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Research Article

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On the existence of nonnegative radial solutions for Dirichlet exterior problems on the Heisenberg group

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Abstract: We investigate the existence and nonexistence of nonnegative radial solutions to exterior problems of the form $\Delta_{\mathbb{H}^m}u(q) + \lambda\psi(q)K(r(q))f(r^{2-Q}(q), u(q)) = 0$ in B_1^c, under the Dirichlet boundary conditions $u = 0$ on ∂B_1 and $\lim_{r(q)\to\infty}u(q) = 0$. Here, $\lambda \geq 0$ is a parameter, $\Delta_{\mathbb{H}^m}$ is the Kohn Laplacian on the Heisenberg group $\mathbb{H}^m = \mathbb{R}^{2m+1}$, $m > 1$, $Q = 2m + 2$, B_1 is the unit ball in $\mathbb{H}^m$, B_1^c is the complement of B_1, and $\psi(q) = \frac{|z|^2}{r^2(q)}$. Namely, under certain conditions on K and f, we show that there exists a critical parameter $\lambda^* \in (0, \infty]$ in the following sense. If $0 \leq \lambda < \lambda^*$, the above problem admits a unique nonnegative radial solution u_λ; if $\lambda^* < \infty$ and $\lambda \geq \lambda^*$, the problem admits no nonnegative radial solution. When $0 \leq \lambda < \lambda^*$, a numerical algorithm that converges to u_λ is provided and the continuity of u_λ with respect to λ, as well as the behavior of u_λ as $\lambda \to \lambda^{*-}$, are studied. Moreover, sufficient conditions on the the behavior of $f(t, s)$ as $s \to \infty$ are obtained, for which $\lambda^* = \infty$ or $\lambda^* < \infty$. Our approach is based on partial ordering methods and fixed point theory in cones.

Keywords: exterior problem, Heisenberg group, nonnegative radial solution, critical value

MSC 2020: 35R03, 35A01, 34A12, 34B40

1 Introduction

This article is concerned with the study of nonnegative radial solutions to Dirichlet exterior problems of the form

$$\begin{cases} \Delta_{\mathbb{H}^m}u(q) + \lambda\psi(q)K(r(q))f(r^{2-Q}(q), u(q)) = 0, & q \in B_1^c, \\ u(q) = 0, & q \in \partial B_1, \\ \lim\limits_{|q|_{\mathbb{H}^m}\to\infty} u(q) = 0, \end{cases} \tag{1.1}$$

where $\lambda \geq 0$ is a parameter, $\Delta_{\mathbb{H}^m}$ is the Kohn Laplacian on the Heisenberg group $\mathbb{H}^m = \mathbb{R}^{2m+1}$, $m > 1$, $Q = 2m + 2$, B_1 is the unit ball in $\mathbb{H}^m$, i.e.,

$$B_1 = \left\{q = (z, \tau) \in \mathbb{H}^m : \ r(q) = |q|_{\mathbb{H}^m} = (|z|^4 + \tau^2)^{\frac{1}{4}} \leq 1\right\},$$

B_1^c is the complement of B_1, and $\psi(q) = \frac{|z|^2}{r^2(q)}$. Problem (1.1) is investigated under the following conditions:

(H1) The function $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous.

(H2) For all $t \in [0, 1]$, the function $f(t, \cdot) : [0, \infty) \to [0, \infty)$ is concave.

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(H3) There exists $\sigma > 0$ such that

$$f(t,0) \geq \sigma, \quad t \in [0,1].$$

(H4) The function $K : [1,\infty) \to (0,\infty)$ is continuous, and $K(r) \sim r^\mu$, as $r \to \infty$, where $\mu < -Q$.

Namely, we show that (1.1) admits a critical value $\lambda^* \in (0,\infty]$ in the following sense:

- For all $0 \leq \lambda < \lambda^*$, (1.1) admits a unique nonnegative radial solution u_λ (i.e., $u_\lambda(q) = u_\lambda(|q\mathrm{x}|_{\mathbb{H}^m})$);
- If $\lambda^* < \infty$ and $\lambda \geq \lambda^*$, then (1.1) has no nonnegative radial solution.

When $0 \leq \lambda < \lambda^*$, a numerical algorithm that converges to u_λ is provided, and the continuity of u_λ with respect to λ as well as the behavior of u_λ as $\lambda \to \lambda^{*-}$ are investigated. Moreover, we obtain sufficient conditions on behavior of $f(t,s)$ as $s \to \infty$, for which $\lambda^* = \infty$ or $\lambda^* < \infty$. Our techniques for proofs are based on partial ordering methods and fixed point theory in cones.

In the Euclidean case, the existence of positive solutions for problems of type

$$\begin{cases} \Delta u + f(x,u) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases}$$

where Ω is an exterior domain of $\mathbb{R}^N$, has been investigated by several authors via different approaches as follows: variational methods (see, e.g., [1–5]), the method of sub- and supersolutions (see, e.g., [6–10]), index theory and the cone expansion fixed point theorem (see, e.g., [11–15]), and the shooting method (see e.g. [16,17]).

In the context of the Heisenberg group, the existence of solutions for nonlinear problems involving the Kohn Laplacian, posed in $\mathbb{H}^m$ or in a bounded domain of $\mathbb{H}^m$, was investigated by several authors via variational methods (see, e.g., [18–23] and the references therein).

On the other hand, due to the lack of compactness in many nonlinear problems appearing in theory and applications, which makes the use of topological methods and variational methods so difficult, since the beginning of the 1980's, Guo et al. have developed various partial ordering methods for studying nonlinear problems without using compactness conditions. By using some inequalities related to some ordering, they have obtained several fixed point results for monotone or mixed monotone operators. For more details, see, e.g., [11,24–30] and the references therein.

Motivated by the above contributions, the existence of nonnegative radial solutions to problem (1.1) is investigated via partial ordering methods.

The rest of the article is organized as follows: in Section 2, we briefly recall some notions related to the Heisenberg group and present our main results; in Section 3, we collect the mathematical tools needed for the proofs of our results; and finally, the proofs are given in Section 4.

2 Main results

First, let us recall some notions related to the Heisenberg group. For more details, see, e.g., [31].

The Heisenberg group, denoted by $\mathbb{H}^m$ (m is a positive natural number), is identified to the Euclidean space $\mathbb{R}^{2m+1}$ with the group law $\circ$ defined as follows:

$$q \circ q' = \left(x + x', y + y', \tau + \tau' + 2\sum_{i=1}^{n} (x_i y_i' - y_i x_i') \right),$$

where

$$q = (z,\tau) = (x,y,\tau) = (x_1,\ldots,x_m, y_1,\ldots,y_m,\tau),$$
$$q' = (z',\tau') = (x',y',\tau') = (x_1',\ldots,x_m', y_1',\ldots,y_m',\tau').$$

In $\mathbb{H}^m$, we define the norm

$$|q|_{\mathbb{H}^m} = (|z|^4 + \tau^2)^{\frac{1}{4}}, \quad q = (z, \tau) \in \mathbb{H}^m, \tag{2.1}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^{2m}$.

For $i = 1, \ldots, m$, consider the vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial \tau}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial \tau},$$

and the associated Heisenberg gradient

$$\nabla_{\mathbb{H}^m} = (X_1, \ldots, X_m, Y_1, \ldots, Y_m).$$

The Kohn Laplacian $\Delta_{\mathbb{H}^m}$ is then the operator defined by

$$\Delta_{\mathbb{H}^m} = \sum_{i=1}^{m} (X_i^2 + Y_i^2).$$

Let $A : q \in \mathbb{H}^m \longmapsto Aq = (A_1 q, \ldots A_{2m} q)$ be a C^1 vector field. The Heisenberg divergence of A is defined as follows:

$$\operatorname{div}_{\mathbb{H}^m} A(q) = \sum_{i=1}^{m} X_i(A_i q) + \sum_{i=1}^{m} Y_i(A_{m+i} q), \quad q \in \mathbb{H}^m.$$

For $\Phi \in C^2(\mathbb{H}^m)$, we have

$$\operatorname{div}_{\mathbb{H}^m}\Big(\nabla_{\mathbb{H}^m}\Phi\Big) = \Delta_{\mathbb{H}^m}\Phi.$$

Let u be a radial regular function, i.e., for all $q = (z, \tau) \in \mathbb{H}^m$,

$$u(q) = u(r(q)), \quad r(q) = r(z, \tau) = (|z|^4 + \tau^2)^{\frac{1}{4}}.$$

Then,

$$\Delta_{\mathbb{H}^m} u(q) = \psi(q)\Big(u''(r) + \frac{Q-1}{r} u'(r)\Big), \tag{2.2}$$

where $Q = 2m + 2$ and $\psi(q) = \frac{|z|^2}{r^2(q)}$.

Since we are interested in radial solutions to (1.1), we assume that $u(q) = u(r(q))$ and $r(q) = r(z, \tau) = (|z|^4 + \tau^2)^{\frac{1}{4}}$, so that (by (2.2)) u solves

$$\begin{cases} u''(r) + \dfrac{Q-1}{r} u'(r) + \lambda K(r) f(r^{2-Q}, u(r)) = 0, & r > 1, \\ u(1) = 0, \\ \lim\limits_{r\to\infty} u(r) = 0. \end{cases} \tag{2.3}$$

Next, after changing variable $u(r) = v(r^{2-Q}) = v(t)$, elementary calculations show that (2.3) reduces to

$$\begin{cases} v''(t) + \lambda h(t) f(t, v(t)) = 0, & 0 < t < 1, \\ v(0) = v(1) = 0, \end{cases} \tag{2.4}$$

where

$$h(t) = t^{\frac{2Q-2}{2-Q}} K\Big(t^{\frac{1}{2-Q}}\Big) > 0, \quad 0 < t \le 1.$$

Remark 2.1. Under condition (H4), it holds that $h \in L^1((0, 1)) \cap C((0, 1])$.

Taking in consideration Remark 2.1, by standard arguments, we can show that, if (H1) and (H4) are satisfied, then for all $\lambda \geq 0$, the following statements are equivalent:

(i) $v_\lambda \in C([0,1]) \cap C^2((0,1))$ is a solution to (2.4).

(ii) $v_\lambda \in C([0,1])$ is a solution to the following integral equation:

$$v_\lambda(t) = \lambda \int_0^1 G(t,s)h(s)f(s, v_\lambda(s))\mathrm{d}s, \quad 0 \leq t \leq 1, \tag{2.5}$$

where

$$G(t,s) = \begin{cases} s(1-t) & \text{if } 0 \leq s \leq t \leq 1, \\ t(1-s) & \text{if } 0 \leq t \leq s \leq 1. \end{cases} \tag{2.6}$$

Our main results are given by the following theorems.

Theorem 2.1. *Suppose that conditions* (H1)–(H4) *are satisfied. The following statements hold*:

(*I*) *There exists a critical parameter* $\lambda^* \in (0, \infty]$ *satisfying*:

(*a*) *For all* $0 \leq \lambda < \lambda^*$, (2.5) *admits a unique nonnegative solution* $v_\lambda \in C([0,1])$.

(*b*) *If* $\lambda^* < \infty$, *for all* $\lambda \geq \lambda^*$, (2.5) *has no nonnegative continuous solution.*

(*II*) *Let* $0 \leq \lambda < \lambda^*$. *Then, the sequence*

$$(v_\lambda{}^{(n)})_{n\geq 0} : \begin{cases} v_\lambda{}^{(0)} \equiv 0, \\ v_\lambda{}^{(n)}(t) = \lambda \displaystyle\int_0^1 G(t,s)h(s)f(s, v_\lambda{}^{(n-1)}(s))\mathrm{d}s, \quad 0 \leq t \leq 1, \ n \geq 1 \end{cases}$$

converges uniformly to v_λ, *i.e.*,

$$\lim_{n\to\infty} \max_{0\leq t\leq 1} |v_\lambda{}^{(n)}(t) - v_\lambda(t)| = 0.$$

(*III*) *For all* $0 \leq \lambda_0 < \lambda^*$,

$$\lim_{\lambda\to\lambda_0,\ \lambda>0} \max_{0\leq t\leq 1} |v_\lambda(t) - v_{\lambda_0}(t)| = 0.$$

(*IV*) *If* $0 \leq \lambda_1 < \lambda_2 < \lambda^*$, *then*

$$v_{\lambda_1}(t) \leq v_{\lambda_2}(t), \quad 0 \leq t \leq 1, \quad \textit{and} \quad v_{\lambda_1} \neq v_{\lambda_2}.$$

(*V*) $\lim_{\lambda\to\lambda^{*-}} \max_{0\leq t\leq 1} v_\lambda(t) = \infty$.

Theorem 2.2. *Suppose that conditions* (H1)–(H4) *are satisfied.*

(*I*) *If* $\lim_{s\to\infty} \sup_{0\leq t\leq 1} \frac{f(t,s)}{s} = 0$, *then* $\lambda^* = \infty$.

(*II*) *If there exist* $c, S > 0$ *such that*

$$f(t,s) \geq cs, \quad 0 \leq t \leq 1, \ s > S,$$

then $\lambda^* < \infty$.

Below are some examples of functions f satisfying conditions (H1)–(H3).

- Let

$$f(t,s) = a(t) + \sum_{i=1}^k \alpha_i s^{p_i}, \quad 0 \leq t \leq 1, \ s \geq 0,$$

where $k \geq 1$, $a \in C([0,1])$, $\min_{0\leq t\leq 1} a(t) > 0$, $\alpha_i \geq 0$, and $0 \leq p_i \leq 1$, for all $i = 1, 2, \ldots, k$. Then, (H1)–(H3) are satisfied with $\sigma = \min_{0\leq t\leq 1} a(t)$.

- Let
$$f(t,s)=\arctan(a(t)+s),\quad 0\le t\le 1,\ s\ge 0,$$
where $a\in C([0,1])$ and $\min_{0\le t\le 1}a(t)>0$. Then, (H1)–(H3) are satisfied with $\sigma=\arctan(\min_{0\le t\le 1}a(t))$. Note that in this case, we have
$$\limsup_{s\to\infty}\sup_{0\le t\le 1}\frac{f(t,s)}{s}=0.$$
Hence, by Theorem 2.2-(I), $\lambda^*=\infty$.
- Let
$$f(t,s)=\ln(a(t)+s)+s,\quad 0\le t\le 1,\ s\ge 0,$$
where $a\in C([0,1])$ and $\min_{0\le t\le 1}a(t)>1$. Then, (H1)–(H3) are satisfied with $\sigma=\ln(\min_{0\le t\le 1}a(t))$. Moreover, we have
$$f(t,s)\ge s,\quad 0\le t\le 1,\ s\ge 0.$$
Hence, by Theorem 2.2-(II), $\lambda^*<\infty$.
- Let
$$f(t,s)=\begin{cases}\displaystyle\int_0^t (t-x)^{\alpha-1}a(x,s)\mathrm{d}x+b & \text{if } 0<t\le 1,\ s\ge 0,\\ b & \text{if } t=0,\ s\ge 0,\end{cases}$$
where $\alpha, b>0$, $a:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous, and $a(x,\cdot):[0,\infty)\to[0,\infty)$ is concave for all $x\in[0,1]$. Then, (H1)–(H3) are satisfied with $\sigma=b$.

3 Preliminaries

Let $(E,\|\cdot\|)$ be a Banach space over $\mathbb{R}$. We denote by 0_E the zero vector in E.

Definition 3.1. Let $C\subset E$ be a nonempty closed convex subset of E ($C\neq\{0_E\}$). We say that C is a cone in E, if the following conditions are satisfied:
(i) $x\in C,\lambda\ge 0\ \Rightarrow\ \lambda x\in C$.
(ii) $x,-x\in C\ \Rightarrow\ x=0_E$.

Moreover, if $\mathring{C}\neq\varnothing$, we say that C is a solid cone.

Let C be a cone in E. We define the partial order $\preccurlyeq$ in E by
$$x\preccurlyeq y\ \Leftrightarrow\ y-x\in C,\quad \text{for all } x,y\in E.$$
If $x,y\in E$, $x\preccurlyeq y$, and $x\neq y$, we denote $x\prec y$. If C is solid and for $x,y\in E$, $y-x\in\mathring{C}$, we denote $x\ll y$.

Definition 3.2. Let C be a cone in E. We say that C is normal, if there exists a constant $k>0$ such that for all $x,y\in E$,
$$0_E\preccurlyeq x\preccurlyeq y\ \Rightarrow\ \|x\|\le k\|y\|.$$
Clearly, if k exists, then $k\ge 1$.

Let C be a cone in E and $e\in C\backslash\{0_E\}$. Let
$$E_e=\{x\in E:\exists\,\eta>0,-\eta e\preccurlyeq x\preccurlyeq \eta e\}$$

and

$$\|x\|_e = \inf\{\eta > 0 : -\eta e \preccurlyeq x \preccurlyeq \eta e\}, \quad x \in E_e.$$

Lemma 3.1. [29] *Assume that C is normal. Then*

(i) $(E_e, \|\cdot\|_e)$ *is a Banach space.*

(ii) *There exists $M > 0$ such that $\|x\| \le M\|x\|_e$, for all $x \in E_e$.*

(iii) *Let $C_e = C \cap E_e$. Then C_e is a normal solid cone in E_e, and*

$$\mathring{C}_e = \{x \in E_e : \exists \tau > 0, x \succcurlyeq \tau e\} = \{x \in E : \exists \eta > \tau > 0, \tau e \preccurlyeq x \preccurlyeq \eta e\}.$$

(iv) *If $e \in \mathring{C}$ (C is solid), then $E_e = E$ and $\|\cdot\| \sim \|\cdot\|_e$.*

Definition 3.3. Let D be a nonempty convex subset of E and $A : D \to E$ be a given operator. Let C be a cone in E. We say that A is concave, if

$$A(\eta x + (1-\eta)y) \succcurlyeq \eta Ax + (1-\eta)Ay, \quad \eta \in (0,1), x, y \in D.$$

Lemma 3.2. [29] *Let C be a normal solid cone and $A : C \to C$ be a concave operator satisfying*

$$0_E \ll A0_E.$$

Then,

(i) *there exists $0 < \lambda^* \le \infty$ such that, for $0 \le \lambda < \lambda^*$, the equation*

$$u = \lambda Au \tag{3.1}$$

admits a unique solution $u_\lambda \in C$. If $\lambda^* < \infty$, for all $\lambda \ge \lambda^*$, (3.1) has no solution in C;

(ii) *if $0 < \lambda < \lambda^*$, then, for any $u^{(0)} \in C$, the sequence $u_\lambda^{(n)} = \lambda Au_\lambda^{(n-1)}$ $(n = 1, 2, \dots, u_\lambda^{(0)} = u^{(0)})$ converges to u_λ;*

(iii) *the function $u_\bullet : \lambda \in [0, \lambda^*) \mapsto u_\lambda \in C$ is continuous and strongly increasing, i.e.,*

$$0 \le \lambda_1 < \lambda_2 < \lambda^* \;\Rightarrow\; u_{\lambda_1} \ll u_{\lambda_2}.$$

(iv) $\lim_{\lambda \to \lambda^{*-}} \|u_\lambda\| = \infty$;

(v) *if there exist $\lambda_0 > 0$ and $v_0 \in C$ such that $v_0 \succcurlyeq \lambda_0 Av_0$, then $\lambda_0 < \lambda^*$.*

For more details on fixed point theory in cones, see also [11] and the references therein.

Lemma 3.3. *Let $F : [0,\infty) \to [0,\infty)$ be a concave function. Then, F is increasing.*

Proof. Let $0 \le x \le y$, $0 < \eta < 1$, and $y_\eta = \frac{1}{1-\eta}y - \frac{\eta}{1-\eta}x$. Then, $y_\eta \ge x$ and $y = \eta x + (1-\eta)y_\eta$. Since F is concave, we obtain

$$F(y) \ge \eta F(x) + (1-\eta)F(y_\eta) \ge \eta F(x).$$

Passing to the limit as $\eta \to 1^-$, it holds that $F(y) \ge F(x)$. This completes the proof. □

4 Proofs of the main results

4.1 Proof of Theorem 2.1

Let $E = C([0,1])$ be the Banach space equipped with the norm

$$\|u\| = \max_{0 \le t \le 1} |u(t)|, \quad u \in E.$$

Let $P = C^+([0,1])$, i.e.,

$$P = \{u \in C([0,1]) : u(t) \geq 0, 0 \leq t \leq 1\}.$$

Then, P is a normal solid cone with

$$\mathring{P} = \{u \in C([0,1]) : u(t) > 0, 0 \leq t \leq 1\}. \tag{4.1}$$

Let $\preccurlyeq$ be the partial order in E induced by P, i.e.,

$$u \preccurlyeq v \iff u(t) \leq v(t), \quad 0 \leq t \leq 1, \quad \text{for all } u, v \in E.$$

Let us introduce the operator $A : P \to P$ defined as follows:

$$(Av)(t) = \int_0^1 G(t,s)h(s)f(s, v(s))\mathrm{d}s, \quad 0 \leq t \leq 1, \quad v \in P.$$

Note that form (H1)–(H4), it is not difficult to show that $A(P) \subset P$. Moreover, from (H2), it follows that A is a concave operator (with respect to the partial order $\preccurlyeq$). Then, in order to apply Lemma 3.2, we have to check whether $0_E \ll A0_E$, where 0_E is the zero function in $[0,1]$. Unfortunately, it is not the case. Namely, we have

$$(A0_E)(t) = \int_0^1 G(t,s)h(s)f(s,0)\mathrm{d}s, \quad 0 \leq t \leq 1.$$

Hence, by (2.6),

$$(A0_E)(0) = 0,$$

which implies by (4.1) that $A0_E \notin \mathring{P}$. Hence, to overcome this difficulty, we have to find an adequate, solid normal cone $C \subset P$ such that $A(C) \subset C$ and $0_E \ll A0_E$. Let

$$e(t) = \int_0^1 G(t,s)h(s)\mathrm{d}s, \quad 0 \leq t \leq 1.$$

We can show easily that $e \in P$ and $e \neq 0_E$. Let

$$E_e = \{u \in C([0,1]) : \exists \eta > 0, -\eta e(t) \leq u(t) \leq \eta e(t), 0 \leq t \leq 1\}$$

and

$$\|u\|_e = \inf\{\eta > 0 : -\eta e(t) \leq u(t) \leq \eta e(t), 0 \leq t \leq 1\}, \quad u \in E_e.$$

Let $C = P \cap E_e$, i.e.,

$$C = \{u \in C([0,1]) : \exists \eta > 0, 0 \leq u(t) \leq \eta e(t), 0 \leq t \leq 1\}.$$

From Lemma 3.1, we know that $(E_e, \|\cdot\|_e)$ is a Banach space and C is a normal solid cone in E_e with

$$\mathring{C} = \{u \in E_e : \exists \tau > 0, u(t) \geq \tau e(t), 0 \leq t \leq 1\}. \tag{4.2}$$

We claim that

$$A(P) \subset C. \tag{4.3}$$

Let $u \in P$. We have

$$(\mathrm{Au})(t) = \int_0^1 G(t,s)h(s)f(s, u(s))\mathrm{d}s, \quad 0 \leq t \leq 1.$$

Since $s \mapsto f(s, u(s))$ is continuous and nonnegative in $[0,1]$ (by (H1)), then

$$0 \leq M = \max_{0 \leq s \leq 1} f(s, u(s)) < \infty.$$

Moreover, since $f(s,\cdot):[0,\infty)\to[0,\infty)$ is concave (by (H2)), then by Lemma 3.3, $f(s,\cdot)$ is increasing. Hence, by (H3), for all $0\le s\le 1$,

$$0<\sigma\le f(s,0)\le f(s,u(s))\le M,$$

which yields

$$0<M<\infty.$$

Next, we deduce that

$$0\le(\mathrm{Au})(t)\le M\int_0^1 G(t,s)h(s)\mathrm{d}s=Me(t),\quad 0\le t\le 1,$$

which proves (4.3). Hence (since $C\subset P$), $A:C\to C$ is well-defined. Moreover, for all $0\le t\le 1$, it follows from (H4) that

$$\left(A0_{E_e}\right)(t)=(A0_E)(t)=\int_0^1 G(t,s)h(s)f(s,0)\mathrm{d}s\ge\sigma\int_0^1 G(t,s)h(s)\mathrm{d}s=\sigma e(t),$$

which yields by (4.2) $A0_{E_e}\in\mathring{C}$, i.e.,

$$0_{E_e}\ll A0_{E_e}.$$

Now, all the assumptions of Lemma 3.2 are satisfied for the operator $A:C\to C$. Then, by Lemma 3.2-(i), we deduce the existence of $\lambda^*\in(0,\infty]$ satisfying the following conditions:
(a) for all $0\le\lambda<\lambda^*$, (2.5) admits a unique solution v_λ in C;
(b) if $\lambda^*<\infty$, for all $\lambda\ge\lambda^*$, then (2.5) has no solution in C.

Note that the above results were obtained in C. We claim that the following statements are equivalent:
(A) v_λ is a solution to (2.5) in C.
(B) v_λ is a solution to (2.5) in P (i.e., v_λ is a nonnegative continuous solution in $[0,1]$).

Observe that (A) $\Rightarrow$ (B) is immediate since $C\subset P$. So, we have to show only that (B) $\Rightarrow$ (A). Let us suppose that v_λ is a solution to (2.5) in P. Then, by (4.3), $v_\lambda=\lambda Av_\lambda\in\lambda C\subset C$ (since $\lambda\ge 0$ and C is a cone). Hence, the equivalence between (A) and (B) is proved. Therefore, part (I) of Theorem 2.1 follows from (a), (b), and (A) $\Leftrightarrow$ (B). Part (II) of Theorem 2.1 follows from Lemma 3.2-(ii) with $u^{(0)}=0_E$. Part (III) of Theorem 2.1 follows from Lemma 3.2-(iii) (namely, form the continuity of the function $u_\bullet:\lambda\in[0,\lambda^*)\mapsto u_\lambda\in C$). Again, by Lemma 3.2-(iii), if $0\le\lambda_1<\lambda_2<\lambda^*$, then $v_{\lambda_2}-v_{\lambda_1}\in\mathring{C}$, i.e., (by (4.2)), there exists $\tau>0$ such that

$$v_{\lambda_2}(t)-v_{\lambda_1}(t)\ge\tau e(t),\quad 0\le t\le 1.$$

Since $e\in P$ and $e\ne 0_E$, it holds that $v_{\lambda_2}(t)\ge v_{\lambda_1}(t)$, for all $0\le t\le 1$, and $v_{\lambda_2}\ne v_{\lambda_1}$. This proves part (IV) of Theorem 2.1. Finally, part (V) of Theorem 2.1 follows from Lemma 3.2-(iv).

4.2 Proof of Theorem 2.2

We continue to use the notations introduced in the proof of Theorem 2.1.

(I) Suppose that

$$\lim_{s\to\infty}\sup_{0\le t\le 1}\frac{f(t,s)}{s}=0.$$

Then, for any $\lambda>0$, we can take ρ sufficiently large such that

$$f(t,\rho)\le(\lambda H)^{-1}\rho,\quad 0\le t\le 1,\tag{4.4}$$

where $H = \frac{1}{4}\|h\|_{L^1((0,1))} > 0$. On the other hand, it is not difficult to show that

$$e(t) \le H, \quad 0 \le t \le 1. \tag{4.5}$$

Let

$$\mu(t) = H^{-1}\rho e(t), \quad 0 \le t \le 1.$$

Then, $\mu \in C$. Using (4.4), (4.5), and the fact that $f(s, \cdot) : [0, \infty) \to [0, \infty)$ is an increasing function, we obtain

$$\begin{aligned}
\lambda(A\mu)(t) &= \lambda \int_0^1 G(t,s)h(s)f(s,\mu(s))\,\mathrm{d}s \\
&= \lambda \int_0^1 G(t,s)h(s)f(s,H^{-1}\rho e(s))\,\mathrm{d}s \\
&\le \lambda \int_0^1 G(t,s)h(s)f(s,\rho)\,\mathrm{d}s \\
&\le \lambda(\lambda H)^{-1}\rho \int_0^1 G(t,s)h(s)\,\mathrm{d}s \\
&= H^{-1}\rho e(t) \\
&= \mu(t),
\end{aligned}$$

for all $0 \le t \le 1$, which yields $\lambda A\mu \preccurlyeq \mu$. Hence, by Lemma 3.2-(v), it holds that $\lambda \le \lambda^*$. Therefore, since $\lambda > 0$ is arbitrary, we deduce that $\lambda^* = \infty$. This proves part (I) of Theorem 2.2.

(II) Suppose that there exist $c, S > 0$ such that

$$f(t,s) \ge cs, \quad 0 \le t \le 1, \ s > S. \tag{4.6}$$

We claim that there exists $\nu > 0$ such that

$$0 \le \frac{s}{f(t,s)} \le \nu, \quad 0 \le t \le 1, s \ge 0. \tag{4.7}$$

Note that from (H3) and the fact that $f(t, \cdot) : [0, \infty) \to [0, \infty)$ is an increasing function, we have

$$f(t,s) \ge f(t,0) \ge \sigma > 0, \quad 0 \le t \le 1, \ s \ge 0.$$

Then,

$$0 \le \frac{s}{f(t,s)} \le \frac{s}{\sigma} \le \frac{S}{\sigma}, \quad 0 \le t \le 1, \ 0 \le s \le S. \tag{4.8}$$

Hence, it follows from (4.6) and (4.8) that

$$0 \le \frac{s}{f(t,s)} \le \max\left\{\frac{1}{c}, \frac{S}{\sigma}\right\}, \quad 0 \le t \le 1, \ s \ge 0.$$

Therefore, (4.7) is proved with $\nu = \max\left\{\frac{1}{c}, \frac{S}{\sigma}\right\} > 0$. Consider now the boundary value problem

$$\begin{cases} -u''(t) = \gamma h(t)F(t,u(t)), & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{4.9}$$

where $\gamma > 0$ is a parameter and

$$F(t,s) = s + 1, \quad 0 \le t \le 1, \ s \ge 0.$$

We can show easily that F satisfies (H1)–(H3). Then, by Theorem 2.1, there exists a critical value $\gamma^* \in (0, \infty]$ such that (4.9) admits a unique solution $u_\gamma \geq 0$, for all $0 < \gamma < \gamma^*$. Let $0 < \gamma_0 < \gamma^*$ be fixed and $u = u_{\gamma_0} \geq 0$ be the corresponding unique solution to (4.9). Let $0 < \lambda < \lambda^*$ and $v_\lambda \in C([0,1])$ be the unique solution to (2.5). Then, v_λ solves the boundary value problem (2.4). Multiplying the first equation in (2.4) by u and integrating over $(0, 1)$, we obtain

$$-\int_0^1 v_\lambda''(t)u(t)\mathrm{d}t = \lambda \int_0^1 h(t)f(t, v_\lambda(t))u(t)\mathrm{d}t.$$

Integrating by parts, it holds that

$$-\int_0^1 v_\lambda(t)u''(t)\mathrm{d}t = \lambda \int_0^1 h(t)f(t, v_\lambda(t))u(t)\mathrm{d}t.$$

Hence, by (4.9), we obtain

$$\gamma_0 \int_0^1 h(t)(u(t)+1)v_\lambda(t)\mathrm{d}t = \lambda \int_0^1 h(t)f(t, v_\lambda(t))u(t)\mathrm{d}t,$$

which yields

$$\int_0^1 h(t)[\gamma_0(u(t)+1)v_\lambda(t) - \lambda f(t, v_\lambda(t))u(t)] = 0. \tag{4.10}$$

We claim that there exists $0 < t_0 < 1$ such that

$$\gamma_0(u(t_0)+1)v_\lambda(t_0) - \lambda f(t_0, v_\lambda(t_0))u(t_0) = 0. \tag{4.11}$$

Suppose that, for all $0 < t < 1$,

$$\gamma_0(u(t)+1)v_\lambda(t) - \lambda f(t, v_\lambda(t))u(t) \neq 0.$$

Then, by continuity, we deduce that

$$\gamma_0(u(t)+1)v_\lambda(t) - \lambda f(t, v_\lambda(t))u(t) > 0, \quad 0 < t < 1$$

or

$$\gamma_0(u(t)+1)v_\lambda(t) - \lambda f(t, v_\lambda(t))u(t) < 0, \quad 0 < t < 1.$$

Since $h \in C((0, 1])$, in both cases, we deduce by (4.10) that $h(t) = 0$, for all $0 < t \leq 1$, which contradicts the fact that $h(t) > 0$, for all $0 < t \leq 1$. Therefore, (4.11) holds. On the other hand,

$$u(t_0) = \gamma_0 \int_0^1 G(t_0, s)h(s)F(s, u(s))\mathrm{d}s \geq \gamma_0 \int_0^1 G(t_0, s)h(s)\mathrm{d}s = \gamma_0 e(t_0) > 0.$$

Hence, by (4.11), we obtain

$$\lambda = \left(\frac{\gamma_0(u(t_0)+1)}{u(t_0)}\right)\frac{v_\lambda(t_0)}{f(t_0, v_\lambda(t_0))}.$$

Next, using (4.7), we deduce that

$$\lambda \leq \frac{\nu\gamma_0(u(t_0)+1)}{u(t_0)}.$$

Since λ is arbitrary, it holds that $\lambda^* \leq \frac{\nu\gamma_0(u(t_0)+1)}{u(t_0)} < \infty$. This proves part (II) of Theorem 2.2.

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