq \gamma \right\} \in I \right\}.$$

Then,

$$\mathcal{B}^c(\Theta, \sigma, \eta, \gamma) := \left\{ q = (q_m) \in l_\infty : \left\{ p \in \mathbb{N} : \frac{1}{p} \{ m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_p^f(q), \sigma) > 1 - \eta \text{ and } \right. \right. \\ \left. \left. \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_p^f(q), \sigma) < \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_p^f(q), \sigma) < \eta \} < \gamma \right\} \in \mathcal{F}(I) \right\}.$$

Presume $q = (q_m) \in \mathcal{B}^c(\Theta, \sigma, \eta, \gamma)$. Then, for

$$\mathcal{P}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(q), \sigma) > 1 - \eta \text{ and } \mathcal{Q}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(q), \sigma) < \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(q), \sigma) < \eta,$$

there exists $\sigma_0 \in (0, \sigma)$ so that

$$\mathcal{P}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(q), \sigma_0) > 1 - \eta \text{ and } \mathcal{Q}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(q), \sigma_0) < \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(q), \sigma_0) < \eta.$$

Putting $\eta_0 = \mathcal{P}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(q), \sigma_0)$ implies $\eta_0 > 1 - \eta$. Then, $\exists r \in (0, 1)$ such that $\eta_0 > 1 - r > 1 - \eta$. For $\eta_0 > 1 - r$, we obtain $\eta_1, \eta_2, \eta_3 \in (0, 1)$ such that $\eta_0 \triangle \eta_1 > 1 - r$, $(1 - \eta_0) \diamond (1 - \eta_2) < r$, and $(1 - \eta_0) \diamond (1 - \eta_3) < r$. Take $\eta_4 = \max\{\eta_1, \eta_2, \eta_3\}$. Now, contemplate the open ball $\mathcal{B}^c(\Theta, \sigma - \sigma_0, 1 - \eta_4, \gamma)$.

We have to denote that

$$\mathcal{B}^c(\Theta, \sigma - \sigma_0, 1 - \eta_4, \gamma) \subset \mathcal{B}^c(\Theta, \sigma, \eta, \gamma).$$

Take $\alpha = (\alpha_m) \in \mathcal{B}^c(q, \sigma - \sigma_0, 1 - \eta_4, \gamma)$. Then,

$$\mathcal{P}(\mathcal{N}_p^f(q) - \mathcal{N}_p^f(\alpha), \sigma - \sigma_0) > \eta_4 \text{ and } \mathcal{Q}(\mathcal{N}_p^f(q) - \mathcal{N}_p^f(\alpha), \sigma - \sigma_0) < 1 - \eta_4 \\ \mathcal{R}(\mathcal{N}_p^f(q) - \mathcal{N}_p^f(\alpha), \sigma - \sigma_0) < 1 - \eta_4.$$

So, we obtain

$$\begin{aligned} \mathcal{P}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\alpha), \sigma) &\geq \mathcal{P}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(q), \sigma_0) \triangle \mathcal{P}(\mathcal{N}_p^f(q) - \mathcal{N}_p^f(\alpha), \sigma - \sigma_0) \\ &\geq \eta_0 \triangle \eta_4 \geq \eta_0 \triangle \eta_1 > 1 - r > 1 - \eta, \end{aligned}$$

$$Q(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\alpha), \sigma) \leq Q(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(q), \sigma_0) \diamond Q(\mathcal{N}_p^f(q) - \mathcal{N}_p^f(\alpha), \sigma - \sigma_0) \leq \eta_0 \triangle \eta_4 \leq \eta_0 \triangle \eta_2 < r < \eta,$$

and

$$\mathcal{R}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\alpha), \sigma) \leq \mathcal{R}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(q), \sigma_0) \diamond \mathcal{R}(\mathcal{N}_p^f(q) - \mathcal{N}_p^f(\alpha), \sigma - \sigma_0) \leq \eta_0 \triangle \eta_4 \leq \eta_0 \triangle \eta_3 < r < \eta.$$

Therefore, we obtain

$$\begin{aligned} &\left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_p^f(\alpha), \sigma) > 1 - \eta \text{ and} \right. \\ &\quad \left. Q(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_p^f(\alpha), \sigma) < \eta, \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_p^f(\alpha), \sigma) < \eta\}| < \gamma \right\} \in \mathcal{F}(I). \end{aligned}$$

Hence, $\alpha = (\alpha_m) \in \mathcal{B}^c(\Theta, \sigma, \eta, \gamma)$. As a result, we obtain $\mathcal{B}^c(\Theta, \sigma - \sigma_0, 1 - \eta_4, \gamma) \subset \mathcal{B}^c(\Theta, \sigma, \eta, \gamma)$. $\square$

Now, we identify a collection $\tau_{\mathcal{S}^H(I)}^{\mathcal{N}^f}$ of a subset of $\mathcal{N}_{\mathcal{S}^H(I)}^f$ as follows:

$$\begin{aligned} \tau_{\mathcal{S}^H(I)}^{\mathcal{N}^f} &= \{T \subset \mathcal{N}_{\mathcal{S}^H(I)}^f : \text{ for all } \Theta = (\Theta_m) \in T \text{ there exist} \\ &\quad \sigma, \gamma > 0 \text{ and } \eta \in (0, 1) \text{ such that } \mathcal{B}(\Theta, \sigma, \eta, \gamma) \subset T\}. \end{aligned}$$

Then, $\tau_{\mathcal{S}^H(I)}^{\mathcal{N}^f}$ determines a topology on the sequence space $\mathcal{N}_{\mathcal{S}^H(I)}^f$. The collection given by

$$\mathcal{B} = \{\mathcal{B}(\Theta, \sigma, \eta, \gamma) : \Theta \in \mathcal{N}_{\mathcal{S}^H(I)}^f, \sigma, \gamma > 0 \text{ and } \eta \in (0, 1)\}$$

is a base for the topology $\tau_{\mathcal{S}^H(I)}^{\mathcal{N}^f}$ on the space $\mathcal{N}_{\mathcal{S}^H(I)}^f$.

Theorem 4. The topology $\tau_{\mathcal{S}^H(I)}^{\mathcal{N}^f}$ on the space of $\mathcal{N}_{\mathcal{S}^H(I)}^f$ is first countable.

Proof. For all $\Theta = (\Theta_m) \in \mathcal{N}_{\mathcal{S}^H(I)}^f$, consider the set $K = \left\{ \mathcal{B}\left(\Theta, \frac{1}{q}, \frac{1}{q}, \frac{1}{q}\right) : q = 1, 2, 3, \dots \right\}$, which is a countable local base at $\Theta = (\Theta_m)$. Hence, the topology $\tau_{\mathcal{S}^H(I)}^{\mathcal{N}^f}$ on the space of $\mathcal{N}_{\mathcal{S}^H(I)}^f$ is first countable. $\square$

Theorem 5. The spaces $\mathcal{N}_{\mathcal{S}^H(I)}^f$ and $\mathcal{N}_{\mathcal{S}^H(I_0)}^f$ are Hausdorff spaces.

Proof. It is obvious that $\mathcal{N}_{\mathcal{S}^H(I_0)}^f \subset \mathcal{N}_{\mathcal{S}^H(I)}^f$. We have to demonstrate the result for only $\mathcal{N}_{\mathcal{S}^H(I)}^f$. Assume $\Theta = (\Theta_m)$ and $\Psi = (\Psi_m) \in \mathcal{N}_{\mathcal{S}^H(I)}^f$ so that $\Theta \neq \Psi$. Then, for all $p \in \mathbb{N}$ and $\sigma > 0$, we obtain

$$\begin{aligned} 0 < \mathcal{P}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \sigma) < 1, \quad 0 < Q(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \sigma) < 1, \\ 0 < \mathcal{R}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \sigma) < 1. \end{aligned}$$

We have

$$\begin{aligned} \eta_1 &= \mathcal{P}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \sigma), \quad \eta_2 = Q(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \sigma), \\ \eta_3 &= \mathcal{R}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \sigma), \end{aligned}$$

and $\eta = \max\{\eta_1, 1 - \eta_2, 1 - \eta_3\}$. At that time, for all $\eta_0 \in (\eta, 1)$, there are $\eta_4, \eta_5, \eta_6 \in (0, 1)$ so that $\eta_4 \triangle \eta_4 \geq \eta_0$, $(1 - \eta_5) \diamond (1 - \eta_5) \leq 1 - \eta_0$, and $(1 - \eta_6) \diamond (1 - \eta_6) \leq 1 - \eta_0$. Again, we take $\eta_7 = \max\{\eta_4, 1 - \eta_5, 1 - \eta_6\}$ and contemplate the open balls $\mathcal{B}\left(\Theta, 1 - \eta_7, \frac{\sigma}{2}, \gamma\right)$ and $\mathcal{B}\left(\Psi, 1 - \eta_7, \frac{\sigma}{2}, \gamma\right)$ centered at Θ and Ψ , respectively. Then, it is obvious that

$$\mathcal{B}^c\left(\Theta, 1 - \eta_7, \frac{\sigma}{2}, \gamma\right) \cap \mathcal{B}^c\left(\Psi, 1 - \eta_7, \frac{\sigma}{2}, \gamma\right) = \emptyset.$$

If possible assume $\alpha = (\alpha_m) \in \mathcal{B}^c\left(\Theta, 1 - \eta_7, \frac{\sigma}{2}, \gamma\right) \cap \mathcal{B}^c\left(\Psi, 1 - \eta_7, \frac{\sigma}{2}, \gamma\right)$. Then, we obtain

$$\begin{aligned}\eta_1 &= \mathcal{P}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \sigma) \\ &\geq \mathcal{P}\left(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\alpha), \frac{\sigma}{2}\right) \triangle \mathcal{P}\left(\mathcal{N}_p^f(\alpha) - \mathcal{N}_p^f(\Psi), \frac{\sigma}{2}\right) \\ &> \eta_7 \triangle \eta_7 \\ &\geq \eta_4 \triangle \eta_4 \\ &\geq \eta_0 > \eta_1, \\ \eta_2 &= \mathcal{Q}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \sigma) \\ &\leq \mathcal{Q}\left(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\alpha), \frac{\sigma}{2}\right) \diamond \mathcal{Q}\left(\mathcal{N}_p^f(\alpha) - \mathcal{N}_p^f(\Psi), \frac{\sigma}{2}\right) \\ &< (1 - \eta_7) \diamond (1 - \eta_7) \\ &\leq (1 - \eta_5) \diamond (1 - \eta_5) \\ &< (1 - \eta_0) < \eta_2,\end{aligned}$$

and

$$\begin{aligned}\eta_3 &= \mathcal{R}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \sigma) \\ &\leq \mathcal{R}\left(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\alpha), \frac{\sigma}{2}\right) \diamond \mathcal{R}\left(\mathcal{N}_p^f(\alpha) - \mathcal{N}_p^f(\Psi), \frac{\sigma}{2}\right) \\ &< (1 - \eta_7) \diamond (1 - \eta_7) \\ &\leq (1 - \eta_6) \diamond (1 - \eta_6) \\ &< (1 - \eta_0) < \eta_3.\end{aligned}$$

From the above equations, we obtain a contradiction. So,

$$\mathcal{B}^c\left(\Theta, 1 - \eta_7, \frac{\sigma}{2}, \gamma\right) \cap \mathcal{B}^c\left(\Psi, 1 - \eta_7, \frac{\sigma}{2}, \gamma\right) = \emptyset.$$

Hence, the space $\mathcal{N}_{S^H(I)}^f$ is a Hausdorff space. $\square$

Theorem 6. Let $\tau_{S^H(I)}^{\mathcal{N}^f}$ be a topology on an NNS $\mathcal{N}_{S^H(I)}^f$, then a sequence $\Theta = (\Theta_m) \in \mathcal{N}_{S^H(I)}^f$ such that $(\mathcal{P}, \mathcal{Q}, \mathcal{R}) - \lim \Theta_m = \beta$ iff $\mathcal{P}(\mathcal{N}_p^f(\Theta) - \beta, \sigma) \rightarrow 1, \mathcal{Q}(\mathcal{N}_p^f(\Theta) - \beta, \sigma) \rightarrow 0$, and $\mathcal{R}(\mathcal{N}_p^f(\Theta) - \beta, \sigma) \rightarrow 0$ as $p \rightarrow \infty$.

Proof. Assume $(\mathcal{P}, \mathcal{Q}, \mathcal{R}) - \lim \Theta_m = \beta$ and take $\sigma_0 > 0$. Then, for $\eta \in (0, 1)$, $\exists m_0 \in \mathbb{N}$, $(\Theta_m) \in \mathcal{B}(\Theta, \sigma, \eta, \gamma)$, $\forall m \geq m_0$ and for a $\sigma > 0$,

$$\begin{aligned}\mathcal{B}(\Theta, \sigma, \eta, \gamma) &:= \left\{ \Theta = (\Theta_m) \in l_\infty : \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \leq 1 - \eta \text{ or} \right. \right. \\ &\quad \left. \left. \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \geq \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \geq \eta\} | \geq \gamma \right\} \right\},\end{aligned}$$

so $\mathcal{B}^c(\Theta, \gamma, \sigma) \in \mathcal{F}(I)$. Then,

$$1 - \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) < \eta, \quad \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) < \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) < \eta.$$

Hence,

$$\mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \rightarrow 1, \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \rightarrow 0 \text{ and } \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \rightarrow 0$$

as $p \rightarrow \infty$.

Conversely, if, for each $\sigma > 0$,

$$\mathcal{P}(\mathcal{N}_p^f(\Theta) - \beta, \sigma) \rightarrow 1, \quad \mathcal{Q}(\mathcal{N}_p^f(\Theta) - \beta, \sigma) \rightarrow 0, \quad \text{and} \quad \mathcal{R}(\mathcal{N}_p^f(\Theta) - \beta, \sigma) \rightarrow 0$$

as $p \rightarrow \infty$, then, for all $\eta \in (0, 1)$, $\exists m_0 \in \mathbb{N}$ such that

$$1 - \mathcal{P}(\mathcal{N}_p^f(\Theta) - \beta, \sigma) < \eta, \quad \mathcal{Q}(\mathcal{N}_p^f(\Theta) - \beta, \sigma) < \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta) - \beta, \sigma) < \eta, \quad \forall m \geq m_0.$$

Hence, we acquire

$$\mathcal{P}(\mathcal{N}_p^f(\Theta) - \beta, \sigma) > 1 - \eta, \quad \mathcal{Q}(\mathcal{N}_p^f(\Theta) - \beta, \sigma) < \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta) - \beta, \sigma) < \eta, \quad \forall m \geq m_0.$$

So, $\Theta = (\Theta_m) \in \mathcal{B}(\Theta, \sigma, \eta, \gamma) \forall m \geq m_0$, and as a result, $(\mathcal{P}, \mathcal{Q}, \mathcal{R}) - \lim \Theta_m = \beta$. $\square$

Theorem 7. Take $\Theta = (\Theta_m) \in \mathcal{N}_{S^H(I)}^f$. When a sequence $\Theta = (\Theta_m)$ is neutrosophic Nörlund $\mathcal{I}$ -statistically convergent, then $\mathcal{N}_p^f - \mathcal{I} - \text{st} \lim \Theta_m$ is unique.

Proof. Assume $\Theta = (\Theta_m)$ is neutrosophic Nörlund $\mathcal{I}$ -statistically convergent. Let, on the contrary, that β_1 and β_2 are two different elements so that $\mathcal{N}_p^f - \mathcal{I} - \text{st} \lim \Theta_m = \beta_1$ and $\mathcal{N}_p^f - \mathcal{I} - \text{st} \lim \Theta_m = \beta_2$. For a given $\eta \in (0, 1)$, select $r > 0$ so that $(1 - r) \triangle (1 - r) > 1 - \eta$, $r \diamond r < \eta$. For $\sigma > 0$, we have to denote $\beta_1 = \beta_2$. We determine the subsequent sets as follows:

$$\begin{aligned} K_1 &:= \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta_1, \sigma) \leq 1 - \eta \text{ or} \right. \\ &\quad \left. \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \beta_1, \sigma) \geq \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta_1, \sigma) \geq \eta\}| \geq \gamma \right\} \in \mathcal{I}; \\ K_2 &:= \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta_2, \sigma) \leq 1 - \eta \text{ or} \right. \\ &\quad \left. \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \beta_2, \sigma) \geq \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta_2, \sigma) \geq \eta\}| \geq \gamma \right\} \in \mathcal{I}. \end{aligned}$$

So, we can write

$$\begin{aligned} K_1^c &:= \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta_1, \sigma) > 1 - \eta \text{ and} \right. \\ &\quad \left. \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \beta_1, \sigma) < \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta_1, \sigma) < \eta\}| < \gamma \right\} \in \mathcal{F}(\mathcal{I}); \\ K_2^c &:= \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta_2, \sigma) > 1 - \eta \text{ and} \right. \\ &\quad \left. \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \beta_2, \sigma) < \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta_2, \sigma) < \eta\}| < \gamma \right\} \in \mathcal{F}(\mathcal{I}). \end{aligned}$$

If we take $K = K_1 \cap K_2$, then $K \in \mathcal{I}$. So, $K^c \in \mathcal{F}(\mathcal{I})$. Then, we obtain $K^c = K_1^c \cap K_2^c \neq \emptyset$. Taking $s \in K_1^c \cap K_2^c$, which means that

$$\mathcal{P}\left(\mathcal{N}_s^f(\Theta_m) - \beta_1, \frac{\sigma}{2}\right) > 1 - r, \quad \mathcal{P}\left(\mathcal{N}_s^f(\Theta_m) - \beta_2, \frac{\sigma}{2}\right) > 1 - r.$$

Therefore, we obtain

$$\mathcal{P}(\beta_1 - \beta_2, \sigma) \geq \mathcal{P}\left(\mathcal{N}_s^f(\Theta_m) - \beta_1, \frac{\sigma}{2}\right) \triangle \mathcal{P}\left(\mathcal{N}_s^f(\Theta_m) - \beta_2, \frac{\sigma}{2}\right) > (1 - r) \triangle (1 - r) > 1 - \eta.$$

Since $\eta > 0$ was arbitrary, $\mathcal{P}(\beta_1 - \beta_2, \sigma) = 1$ for all $\eta > 0$. As a result, we obtain $\beta_1 = \beta_2$, which is a contradiction.

If $s \in K^c$, then we have

$$Q\left(\mathcal{N}_s^f(\Theta_m) - \beta_1, \frac{\sigma}{2}\right) < \eta, \quad Q\left(\mathcal{N}_s^f(\Theta_m) - \beta_2, \frac{\sigma}{2}\right) < \eta.$$

Therefore, we have

$$Q(\beta_1 - \beta_2, \sigma) \leq Q\left(\mathcal{N}_s^f(\Theta_m) - \beta_1, \frac{\sigma}{2}\right) \diamond Q\left(\mathcal{N}_s^f(\Theta_m) - \beta_2, \frac{\sigma}{2}\right) < r \diamond r < \eta.$$

Since $\eta > 0$ was arbitrary, $Q(\beta_1 - \beta_2, \sigma) = 0$ for all $\eta > 0$. As a result, we obtain $\beta_1 = \beta_2$, which is a contradiction.

If $s \in K^c$, then we have

$$\mathcal{R}\left(\mathcal{N}_s^f(\Theta_m) - \beta_1, \frac{\sigma}{2}\right) < \eta, \quad \mathcal{R}\left(\mathcal{N}_s^f(\Theta_m) - \beta_2, \frac{\sigma}{2}\right) < \eta.$$

Therefore, we have

$$\mathcal{R}(\beta_1 - \beta_2, \sigma) \leq \mathcal{R}\left(\mathcal{N}_s^f(\Theta_m) - \beta_1, \frac{\sigma}{2}\right) \diamond \mathcal{R}\left(\mathcal{N}_s^f(\Theta_m) - \beta_2, \frac{\sigma}{2}\right) < r \diamond r < \eta.$$

Since $\eta > 0$ was arbitrary, $\mathcal{R}(\beta_1 - \beta_2, \sigma) = 0$ for all $\eta > 0$. As a result, we obtain $\beta_1 = \beta_2$, which is a contradiction. For all cases, we obtain $\beta_1 = \beta_2$. We demonstrate that $\mathcal{N}_p^f - \mathcal{I} - st \lim \Theta_m$ is unique. $\square$

Theorem 8. A sequence $\Theta = (\Theta_m)$ is neutrosophic Nörlund $\mathcal{I}$ -statistically convergent w.r.t. $NN\mathcal{H} = (\mathcal{P}, Q, \mathcal{R})$ iff it is neutrosophic Nörlund $\mathcal{I}$ -statistically Cauchy w.r.t. the same norms.

Proof. Assume $\Theta = (\Theta_m)$ be neutrosophic Nörlund $\mathcal{I}$ -statistically convergent w.r.t. $NN\mathcal{H} = (\mathcal{P}, Q, \mathcal{R})$ so that $\mathcal{N}_p^f - \mathcal{I} - st \lim \Theta_m = \beta$. For a given $\eta \in (0, 1)$, there exists $r_1 \in (0, 1)$ such that $(1 - r_1) \triangle (1 - r_1) > 1 - \eta$ and $r_1 \diamond r_1 < \eta$. Since $\mathcal{N}_p^f - \mathcal{I} - st \lim \Theta_m = \beta$, therefore, for each $\sigma, \gamma > 0$,

$$K_1 := \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \leq 1 - r_1 \text{ or } Q(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \geq r_1, \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \geq r_1\}| \geq \gamma \right\} \in \mathcal{I};$$

which implies

$$K_1^c := \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) > 1 - r_1 \text{ and } Q(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) < r_1, \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) < r_1\}| < \gamma \right\} \in \mathcal{F}(\mathcal{I}).$$

For $s \in K_1^c$, we obtain

$$\mathcal{P}(\mathcal{N}_s^f(\Theta_m) - \beta, \sigma) > 1 - r_1 \text{ and } Q(\mathcal{N}_s^f(\Theta_m) - \beta, \sigma) < r_1, \quad \mathcal{R}(\mathcal{N}_s^f(\Theta_m) - \beta, \sigma) < r_1.$$

For fix $s \in K_1^c$, let

$$K_2 := \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \leq 1 - \eta \text{ or } Q(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \geq \eta, \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \geq \eta\}| \geq \gamma \right\} \in \mathcal{I}.$$

We have to demonstrate that $K_2 \subset K_1$. Take $p \in K_2$, then we obtain

$$\mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \leq 1 - \eta \quad \text{or} \quad Q(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \geq \eta,$$

$$\mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \geq \eta.$$

We obtain the following possible cases.

Initially, consider $\mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \leq 1 - \eta$. Then $\mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}) \leq 1 - \eta$.

If possible, assume $\mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}) > 1 - r_1$. Then, we can write

$$\begin{aligned} 1 - \eta &\geq \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \\ &\geq \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}) \triangle \mathcal{P}(\mathcal{N}_s^f(\Theta_m) - \beta, \frac{\sigma}{2}) \\ &> (1 - r_1) \triangle (1 - r_1) \\ &> 1 - \eta, \end{aligned}$$

which is a contradiction. So, $\mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}) \leq 1 - r_1$ holds.

In the same way, consider $\mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \geq \eta$. Then, $\mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}) \geq \eta$.

If possible assume $\mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}) < r_1$. Then, we can write

$$\eta \leq \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \leq \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}) \diamond \mathcal{Q}(\mathcal{N}_s^f(\Theta_m) - \beta, \frac{\sigma}{2}) < r_1 \diamond r_1 < \eta,$$

which is a contradiction. So, $\mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}) \geq r_1$ supplies.

Similarly, consider $\mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \geq \eta$. Then, $\mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}) \geq \eta$.

If possible, let $\mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}) < r_1$. Then, we can write

$$\eta \leq \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \leq \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}) \diamond \mathcal{R}(\mathcal{N}_s^f(\Theta_m) - \beta, \frac{\sigma}{2}) < r_1 \diamond r_1 < \eta,$$

which is a contradiction. So, $\mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}) \geq r_1$ holds. So, for $p \in K_2$, we obtain

$$\mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \leq 1 - r_1 \text{ or } \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \geq r_1, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \geq r_1.$$

Therefore, $p \in K_1$. Hence, $K_2 \subset K_1$. Since $K_1 \in \mathcal{I}$, we obtain $K_2 \in \mathcal{I}$. As a result, $\Theta = (\Theta_m)$ is neutrosophic Nörlund $\mathcal{I}$ -statistically Cauchy w.r.t. NN $\mathcal{H} = (\mathcal{P}, \mathcal{Q}, \mathcal{R})$.

Conversely, assume the sequence $\Theta = (\Theta_m)$ is neutrosophic Nörlund $\mathcal{I}$ -statistically Cauchy w.r.t. NN $\mathcal{H} = (\mathcal{P}, \mathcal{Q}, \mathcal{R})$. Let, on the contrary, the sequence $\Theta = (\Theta_m)$ is not neutrosophic Nörlund $\mathcal{I}$ -statistically convergent indicated by S_2 . Then, there exists $s \in \mathbb{N}$ so that

$$\begin{aligned} S_1 = \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \leq 1 - \eta \text{ or } \right. \\ \left. \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \geq \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \geq \eta\}| \geq \gamma \right\} \in \mathcal{I}. \end{aligned}$$

Let, on the contrary,

$$\begin{aligned} S_2 = \left\{ p \in \mathbb{N} : \frac{1}{p} \left| \left\{ m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}) > 1 - r_1 \text{ and } \right. \right. \right. \\ \left. \left. \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}) < r_1, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}) < r_1 \right\} \right| < \gamma \right\} \in \mathcal{I}. \end{aligned}$$

So, we have

$$\begin{aligned} 1 - \eta &\geq \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \\ &\geq \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}) \triangle \mathcal{P}(\mathcal{N}_s^f(\Theta_m) - \beta, \frac{\sigma}{2}) \\ &> (1 - r_1) \triangle (1 - r_1) \\ &> 1 - \eta, \end{aligned}$$

which is a contradiction. Now

$$\eta \leq Q(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \leq Q\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) \diamond Q\left(\mathcal{N}_s^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < r_1 \diamond r_1 < \eta,$$

which is a contradiction. Furthermore,

$$\eta \leq \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_s^f(\Theta_m), \sigma) \leq \mathcal{R}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) \diamond \mathcal{R}\left(\mathcal{N}_s^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < r_1 \diamond r_1 < \eta,$$

which is a contradiction. So, $S_2 \in \mathcal{F}(I)$ and thus $\Theta = (\Theta_m)$ is neutrosophic Nörlund I -statistically convergent w.r.t. $\text{NN } \mathcal{H} = (\mathcal{P}, \mathcal{Q}, \mathcal{R})$. $\square$

Theorem 9. A sequence $\Theta = (\Theta_m) \in \mathcal{N}_{S^{\mathcal{H}}(I)}^f$ is neutrosophic Nörlund I -statistically convergent w.r.t. $\text{NN } \mathcal{H} = (\mathcal{P}, \mathcal{Q}, \mathcal{R})$. Then, for some $\beta \in \mathbb{R}$, $\mathcal{N}_p^f - I - \text{st } \lim \Theta_m = \beta$ iff for each $\eta \in (0, 1)$ and $\sigma, \gamma > 0$, there are positive integers $T = T(\Theta, \sigma, \eta, \gamma)$ such that

$$\left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \sigma) \leq 1 - \eta \text{ or } Q(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \sigma) \geq \eta, \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \sigma) \geq \eta\}| \geq \gamma \right\} \in I.$$

Proof. Assume $\mathcal{N}_p^f - I - \text{st } \lim \Theta_m = \beta$, for some $\beta \in \mathbb{R}$. For given $r > 0$, there exists $\eta \in (0, 1)$ such that $(1 - r) \triangle (1 - r) > 1 - \eta$ and $r \diamond r < \eta$. Since $\mathcal{N}_p^f - I - \text{st } \lim \Theta_m = \beta$, for all $\sigma, \gamma > 0$,

$$\mathcal{A} = \left\{ p \in \mathbb{N} : \frac{1}{p} \left| \left\{ m \leq p : \mathcal{P}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) \leq 1 - r \text{ or } Q\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) \geq r, \mathcal{R}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) \geq r \right\} \right| \geq \gamma \right\} \in I,$$

which means that

$$\mathcal{A}^c = \left\{ p \in \mathbb{N} : \frac{1}{p} \left| \left\{ m \leq p : \mathcal{P}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) > 1 - r \text{ and } Q\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < r, \mathcal{R}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < r \right\} \right| < \gamma \right\} \in \mathcal{F}(I).$$

Selecting a natural number $T \in \mathcal{A}^c$, we obtain

$$\mathcal{P}\left(\mathcal{N}_T^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) > 1 - r \text{ and } Q\left(\mathcal{N}_T^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < r, \mathcal{R}\left(\mathcal{N}_T^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < r.$$

We denote that there is a positive integer $T = T(\Theta, \sigma, \eta, \gamma)$ such that

$$\mathcal{B} = \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \sigma) \leq 1 - \eta \text{ or } Q(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \sigma) \geq \eta, \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \sigma) \geq \eta\}| \geq \gamma \right\}.$$

So, for $\Theta = (\Theta_m) \in \mathcal{N}_{S^{\mathcal{H}}(I)}^f$, we have to denote that $\mathcal{B} \subseteq \mathcal{A}$.

Let, on the contrary, $\mathcal{B} \not\subseteq \mathcal{A}$. Then, there exists $q \in \mathcal{B}$; however, $q \notin \mathcal{A}$. Hence,

$$\mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \sigma) \leq 1 - \eta \text{ and } \mathcal{P}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) > 1 - r.$$

Especially, $\mathcal{P}\left(\mathcal{N}_T^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) > 1 - r$. So, we obtain

$$1 - \eta \geq \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \sigma) \geq \mathcal{P}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) \triangle \mathcal{P}\left(\mathcal{N}_T^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) > (1 - r) \triangle (1 - r) > 1 - \eta,$$

which is a contradiction. In the same way,

$$\mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \sigma) \geq \eta \text{ and } \mathcal{Q}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < r.$$

In particular, $\mathcal{Q}\left(\mathcal{N}_T^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < r$. Hence, we acquire

$$\eta \leq \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \sigma) \leq \mathcal{Q}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) \diamond \mathcal{Q}\left(\mathcal{N}_T^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < r \diamond r < \eta,$$

which is a contradiction. Similarly,

$$\mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \sigma) \geq \eta \text{ and } \mathcal{R}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < r.$$

Particularly, $\mathcal{R}\left(\mathcal{N}_T^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < r$. Therefore, we obtain

$$\eta \leq \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \sigma) \leq \mathcal{R}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) \diamond \mathcal{R}\left(\mathcal{N}_T^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < r \diamond r < \eta,$$

which is a contradiction. Hence, we have $\mathcal{B} \subseteq \mathcal{A}$ and since $\mathcal{A} \in \mathcal{I}$, so we obtain $\mathcal{B} \in \mathcal{I}$.

Conversely, let, on the contrary, $\Theta = (\Theta_m)$ is not neutrosophic Nörlund $\mathcal{I}$ -statistically convergent w.r.t. NN $\mathcal{H} = (\mathcal{P}, \mathcal{Q}, \mathcal{R})$ and $\mathcal{B}$ holds, then

$$\mathcal{Y} = \left\{ p \in \mathbb{N} : \frac{1}{p} \left| \left\{ m \leq p : \mathcal{P}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) > 1 - \eta \text{ and } \right. \right. \right. \\ \left. \left. \left. \mathcal{Q}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < \eta, \quad \mathcal{R}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < \eta \right\} \right| < \gamma \right\} \in \mathcal{I},$$

which implies that $\mathcal{Y}^c \in \mathcal{F}(\mathcal{I})$. Since $\mathcal{B}$ holds, then there is an $T = T(\Theta, \sigma, \gamma)$ so that

$$\mathcal{B} = \left\{ p \in \mathbb{N} : \frac{1}{p} \left| \left\{ m \leq p : \mathcal{P}\left(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \frac{\sigma}{2}\right) \leq 1 - \eta \text{ or } \right. \right. \right. \\ \left. \left. \left. \mathcal{Q}\left(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \frac{\sigma}{2}\right) \geq \eta, \quad \mathcal{R}\left(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \frac{\sigma}{2}\right) \geq \eta \right\} \right| \geq \gamma \right\} \in \mathcal{I}.$$

As

$$\mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \sigma) \geq 2\mathcal{P}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) > 1 - \eta \\ \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \sigma) \leq 2\mathcal{Q}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < \eta \text{ and} \\ \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \sigma) \leq 2\mathcal{R}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < \eta,$$

then

$$\mathcal{P}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) > \frac{1 - \eta}{2}, \mathcal{Q}\left(\mathcal{N}_p^f(\Theta_m) - \beta, \frac{\sigma}{2}\right) < \frac{\eta}{2} \text{ and } \mathcal{R}\left(\mathcal{N}_p^f(\Theta_m) - \mathcal{N}_T^f(\Theta_m), \frac{\sigma}{2}\right) < \frac{\eta}{2}.$$

So, we obtain $\mathcal{B}^c \in \mathcal{I}$. Equivalently, $\mathcal{B} \in \mathcal{F}(\mathcal{I})$, which is a contradiction, as $\mathcal{B}$ holds. $\square$

Now $l_{\mathcal{H}}^{\infty}(\mathcal{N}^f)$ shows the space of all sequences, whose $\mathcal{N}^f$ -transform is neutrosophic bounded sequence.

$\mathcal{N}_{l_{\mathcal{H}}^{\infty}}^f(S(I))$ denotes the space of all sequences, whose $\mathcal{N}^f$ -transform is neutrosophic bounded and neutrosophic I -statistically convergent sequence.

Theorem 10. $\mathcal{N}_{l_{\mathcal{H}}^{\infty}}^f(S(I))$ is closed linear space of $l_{\mathcal{H}}^{\infty}(\mathcal{N}^f)$.

Proof. It is obvious that $\mathcal{N}_{l_{\mathcal{H}}^{\infty}}^f(S(I)) \subset l_{\mathcal{H}}^{\infty}(\mathcal{N}^f)$. We have to demonstrate that $\mathcal{N}_{l_{\mathcal{H}}^{\infty}}^f(S(I))$ is closed, i.e., $\overline{\mathcal{N}_{l_{\mathcal{H}}^{\infty}}^f(S(I))} = \mathcal{N}_{l_{\mathcal{H}}^{\infty}}^f(S(I))$ (where $\overline{\mathcal{N}_{l_{\mathcal{H}}^{\infty}}^f(S(I))}$ shows the closure of $\mathcal{N}_{l_{\mathcal{H}}^{\infty}}^f(S(I))$). It is clear that $\mathcal{N}_{l_{\mathcal{H}}^{\infty}}^f(S(I)) \subset \overline{\mathcal{N}_{l_{\mathcal{H}}^{\infty}}^f(S(I))}$. Conversely, we denote that $\overline{\mathcal{N}_{l_{\mathcal{H}}^{\infty}}^f(S(I))} \subset \mathcal{N}_{l_{\mathcal{H}}^{\infty}}^f(S(I))$. Let $\Theta = (\Theta_m) \in \overline{\mathcal{N}_{l_{\mathcal{H}}^{\infty}}^f(S(I))}$. Then, $\mathcal{B}(\Theta, \sigma, \eta, \gamma) \cap \mathcal{N}_{l_{\mathcal{H}}^{\infty}}^f(S(I)) \neq \emptyset$ for all open ball $\mathcal{B}(\Theta, \sigma, \eta, \gamma)$ of any radius $\eta > 0$ and $\sigma, \gamma > 0$ centered at Θ .

So, take $\Theta \in \mathcal{B}(\Theta, \sigma, \eta, \gamma) \cap \mathcal{N}_{l_{\mathcal{H}}^{\infty}}^f(S(I))$. For $r \in (0, 1)$ and $\eta \in (0, 1)$, select $(1-r) \triangle (1-r) > 1-\eta$ and $r \diamond r < \eta$. Since $\Psi \in \mathcal{B}(\Theta, \sigma, \eta, \gamma) \cap \mathcal{N}_{l_{\mathcal{H}}^{\infty}}^f(S(I))$, there is a subset T of $\mathbb{N}$ such that $T \in \mathcal{F}(I)$, and for each $p \in T$, we obtain

$$\mathcal{P}\left(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \frac{\sigma}{2}\right) > 1-r, \quad \mathcal{Q}\left(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \frac{\sigma}{2}\right) < r \quad \text{and} \quad \mathcal{R}\left(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \frac{\sigma}{2}\right) < r$$

and

$$\mathcal{P}\left(\mathcal{N}_p^f(\Psi) - \beta, \frac{\sigma}{2}\right) > 1-r, \quad \mathcal{Q}\left(\mathcal{N}_p^f(\Psi) - \beta, \frac{\sigma}{2}\right) < r \quad \text{and} \quad \mathcal{R}\left(\mathcal{N}_p^f(\Psi) - \beta, \frac{\sigma}{2}\right) < r.$$

So, for all $p \in T$, we have

$$\begin{aligned} \mathcal{P}(\mathcal{N}_p^f(\Theta) - \beta, \sigma) &= \mathcal{P}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi) + \mathcal{N}_p^f(\Psi) - \beta, \sigma) \\ &\geq \mathcal{P}\left(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \frac{\sigma}{2}\right) \triangle \mathcal{P}\left(\mathcal{N}_p^f(\Psi) - \beta, \frac{\sigma}{2}\right) \\ &> (1-r) \triangle (1-r) \\ &> 1-\eta, \end{aligned}$$

$$\begin{aligned} \mathcal{Q}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \sigma) &= \mathcal{Q}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi) + \mathcal{N}_p^f(\Psi) - \beta, \sigma) \\ &\leq \mathcal{Q}\left(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \frac{\sigma}{2}\right) \diamond \mathcal{Q}\left(\mathcal{N}_p^f(\Psi) - \beta, \frac{\sigma}{2}\right) \\ &< r \diamond r \\ &< r, \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \sigma) &= \mathcal{R}(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi) + \mathcal{N}_p^f(\Psi) - \beta, \sigma) \\ &\leq \mathcal{R}\left(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \frac{\sigma}{2}\right) \diamond \mathcal{R}\left(\mathcal{N}_p^f(\Psi) - \beta, \frac{\sigma}{2}\right) \\ &< r \diamond r \\ &< r. \end{aligned}$$

As a result, we obtain

$$\begin{aligned} T \subset \left\{ p \in \mathbb{N} : \frac{1}{p} |\{m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) > 1-\eta \text{ and} \right. \\ \left. \mathcal{Q}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) < \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) < \eta\}| < \gamma \right\}. \end{aligned}$$

Since $T \in \mathcal{F}(I)$, we obtain

$$\left\{ p \in \mathbb{N} : \frac{1}{p} \left| \{ m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) > 1 - \eta \text{ and } \right. \right. \\ \left. \left. Q(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) < \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) < \eta \} \right| < \gamma \right\} \in \mathcal{F}(\mathcal{I}).$$

So, $\Theta \in \mathcal{N}_{l_{\mathcal{H}}}^f(S(\mathcal{I}))$. Hence, $\overline{\mathcal{N}_{l_{\mathcal{H}}}^f(S(\mathcal{I}))} = \mathcal{N}_{l_{\mathcal{H}}}^f(S(\mathcal{I}))$. $\square$

Theorem 11. Assume $\Theta = (\Theta_m) \in l_{\infty}$ be a sequence. If there is a sequence $\Psi = (\Psi_m) \in \mathcal{N}_{S(\mathcal{I})}^f$ so that $\mathcal{N}_p^f(\Theta_m) = \mathcal{N}_p^f(\Psi_m)$ for almost all p relative to neutrosophic $\mathcal{I}$, then $\Theta = (\Theta_m) \in \mathcal{N}_{S(\mathcal{I})}^f$.

Proof. Consider $\mathcal{N}_p^f(\Theta_m) = \mathcal{N}_p^f(\Psi_m)$ for almost all p relative to neutrosophic $\mathcal{I}$. Then,

$$\{p \in \mathbb{N} : \mathcal{N}_p^f(\Theta_m) \neq \mathcal{N}_p^f(\Psi_m)\} \in \mathcal{I},$$

which implies

$$\{p \in \mathbb{N} : \mathcal{N}_p^f(\Theta_m) = \mathcal{N}_p^f(\Psi_m)\} \in \mathcal{F}(\mathcal{I}).$$

So, for all $p \in \mathcal{F}(\mathcal{I})$ for all $\sigma > 0$, we obtain

$$\mathcal{P}\left(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \frac{\sigma}{2}\right) = 1, \quad Q\left(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \frac{\sigma}{2}\right) = 0 \quad \text{and} \quad \mathcal{R}\left(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \frac{\sigma}{2}\right) = 0.$$

Since $\Psi = (\Psi_m) \in \mathcal{N}_{S(\mathcal{I})}^f$, suppose $\mathcal{N}_p^f - \mathcal{I} - st \lim_m \Psi = \beta$. Then, for all $\sigma, \gamma > 0$ and $\eta \in (0, 1)$,

$$K_1 = \left\{ p \in \mathbb{N} : \frac{1}{p} \left| \left\{ m \leq p : \mathcal{P}\left(\mathcal{N}_p^f(\Psi_m) - \beta, \frac{\sigma}{2}\right) > 1 - \eta \text{ and } \right. \right. \right. \\ \left. \left. Q\left(\mathcal{N}_p^f(\Psi_m) - \beta, \frac{\sigma}{2}\right) < \eta, \quad \mathcal{R}\left(\mathcal{N}_p^f(\Psi_m) - \beta, \frac{\sigma}{2}\right) < \eta \right\} \right| < \gamma \right\} \in \mathcal{F}(\mathcal{I}).$$

Consider the following set:

$$K_2 = \left\{ p \in \mathbb{N} : \frac{1}{p} \left| \{ m \leq p : \mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) > 1 - \eta \text{ and } \right. \right. \\ \left. \left. Q(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) < \eta, \quad \mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) < \eta \} \right| < \gamma \right\}.$$

We denote that $K_1 \subset K_2$. So, for all $p \in K_1$, we obtain

$$\mathcal{P}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \geq \mathcal{P}\left(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \frac{\sigma}{2}\right) \triangle \mathcal{P}\left(\mathcal{N}_p^f(\Psi_m) - \beta, \frac{\sigma}{2}\right) > 1 \triangle (1 - \eta) = 1 - \eta,$$

$$Q(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \leq Q\left(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \frac{\sigma}{2}\right) \diamond Q\left(\mathcal{N}_p^f(\Psi_m) - \beta, \frac{\sigma}{2}\right) < r \diamond r, < r$$

and

$$\mathcal{R}(\mathcal{N}_p^f(\Theta_m) - \beta, \sigma) \leq \mathcal{R}\left(\mathcal{N}_p^f(\Theta) - \mathcal{N}_p^f(\Psi), \frac{\sigma}{2}\right) \diamond \mathcal{R}\left(\mathcal{N}_p^f(\Psi_m) - \beta, \frac{\sigma}{2}\right) < r \diamond r < r.$$

This means that $p \in K_2$ and thus $K_1 \subset K_2$. Since $K_1 \in \mathcal{F}(\mathcal{I})$, we obtain $K_2 \in \mathcal{F}(\mathcal{I})$. As a result $\Theta = (\Theta_m) \in \mathcal{N}_{S(\mathcal{I})}^f$. $\square$

4 Conclusion

NS was first introduced in 1998 by Smarandache [14]. A unified idea of the NSS has been developed by Maji [15]. The notion of NSNLS was then examined, and some of its features were suggested in [17,18]. The NMS was studied by Kirişçi and Şimşek [45] using continuous t-norms and continuous t-conorms. NNS and statistical convergence in NNS were proposed by Kirişçi and Şimşek [46]. Even though certain features in neutrosophic Nörlund convergent sequence spaces have been examined, it is yet open to explore further properties in neutrosophic Nörlund $\mathcal{I}$ -statistically convergent sequence spaces, such as $\mathcal{N}_{S(I_0)}^f$, $\mathcal{N}_{S(I)}^f$, $\mathcal{N}_{S^{\infty}(I)}^f$. So, the main results of the present article fill up the gap in the existing literature. On the basis of this idea, we anticipate further research on probabilistic metric spaces employing neutrosophic probability.

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