Research Article

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A novel class of bipolar soft separation axioms concerning crisp points

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Abstract: The main objective of this study is to define a new class of bipolar soft (BS) separation axioms known as BS $\tilde{\tilde{T_i}}$-space ($i = 0, 1, 2, 3, 4$). This type is defined in terms of ordinary points. We prove that BS $\tilde{\tilde{T_i}}$-space implies BS $\tilde{\tilde{T}}_{i-1}$-space for $i = 1, 2$; however, the opposite is incorrect, as demonstrated by an example. For $i = 0, 1, 2, 3, 4$, we investigate that every BS $\tilde{\tilde{T_i}}$-space is soft $\tilde{T_i}$-space; and we set up a condition in which the reverse is true. Moreover, we point out that a BS subspace of a BS $\tilde{\tilde{T_i}}$-space is a BS $\tilde{\tilde{T_i}}$-space for $i = 0, 1, 2, 3$.

Keywords: bipolar soft separation axioms, bipolar soft topology, soft topology, bipolar soft sets, soft sets

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1 Introduction

Set theory is the fundamental theory of mathematics. However, the concept of uncertainty is linked to the concept of a set. Because of this ambiguity, academics and mathematicians have struggled for a long time to tackle difficult issues in disciplines including medicine, engineering, economics, decision-making, social sciences, and artificial intelligence. Researchers in mathematics and a variety of other subjects have suggested theories such as probability theory, fuzzy set theory [1], rough set theory [2], vague set theory, decision-making theory, and graph theory to eliminate this ambiguity. However, as indicated in [3], each of these theories has its own series of challenges, which may be attributable to the insufficiency of the theories' parametrization tools.

In 1999, Molodtsov [3] proposed the soft set as a novel mathematical strategy for dealing with ambiguity and vagueness. Soft set theory has a wide range of potential applications. Many interesting applications of soft set theory can be seen in [4–6]. Maji et al. [7] developed various soft set operations and conducted a theoretical study of soft sets. Ali et al. [8] suggested a variety of new operations on soft sets based on [7] and improved some concepts of soft sets. Soft set theory is becoming increasingly popular, see [8–13]. In 2011, Shabir and Naz [14] defined soft topologies as a mix of classical topology and soft set theory. The topological concepts, characteristics, and results in soft topologies have since been addressed by a number of researchers, see [15–23]. Separation axioms, with regard to both ordinary and soft points, are among the most essential notions in soft topological spaces (STSs). The authors of [24–31] investigated these ideas and determined their main characteristics.

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Shabir and Naz [32] proposed the concept of BS sets in 2013, in response to the necessity to convey both positive and negative aspects of data. They utilized BS sets to define some set-theoretic operations including union, intersection, and complement, and discussed its application to decision-making problems. In 2017, Shabir and Bakhtawar [33] initially presented the notions of connectedness, compactness, and separation axioms via BS sets. Later on, Öztürk [34] further studied the concepts of closure and interior operators in the bipolar soft topological spaces (BSTSs). Many researchers have contributed to BS sets and their topological structures (see, for example, [35–43]). The main aim of this research is to utilize crisp points to define another interesting and novel form of BS separation axioms.

The following is a breakdown of the article's structure. Section 2 provides an outline of several fundamental concepts that are necessary for understanding our research. Section 3 initiates the concepts of BS $\widetilde{\widetilde{T_i}}$-space for $i = 0, 1, 2, 3, 4$. Furthermore, we discuss the relationship between these soft spaces and their counterparts of soft topology and classical topology. Section 4 discusses the notions of BS regular and BS normal spaces. Section 5 concludes with a summary of the current work as well as a suggestion for future research.

2 Preliminaries

This section is focused on recalling certain key concepts and ideas that will be used in the next sections.

Throughout this article, the notations $\aleph$ and $2^{\aleph}$ will be used to denote the initial universe and the power set of $\aleph$, respectively. Let Λ, B, and Σ be the parameters set with $\Lambda, B \subseteq \Sigma$.

Definition 2.1. [7] Let $\Sigma = \{\ell_1, \ell_2, \ldots, \ell_n\}$ be a set of parameters. The NOT set of Σ denoted by $\neg\Sigma$ is defined by $\neg\Sigma = \{\neg\ell_1, \neg\ \ell_2, \ldots, \neg\ell_n\}$, where $\neg\ell_i=$ not ℓ_i for $i = 1, 2, \ldots, n$.

Definition 2.2. [32] Let the mappings g and $\widehat{g}$ be defined by $g : \Sigma \to 2^{\aleph}$ and $\widehat{g} : \neg\ \Sigma \to 2^{\aleph}$ with $g(\ell) \cap \widehat{g}(\neg\ell) = \varnothing$ for all $\ell \in \Sigma$. Then, $(g, \widehat{g}, \Sigma)$ is said to be a BS set over $\aleph$.

A BS set $(g, \widehat{g}, \Sigma)$ can alternatively be represented as follows:

$$(g, \widehat{g}, \Sigma) = \{(\ell, g(\ell), \widehat{g}(\neg\ell)) : \ell \in \Sigma, \quad \text{and} \quad g(\ell) \cap \widehat{g}(\neg\ell) = \varnothing\}.$$

Definition 2.3. [32] A BS set $(g_1, \widehat{g}_1, \Lambda)$ is a BS subset of BS set $(g_2, \widehat{g}_2, B)$, symbolized by $(g_1, \widehat{g}_1, \Lambda) \widetilde{\widetilde{\sqsubseteq}} (g_2, \widehat{g}_2, B)$, if, (i) $\Lambda \subseteq B$ and (ii) for all $\ell \in \Lambda$, $g_1(\ell) \subseteq g_2(\ell)$ and $\widehat{g}_2(\neg\ell) \subseteq \widehat{g}_1(\neg\ell)$.

A BS set $(g_1, \widehat{g}_1, \Lambda)$ is said to be a BS superset of $(g_2, \widehat{g}_2, B)$, symbolized by $(g_1, \widehat{g}_1, \Lambda) \widetilde{\widetilde{\sqsupseteq}} (g_2, \widehat{g}_2, B)$, if $(g_2, \widehat{g}_2, B)$ be a BS subset of $(g_1, \widehat{g}_1, \Lambda)$.

Definition 2.4. [32] Two BS sets $(g_1, \widehat{g}_1, \Lambda)$ and $(g_2, \widehat{g}_2, B)$ are said to be BS equal if $(g_1, \widehat{g}_1, \Lambda) \widetilde{\widetilde{\sqsubseteq}} (g_2, \widehat{g}_2, B)$ and $(g_2, \widehat{g}_2, \Lambda) \widetilde{\widetilde{\sqsubseteq}} (g_1, \widehat{g}_1, B)$.

Definition 2.5. [32] A BS set $(g, \widehat{g}, \Lambda)^c$ is the complement of a BS set $(g, \widehat{g}, \Lambda)$ and is defined by $(g^c, \widehat{g}^c, \Lambda)$, where g^c and $\widehat{g}^c$ are mappings given by $g^c(\ell) = \widehat{g}(\neg\ell)$ and $\widehat{g}^c(\neg\ell) = g(\ell)$ for all $\ell \in \Lambda$.

Definition 2.6. [32] A relative null BS set of a BS set $(g, \widehat{g}, \Lambda)$, denoted by $(\widehat{\Phi}, \widehat{\aleph}, \Lambda)$, is defined as $\widehat{\Phi}(\ell) = \varnothing$ and $\widehat{\aleph}(\neg\ell) = \aleph$ for all $\ell \in \Lambda$.

We symbolized the absolute null BS set over $\aleph$ by $(\widehat{\Phi}, \widehat{\aleph}, \Sigma)$.

Definition 2.7. [32] A relative whole BS set of a BS set $(g, \widehat{g}, \Lambda)$, denoted by $(\widehat{\aleph}, \widehat{\Phi}, \Lambda)$, is defined as $\widehat{\aleph}(\ell) = \aleph$ and $\widehat{\Phi}(\neg\ell) = \varnothing$ for all $\ell \in \Lambda$.

We symbolized the absolute whole BS set over $\aleph$ by $(\widehat{\aleph}, \widehat{\Phi}, \Sigma)$.

Definition 2.8. [32] The union of two BS sets $(g_1, \widehat{g}_1, \Lambda)$ and $(g_2, \widehat{g}_2, B)$ is a BS set $(f, \widehat{f}, \Gamma)$, symbolized by $(g_1, \widehat{g}_1, \Lambda)\,\widetilde{\widetilde{\sqcup}}\,(g_2, \widehat{g}_2, B) = (f, \widehat{f}, \Gamma)$, where $\Gamma = \Lambda \cap B$ and for all $\ell \in \Gamma$, $f(\ell) = g_1(\ell) \cup g_2(\ell)$ and $\widehat{f}(\neg\ell) = \widehat{g}_1(\neg\ell) \cap \widehat{g}_2(\neg\ell)$.

Definition 2.9. [32] The intersection of two BS sets $(g_1, \widehat{g}_1, \Lambda)$ and $(g_2, \widehat{g}_2, B)$ is a BS set $(f, \widehat{f}, \Gamma)$, symbolized by $(g_1, \widehat{g}_1, \Lambda)\,\widetilde{\widetilde{\sqcap}}\,(g_2, \widehat{g}_2, B) = (f, \widehat{f}, \Gamma)$, where $\Gamma = \Lambda \cap B$ and for all $\ell \in \Gamma$, $f(\ell) = g_1(\ell) \cap g_2(\ell)$ and $\widehat{f}(\neg\ell) = \widehat{g}_1(\neg\ell) \cup \widehat{g}_2(\neg\ell)$.

Definition 2.10. [38] The difference of two BS sets $(g_1, \widehat{g}_1, \Lambda)$ and $(g_2, \widehat{g}_2, B)$ is a BS set $(f, \widehat{f}, \Gamma)$ where $\Gamma = \Lambda \cap B$ and $(g_1, \widehat{g}_1, \Lambda)\backslash(g_2, \widehat{g}_2, B) = (g_1, \widehat{g}_1, \Lambda)\widetilde{\widetilde{\sqcap}}(g_2, \widehat{g}_2, B)^c = (f, \widehat{f}, \Gamma)$.

Definition 2.11. [33] Let $(g, \widehat{g}, \Sigma)$ be a BS set over $\aleph$ and $\eta \in \aleph$. Then, $\eta \in (g, \widehat{g}, \Sigma)$ if $\eta \in g(\ell)$ for all $\ell \in \Sigma$. For any $\eta \in \aleph$, $\eta \notin (g, \widehat{g}, \Sigma)$ if $\eta \notin g(\ell)$ for some $\ell \in \Sigma$.

Definition 2.12. [33] Two BS sets $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ are said to be disjoint BS sets if for all $\ell \in \Sigma$, $g_1(\ell) \cap g_2(\ell) = \varnothing$. We symbolized by $(g_1, \widehat{g}_1, \Sigma)\widetilde{\widetilde{\sqcap}}(g_2, \widehat{g}_2, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$, where $\widehat{\Phi}(\ell) = \varnothing$ for all $\ell \in \Sigma$ and $\widehat{g}(\neg\ell) \subseteq \aleph$ for all $\neg\ell \in \neg\Sigma$.

Definition 2.13. [33] Let $(g, \widehat{g}, \Sigma)$ be a BS set over $\aleph$ and $\Upsilon \subseteq \aleph$. The sub BS set of $(g, \widehat{g}, \Sigma)$ over Υ, denoted by $(g_\Upsilon, \widehat{g}_\Upsilon, \Sigma)$, is defined as $g_\Upsilon(\ell) = \Upsilon \cap g(\ell)$ and $\widehat{g}_\Upsilon(\neg\ell) = \Upsilon \cap \widehat{g}(\neg\ell)$ for each $\ell \in \Sigma$.

Definition 2.14. [33] Let $\eta \in \aleph$, then $(g_\eta, \widehat{g}_\eta, \Sigma)$ denotes the BS set over $\aleph$ defined by $g_\eta(\ell) = \{\eta\}$ and $\widehat{g}_\eta(\neg\ell) = \aleph\backslash\{\eta\}$ for all $\ell \in \Sigma$.

Definition 2.15. [33] Let $T_{\mathcal{BS}}$ be the collection of BS sets over $\aleph$, then $T_{\mathcal{BS}}$ is said to be a bipolar soft topology (BST) on $\aleph$ if

(1) $(\widehat{\Phi}, \widehat{\aleph}, \Sigma)$, $(\widehat{\aleph}, \widehat{\Phi}, \Sigma)$ belong to $T_{\mathcal{BS}}$;
(2) the intersection of any two BS sets in $T_{\mathcal{BS}}$ belongs to $T_{\mathcal{BS}}$;
(3) the union of any number of BS sets in $T_{\mathcal{BS}}$ belongs to $T_{\mathcal{BS}}$.

We termed $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ a BSTS over $\aleph$. The members of $T_{\mathcal{BS}}$ are said to be BS open sets in $\aleph$, while its complements are said to be BS closed sets.

Definition 2.16. [33] Let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BSTS over $\aleph$ and $\Upsilon \subseteq \aleph$. Then, $T_{\mathcal{BS}_\Upsilon} = \{(g_\Upsilon, \widehat{g}_\Upsilon, \Sigma) | (g, \widehat{g}, \Sigma)\ \widetilde{\widetilde{\in}}\ T_{\mathcal{BS}}\}$ is said to be a relative BST on Υ and $(\Upsilon, T_{\mathcal{BS}_\Upsilon}, \Sigma, \neg\Sigma)$ is called a BS subspace of $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$

Proposition 2.17. [34] *Let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BSTS. Then, the following collections define the soft topology on $\aleph$.*

(1) $T_{\mathcal{S}} = \{(g, \Sigma) | (g, \widehat{g}, \Sigma)\ \widetilde{\widetilde{\in}}\ T_{\mathcal{BS}}\}$.
(2) $\neg T_{\mathcal{S}} = \{(\widehat{g}, \neg\Sigma) | (g, \widehat{g}, \Sigma)\ \widetilde{\widetilde{\in}}\ T_{\mathcal{BS}}\}$ (*provided that $\aleph$ is finite*).

Proposition 2.18. [34,33] *Let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BSTS. Then, the following collections define topology on $\aleph$.*

(1) $T_\ell = \{g(\ell) | (g, \widehat{g}, \Sigma)\ \widetilde{\widetilde{\in}}\ T_{\mathcal{BS}}\}$ *for each* $\ell \in \Sigma$.
(2) $T_{\neg\ell} = \{\widehat{g}(\neg\ell) | (g, \widehat{g}, \Sigma)\ \widetilde{\widetilde{\in}}\ T_{\mathcal{BS}}\}$ *for each* $\neg\ell \in \neg\Sigma$. (*provided that $\aleph$ is finite*).

Proposition 2.19. [33] *Let* $(\aleph, T_{\mathcal{S}}, \Sigma)$ *be an STS. Then, the collection* $T_{\mathcal{BS}}$ *defines a BST over* $\aleph$ *if it consists of BS sets* $(g, \widehat{g}, \Sigma)$ *with* $(g, \Sigma) \tilde{\in} T_{\mathcal{S}}$ *and* $\widehat{g}(\neg\ell) = \aleph \backslash g(\ell)$ *for all* $\ell \in \Sigma$.

Definition 2.20. [14] An STS $(\aleph, T_{\mathcal{S}}, \Sigma)$ is said to be:

(i) a soft $\widetilde{T}_0$-space if for every $\eta \neq \kappa \in \aleph$, there is a soft open set (g, Σ) such that $\eta \in (g, \Sigma)$ but $\kappa \notin (g, \Sigma)$ or $\kappa \in (g, \Sigma)$ but $\eta \notin (g, \Sigma)$.

(ii) a soft $\widetilde{T}_1$-space if for every $\eta \neq \kappa \in \aleph$, there are soft open sets (g_1, Σ) and (g_2, Σ) such that $\eta \in (g_1, \Sigma)$ but $\kappa \notin (g_1, \Sigma)$ and $\kappa \in (g_2, \Sigma)$ but $\eta \notin (g_2, \Sigma)$.

(iii) a soft $\widetilde{T}_2$-space if for every $\eta \neq \kappa \in \aleph$, there are soft open sets (g_1, Σ) and (g_2, Σ) such that $\eta \in (g_1, \Sigma)$ but $\kappa \in (g_2, \Sigma)$ and $(g_1, \Sigma) \tilde{\sqcap} (g_2, \Sigma) = (\widehat{\Phi}, \Sigma)$.

Corollary 2.21. *Let* $(g, \widehat{g}, \Sigma)$ *be a BS set over* $\aleph$ *and* $\eta \in \aleph$. *Then,*

(i) $\eta \in (g, \widehat{g}, \Sigma)$ *if and only if* $(g_\eta, \widehat{g}_\eta, \Sigma) \tilde{\tilde{\sqsubseteq}} (g, \widehat{g}, \Sigma)$.

(ii) If $(g_\eta, \widehat{g}_\eta, \Sigma) \tilde{\tilde{\sqcap}} (g, \widehat{g}, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$, *then* $\eta \notin (g, \widehat{g}, \Sigma)$.

Proof. Straightforward. □

Remark 2.22. The converse of Corollary 2.21 (ii) is not true.

Example 2.23. Suppose that $\aleph = \{\eta_1, \eta_2\}$, $\Sigma = \{\ell_1, \ell_2, \ell_3\}$, and $(g, \widehat{g}, \Sigma) = \{(\ell_1, \{\eta_1\}, \{\eta_2\}), (\ell_2, \{\eta_1\}, \{\eta_2\}), (\ell_3, \aleph, \varnothing)\}$. Then, $\eta_2 \notin (g, \widehat{g}, \Sigma)$ but $(g_{\eta_2}, \widehat{g}_{\eta_2}, \Sigma) \tilde{\tilde{\sqcap}} (g, \widehat{g}, \Sigma) \neq (\widehat{\Phi}, \widehat{g}, \Sigma)$.

Proposition 2.24. *Let* $(\Upsilon, T_{\mathcal{BS}_\Upsilon}, \Sigma, \neg\Sigma)$ *be a BS subspace of* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$. *Then,*

(i) $(g_\Upsilon, \widehat{g}_\Upsilon, \Sigma)$ *is a BS open set in* Υ *if and only if* $g_\Upsilon(\ell) = \Upsilon \cap g(\ell)$ *and* $\widehat{g}_\Upsilon(\neg\ell) = \Upsilon \cap \widehat{g}(\neg\ell)$ *for some BS open set* $(g, \widehat{g}, \Sigma)$ *in* $\aleph$.

(ii) $(f_\Upsilon, \widehat{f}_\Upsilon, \Sigma)$ *is a BS closed set in* Υ *if and only if* $f_\Upsilon(\ell) = \Upsilon \cap f(\ell)$ *and* $\widehat{f}_\Upsilon(\neg\ell) = \Upsilon \cap \widehat{f}(\neg\ell)$ *for some BS closed set* $(f, \widehat{f}, \Sigma)$ *in* $\aleph$.

Proof.

(i) Follows from Definition 2.16.

(ii) $(f_\Upsilon, \widehat{f}_\Upsilon, \Sigma)$ is a BS closed set in Υ *iff* $(f_\Upsilon, \widehat{f}_\Upsilon, \Sigma)^c$ is a BS open set in Υ *iff* $f_\Upsilon{}^c(\ell) = \Upsilon \cap g(\ell)$ and $\widehat{f}_\Upsilon{}^c(\neg\ell) = \Upsilon \cap \widehat{g}(\neg\ell)$ for some BS open set $(g, \widehat{g}, \Sigma)$ in $\aleph$ *iff* $\Upsilon \backslash f_\Upsilon{}^c(\ell) = \Upsilon \backslash [\Upsilon \cap g(\ell)]$ and $\Upsilon \backslash \widehat{f}_\Upsilon{}^c(\neg\ell) = \Upsilon \backslash [\Upsilon \cap \widehat{g}(\neg\ell)]$ *iff* $f_\Upsilon(\ell) = \Upsilon \cap g^c(\ell)$ and $\widehat{f}_\Upsilon(\neg\ell) = \Upsilon \cap \widehat{g}^c(\neg\ell)$ *iff* $f_\Upsilon(\ell) = \Upsilon \cap f(\ell)$ and $\widehat{f}_\Upsilon(\neg\ell) = \Upsilon \cap \widehat{f}(\neg\ell)$, where $(f, \widehat{f}, \Sigma) = (g, \widehat{g}, \Sigma)^c$ is a BS closed set in $\aleph$ since $(g, \widehat{g}, \Sigma)$ is a BS open set in $\aleph$. □

3 Bipolar soft separation axioms

The definitions of BS $\tilde{\tilde{T}}_i$-space $(i = 0, 1, 2, 3, 4)$ using ordinary points are given in this section. Basic properties of these notions are established and the relationships between them and other spaces are demonstrated.

Definition 3.1. A BSTS $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is said to be:

(i) a BS $\tilde{\tilde{T}}_0$-space if for every $\eta \neq \kappa \in \aleph$, there is a BS open set $(g, \widehat{g}, \Sigma)$ with $\eta \in (g, \widehat{g}, \Sigma)$ but $\kappa \notin (g, \widehat{g}, \Sigma)$ or $\kappa \in (g, \widehat{g}, \Sigma)$ but $\eta \notin (g, \widehat{g}, \Sigma)$.

(ii) a BS $\widetilde{\widetilde{T}}_1$-space if for every $\eta \neq \kappa \in \aleph$, there are BS open sets $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ with $\eta \in (g_1, \widehat{g}_1, \Sigma)$ but $\kappa \notin (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ but $\eta \notin (g_2, \widehat{g}_2, \Sigma)$.

(iii) a BS $\widetilde{\widetilde{T}}_2$-space if for every $\eta \neq \kappa \in \aleph$, there are BS open sets $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ with $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ and $(g_1, \widehat{g}_1, \Sigma)\widetilde{\widetilde{\sqcap}}(g_2, \widehat{g}_2, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$.

Furthermore, we examine some results related to BS $\widetilde{\widetilde{T}}_0$-space.

Proposition 3.2. *Let* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *be a BSTS and* $\eta \neq \kappa \in \aleph$. *If there is a BS open set* $(g, \widehat{g}, \Sigma)$ *with* $\eta \in (g, \widehat{g}, \Sigma)$ *and* $\kappa \in (g, \widehat{g}, \Sigma)^c$ *or* $\kappa \in (g, \widehat{g}, \Sigma)$ *and* $\eta \in (g, \widehat{g}, \Sigma)^c$, *then* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *is a BS* $\widetilde{\widetilde{T}}_0$*-space.*

Proof. Let $\eta \neq \kappa \in \aleph$ and $(g, \widehat{g}, \Sigma)$ be a BS open set with $\eta \in (g, \widehat{g}, \Sigma)$ and $\kappa \in (g, \widehat{g}, \Sigma)^c$. If $\kappa \in (g, \widehat{g}, \Sigma)^c$, then $\kappa \in g^c(\ell)$ for all $\ell \in \Sigma$. This means that $\kappa \notin g(\ell)$ for all $\ell \in \Sigma$. Hence, $\kappa \notin (g, \widehat{g}, \Sigma)$. Similarly, we may verify $\kappa \in (g, \widehat{g}, \Sigma)$ and $\eta \notin (g, \widehat{g}, \Sigma)$. Therefore, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_0$-space. □

Proposition 3.3. *If* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *be a BS* $\widetilde{\widetilde{T}}_0$*-space, then* $(\aleph, T_{\mathcal{S}}, \Sigma)$ *is a soft* $\widetilde{T}_0$*-space.*

Proof. Let $\eta \neq \kappa \in \aleph$. Since $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_0$-space. Then, there is a BS open set $(g, \widehat{g}, \Sigma)$ with $\eta \in (g, \widehat{g}, \Sigma)$ but $\kappa \notin (g, \widehat{g}, \Sigma)$ or $\kappa \in (g, \widehat{g}, \Sigma)$ but $\eta \notin (g, \widehat{g}, \Sigma)$. Say, $\eta \in (g, \widehat{g}, \Sigma)$ but $\kappa \notin (g, \widehat{g}, \Sigma)$. This means that $\eta \in g(\ell)$ for all $\ell \in \Sigma$ and $\kappa \notin g(\ell)$ for some $\ell \in \Sigma$. Hence, $\eta \in (g, \Sigma)$ and $\kappa \notin (g, \Sigma)$. Similarly, we may verify $\kappa \in (g, \Sigma)$ and $\eta \notin (g, \Sigma)$. Therefore, $(\aleph, T_{\mathcal{S}}, \Sigma)$ is a soft $\widetilde{T}_0$-space. □

Remark 3.4. If $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BS $\widetilde{\widetilde{T}}_0$-space. Then, $(\aleph, T_\ell)$ need not be T_0-space for each $\ell \in \Sigma$.

Example 3.5. Let $\aleph = \{\eta_1, \eta_2\}$ and $\Sigma = \{\ell_1, \ell_2\}$. Let $T_{\mathcal{BS}} = \{(\widehat{\Phi}, \widehat{\aleph}, \Sigma), (\widehat{\aleph}, \widehat{\Phi}, \Sigma), (g, \widehat{g}, \Sigma)\}$, where $(g, \widehat{g}, \Sigma) = \{(\ell_1, \{\eta_1\}, \{\eta_2\}), (\ell_2, \aleph, \varnothing)\}$. Then, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_0$-space. Also, we have

$T_{\ell_1} = \{\varnothing, \aleph, \{\eta_1\}\}$, and

$T_{\ell_2} = \{\varnothing, \aleph\}$.

Then, T_{ℓ_2} is not T_0-space.

To fix the problem in Remark 3.4, we propose the following condition.

Proposition 3.6. *Let* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *be a BSTS and* $\eta \neq \kappa \in \aleph$. *If there is a BS open set* $(g, \widehat{g}, \Sigma)$ *with* $\eta \in (g, \widehat{g}, \Sigma)$ *and* $\kappa \in (g, \widehat{g}, \Sigma)^c$ or $\kappa \in (g, \widehat{g}, \Sigma)$ *and* $\eta \in (g, \widehat{g}, \Sigma)^c$, *then* $(\aleph, T_{\mathcal{S}}, \Sigma)$ *is a soft* $\widetilde{T}_0$*-space and* $(\aleph, T_\ell)$ *is a* T_0*-space for each* $\ell \in \Sigma$.

Proof. By Propositions 3.2 and 3.3, it is obvious that $(\aleph, T_{\mathcal{S}}, \Sigma)$ is a soft $\widetilde{T}_0$-space. Now, for any $\ell \in \Sigma$, $(\aleph, T_\ell)$ is a topological space and since $\eta \in (g, \widehat{g}, \Sigma)$ and $\kappa \in (g, \widehat{g}, \Sigma)^c$ or $\kappa \in (g, \widehat{g}, \Sigma)$ and $\eta \in (g, \widehat{g}, \Sigma)^c$, then $\eta \in g(\ell)$ but $\kappa \notin g(\ell)$ or $\kappa \in g(\ell)$ but $\eta \notin g(\ell)$. Hence, $(\aleph, T_\ell)$ is a T_0-space. □

Proposition 3.7. *Let* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *be a BSTS and* $\eta \neq \kappa \in \aleph$ *and let* $\aleph$ *be a finite set. If there is a BS open set* $(g, \widehat{g}, \Sigma)$ *with* $\eta \in (g, \widehat{g}, \Sigma)$ *and* $\kappa \in (g, \widehat{g}, \Sigma)^c$ *or* $\kappa \in (g, \widehat{g}, \Sigma)$ *and* $\eta \in (g, \widehat{g}, \Sigma)^c$, *then* $(\aleph, \neg T_{\mathcal{S}}, \neg\Sigma)$ *is a soft* $\widetilde{T}_0$*-space and* $(\aleph, T_{\neg\ell})$ *is a* T_0*-space for all* $\neg\ell \in \neg\Sigma$.

Proof. Let $\eta \in (g, \widehat{g}, \Sigma)$ and $\kappa \in (g, \widehat{g}, \Sigma)^c$. This means that $\eta \in g(\ell)$ and $\kappa \in g^c(\ell)$ for all $\ell \in \Sigma$. Then, $\eta \notin \widehat{g}(\neg\ell)$ and $\kappa \in \widehat{g}(\neg\ell)$ for all $\ell \in \Sigma$. So, $\kappa \in (\widehat{g}, \neg\Sigma)$ and $\eta \notin (\widehat{g}, \neg\Sigma)$. Similarly, we may verify $\eta \in (\widehat{g}, \neg\Sigma)$ and $\kappa \notin (\widehat{g}, \neg\Sigma)$. Hence, $(\aleph, \neg T_{\mathcal{S}}, \neg\Sigma)$ is a soft $\widetilde{T}_0$-space. Now, for any $\neg\ell \in \neg\Sigma$, $(\aleph, T_{\neg\ell})$ is a topological space, and since $\eta \in (g, \widehat{g}, \Sigma)$ and $\kappa \in (g, \widehat{g}, \Sigma)^c$ or $\kappa \in (g, \widehat{g}, \Sigma)$ and $\eta \in (g, \widehat{g}, \Sigma)^c$, then $\eta \notin \widehat{g}(\neg\ell)$ and $\kappa \in \widehat{g}(\neg\ell)$ or $\kappa \notin \widehat{g}(\neg\ell)$ and $\eta \in \widehat{g}(\neg\ell)$. Hence, $(\aleph, T_{\neg\ell})$ is T_0-space. □

Proposition 3.8. *Let $(\aleph, T_{\mathcal{S}}, \Sigma)$ be an STS and $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BSTS constructed from $(\aleph, T_{\mathcal{S}}, \Sigma)$ as in Proposition 2.19. If $(\aleph, T_{\mathcal{S}}, \Sigma)$ be a soft $\widetilde{T}_0$-space, then $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_0$-space.*

Proof. Let $\eta \neq \kappa \in \aleph$. Since $(\aleph, T_{\mathcal{S}}, \Sigma)$ is a soft $\widetilde{T}_0$-space, then there is a soft open set (g, Σ) with $\eta \in (g, \Sigma)$ but $\kappa \notin (g, \Sigma)$ or $\kappa \in (g, \Sigma)$ but $\eta \notin (g, \Sigma)$. Say, $\eta \in (g, \Sigma)$ and $\kappa \notin (g, \Sigma)$. This means that $\eta \in g(\ell)$ for all $\ell \in \Sigma$ and $\kappa \notin g(\ell)$ for some $\ell \in \Sigma$. Since $\widehat{g}(\neg\ell) = \aleph \backslash g(\ell)$ for all $\ell \in \Sigma$, then $\eta \notin \widehat{g}(\neg\ell)$ for all $\ell \in \Sigma$ and $\kappa \in \widehat{g}(\neg\ell)$ for some $\ell \in \Sigma$. Therefore, $\eta \in (g, \widehat{g}, \Sigma)$ and $\kappa \notin (g, \widehat{g}, \Sigma)$. Similarly, we may verify $\kappa \in (g, \widehat{g}, \Sigma)$ but $\eta \notin (g, \widehat{g}, \Sigma)$. Hence, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_0$-space. □

Proposition 3.9. *Let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BSTS and $\Upsilon \subseteq \aleph$. If $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BS $\widetilde{\widetilde{T}}_0$-space, then $(\Upsilon, T_{\mathcal{BS}_\Upsilon}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_0$-space.*

Proof. Let $\eta \neq \kappa \in \Upsilon$. Since $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_0$-space, then there is a BS open set $(g, \widehat{g}, \Sigma)$ with $\eta \in (g, \widehat{g}, \Sigma)$ but $\kappa \notin (g, \widehat{g}, \Sigma)$ or $\kappa \in (g, \widehat{g}, \Sigma)$ but $\eta \notin (g, \widehat{g}, \Sigma)$. Say, $\eta \in (g, \widehat{g}, \Sigma)$ and $\kappa \notin (g, \widehat{g}, \Sigma)$. As, $\eta \in (g, \widehat{g}, \Sigma)$, then $\eta \in g(\ell)$ and $\eta \notin \widehat{g}(\neg\ell)$ for all $\ell \in \Sigma$. Since $\eta \in \Upsilon$, then $\eta \in \Upsilon \cap g(\ell) = g_\Upsilon(\ell)$ and $\kappa \notin \Upsilon \cap \widehat{g}(\neg\ell) = \widehat{g}_\Upsilon(\neg\ell)$ for all $\ell \in \Sigma$. Hence, $\eta \in (g_\Upsilon, \widehat{g}_\Upsilon, \Sigma)$. Now, if $\kappa \notin (g, \widehat{g}, \Sigma)$, then $\kappa \notin g(\ell)$ for some $\ell \in \Sigma$. This means that $\kappa \notin \Upsilon \cap g(\ell) = g_\Upsilon(\ell)$ for some $\ell \in \Sigma$. Hence, $\kappa \notin (g_\Upsilon, \widehat{g}_\Upsilon, \Sigma)$. Similarly, we may verify $\kappa \in (g_\Upsilon, \widehat{g}_\Upsilon, \Sigma)$ but $\eta \notin (g_\Upsilon, \widehat{g}_\Upsilon, \Sigma)$. Hence, $(\Upsilon, T_{\mathcal{BS}_\Upsilon}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_0$-space. □

In the following result, we present a complete description of a BS $\widetilde{\widetilde{T}}_1$-space and then establish several characteristics of this space.

Proposition 3.10. *If $(g_\eta, \widehat{g}_\eta, \Sigma)$ be a BS closed set of $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ for each $\eta \in \aleph$, then $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_1$-space.*

Proof. Let for each $\eta \in \aleph$, $(g_\eta, \widehat{g}_\eta, \Sigma)$ is a BS closed set of $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$. Then, $(g_\eta, \widehat{g}_\eta, \Sigma)^c$ is a BS open set in $T_{\mathcal{BS}}$. Let $\eta \neq \kappa \in \aleph$. For $\eta \in \aleph$, $(g_\eta, \widehat{g}_\eta, \Sigma)^c$ is a BS open set such that $\kappa \in (g_\eta, \widehat{g}_\eta, \Sigma)^c$ and $\eta \notin (g_\eta, \widehat{g}_\eta, \Sigma)^c$. Similarly, $(g_\kappa, \widehat{g}_\kappa, \Sigma)^c \widetilde{\widetilde{\in}} T_{\mathcal{BS}}$ is such that $\eta \in (g_\kappa, \widehat{g}_\kappa, \Sigma)^c$ and $\kappa \notin (g_\kappa, \widehat{g}_\kappa, \Sigma)^c$. Thus, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_1$-space. □

Remark 3.11. The converse of Proposition 3.10 is incorrect as the next example illustrates.

Example 3.12. Let $\aleph = \{\eta_1, \eta_2\}$ and $\Sigma = \{\ell_1, \ell_2\}$. Let $T_{\mathcal{BS}} = \{(\widehat{\Phi}, \widehat{\aleph}, \Sigma), (\widehat{\aleph}, \widehat{\Phi}, \Sigma), (g_1, \widehat{g}_1, \Sigma), (g_2, \widehat{g}_2, \Sigma), (g_3, \widehat{g}_3, \Sigma)\}$ be a BST defined on $\aleph$, where

$$(g_1, \widehat{g}_1, \Sigma) = \{(\ell_1, \{\eta_1\}, \{\eta_2\}), (\ell_2, \{\eta_2\}, \{\eta_1\})\},$$
$$(g_2, \widehat{g}_2, \Sigma) = \{(\ell_1, \{\eta_1\}, \{\eta_2\}), (\ell_2, \aleph, \varnothing)\}, \text{ and}$$
$$(g_3, \widehat{g}_3, \Sigma) = \{(\ell_1, \aleph, \varnothing), (\ell_2, \{\eta_2\}, \{\eta_1\})\}.$$

Then, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_1$-space.

We note that for $(g_{\eta_1}, \widehat{g}_{\eta_1}, \Sigma)$, $(g_{\eta_2}, \widehat{g}_{\eta_2}, \Sigma)$ over $\aleph$, where

$$\left(g_{\eta_1}, \widehat{g}_{\eta_1}, \Sigma\right) = \{(\ell_1, \{\eta_1\}, \{\eta_2\}), (\ell_2, \{\eta_1\}, \{\eta_2\})\}, \text{ and}$$
$$\left(g_{\eta_2}, \widehat{g}_{\eta_2}, \Sigma\right) = \{(\ell_1, \{\eta_2\}, \{\eta_1\}), (\ell_2, \{\eta_2\}, \{\eta_1\})\},$$

the BS complements $(g_{\eta_1}, \widehat{g}_{\eta_1}, \Sigma)^c$, and $(g_{\eta_2}, \widehat{g}_{\eta_2}, \Sigma)^c$ over $\aleph$ are defined as follows:

$$\left(g_{\eta_1}, \widehat{g}_{\eta_1}, \Sigma\right)^c = \{(\ell_1, \{\eta_2\}, \{\eta_1\}), (\ell_2, \{\eta_2\}, \{\eta_1\})\}, \text{ and}$$
$$\left(g_{\eta_2}, \widehat{g}_{\eta_2}, \Sigma\right)^c = \{(\ell_1, \{\eta_1\}, \{\eta_2\}), (\ell_2, \{\eta_1\}, \{\eta_2\})\}.$$

Neither $(g_{\eta_1}, \widehat{g}_{\eta_1}, \Sigma)^c$ nor $(g_{\eta_2}, \widehat{g}_{\eta_2}, \Sigma)^c$ belongs to $T_{\mathcal{BS}}$. Thus, $(g_{\eta_1}, \widehat{g}_{\eta_1}, \Sigma)$ and $(g_{\eta_2}, \widehat{g}_{\eta_2}, \Sigma)$ are not BS closed sets of $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\, \Sigma)$.

Proposition 3.13. *Let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\, \Sigma)$ be a BSTS and $\eta \in \aleph$. If $\aleph$ be a BS $\widetilde{\widetilde{T}}_1$-space, then for every BS open set $(g, \widehat{g}, \Sigma)$ such that $\eta \in (g, \widehat{g}, \Sigma)$:*

(i) $(g_\eta, \widehat{g}_\eta, \Sigma)\widetilde{\widetilde{\sqsubseteq}}[\widetilde{\widetilde{\sqcap}}(g, \widehat{g}, \Sigma)]$;

(ii) *For all $\kappa \neq \eta$, $\kappa \notin \widetilde{\widetilde{\sqcap}}(g, \widehat{g}, \Sigma)$.*

Proof.

(*i*) Since $\eta \in \widetilde{\widetilde{\sqcap}}(g, \widehat{g}, \Sigma)$, then $(g_\eta, \widehat{g}_\eta, \Sigma)\widetilde{\widetilde{\sqsubseteq}}[\widetilde{\widetilde{\sqcap}}(g, \widehat{g}, \Sigma)]$ by Corollary 2.21.

(*ii*) For $\eta, \kappa \in \aleph$ with $\eta \neq$, then there are BS open sets $(g, \widehat{g}, \Sigma)$ such that $\eta \in (g, \widehat{g}, \Sigma)$ and $\kappa \notin (g, \widehat{g}, \Sigma)$. So, for some $\ell \in \Sigma$, $\kappa \notin g(\ell)$, and hence, $\kappa \notin \cap_{\ell\in\Sigma} g(\ell)$. Thus, $\kappa \notin \widetilde{\widetilde{\sqcap}}(g, \widehat{g}, \Sigma)$. □

Remark 3.14. The equality in Proposition 3.13 (i) is false in general.

Example 3.15. Let $\aleph = \{\eta_1, \eta_2\}$, and $\Sigma = \{\ell_1, \ell_2, \ell_3, \ell_4,\}$. Let $T_{\mathcal{BS}} = \{(\widehat{\Phi}, \widehat{\aleph}, \Sigma), (\widehat{\aleph}, \widehat{\Phi}, \Sigma), (g_1, \widehat{g}_1, \Sigma), (g_2, \widehat{g}_2, \Sigma), (g_3, \widehat{g}_3, \Sigma), (g_4, \widehat{g}_4, \Sigma), (g_5, \widehat{g}_5, \Sigma)\}$ be a BST defined on $\aleph$, where

$$\begin{aligned}
(g_1, \widehat{g}_1, \Sigma) &= \{(\ell_1, \aleph, \varnothing), (\ell_2, \{\eta_1\}, \{\eta_2\}), (\ell_3, \{\eta_1\}, \{\eta_2\}), (\ell_4, \{\eta_1\}, \{\eta_2\})\},\\
(g_2, \widehat{g}_2, \Sigma) &= \{(\ell_1, \{\eta_2\}, \{\eta_1\}), (\ell_2, \{\eta_2\}, \{\eta_1\}), (\ell_3, \{\eta_2\}, \{\eta_1\}), (\ell_4, \aleph, \varnothing)\},\\
(g_3, \widehat{g}_3, \Sigma) &= \{(\ell_1, \aleph, \varnothing), (\ell_2, \aleph, \varnothing), (\ell_3, \{\eta_1\}, \{\eta_2\}), (\ell_4, \{\eta_1\}, \{\eta_2\})\},\\
(g_4, \widehat{g}_4, \Sigma) &= \{(\ell_1, \{\eta_2\}, \{\eta_1\}), (\ell_2, \varnothing, \aleph), (\ell_3, \varnothing, \aleph), (\ell_4, \{\eta_1\}, \{\eta_2\})\}, \text{ and}\\
(g_5, \widehat{g}_5, \Sigma) &= \{(\ell_1, \{\eta_2\}, \{\eta_1\}), (\ell_2, \{\eta_2\}, \{\eta_1\}), (\ell_3, \varnothing, \aleph), (\ell_4, \{\eta_1\}, \{\eta_2\})\}.
\end{aligned}$$

Then, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_1$-space. But, for all BS open sets $\eta_1 \in (g_1, \widehat{g}_1, \Sigma)$ and $\eta_1 \in (g_3, \widehat{g}_3, \Sigma)$, we have $(g_1, \widehat{g}_1, \Sigma)\widetilde{\widetilde{\sqcap}}(g_3, \widehat{g}_3, \Sigma) = (g_1, \widehat{g}_1, \Sigma) \neq (g_{\eta_1}, \widehat{g}_{\eta_1}, \Sigma)$.

Proposition 3.16. *Let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BSTS and $\eta \neq \kappa \in \aleph$. If there are BS open sets $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ with $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_1, \widehat{g}_1, \Sigma)^c$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ and $\eta \in (g_2, \widehat{g}_2, \Sigma)^c$, then $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_1$-space.*

Proof. Similar to the proof of Proposition 3.2. □

Proposition 3.17. *If $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BS $\widetilde{\widetilde{T}}_1$-space, then $(\aleph, T_{\mathcal{S}}, \Sigma)$ is a soft $\widetilde{T}_1$-space.*

Proof. Similar to the proof of Proposition 3.3. □

Remark 3.18. If $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BS $\widetilde{\widetilde{T}}_1$-space, then $(\aleph, T_\ell)$ need not be $\widetilde{T}_1$-space for each $\ell \in \Sigma$.

Example 3.19. Assume that $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be the same as in Example 3.12. There we showed that $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_1$-space. In addition, we have

$$T_{\ell_1} = \{\varnothing, \aleph, \{\eta_1\}\} \text{ and } T_{\ell_2} = \{\varnothing, \aleph, \{\eta_2\}\}.$$

Then, T_{ℓ_1} and T_{ℓ_2} are not $\widetilde{T}_1$-spaces.

The next proposition gives us the conditions to fix the problem in Remark 3.18.

Proposition 3.20. *Let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BSTS and $\eta \neq \kappa \in \aleph$. If there is a BS open set $(g, \widehat{g}, \Sigma)$ with $\eta \in (g, \widehat{g}, \Sigma)$ and $\kappa \in (g, \widehat{g}, \Sigma)^c$ and $\kappa \in (g, \widehat{g}, \Sigma)$ and $\eta \in (g, \widehat{g}, \Sigma)^c$, then $(\aleph, T_{\mathcal{S}}, \Sigma)$ is a soft $\widetilde{T}_1$-space and $(\aleph, T_\ell)$ is a $\widetilde{T}_1$-space for each $\ell \in \Sigma$.*

Proof. By Propositions 3.16 and 3.17, it is obvious that $(\aleph, T_{\mathcal{S}}, \Sigma)$ is a soft $\widetilde{T}_1$-space. Now, for any $\ell \in \Sigma$, $(\aleph, T_\ell)$ is a topological space, and since $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_1, \widehat{g}_1, \Sigma)^c$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ and $\eta \in (g_2, \widehat{g}_2, \Sigma)^c$, then $\eta \in g_1(\ell)$ but $\kappa \notin g_1(\ell)$ and $\kappa \in g_2(\ell)$ but $\eta \notin g_2(\ell)$. Hence, $(\aleph, T_\ell)$ is a $\widetilde{T}_1$-space. □

Proposition 3.21. *Let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BSTS and $\eta \neq \kappa \in \aleph$ and let $\aleph$ be a finite set. If there is a BS open set $(g, \widehat{g}, \Sigma)$ with $\eta \in (g, \widehat{g}, \Sigma)$ and $\kappa \in (g, \widehat{g}, \Sigma)^c$* or *$\kappa \in (g, \widehat{g}, \Sigma)$ and $\eta \in (g, \widehat{g}, \Sigma)^c$, then $(\aleph, \neg T_{\mathcal{S}}, \neg\Sigma)$ is a soft $\widetilde{T}_1$-space and $(\aleph, T_{\neg\ell})$ is a $\widetilde{T}_1$-space for each $\neg\ell \in \neg\Sigma$.*

Proof. Similar to the proof of Proposition 3.7. □

Proposition 3.22. *Let $(\aleph, T_{\mathcal{S}}, \Sigma)$ be an STS and $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BSTS constructed from $(\aleph, T_{\mathcal{S}}, \Sigma)$ as in* Proposition 2.19. *If $(\aleph, T_{\mathcal{S}}, \Sigma)$ be a soft $\widetilde{T}_1$-space over $\aleph$, then $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BS $\widetilde{\widetilde{T}}_1$-space over $\aleph$.*

Proof. Similar to the proof of Proposition 3.8. □

Proposition 3.23. *Let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be an BSTS and $\Upsilon \subseteq \aleph$. If $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BS $\widetilde{\widetilde{T}}_1$-space, then $(\Upsilon, T_{\mathcal{BS}_\Upsilon}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_1$-space.*

Proof. Similar to the proof of Proposition 3.9. □

In the following results, we characterize a BS $\widetilde{\widetilde{T}}_2$-space and investigate some of its properties.

Proposition 3.24. *Let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BSTS and $\eta \in \aleph$. If $\aleph$ be a BS $\widetilde{\widetilde{T}}_2$-space, then for each BS open set $(g, \widehat{g}, \Sigma)$ such that $\eta \in (g, \widehat{g}, \Sigma)$:*
(i) $(g_\eta, \widehat{g}_\eta, \Sigma) \widetilde{\widetilde{\sqsubseteq}} [\widetilde{\widetilde{\sqcap}}(g, \widehat{g}, \Sigma)]$;
(ii) $\kappa \notin \widetilde{\widetilde{\sqcap}}(g, \widehat{g}, \Sigma)$ *for all* $\kappa \neq \eta$.

Proof. Similar to the proof of Proposition 3.13. □

Remark 3.25. The equality in Proposition 3.24 (i) is incorrect in general.

Example 3.26. Let $\aleph = \{\eta_1, \eta_2\}$, and $\Sigma = \{\ell_1, \ell_2\}$. Let $T_{\mathcal{BS}} = \{(\widehat{\Phi}, \widehat{\aleph}, \Sigma), (\widehat{\aleph}, \widehat{\Phi}, \Sigma), (g_1, \widehat{g}_1, \Sigma), (g_2, \widehat{g}_2, \Sigma), (g_3, \widehat{g}_3, \Sigma)\}$ be a BST defined on $\aleph$, where

$$\begin{aligned}
&(g_1, \widehat{g}_1, \Sigma) = \{(\ell_1, \{\eta_1\}, \{\eta_2\}), (\ell_2, \{\eta_1\}, \varnothing)\},\\
&(g_2, \widehat{g}_2, \Sigma) = \{(\ell_1, \{\eta_2\}, \{\eta_1\}), (\ell_2, \{\eta_2\}, \varnothing)\}, \text{ and}\\
&(g_3, \widehat{g}_3, \Sigma) = \{(\ell_1, \varnothing, \aleph), (\ell_2, \varnothing, \varnothing)\}.
\end{aligned}$$

Then, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_2$-space. We note that $(g_1, \widehat{g}_1, \Sigma)$ is the only BS open set with $\eta_1 \in (g_1, \widehat{g}_1, \Sigma)$, but $(g_{\eta_1}, \widehat{g}_{\eta_1}, \Sigma) \neq (g_1, \widehat{g}_1, \Sigma)$.

Proposition 3.27. *Let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BSTS and $\eta \neq \kappa \in \aleph$. If there are BS open sets $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ with $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_1, \widehat{g}_1, \Sigma)^c$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ and $\eta \in (g_2, \widehat{g}_2, \Sigma)^c$, then $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_2$-space.*

Proof. Let $\eta \neq \kappa \in \aleph$ and $(g_1, \widehat{g}_1, \Sigma)$, and $(g_2, \widehat{g}_2, \Sigma)$ be two BS open sets with $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_1, \widehat{g}_1, \Sigma)^c$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ and $\eta \in (g_2, \widehat{g}_2, \Sigma)^c$. This means that $\eta \in g_1(\ell)$, $\kappa \in g_1^c(\ell)$ and $\kappa \in g_2(\ell)$, $\eta \in g_2^c(\ell)$ for all $\ell \in \Sigma$. Then, $\eta \in g_1(\ell)$ but $\kappa \notin g_1(\ell)$ and $\kappa \in g_2(\ell)$ but $\eta \notin g_2(\ell)$ with $g_1(\ell) \cap g_2(\ell) = \varnothing$. So that $\eta \notin \widehat{g}_1(\neg\ell)$ and $\kappa \in \widehat{g}_1(\neg\ell)$ and $\kappa \notin \widehat{g}_2(\neg\ell)$ and $\eta \in \widehat{g}_2(\neg\ell)$ with $\widehat{g}_1(\neg\ell) \cup \widehat{g}_2(\neg\ell) = \aleph$ for all $\neg\ell \in \neg\Sigma$. Then, we have $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ with $(g_1, \widehat{g}_1, \Sigma)\widetilde{\widetilde{\sqcap}}(g_2, \widehat{g}_2, \Sigma) = (\widehat{\Phi}, \widehat{\aleph}, \Sigma)$. Thus, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_2$-space. □

Proposition 3.28. *If $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BS $\widetilde{\widetilde{T}}_2$-space, then $(\aleph, T_{\mathcal{S}}, \Sigma)$ is a soft $\widetilde{T}_2$-space and $(\aleph, T_\ell)$ is T_2-space for each $\ell \in \Sigma$.*

Proof. Let $\eta \neq \kappa \in \aleph$. Since $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_2$-space, then there are BS open sets $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ such that $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ with $(g_1, \widehat{g}_1, \Sigma)\,\widetilde{\widetilde{\sqcap}}\,(g_2, \widehat{g}_2, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$. This means that $\eta \in g_1(\ell)$ and $\kappa \in g_2(\ell)$ with $g_1(\ell) \cap g_2(\ell) = \varnothing$ for all $\ell \in \Sigma$. Consequently, $\eta \in (g_1, \Sigma)$ and $\kappa \in (g_2, \Sigma)$ with $(g_1, \Sigma)\,\widetilde{\sqcap}\,(g_2, \Sigma) = (\widehat{\Phi}, \Sigma)$. Hence, $(\aleph, T_{\mathcal{S}}, \Sigma)$ is a soft $\widetilde{T}_2$-space. Now, for any $\ell \in \Sigma$, $(\aleph, T_\ell)$ is a topological space and since $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ with $(g_1, \widehat{g}_1, \Sigma)\ \widetilde{\widetilde{\sqcap}}(g_2, \widehat{g}_2, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$. Then, we have $\eta \in g_1(\ell)$ and $\kappa \in g_2(\ell)$ with $g_1(\ell) \cap g_2(\ell) = \varnothing$ for all $\ell \in \Sigma$. Thus, $(\aleph, T_\ell)$ is T_2-space for each $\ell \in \Sigma$. □

Proposition 3.29. *Let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be an BSTS and $\eta \neq \kappa \in \aleph$. If there are BS open sets $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ with $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_1, \widehat{g}_1, \Sigma)^c$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ and $\eta \in (g_2, \widehat{g}_2, \Sigma)^c$, then $(\aleph, T_{\mathcal{S}}, \Sigma)$ is soft $\widetilde{T}_2$-space and $(\aleph, T_\ell)$ is T_2-space for all $\ell \in \Sigma$.*

Proof. Follows from Propositions 3.27 and 3.28. □

Proposition 3.30. *Let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BSTS and $\eta \neq \kappa \in \aleph$ and let $\aleph$ be a finite set. If there are BS open sets $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ with $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_1, \widehat{g}_1, \Sigma)^c$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ and $\eta \in (g_2, \widehat{g}_2, \Sigma)^c$, then $(\aleph, \neg\, T_{\mathcal{S}}, \neg\Sigma)$ is soft $\widetilde{T}_2$-space and $(\aleph, T_{\neg\ell})$ is T_2-space for each $\neg\ell \in \neg\Sigma$.*

Proof. Let $\eta \neq \kappa \in \aleph$ and $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ be two BS open sets with $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_1, \widehat{g}_1, \Sigma)^c$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ and $\eta \in (g_2, \widehat{g}_2, \Sigma)^c$. This means that $\eta \notin \widehat{g}_1(\neg\ell)$ and $\kappa \in \widehat{g}_1(\neg\ell)$ and $\kappa \notin \widehat{g}_2(\neg\ell)$ and $\eta \in \widehat{g}_2(\neg\ell)$ with $\widehat{g}_1(\neg\ell) \cap \widehat{g}_2(\neg\ell) = \varnothing$. Then, we have $\kappa \in (\widehat{g}_1, \neg\Sigma)$ and $\eta \in (\widehat{g}_2, \neg\Sigma)$ with $(\widehat{g}_1, \neg\Sigma)\,\widetilde{\sqcap}\,(\widehat{g}_2, \neg\Sigma) = (\widehat{\Phi}, \neg\Sigma)$. Thus, $(\aleph, \neg\, T_{\mathcal{S}}, \neg\Sigma)$ is a soft $\widetilde{T}_2$-space. Now, for any $\neg\ell \in \neg\Sigma$, $(\aleph, T_{\neg\ell})$ is a topological space and since $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_1, \widehat{g}_1, \Sigma)^c$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ and $\eta \in (g_2, \widehat{g}_2, \Sigma)^c$. Then, we have $\eta \notin \widehat{g}_1(\neg\ell)$ and $\kappa \in \widehat{g}_1(\neg\ell)$ and $\kappa \notin \widehat{g}_2(\neg\ell)$ and $\eta \in \widehat{g}_2(\neg\ell)$ with $\widehat{g}_1(\neg\ell) \cap \widehat{g}_2(\neg\ell) = \varnothing$. Hence, $(\aleph, T_{\neg\ell})$ is T_2-space. □

Proposition 3.31. *Let $(\aleph, T_{\mathcal{S}}, \Sigma)$ be an STS and let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BSTS over $\aleph$ constructed from $(\aleph, T_{\mathcal{S}}, \Sigma)$ as in Proposition 2.19. If $(\aleph, T_{\mathcal{S}}, \Sigma)$ be a soft $\widetilde{T}_2$-space, then $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_2$-space.*

Proof. Let $\eta \neq \kappa \in \aleph$. Since $(\aleph, T_{\mathcal{S}}, \Sigma)$ is a soft $\widetilde{T}_2$-space, then there are soft open sets (g_1, Σ) and (g_2, Σ) such that $\eta \in (g_1, \Sigma)$ and $\kappa \in (g_2, \Sigma)$ with $(g_1, \Sigma)\widetilde{\sqcap}(g_2, \Sigma) = (\widehat{\Phi}, \Sigma)$. This means that $\eta \in g_1(\ell)$ and $\kappa \in g_2(\ell)$ with $g_1(\ell) \cap g_2(\ell) = \varnothing$ for all $\ell \in \Sigma$. Then, we have $\eta \notin \widehat{g}_1(\neg\ell)$ and $\kappa \notin \widehat{g}_2(\neg\ell)$ with $\widehat{g}_1(\neg\ell) \cup \widehat{g}_2(\neg\ell) = \aleph$ for all $\neg\ell \in \neg\,\Sigma$. Therefore, $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ with $(g_1, \widehat{g}_1, \Sigma)\widetilde{\widetilde{\sqcap}}(g_2, \widehat{g}_2, \Sigma) = (\widehat{\Phi}, \widehat{\aleph}, \Sigma)$. Thus, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\,\Sigma)$ is a BS $\widetilde{\widetilde{T}}_2$-space. □

Proposition 3.32. *Let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BSTS and $\Upsilon \subseteq \aleph$. If $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BS $\widetilde{\widetilde{T}}_2$-space, then $(\Upsilon, T_{\mathcal{BS}_\Upsilon}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_2$-space.*

Proof. Let $\eta \neq \kappa \in \Upsilon$. Since $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_2$-space, then there are BS open sets $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ such that $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ with $(g_1, \widehat{g}_1, \Sigma)\widetilde{\widetilde{\sqcap}}(g_2, \widehat{g}_2, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$. This means that $\eta \in g_1(\ell)$ and $\kappa \in g_2(\ell)$ with $g_1(\ell) \cap g_2(\ell) = \varnothing$ for all $\ell \in \Sigma$. Then, we have $\eta \notin \widehat{g}_1(\neg\ell)$ and $\kappa \notin \widehat{g}_2(\neg\ell)$ with

$\widehat{g}_1(\neg\ell)\cup\widehat{g}_2(\neg\ell) = \aleph$ for all $\ell \in \Sigma$. Since $\eta, \kappa \in \Upsilon$, then $\eta \in \Upsilon \cap g_1(\ell) = g_{1_\Upsilon}(\ell)$ and $\kappa \in \Upsilon \cap g_2(\ell) = g_{2_\Upsilon}(\ell)$ with $g_{1_\Upsilon}(\ell)\cap g_{2_\Upsilon}(\ell) = \varnothing$ for all $\ell \in \Sigma$. In addition, $\eta \notin \Upsilon \cap \widehat{g}_1(\neg\ell) = \widehat{g}_{1_\Upsilon}(\neg\ell)$ and $\kappa \notin \Upsilon \cap \widehat{g}_2(\neg\ell) = \widehat{g}_{2_\Upsilon}(\neg\ell)$ with $\widehat{g}_{1_\Upsilon}(\neg\ell) \cup\widehat{g}_{2_\Upsilon}(\neg\ell) = \aleph$ for all $\ell \in \Sigma$. Then, $\eta \in (g_{1_\Upsilon}, \widehat{g}_{1_\Upsilon}, \Sigma)$ and $\kappa \in (g_{2_\Upsilon}, \widehat{g}_{2_\Upsilon}, \Sigma)$ with $(g_{1_\Upsilon}, \widehat{g}_{1_\Upsilon}, \Sigma)\widetilde{\widetilde{\sqcap}}(g_{2_\Upsilon}, \widehat{g}_{2_\Upsilon}, \Sigma) = (\widehat{\Phi}, \widehat{\aleph}, \Sigma)$. Hence, $(\Upsilon, T_{\mathcal{BS}_\Upsilon}, \Sigma, \neg \Sigma)$ is a BS $\widetilde{\widetilde{T}}_2$-space. □

Proposition 3.33. *Every BS $\widetilde{\widetilde{T}}_i$-space is BS $\widetilde{\widetilde{T}}_{i-1}$-space, for $i = 1, 2$.*

Proof. Let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BSTS and $\eta \neq \kappa \in \aleph$. For the case $i = 1$, let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BS $\widetilde{\widetilde{T}}_1$-space, then there are BS open sets $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ with $\eta \in (g_1, \widehat{g}_1, \Sigma)$ but $\kappa \notin (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ but $\eta \notin (g_2, \widehat{g}_2, \Sigma)$. Obviously, we have $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \notin (g_1, \widehat{g}_1, \Sigma)$ or $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ and $\eta \notin (g_2, \widehat{g}_2, \Sigma)$. Thus, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_0$-space. Now, for the case $i = 2$, let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BS $\widetilde{\widetilde{T}}_2$-space, then there are BS open sets $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ with $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ and $(g_1, \widehat{g}_1, \Sigma)\widetilde{\widetilde{\sqcap}}(g_2, \widehat{g}_2, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$. This means that $\eta \in g_1(\ell)$ and $\kappa \in g_2(\ell)$ and $g_1(\ell)\cap g_2(\ell) = \varnothing$ for all $\ell \in \Sigma$. Then, we have $\eta \in g_1(\ell)$ but $\kappa \notin g_1(\ell)$ and $\kappa \in g_2(\ell)$ but $\eta \notin g_2(\ell)$ for all $\ell \in \Sigma$. Thus, $\eta \in (g_1, \widehat{g}_1, \Sigma)$ but $\kappa \notin (g_1, \widehat{g}_1, \Sigma)$ and $\kappa \in (g_2, \widehat{g}_2, \Sigma)$ but $\eta \notin (g_2, \widehat{g}_2, \Sigma)$. Therefore, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_1$-space. □

Remark 3.34. The opposite of Proposition 3.33 does not hold.

Example 3.35. Assume that $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is the same as in Example 3.12. Then, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_1$-space but not a BS $\widetilde{\widetilde{T}}_2$-space since, for $\eta_1, \eta_2 \in \aleph$, there are not any two BS open sets $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ with $\eta_1 \in (g_1, \widehat{g}_1, \Sigma)$ and $\eta_2 \in (g_2, \widehat{g}_2, \Sigma)$ and $(g_1, \widehat{g}_1, \Sigma) \widetilde{\widetilde{\sqcap}} (g_2, \widehat{g}_2, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$.

Now, if we consider $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ as in Example 3.5. Then, we saw that $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_0$-space. But, for $\eta_1, \eta_2 \in \aleph$, there do not exist BS open sets $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ with $\eta_1 \in (g_1, \widehat{g}_1, \Sigma)$ but $\eta_2 \notin (g_1, \widehat{g}_1, \Sigma)$ and $\eta_2 \in (g_2, \widehat{g}_2, \Sigma)$ but $\eta_1 \notin (g_2, \widehat{g}_2, \Sigma)$. Hence, it is not a BS $\widetilde{\widetilde{T}}_1$-space

4 Bipolar soft regular and bipolar soft normal spaces

We define and study in detail the BS regular and BS normal spaces in this section.

Definition 4.1. A BSTS $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is said to be BS regular space if for every BS closed set $(f, \widehat{f}, \Sigma)$ with $\eta \notin (f, \widehat{f}, \Sigma)$, there are BS open sets $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ such that $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $(f, \widehat{f}, \Sigma) \widetilde{\widetilde{\sqsubseteq}} (g_2, \widehat{g}_2, \Sigma)$ with $(g_1, \widehat{g}_1, \Sigma)\widetilde{\widetilde{\sqcap}}(g_2, \widehat{g}_2, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$.

Corollary 4.2. *Let $(f, \widehat{f}, \Sigma)$ be a BS closed subset of BSTS $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg \Sigma)$ with $\eta \notin (f, \widehat{f}, \Sigma)$. If $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg \Sigma)$ be a BS regular space, then there is a BS open set $(g, \widehat{g}, \Sigma)$ such that $\eta \in (g, \widehat{g}, \Sigma)$ and $(g, \widehat{g}, \Sigma)\widetilde{\widetilde{\sqcap}}(f, \widehat{f}, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$.*

Proposition 4.3. *Let $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg \Sigma)$ be a BSTS and $\eta \in \aleph$. If $\aleph$ be a BS regular space, then*:

(i) *for a BS closed set $(f, \widehat{f}, \Sigma)$, $\eta \notin (f, \widehat{f}, \Sigma)$ if and only if $(g_\eta, \widehat{g}_\eta, \Sigma)\widetilde{\widetilde{\sqcap}}(f, \widehat{f}, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$;*

(ii) *for a BS open set $(g, \widehat{g}, \Sigma)$, $\eta \notin (g, \widehat{g}, \Sigma)$ if and only if $(g_\eta, \widehat{g}_\eta, \Sigma)\widetilde{\widetilde{\sqcap}}(g, \widehat{g}, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$.*

Proof.

(i) Let $\eta \notin (f, \widehat{f}, \Sigma)$. Then, by Corollary 4.2, there is a BS open set $(g, \widehat{g}, \Sigma)$ with $\eta \in (g, \widehat{g}, \Sigma)$ and $(g, \widehat{g}, \Sigma)\widetilde{\widetilde{\sqcap}}(f, \widehat{f}, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$. Since $\eta \in (g, \widehat{g}, \Sigma)$, then $(g_\eta, \widehat{g}_\eta, \Sigma)\widetilde{\widetilde{\sqsubseteq}}(g, \widehat{g}, \Sigma)$ by Corollary 2.21 (i). Hence,

$(g_\eta, \widehat{g}_\eta, \Sigma)\widetilde{\widetilde{\sqcap}}(f, \widehat{f}, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$.

The converse is obtained by Corollary 2.21 (ii).

(ii) Let $\eta \notin (g, \widehat{g}, \Sigma)$. Then, we have two cases: (1) for all $\ell \in \Sigma$, $\eta \notin g(\ell)$ and (2) for some $\ell, \beta \in \Sigma$, $\eta \notin g(\ell)$ and $\eta \in g(\beta)$. In case (1), we have $(g_\eta, \widehat{g}_\eta, \Sigma)\widetilde{\widetilde{\sqcap}}(g, \widehat{g}, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$. In case (2), $\eta \in g(\beta)$ implies $\eta \notin g^c(\beta)$ for some $\beta \in \Sigma$. Hence, $(g, \widehat{g}, \Sigma)^c$ is a BS closed set such that $\eta \notin (g, \widehat{g}, \Sigma)^c$, and by (i), $(g_\eta, \widehat{g}_\eta, \Sigma)\widetilde{\widetilde{\sqcap}}(g, \widehat{g}, \Sigma)^c = (\widehat{\Phi}, \widehat{g}, \Sigma)$. So $(g_\eta, \widehat{g}_\eta, \Sigma)\widetilde{\widetilde{\sqsubseteq}}(g, \widehat{g}, \Sigma)$ but this is contradiction. Hence, $(g_\eta, \widehat{g}_\eta, \Sigma)\widetilde{\widetilde{\sqcap}}(g, \widehat{g}, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$. The converse is obvious. □

Proposition 4.4. *Let* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *be a BSTS over* $\aleph$ *and* $\eta \in \aleph$. *Then, the following are equivalent*:

(*i*) $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *is a BS regular space.*

(*ii*) *For each BS closed set* $(f, \widehat{f}, \Sigma)$ *with* $(g_\eta, \widehat{g}_\eta, \Sigma)\widetilde{\widetilde{\sqcap}}(f, \widehat{f}, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$, *there are BS open sets* $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ *such that* $(g_\eta, \widehat{g}_\eta, \Sigma)\widetilde{\widetilde{\sqsubseteq}}(g_1, \widehat{g}_1, \Sigma)$ and $(f, \widehat{f}, \Sigma)\widetilde{\widetilde{\sqsubseteq}}(g_2, \widehat{g}_2, \Sigma)$ *and* $(g_1, \widehat{g}_1, \Sigma)\widetilde{\widetilde{\sqcap}}(g_2, \widehat{g}_2, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$.

Proof. Follows from Proposition 4.3 (i) and Corollary 2.21 (i). □

Proposition 4.5. *Let* $(g, \widehat{g}, \Sigma)$ *be a BS open subset of* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *and* $\eta \in \aleph$. *If* $\aleph$ *be a BS regular space, then* $\eta \in (g, \widehat{g}, \Sigma)$ *if and only if* $\eta \in g(\ell)$ *for some* $\ell \in \Sigma$.

Proof. Suppose that $\eta \in g(\ell)$ for some $\ell \in \Sigma$, and $\eta \notin (g, \widehat{g}, \Sigma)$. Then, $(g_\eta, \widehat{g}_\eta, \Sigma)\widetilde{\widetilde{\sqcap}}(g, \widehat{g}, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$ by Proposition 4.3 (ii). But this contradicts our assumption and so $\eta \in (g, \widehat{g}, \Sigma)$. The converse is obvious. □

Proposition 4.6. *If* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *be a BS regular space, then the following are equivalent*:

(*i*) $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *is a BS* $\widetilde{\widetilde{T}}_1$*-space.*

(*ii*) *For* $\eta \neq \kappa \in \aleph$, *there are BS open sets* $(g_1, \widehat{g}_1, \Sigma)$ *and* $(g_2, \widehat{g}_2, \Sigma)$ *such that* $(g_\eta, \widehat{g}_\eta, \Sigma)\widetilde{\widetilde{\sqsubseteq}}(g_1, \widehat{g}_1, \Sigma)$ *and* $(g_\kappa, \widehat{g}_\kappa, \Sigma)\widetilde{\widetilde{\sqcap}}(g_1, \widehat{g}_1, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$, *and* $(g_\kappa, \widehat{g}_\kappa, \Sigma)\widetilde{\widetilde{\sqsubseteq}}(g_2, \widehat{g}_2, \Sigma)$ *and* $(g_\eta, \widehat{g}_\eta, \Sigma)\widetilde{\widetilde{\sqcap}}(g_2, \widehat{g}_2, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$.

Proof. $\eta \in (g, \widehat{g}, \Sigma)$ if and only if $(g_\eta, \widehat{g}_\eta, \Sigma)\widetilde{\widetilde{\sqsubseteq}}(g, \widehat{g}, \Sigma)$ and $\eta \notin (g, \widehat{g}, \Sigma)$ by Proposition 4.3 (ii) if and only if $(g_\eta, \widehat{g}_\eta, \Sigma)\widetilde{\widetilde{\sqcap}}(g, \widehat{g}, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$. Hence, the aforementioned statements are equivalent. □

Definition 4.7. A BSTS $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is said to be BS $\widetilde{\widetilde{T}}_3$-space if it is BS regular and BS $\widetilde{\widetilde{T}}_1$-space.

Proposition 4.8. *If* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *be a BS* $\widetilde{\widetilde{T}}_3$*-space, then* $(\aleph, T_{\mathcal{S}}, \Sigma)$ *is a soft* $\widetilde{T}_3$*-space.*

Proof. Since $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BS $\widetilde{\widetilde{T}}_3$-space, then it is BS $\widetilde{\widetilde{T}}_1$-space. By Proposition 3.17, $(\aleph, T_{\mathcal{S}}, \Sigma)$ is soft $\widetilde{T}_1$-space. Now, let $(f, \widehat{f}, \Sigma)$ be a BS closed set such that $\eta \notin (f, \widehat{f}, \Sigma)$. This implies $\eta \notin f(\ell)$ for some $\ell \in \Sigma$, and hence, $\eta \notin (f, \Sigma)$ as a soft closed set. As $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is BS regular, then there are BS open sets $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ such that $\eta \in (g_1, \widehat{g}_1, \Sigma)$ and $(f, \widehat{f}, \Sigma)\widetilde{\widetilde{\sqsubseteq}}(g_2, \widehat{g}_2, \Sigma)$ with $(g_1, \widehat{g}_1, \Sigma)\widetilde{\widetilde{\sqcap}}(g_2, \widehat{g}_2, \Sigma) = (\widehat{\Phi}, \widehat{g}, \Sigma)$. Then, we have $\eta \in g_1(\ell)$ and $f(\ell) \subseteq g_2(\ell)$ with $g_1(\ell) \cap g_2(\ell) = \varnothing$ for all $\ell \in \Sigma$. It follows that for a soft closed set (f, Σ) with $\eta \notin (f, \Sigma)$, we have $\eta \in (g_1, \Sigma)$ and $(f, \Sigma)\widetilde{\sqsubseteq}(g_2, \Sigma)$ with $(g_1, \Sigma)\widetilde{\sqcap}(g_2, \Sigma) = (\widehat{\Phi}, \Sigma)$. Thus, $(\aleph, T_{\mathcal{S}}, \Sigma)$ is soft regular, and hence, $(\aleph, T_{\mathcal{S}}, \Sigma)$ is soft $\widetilde{T}_3$-space. □

Proposition 4.9. *Let* $(\aleph, T_{\mathcal{S}}, \Sigma)$ *be an STS and let* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *be a BSTS constructed from* $(\aleph, T_{\mathcal{S}}, \Sigma)$ *as in Proposition* 2.19. *If* $(\aleph, T_{\mathcal{S}}, \Sigma)$ *be a soft* $\widetilde{T}_3$*-space, then* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *is a BS* $\widetilde{\widetilde{T}}_3$*-space.*

Proof. Since $(\aleph, T_{\mathcal{S}}, \Sigma)$ be a soft $\widetilde{T}_3$-space, then it is soft $\widetilde{T}_1$-space. By Proposition 3.22, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is BS $\widetilde{\widetilde{T}}_1$-space. Now, let (f, Σ) be a soft closed set such that $\eta \notin (f, \Sigma)$. This implies that $\eta \notin f(\ell)$ for some $\ell \in \Sigma$, and hence, $\eta \notin (f, \widehat{f}, \Sigma)$ as a BS closed set. As $(\aleph, T_{\mathcal{S}}, \Sigma)$ is soft regular, then there are soft open sets (g_1, Σ)

and (g_2, Σ) such that $\eta \tilde{\in} (g_1, \Sigma)$ and $(f, \Sigma) \tilde{\sqsubseteq} (g_2, \Sigma)$ with $(g_1, \Sigma) \tilde{\sqcap} (g_2, \Sigma) = (\widehat{\Phi}, \Sigma)$. Then, $\eta \in g_1(\ell)$ and $f(\ell) \subseteq g_2(\ell)$ with $g_1(\ell) \cap g_2(\ell) = \varnothing$ for all $\ell \in \Sigma$. In addition, we have $\eta \notin \hat{g}_1(\neg\ell)$ and $\hat{g}_2(\neg\ell) \subseteq \hat{f}(\neg\ell)$ with $\hat{g}_1(\neg\ell) \cup \hat{g}_2(\neg\ell) = \aleph$ for all $\ell \in \Sigma$. It follows that for a BS closed set $(f, \hat{f}, \Sigma)$ with $\eta \notin (f, \hat{f}, \Sigma)$, we have $\eta \tilde{\in} (g_1, \hat{g}_1, \Sigma)$ and $(f, \hat{f}, \Sigma) \tilde{\tilde{\sqsubseteq}} (g_2, \hat{g}_2, \Sigma)$ with $(g_1, \hat{g}_1, \Sigma) \tilde{\tilde{\sqcap}} (g_2, \hat{g}_2, \Sigma) = (\widehat{\Phi}, \widehat{\aleph}, \Sigma)$. Thus, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is BS regular, and hence, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is BS $\tilde{\tilde{T}}_3$-space. □

Proposition 4.10. *Let* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *be a BSTS and* $\Upsilon \subseteq \aleph$. *If* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *be a BS* $\tilde{\tilde{T}}_3$*-space, then* $(\Upsilon, T_{\mathcal{BS}_\Upsilon}, \Sigma, \neg\Sigma)$ *is a BS* $\tilde{\tilde{T}}_3$*-space.*

Proof. Since $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BS $\tilde{\tilde{T}}_3$-space, then it is BS $\tilde{\tilde{T}}_1$-space. By Proposition 3.23, $(\Upsilon, T_{\mathcal{BS}_\Upsilon}, \Sigma, \neg\Sigma)$ is a BS $\tilde{\tilde{T}}_1$-space. Let $\eta \in \Upsilon$ and let $(f, \hat{f}, \Sigma)$ be a BS closed set in Υ with $\eta \tilde{\notin} (f, \hat{f}, \Sigma)$. Then, $\eta \notin f(\ell)$ for some $\ell \in \Sigma$. Since $(f, \hat{f}, \Sigma)$ be a BS closed set in Υ, then there is a BS closed set $(h, \hat{h}, \Sigma)$ in $\aleph$ such that $f(\ell) = h(\ell) \cap \Upsilon$ and $\hat{f}(\neg\ell) = \hat{h}(\neg\ell) \cap \Upsilon$. Since $\eta \notin f(\ell)$ for some $\ell \in \Sigma$, then $\eta \notin h(\ell) \cap \Upsilon = f(\ell)$, and hence, $\eta \notin (h, \hat{h}, \Sigma)$. As $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is BS regular space, there are BS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ such that $\eta \in (g_1, \hat{g}_1, \Sigma)$ and $(h, \hat{h}, \Sigma) \tilde{\tilde{\sqsubseteq}} (g_2, \hat{g}_2, \Sigma)$ with $(g_1, \hat{g}_1, \Sigma) \tilde{\tilde{\sqcap}} (g_2, \hat{g}_2, \Sigma) = (\widehat{\Phi}, \hat{g}, \Sigma)$. Now, if we take $(g_{1_\Upsilon}, \hat{g}_{1_\Upsilon}, \Sigma)$ and $(g_{2_\Upsilon}, \hat{g}_{2_\Upsilon}, \Sigma)$ as two BS open sets in Υ, then $g_{1_\Upsilon}(\ell) = g_1(\ell) \cap \Upsilon$, $\hat{g}_{1_\Upsilon}(\neg\ell) = \hat{g}_1(\neg\ell) \cap \Upsilon$, and $g_{2_\Upsilon}(\ell) = g_2(\ell) \cap \Upsilon$, $\hat{g}_{2_\Upsilon}(\neg\ell) = \hat{g}_2(\neg\ell) \cap \Upsilon$. This implies $\eta \in (g_{1_\Upsilon}, \hat{g}_{1_\Upsilon}, \Sigma)$ and $(f, \hat{f}, \Sigma) \tilde{\tilde{\sqsubseteq}} (g_{2_\Upsilon}, \hat{g}_{2_\Upsilon}, \Sigma)$ with $(g_{1_\Upsilon}, \hat{g}_{1_\Upsilon}, \Sigma) \tilde{\tilde{\sqcap}} (g_{2_\Upsilon}, \hat{g}_{2_\Upsilon}, \Sigma) = (\widehat{\Phi}, \hat{g}, \Sigma)$. Thus, $(\Upsilon, T_{\mathcal{BS}_\Upsilon}, \Sigma, \neg\Sigma)$ is a BS regular space, and hence, $(\Upsilon, T_{\mathcal{BS}_\Upsilon}, \Sigma, \neg\Sigma)$ is a BS $\tilde{\tilde{T}}_3$-space. □

Definition 4.11. A BSTS $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is said to be BS normal space if for every BS closed sets $(f_1, \hat{f}_1, \Sigma)$ and $(f_2, \hat{f}_2, \Sigma)$ with $(f_1, \hat{f}_1, \Sigma) \tilde{\tilde{\sqcap}} (f_2, \hat{f}_2, \Sigma) = (\widehat{\Phi}, \hat{g}, \Sigma)$, there are BS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ such that $(f_1, \hat{f}_1, \Sigma) \tilde{\tilde{\sqsubseteq}} (g_1, \hat{g}_1, \Sigma)$ and $(f_2, \hat{f}_2, \Sigma) \tilde{\tilde{\sqsubseteq}} (g_2, \hat{g}_2, \Sigma)$ with $(g_1, \hat{g}_1, \Sigma) \tilde{\tilde{\sqcap}} (g_2, \hat{g}_2, \Sigma) = (\widehat{\Phi}, \hat{g}, \Sigma)$.

Proposition 4.12. *Let* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *be a BSTS. If* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *be a BS normal space and if* $(g_\eta, \hat{g}_\eta, \Sigma)$ *be a BS closed set for each* $\eta \in \aleph$, *then* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *is a BS* $\tilde{\tilde{T}}_3$*-space.*

Proof. Since $(g_\eta, \hat{g}_\eta, \Sigma)$ is a BS closed set for each $\eta \in \aleph$, then by Proposition 3.10, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\tilde{\tilde{T}}_1$-space. It is also BS regular space by Propositions 4.4 and 4.11. Hence, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\tilde{\tilde{T}}_3$-space. □

Definition 4.13. A BSTS $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is said to be BS $\tilde{\tilde{T}}_4$-space if it is BS normal and BS $\tilde{\tilde{T}}_1$-space.

Proposition 4.14. *If* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *be a BS* $\tilde{\tilde{T}}_4$*-space, then* $(\aleph, T_{\mathcal{S}}, \Sigma)$ *is a soft* $\tilde{T}_4$*-space.*

Proof. Since $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BS $\tilde{\tilde{T}}_4$-space, then it is BS $\tilde{\tilde{T}}_1$-space, and hence, $(\aleph, T_{\mathcal{S}}, \Sigma)$ is soft $\tilde{T}_1$-space by Proposition 3.17. Furthermore, since $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ be a BS normal space, then for every BS closed sets $(f_1, \hat{f}_1, \Sigma)$ and $(f_2, \hat{f}_2, \Sigma)$ with $(f_1, \hat{f}_1, \Sigma) \tilde{\tilde{\sqcap}} (f_2, \hat{f}_2, \Sigma) = (\widehat{\Phi}, \hat{g}, \Sigma)$, there are BS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ such that $(f_1, \hat{f}_1, \Sigma) \tilde{\tilde{\sqsubseteq}} (g_1, \hat{g}_1, \Sigma)$ and $(f_2, \hat{f}_2, \Sigma) \tilde{\tilde{\sqsubseteq}} (g_2, \hat{g}_2, \Sigma)$ with $(g_1, \hat{g}_2, \Sigma) \tilde{\tilde{\sqcap}} (g_2, g_2, \Sigma) = (\widehat{\Phi}, \hat{g}, \Sigma)$. This implies that, for all $\ell \in \Sigma$, $f_1(\ell) \cap f_2(\ell) = \varnothing$ and $f_1(\ell) \subseteq g_1(\ell)$, $f_2(\ell) \subseteq g_2(\ell)$ with $g_1(\ell) \cap g_2(\ell) = \varnothing$. Then, for two soft closed sets (f_1, Σ) and (f_2, Σ) with $(f_1, \Sigma) \tilde{\sqcap} (f_2, \Sigma) = (\widehat{\Phi}, \Sigma)$, there are soft open sets (g_1, Σ) and (g_2, Σ) such that $(f_1, \Sigma) \tilde{\sqsubseteq} (g_1, \Sigma)$ and $(f_2, \Sigma) \tilde{\sqsubseteq} (g_2, \Sigma)$ with $(g_1, \Sigma) \tilde{\sqcap} (g_2, \Sigma) = (\widehat{\Phi}, \Sigma)$. Thus, $(\aleph, T_{\mathcal{S}}, \Sigma)$ is soft normal, and hence, $(\aleph, T_{\mathcal{S}}, \Sigma)$ is soft $\tilde{T}_4$-space. □

Proposition 4.15. *Let* $(\aleph, T_{\mathcal{S}}, \Sigma)$ *be an STS and let* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *be a BSTS constructed from* $(\aleph, T_{\mathcal{S}}, \Sigma)$ *as in* Proposition 2.19. *If* $(\aleph, T_{\mathcal{S}}, \Sigma)$ *be a soft* $\tilde{T}_4$*-space over* $\aleph$, *then* $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ *is a BS* $\tilde{\tilde{T}}_4$*-space over* $\aleph$.

Proof. Since $(\aleph, T_{\mathcal{S}}, \Sigma)$ be a soft $\widetilde{T}_4$-space, then it is soft $\widetilde{T}_1$-space, and so, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is a BS $\widetilde{\widetilde{T}}_1$-space by Proposition 3.22. Furthermore, since $(\aleph, T_{\mathcal{S}}, \Sigma)$ be a soft normal space, then for every soft closed sets (f_1, Σ) and (f_2, Σ) with $(f_1, \Sigma)\widetilde{\sqcap}(f_2, \Sigma) = (\widehat{\Phi}, \Sigma)$, there are soft open sets (g_1, Σ) and (g_2, Σ) such that $(f_1, \Sigma)\widetilde{\sqsubseteq}(g_1, \Sigma)$ and $(f_2, \Sigma)\widetilde{\sqsubseteq}(g_2, \Sigma)$ with $(g_1, \Sigma)\widetilde{\sqcap}(g_2, \Sigma) = (\widehat{\Phi}, \Sigma)$. This implies that $f_1(\ell)\cap f_2(\ell) = \varnothing$ and $f_1(\ell) \subseteq g_1(\ell)$, $f_2(\ell) \subseteq g_2(\ell)$ with $g_1(\ell)\cap g_2(\ell) = \varnothing$. Then, we have $\widehat{f}_1(\neg\ell)\cup\widehat{f}_2(\neg\ell) = \aleph$ and $\widehat{g}_1(\neg\ell) \subseteq \widehat{f}_1(\neg\ell)$, $\widehat{g}_2(\neg\ell) \subseteq \widehat{f}_2(\neg\ell)$ with $\widehat{g}_1(\neg\ell) \cup\widehat{g}_2(\neg\ell) = \aleph$. It follows that for two BS closed sets $(f_1, \widehat{f}_1, \Sigma)$ and $(f_2, \widehat{f}_2, \Sigma)$ with $(f_1, \widehat{f}_1, \Sigma)\widetilde{\widetilde{\sqcap}}(f_2, \widehat{f}_2, \Sigma) = (\widehat{\Phi}, \widehat{\aleph}, \Sigma)$, there are BS open sets $(g_1, \widehat{g}_1, \Sigma)$ and $(g_2, \widehat{g}_2, \Sigma)$ such that $(f_1, \widehat{f}_1, \Sigma)\widetilde{\widetilde{\sqsubseteq}}(g_1, \widehat{g}_1, \Sigma)$ and $(f_2, \widehat{f}_2, \Sigma)\widetilde{\widetilde{\sqsubseteq}}(g_2, \widehat{g}_2, \Sigma)$ with $(g_1, \widehat{g}_1, \Sigma)\widetilde{\widetilde{\sqcap}}(g_2, \widehat{g}_2, \Sigma) = (\widehat{\Phi}, \widehat{\aleph}, \Sigma)$. Thus, $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is BS normal, and hence, $(\aleph, T_{\mathcal{S}}, \Sigma)$ is BS $\widetilde{\widetilde{T}}_4$-space. □

Remark 4.16. A BS $\widetilde{\widetilde{T}}_4$-space need not be BS $\widetilde{\widetilde{T}}_3$-space.

Example 4.17. Let $\aleph = \{\eta_1, \eta_2\}$, and $\Sigma = \{\ell_1, \ell_2, \ell_3\}$. Let $T_{\mathcal{BS}} = \{(\widehat{\Phi}, \widehat{\aleph}, \Sigma), (\widehat{\aleph}, \widehat{\Phi}, \Sigma), (g_1, \widehat{g}_1, \Sigma), (g_2, \widehat{g}_2, \Sigma), (g_3, \widehat{g}_3, \Sigma), (g_4, \widehat{g}_4, \Sigma), (g_5, \widehat{g}_5, \Sigma), (g_6, \widehat{g}_6, \Sigma), (g_7, \widehat{g}_7, \Sigma)\}$ be a BST defined on $\aleph$, where

$$
\begin{aligned}
(g_1, \widehat{g}_1, \Sigma) &= \{(\ell_1, \{\eta_1\}, \{\eta_2\}), (\ell_2, \{\eta_1\}, \{\eta_2\}), (\ell_3, \{\eta_1\}, \{\eta_2\})\},\\
(g_2, \widehat{g}_2, \Sigma) &= \{(\ell_1, \{\eta_2\}, \{\eta_1\}), (\ell_2, \{\eta_2\}, \{\eta_1\}), (\ell_3, \{\eta_2\}, \{\eta_1\})\},\\
(g_3, \widehat{g}_3, \Sigma) &= \{(\ell_1, \varnothing, \aleph), (\ell_2, \{\eta_1\}, \{\eta_2\}), (\ell_3, \{\eta_1\}, \{\eta_2\})\},\\
(g_4, \widehat{g}_4, \Sigma) &= \{(\ell_1, \varnothing, \aleph), (\ell_2, \{\eta_2\}, \{\eta_1\}), (\ell_3, \{\eta_2\}, \{\eta_1\})\},\\
(g_5, \widehat{g}_5, \Sigma) &= \{(\ell_1, \{\eta_1\}, \{\eta_2\}), (\ell_2, \aleph, \varnothing), (\ell_3, \aleph, \varnothing)\},\\
(g_6, \widehat{g}_6, \Sigma) &= \{(\ell_1, \{\eta_2\}, \{\eta_1\}), (\ell_2, \aleph, \varnothing), (\ell_3, \aleph, \varnothing)\}, \quad \text{and}\\
(g_7, \widehat{g}_7, \Sigma) &= \{(\ell_1, \varnothing, \aleph), (\ell_2, \aleph, \varnothing), (\ell_3, \aleph, \varnothing)\}.
\end{aligned}
$$

Then, it is easy to see that $(\aleph, T_{\mathcal{BS}}, \Sigma, \neg\Sigma)$ is BS $\widetilde{\widetilde{T}}_4$-space but not BS $\widetilde{\widetilde{T}}_3$-space.

5 Conclusion

This study is dedicated to introducing new BS separation axioms called BS $\widetilde{\widetilde{T}}_i$-space ($i = 0, 1, 2, 3, 4$). This type is formulated with respect to ordinary points. We investigated the relationship between these spaces and their counterparts in both soft topology and classical topology. We also studied how these new spaces behave in terms of BS subspace. In the literature (see, for example, [33,35]), there are different types of belong and non-belong relations between ordinary points and BS sets, so some other types of BS separation axioms can be studied.

Conflict of interest: The authors state that there is no conflict of interest.

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