Research Article

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Asymptotic behavior of resolvents of equilibrium problems on complete geodesic spaces

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Abstract: In this article, we discuss equilibrium problems and their resolvents on complete geodesic spaces. In particular, we consider asymptotic behavior and continuity of resolvents with positive parameter in a complete geodesic space whose curvature is bounded above. Furthermore, we apply these results to resolvents of convex functions.

Keywords: asymptotic behavior, geodesic space, convex function, resolvent

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1 Introduction

The geodesic spaces are a class of metric spaces with some convex structures and have been actively researched in recent years. A complete CAT(0) space is an example of such spaces and it is a generalization of a Hilbert space. A complete CAT(1) space is also a geodesic space and can be seen locally as a generalization of a complete CAT(0) space. Many researchers have been tried to apply theories of linear spaces such as Hilbert and Banach spaces to these geodesic spaces. The concept of resolvent is one of them. For example, we can define various types of resolvent maps for a convex function on complete CAT(0) spaces and complete CAT(1) spaces. In addition, it is defined for a bifunction of the equilibrium problems to be introduced later. The problem of asymptotic behavior of resolvents is one of the important subjects to know the property of resolvents. We describe famous results about asymptotic behavior of a resolvent for a convex function in a Hilbert space below.

Theorem 1.1. (See [1]) *Let X be a Hilbert space, $g : X \to]-\infty, \infty]$ a proper lower semicontinuous convex function, and $x \in X$. Define x_λ by*

$$x_\lambda = J_{\lambda g}x = \operatorname*{argmin}_{y\in X}\{\lambda g(y) + \|y - x\|^2\}$$

for a positive number λ. If there exists $\{\mu_n\} \subset]0, \infty[$ such that $\mu_n \to \infty$ and $\{d(x_{\mu_n}, x)\}$ is bounded, then $\operatorname{argmin} g \neq \emptyset$ *and*

$$\lim_{\lambda\to\infty} x_\lambda = P_{\operatorname{argmin} g}x,$$

where P_K is a metric projection onto K for a nonempty closed convex subset K of X.

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Theorem 1.2. (See [1]) *Let X be a Hilbert space, $g : X \to]-\infty, \infty]$ a proper lower semicontinuous convex function, and $x \in X$. Define x_λ by*

$$x_\lambda = J_{\lambda g}x = \operatorname*{argmin}_{y \in X}\{\lambda g(y) + d(x, y)^2\}$$

for a positive number λ. Then

$$\lim_{\lambda \to 0} x_\lambda = P_{\overline{\mathrm{dom} g}}x.$$

Furthermore, the convergence of resolvents of maximal monotone operators has been considered in many papers. For example, see [2–8].

The equilibrium problem is an important problem which contains optimization problems, complementarity problems, fixed point problems, variational inequalities, and Nash equilibria. Let K be a set and $f : K \times K \to \mathbb{R}$ a bifunction. An equilibrium problem for a bifunction $f : K \times K \to \mathbb{R}$ is a problem of finding $z \in K$ such that $f(z, y) \geq 0$ for all $y \in K$. Resolvents of equilibrium problems are considered to understand this problem and to find their solutions. For example, for equilibrium problems in CAT(0) spaces, a resolvent is defined and studied in [9]. In CAT (1) spaces, it is studied in [10].

In this article, we study properties of resolvents of equilibrium problems with positive parameter λ. In Section 2, we introduce basic concepts. In Section 3, we consider asymptotic behavior and continuity of a resolvent on a complete CAT(0) space for λ. In Section 4, we study similar properties of resolvent on a complete CAT(1) space. In Section 5, we discuss the relationship between resolvent of a convex function and that of equilibrium problems.

2 Preliminaries

Let (X, d) be a metric space and $x, y \in X$. A mapping $c_{xy} : [0, l] \to X$ is called a geodesic with endpoints x, y if it satisfies

$$c(0) = x, \quad c(l) = y, \quad \text{and} \quad |c(u) - c(v)| = d(u, v) \quad \text{for all } u, v \in [0, l],$$

where l is a distance between x and y. X is called a geodesic space if there exists a geodesic c_{xy} for any two points $x, y \in X$. Furthermore, X is called a uniquely geodesic space if a geodesic exists uniquely for each two points. Let X be a uniquely geodesic space. Then the geodesic segment joining x and y is defined as the image of the geodesic c_{xy} and denoted by $[x, y]$. That is, $[x, y] = c_{xy}([0, l])$. Also, we define convex combination $tx \oplus (1 - t)y$ for $t \in [0, 1]$ as follows:

$$tx \oplus (1 - t)y = c_{xy}((1 - t)l).$$

Let $x_1, x_2, x_3 \in X$. A geodesic triangle $\triangle(x_1, x_2, x_3)$ is defined by

$$\triangle(x_1, x_2, x_3) = [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1].$$

For this triangle, a comparison triangle $\overline{\triangle}(\overline{x}_1, \overline{x}_2, \overline{x}_3) \subset \mathbb{E}^2$ is a triangle with the same side length as $\triangle(x_1, x_2, x_3)$ in the two-dimensional Euclidian space. Let p be a point on $[x_i, x_j](i, j \in \{1, 2, 3\}, i \neq j)$. For p, a comparison point $\overline{p}$ is a point on $[\overline{x}_i, \overline{x}_j]$ with $d(\overline{p}, \overline{x}_i)_{\mathbb{E}^2} = d(p, x_i)$. If for any $x_1, x_2, x_3 \in X$, $p, q \in \triangle(x_1, x_2, x_3)$, and their comparison points $\overline{p}, \overline{q} \in \overline{\triangle}(\overline{x}_1, \overline{x}_2, \overline{x}_3)$, it holds that

$$d(\overline{p}, \overline{q})_{\mathbb{E}^2} \leq d(p, q),$$

then X is called a CAT(0) space. We know the following equivalence condition. A uniquely geodesic space X is a CAT(0) space if and only if it holds that

$$d(tx \oplus (1 - t)y, z)^2 \leq td(x, z)^2 + (1 - t)d(y, z)^2 - t(1 - t)d(x, y)^2$$

for all $x, y, z \in X$ and $t \in [0, 1]$. This inequality is called a parallelogram law in CAT(0) space.

Next, we define CAT(1) space. Let X be a uniquely geodesic space and $x_1, x_2, x_3 \in X$ with $d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1) < 2\pi$. We consider a geodesic triangle $\triangle(x_1, x_2, x_3)$, its comparison triangle $\overline{\triangle}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the two-dimensional unit sphere $\mathbb{S}^2$, and a comparison point in a similar way. If for any $x_1, x_2, x_3 \in X$ with $d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1) < 2\pi$, $p, q \in \triangle(x_1, x_2, x_3)$, and their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, it holds that

$$d(\bar{p}, \bar{q})_{\mathbb{S}^2} \le d(p, q),$$

then X is called a CAT(1) space. We also know that a uniquely geodesic space X is CAT(1) space if and only if for all $x, y, z \in X$ with $d(x, y) + d(y, z) + d(z, x) < 2\pi$ and $t \in [0, 1]$, it holds that

$$\cos d(tx \oplus (1-t)y, z) \sin d(x, y) \ge \cos d(x, z) \sin t d(x, y) + \cos d(y, z) \sin(1-t) d(x, y),$$

which is called a parallelogram law in CAT(1) space. From the above, considering the case that $t = 1/2$, we have

$$\cos d\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \cos \frac{d(x, y)}{2} \ge \frac{1}{2} \cos d(x, z) + \frac{1}{2} \cos d(y, z)$$

if X is a CAT(1) space. This inequality is equivalent to

$$-\log \cos d\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \le -\frac{1}{2} \log \cos d(x, z) - \frac{1}{2} \log \cos d(y, z) + \log \cos \frac{d(x, y)}{2}.$$

A CAT(1) space X is said to be admissible if

$$d(x, y) < \frac{\pi}{2}$$

for all $x, y \in X$.

Let X be a uniquely geodesic space. A function $g : X \to \mathbb{R}$ is said to be convex if

$$g(tx \oplus (1-t)y) \le t g(x) + (1-t) g(y)$$

for all $x, y \in X$ and $t \in [0, 1]$. Furthermore, $g : X \to \mathbb{R}$ is said to be upper hemicontinuous if

$$\limsup_{t \to 0} g(tx \oplus (1-t)y) \le g(y)$$

for all $x, y \in X$. We say a subset C of X is convex if $tx \oplus (1-t)y \in C$ whenever $x, y \in C$. For a subset A of X, the convex hull $\operatorname{co} A$ of A is defined by

$$\operatorname{co} A = \bigcup_{i=1}^{\infty} A_i,$$

where $A_1 = A$ and $A_{i+1} = \{tx \oplus (1-t)y \mid x, y \in A_i, t \in [0, 1]\}$ for $i \in \mathbb{N}$. We say X has the convex hull finite property if for every finite subset A of X, continuous self-mapping T on $\overline{\operatorname{co} A}$ has a fixed point. For the example of geodesic spaces satisfying this property, see [10]. Let X be a complete CAT(0) space or a complete admissible CAT(1) space, C a nonempty closed convex subset of X, and $x \in X$. Then, there exists a unique nearest point $p \in C$, which satisfies

$$d(x, p) \le d(x, y)$$

for all $y \in C$. Therefore, for a closed convex subset C of X, we can define the metric projection P_C onto C by

$$P_C x = \operatorname*{argmin}_{y \in C} d(x, y)$$

for $x \in X$. The metric projection P_C has the following properties:

- in a complete CAT(0) space

$$d(x, P_C x)^2 + d(P_C x, y)^2 \le d(x, y)^2$$

  for all $x \in X$ and $y \in C$;

- in a complete admissible CAT(1) space,

$$\cos d(x, P_C x) \cos d(P_C x, y) \leq \cos d(x, y)$$

for all $x \in X$ and $y \in C$.

Let X be a complete CAT(0) space or complete admissible CAT(1) space and K a nonempty closed convex subset of X. We consider an equilibrium problem for a bifunction $f : K \times K \to \mathbb{R}$. We denote the set of solutions of an equilibrium problem for f by $\mathrm{Equil} f$. That is,

$$\mathrm{Equil} f = \left\{ z \in K \,\middle|\, \inf_{y \in K} f(z, y) \geq 0 \right\}.$$

In what follows, we suppose f satisfies the following four conditions:
(E1) $f(x, x) = 0$ for all $x \in K$;
(E2) $f(x, y) + f(y, x) \leq 0$ for all $x, y \in K$;
(E3) $f(x, \cdot) : K \to \mathbb{R}$ is convex and lower semicontinuous for all $x \in K$;
(E4) $f(\cdot, x) : K \to \mathbb{R}$ is upper hemicontinuous for all $x \in K$.

Minimization problem is an example of the equilibrium problems. Let $g : K \to \mathbb{R}$ be a convex lower semicontinuous function. Define $f : K \times K \to \mathbb{R}$ by

$$f(z, y) = g(y) - g(z)$$

for all $y, z \in K$. Then f satisfies (E1)–(E4) and the solutions of the equilibrium problem for f coincide with the solutions of the minimization problem for g, and hence

$$\mathrm{Equil} f = \operatorname{argmin} g.$$

Let X be a complete CAT(0) space having the convex hull finite property, K a nonempty closed convex subset of X, and f a bifunction $K \times K \to \mathbb{R}$ satisfying (E1)–(E4). A resolvent Q_f of an equilibrium problem for f is defined by

$$Q_f x = \left\{ z \in K \,\middle|\, \inf_{y \in K} (f(z, y) + d(y, x)^2 - d(z, x)^2) \geq 0 \right\}$$

for $x \in X$. Then $Q_f x$ is a singleton; see [9]. Hence, we can consider Q_f a single-valued mapping. We know that Q_f is nonexpansive and the set of fixed points coincides with the solution set of the equilibrium problem. Therefore, approximation methods of a fixed point of nonexpansive mapping can be diverted to approximate a solution to the original problem; for instance, see [11–15].

Let X be a complete admissible CAT(1) space having the convex hull finite property, K a nonempty closed convex subset of X, and f a bifunction $K \times K \to \mathbb{R}$ satisfying (E1)–(E4). A resolvent R_f of equilibrium problems for f is defined by

$$R_f x = \left\{ z \in K \,\middle|\, \inf_{y \in K} (f(z, y) - \log \cos d(y, x) + \log \cos d(z, x)) \geq 0 \right\}$$

for $x \in X$. Similarly, we know $R_f x$ is to be singleton (see [10]), and R_f can be considered as a single-valued mapping.

3 Resolvent on CAT(0) space

In a complete CAT(0) space, we consider equilibrium problems for a bifunction and its resolvent. Let X be a complete CAT(0) space having the convex hull finite property and f a bifunction on K satisfying (E1)–(E4). For a positive number λ, we define a resolvent $Q_{\lambda f}$ as follows:

$$Q_{\lambda f}x = \left\{ z \in K \,|\, \inf_{y \in K}(\lambda f(z, y) + (d(y, x)^2 - d(z, x)^2)) \geq 0 \right\}$$

for $x \in X$. We consider the asymptotic behavior and continuity of this resolvent $J_{\lambda f}x$ for λ. First we introduce useful lemmas to discuss the resolvent with parameter λ.

Lemma 3.1. *Let X be a complete* CAT(0) *space having the convex hull finite property, K a nonempty closed convex subset of X, and $f : K \times K \to \mathbb{R}$ a bifunction satisfying* (E1)–(E4). *Let $x \in X$ and $\lambda, \mu \in \mathbb{R}$ with $0 \leq \lambda \leq \mu$. Then*

$$d(Q_{\lambda f}x, x) \leq d(Q_{\mu f}x, x), \quad f(Q_{\mu f}x, Q_{\lambda f}x) \geq 0, \quad \text{and } f(Q_{\lambda f}x, Q_{\mu f}x) \leq 0.$$

Proof. If $\lambda = \mu$, it is obvious. Suppose $\lambda < \mu$. Put $x_\lambda = Q_{\lambda f}x$ and $x_\mu = Q_{\mu f}x$. From the definition of the resolvent, we have

$$0 \leq f(x_\lambda, x_\mu) + \frac{1}{\lambda}\{d(x_\mu, x)^2 - d(x_\lambda, x)^2\}, \tag{1}$$

$$0 \leq f(x_\mu, x_\lambda) + \frac{1}{\mu}\{d(x_\lambda, x)^2 - d(x_\mu, x)^2\}. \tag{2}$$

From these inequalities and (E2), we obtain

$$0 \leq f(x_\lambda, x_\mu) + f(x_\mu, x_\lambda) + \left(\frac{1}{\lambda} - \frac{1}{\mu}\right)\{d(x_\mu, x)^2 - d(x_\lambda, x)^2\} \leq 0 + \left(\frac{1}{\lambda} - \frac{1}{\mu}\right)\{d(x_\mu, x)^2 - d(x_\lambda, x)^2\}.$$

Since $1/\lambda > 1/\mu$, we obtain

$$d(x_\lambda, x) \leq d(x_\mu, x).$$

From (1) and (E2), we have

$$0 \leq f(x_\mu, x_\lambda) + \frac{1}{\mu}\{d(x_\lambda, x)^2 - d(x_\mu, x)^2\} \leq f(x_\mu, x_\lambda) \leq -f(x_\lambda, x_\mu).$$

It completes the proof. □

Lemma 3.2. *Let X be a complete* CAT(0) *space having the convex hull finite property, K a nonempty closed convex subset of X, and $f : K \times K \to \mathbb{R}$ a bifunction satisfying* (E1)–(E4). *Let $x \in X$, $\lambda, \mu \in \mathbb{R}$ with $0 < \lambda \leq \mu$, and $t \in [0, 1]$. Then*

$$d(Q_{\lambda f}x, x) \leq d(tQ_{\lambda f}x + (1 - t)Q_{\mu f}x, x) \leq d(Q_{\mu f}x, x).$$

Proof. Put $x_\lambda = Q_{\lambda f}x$ and $x_\mu = Q_{\mu f}x$. Since K is convex, we have $tx_\lambda \oplus (1 - t)x_\mu \in K$ for $t \in [0, 1]$. From (E3), (E1), and Lemma 3.1, we have

$$\begin{aligned} 0 &\leq \lambda f(x_\lambda, tx_\lambda \oplus (1 - t)x_\mu) + d(tx_\lambda \oplus (1 - t)x_\mu, x)^2 - d(x_\lambda, x)^2 \\ &\leq \lambda t f(x_\lambda, x_\lambda) + \lambda(1 - t)f(x_\lambda, x_\mu) + d(tx_\lambda \oplus (1 - t)x_\mu, x)^2 - d(x_\lambda, x)^2 \\ &\leq d(tx_\lambda \oplus (1 - t)x_\mu, x)^2 - d(x_\lambda, x)^2. \end{aligned}$$

Therefore, we obtain

$$d(x_\lambda, x) \leq d(tx_\lambda \oplus (1 - t)x_\mu, x) \leq td(x_\lambda, x) + (1 - t)d(x_\mu, x) \leq d(x_\mu, x).$$

This is the desired result. □

Lemma 3.3. *Let X be a complete* CAT(0) *space having the convex hull finite property, K a nonempty closed convex subset of X, and $f : K \times K \to \mathbb{R}$ a bifunction satisfying* (E1)–(E4). *Let $x \in X$ and $\lambda, \mu \in \mathbb{R}$ with $0 < \lambda \leq \mu$. Then*

$$d(Q_{\lambda f}x, Q_{\mu f}x)^2 \le d(Q_{\mu f}x, x)^2 - d(Q_{\lambda f}x, x)^2.$$

Proof. Put $x_\lambda = Q_\lambda x$ and $x_\mu = Q_\mu x$. By Lemma 3.2 and the parallelogram law, we have

$$d(x_\lambda, x)^2 \le d(tx_\lambda \oplus (1-t)x_\mu, x)^2 \le td(x_\lambda, x)^2 + (1-t)d(x_\mu, x)^2 - t(1-t)d(x_\lambda, x_\mu)^2,$$

and hence

$$t(1-t)d(x_\lambda, x_\mu)^2 \le (1-t)d(x_\mu, x)^2 - (1-t)d(x_\lambda, x)^2.$$

Dividing by $1-t$ and letting $t \to 1$, we obtain

$$d(x_\lambda, x_\mu)^2 \le d(x_\mu, x)^2 - d(x_\lambda, x)^2,$$

which is the desired result. □

From these lemmas, we consider the asymptotic behavior of resolvent at $\lambda \to \infty$.

Theorem 3.4. *Let X be a complete* CAT(0) *space having the convex hull finite property, K a nonempty closed convex subset of X, and $f : K \times K \to \mathbb{R}$ a bifunction satisfying* (E1)–(E4). *Let $x \in X$ and define x_λ by*

$$x_\lambda = Q_{\lambda f}x = \left\{ z \in K \,\middle|\, \inf_{y \in K}(\lambda f(z, y) + d(y, x)^2 - d(z, x)^2) \ge 0 \right\}$$

for a positive number λ. If there exists $\{\mu_n\} \subset]0, \infty[$ such that $\mu_n \to \infty$ and $\{d(x_{\mu_n}, x)\}$ is bounded, then $\operatorname{Equil} f \ne \emptyset$ *and*

$$\lim_{\lambda \to \infty} x_\lambda = P_{\operatorname{Equil} f}x,$$

where P_K is a metric projection onto K for a nonempty closed convex subset K of X.

Proof. Let $\{\lambda_n\}$, $\{\mu_n\}$ be positive sequences diverging to ∞. Assume that $\{d(x_{\mu_n}, x)\}$ is bounded and $\{\lambda_n\}$ is increasing. Then $\{d(x_{\lambda_n}, x)\}$ is bounded. In fact, suppose $\{d(x_{\lambda_n}, x)\}$ is not bounded. Since $\{d(x_{\mu_n}, x)\}$ is bounded, there exists $M > 0$ such that

$$d\Big(x_{\mu_n}, x\Big) \le M$$

for all $n \in \mathbb{N}$. On the other hand, there exists a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ such that

$$d\Big(x_{\lambda_{n_i}}, x\Big) > M$$

for all $i \in \mathbb{N}$. Since $\{\mu_n\}$ diverges to ∞, we can find $l \in \mathbb{N}$, such that $\lambda_{n_1} \le \mu_l$. From Lemma 3.2, we obtain

$$M < d\Big(x_{\lambda_{n_1}}, x\Big) \le d\Big(x_{\mu_l}, x\Big) \le M.$$

This is a contradiction. Put $x_n = x_{\lambda_n}$ for $n \in \mathbb{N}$. Let $n, m \in \mathbb{N}$ satisfy $n \le m$. From Lemma 3.3, we have

$$d(x_n, x_m)^2 \le d(x_m, x)^2 - d(x_n, x)^2.$$

By Lemma 3.2, $\{d(x_n, x)\}$ is bounded and increasing. Therefore, $\{x_n\}$ is a Cauchy sequence. Since $\{x_n\} \subset K$ and K is closed, there exists $p \in K$ such that $x_n \to p$. By the definition of x_n, we obtain

$$0 \le f(x_n, y) + \frac{1}{\lambda_n}\{d(y, x)^2 - d(x_n, x)^2\}$$

for all $y \in X$. Since $\lambda_n \to \infty$ as $n \to \infty$, from (E2) and (E3), we have

$$0 \le \limsup_{n \to \infty} f(x_n, y) \le \limsup_{n \to \infty}(-f(y, x_n)) = -\liminf_{n \to \infty} f(y, x_n) \le -f(y, p).$$

That is, $f(y, p) \leq 0$ for all $y \in K$. Let $w \in K$ and $t \in]0, 1[$. Since K is closed, $f(tw \oplus (1-t)p, p) \leq 0$. From (E1) and (E2), we have

$$0 = f(tw \oplus (1-t)p, tw \oplus (1-t)p) \leq tf(tw \oplus (1-t)p, w) + (1-t)f(tw \oplus (1-t)p, p) \leq tf(tw \oplus (1-t)p, w).$$

Dividing by t and letting $t \to 0$, we have $0 \leq f(p, w)$ for all $w \in K$ from (E4). Therefore, we obtain

$$p \in \operatorname{Equil} f \neq \emptyset.$$

For all $z \in \operatorname{Equil} f$, we have

$$0 \leq \lambda_n f(x_n, z) + d(z, x)^2 - d(x_n, x)^2 \leq -\lambda_n f(z, x_n) + d(z, x)^2 - d(x_n, x)^2 \leq d(z, x)^2 - d(x_n, x)^2,$$

and hence $d(x_n, x) \leq d(z, x)$. Letting $n \to \infty$, we obtain

$$d(p, x) \leq \liminf_{n\to\infty} d(x_n, x) \leq d(z, x)$$

for all $z \in \operatorname{Equil} f$. This implies $p = P_{\operatorname{Equil} f} x$. From the above, we know that $x_n \to P_{\operatorname{Equil} f} x$ for every increasing sequence $\{\lambda_n\}$ diverging to infinity. Hence, we obtain

$$\lim_{\lambda\to\infty} x_\lambda = P_{\operatorname{Equil} f} x.$$

This is the desired result. □

Furthermore, we discuss the asymptotic behavior of resolvent when $\lambda \to 0$.

Theorem 3.5. *Let X be a complete* CAT(0) *space having the convex hull finite property, K a nonempty closed convex subset of X, and $f : K \times K \to \mathbb{R}$ a bifunction satisfying* (E1)–(E4)*. Let $x \in X$ and define x_λ by*

$$x_\lambda = Q_{\lambda f} x = \left\{ z \in K \,\middle|\, \inf_{y\in K} (\lambda f(z, y) + d(y, x)^2 - d(z, x)^2) \geq 0 \right\}$$

for a positive number λ. Then

$$\lim_{\lambda\to 0} x_\lambda = P_K x.$$

Proof. Let $\{\lambda_n\}$ be a decreasing positive sequence converging to 0 and put $x_n = x_{\lambda_n}$ for $n \in \mathbb{N}$. By Lemma 3.3,

$$d(x_n, x_m)^2 \leq |d(x_m, x)^2 - d(x_n, x)^2|$$

for $n, m \in \mathbb{N}$. On the other hand, from Lemma 3.2, $\{d(x_n, x)\}$ is decreasing and bounded below. Therefore, $\{x_n\}$ is a Cauchy sequence, and hence $\{x_n\}$ converges to some $p \in K$. By the definition of the resolvent, we have

$$0 \leq \lambda_n f(x_n, y) + d(y, x)^2 - d(x_n, x)^2 \leq -\lambda_n f(y, x_n) + d(y, x)^2 - d(x_n, x)^2.$$

From the lower semicontinuity of f for the second argument, letting $n \to \infty$, we obtain $0 \leq d(y, x)^2 - d(p, x)^2$, and hence

$$d(p, x) \leq d(y, x)$$

for all $y \in X$. This implies $p = P_K x$. Hence, for every decreasing sequence $\{\lambda_n\}$ convergent to 0, $\{x_n\}$ converges to $P_K x$. From this fact, we conclude that

$$\lim_{\lambda\to 0} x_\lambda = P_K x,$$

which is the desired result. □

Next, we consider the continuity of the function $\lambda \mapsto Q_{\lambda f} x$.

Theorem 3.6. *Let X be a complete* CAT(0) *space having the convex hull finite property, K a nonempty closed convex subset of X, and $f : K \times K \to \mathbb{R}$ a bifunction satisfying* (E1)–(E4). *Let $x \in X$ and define x_λ by*

$$x_\lambda = Q_{\lambda f} x = \left\{ z \in K \,\middle|\, \inf_{y \in K} (\lambda f(z, y) + d(y, x)^2 - d(z, x)^2) \geq 0 \right\}$$

for a positive number λ. Then

$$\lim_{\lambda \to \lambda_0} x_\lambda = x_{\lambda_0},$$

where λ_0 is a positive number.

Proof. Let $\{\lambda_n\}$ be an increasing or decreasing positive sequence converging to λ_0 and put $x_n = x_{\lambda_n}$ for $n \in \mathbb{N}$. We know by Lemma 3.3,

$$d(x_n, x_m)^2 \leq |d(x_m, x)^2 - d(x_n, x)^2|$$

for $n, m \in \mathbb{N}$ and $\{d(x_n, x)\}$ is convergent. Hence, x_n is a Cauchy sequence and we can find $x_0 \in K$ such that $x_n \to x_0$. From the definition of x_n, we obtain

$$0 \leq \lambda_n f(x_n, y) + d(y, x)^2 - d(x_n, x)^2 \leq -\lambda_n f(y, x_n) + d(y, x)^2 - d(x_n, x)^2$$

for all $y \in K$. By the lower semicontinuity of $f(y, \cdot)$, we have

$$\limsup_{n \to \infty} (-f(y, x_n)) = -\liminf_{n \to \infty} f(y, x_n) \leq -f(y, x_0),$$

and hence

$$\begin{aligned} 0 &\leq \limsup_{n \to \infty} \{-\lambda_n f(x_n, y) + d(y, x)^2 - d(x_n, x)^2\} \\ &\leq \lambda_0 \limsup_{n \to \infty} (-f(y, x_n)) + d(y, x)^2 - d(x_0, x)^2 \\ &\leq -\lambda_0 f(y, x_0) + d(y, x)^2 - d(x_0, x)^2, \end{aligned}$$

for all $y \in K$. Let $w \in K$ and $t \in]0, 1[$. From (E1) and (E2), we have

$$\begin{aligned} 0 &= \lambda_0 f(tw \oplus (1-t)x_0, tw \oplus (1-t)x_0) \\ &\leq \lambda_0 t f(tw \oplus (1-t)x_0, w) + \lambda_0 (1-t) f(tw \oplus (1-t)x_0, x_0) \\ &\leq \lambda_0 t f(tw \oplus (1-t)x_0, w) + (1-t)\{d(tw \oplus (1-t)x_0, x)^2 - d(x_0, x)^2\} \\ &\leq \lambda_0 t f(tw \oplus (1-t)x_0, w) + (1-t)\{t d(w, x)^2 + (1-t) d(x_0, x)^2 - d(x_0, x)^2\} \\ &= \lambda_0 t f(tw \oplus (1-t)x_0, w) + t(1-t) d(w, x)^2 - t(1-t) d(x_0, x)^2. \end{aligned}$$

Dividing by t and letting $t \to 0$, we obtain

$$0 \leq \lambda_0 f(x_0, w) + d(w, x)^2 - d(x_0, x)^2$$

for all $w \in K$. This implies $x_0 = x_{\lambda_0}$, and hence we conclude that

$$\lim_{\lambda \to \lambda_0} x_\lambda = x_{\lambda_0}.$$

This completes the proof. □

4 Resolvent on CAT(1) space

In a complete admissible CAT(1) space, we consider a resolvent with parameter λ and its convergence. Let X be a complete admissible CAT(1) space having the convex hull finite property and f a bifunction on K satisfying (E1)–(E4). For a positive number λ, we define a resolvent $R_{\lambda f}$ as follows:

$$R_{\lambda f}x = \left\{ z \in K | \inf_{y \in K}(\lambda f(z,y) - \log\cos d(y,x) + \log\cos d(z,x)) \geq 0 \right\}$$

for $x \in X$. We describe some lemmas and show the result for CAT(1) spaces.

Lemma 4.1. *Let X be a complete admissible* CAT(1) *space satisfying the convex hull finite property, K a nonempty closed convex subset of X, and $f: K \times K \to \mathbb{R}$ a bifunction satisfying* (E1)–(E4). *Let $x \in X$ and $\lambda, \mu \in \mathbb{R}$ with $0 < \lambda \leq \mu$. Then*

$$d(Q_{\lambda f}x, x) \leq d(Q_{\mu f}x, x), \quad f(Q_{\mu f}x, Q_{\lambda f}x) \geq 0, \quad \textit{and } f(Q_{\lambda f}x, Q_{\mu f}x) \leq 0.$$

Proof. Put $x_\lambda = R_{\lambda f}x$ and $x_\mu = R_{\mu f}x$. Then, we have

$$0 \leq f(x_\lambda, x_\mu) + \frac{1}{\lambda}\{-\log\cos d(x_\mu, x) + \log\cos d(x_\lambda, x)\} \tag{3}$$

and

$$0 \leq f(x_\mu, x_\lambda) + \frac{1}{\mu}\{-\log\cos d(x_\lambda, x) + \log\cos d(x_\mu, x)\}. \tag{4}$$

From these inequalities and (E2), we obtain

$$\begin{aligned} 0 &\leq f(x_\lambda, x_\mu) + f(x_\mu, x_\lambda) + \left(\frac{1}{\lambda} - \frac{1}{\mu}\right)\{-\log\cos d(x_\mu, x) + \log\cos d(x_\lambda, x)\} \\ &\leq 0 + \left(\frac{1}{\lambda} - \frac{1}{\mu}\right)\{-\log\cos d(x_\mu, x) + \log\cos d(x_\lambda, x)\}. \end{aligned}$$

Assume $\lambda < \mu$. From the inequality above, we obtain

$$-\log\cos d(x_\lambda, x) \leq -\log\cos d(x_\mu, x),$$

which is equivalent to $d(x_\lambda, x) \leq d(x_\mu, x)$. Therefore, we obtain

$$0 \leq f(x_\mu, x_\lambda) + \frac{1}{\mu}\{-\log\cos d(x_\lambda, x) + \log\cos d(x_\mu, x)^2\} \leq f(x_\mu, x_\lambda) \leq -f(x_\lambda, x_\mu).$$

It completes the proof. □

Lemma 4.2. *Let X be a complete admissible* CAT(1) *space satisfying the convex hull finite property, K a nonempty closed convex subset of X, and $f: K \times K \to \mathbb{R}$ a bifunction satisfying* (E1)–(E4). *Let $x \in X$, $\lambda, \mu \in \mathbb{R}$ with $0 < \lambda \leq \mu$, and $t \in [0,1]$. Then*

$$d(R_\lambda x, x) \leq d(tR_\lambda x + (1-t)R_\mu x, x) \leq d(R_\mu x, x).$$

Proof. Put $x_\lambda = R_\lambda x$ and $x_\mu = R_\mu x$. From Lemma 4.1, we have

$$\begin{aligned} 0 &\leq \lambda f(x_\lambda, tx_\lambda \oplus (1-t)x_\mu) - \log\cos d(tx_\lambda \oplus (1-t)x_\mu, x) + \log\cos d(x_\lambda, x) \\ &\leq \lambda t f(x_\lambda, x_\lambda) + \lambda(1-t)f(x_\lambda, x_\mu) - \log\cos d(tx_\lambda \oplus (1-t)x_\mu, x) + \log\cos d(x_\lambda, x) \\ &\leq -\log\cos d(tx_\lambda \oplus (1-t)x_\mu, x) + \log\cos d(x_\lambda, x), \end{aligned}$$

and hence $-\log\cos d(x_\lambda, x) \leq -\log\cos d(tx_\lambda \oplus (1-t)x_\mu, x)$. From the convexity of the function $-\log\cos d(\cdot, x)$, we have

$$\begin{aligned} -\log\cos d(x_\lambda, x) &\leq -\log\cos d(tx_\lambda \oplus (1-t)x_\mu, x) \\ &\leq -t\log\cos d(x_\lambda, x) - (1-t)\log\cos d(x_\mu, x) \\ &\leq -\log\cos d(x_\mu, x). \end{aligned}$$

This implies

$$d(x_\lambda, x) \le d(tx_\lambda \oplus (1-t)x_\mu, x) \le d(x_\mu, x),$$

which is the desired result. □

Lemma 4.3. *Let X be a complete admissible* CAT(1) *space satisfying the convex hull finite property, K a nonempty closed convex subset of X, and $f : K \times K \to \mathbb{R}$ a bifunction satisfying* (E1)–(E4). *Let $x \in X$ and $\lambda, \mu \in \mathbb{R}$ with $0 < \lambda \le \mu$. Then*

$$-2\log\cos\frac{d(R_{\lambda f}x, R_{\mu f}x)}{2} \le -\log\cos d(R_{\mu f}x, x) + \log\cos d(R_{\lambda f}x, x).$$

Proof. Put $x_\lambda = R_{\lambda f}x$ and $x_\mu = R_{\mu f}x$. By Lemma 4.2 and the parallelogram law at $t = 1/2$, we have

$$\begin{aligned}-\log\cos d(x_\lambda, x) &\le -\log\cos d\left(\frac{1}{2}x_\lambda \oplus \frac{1}{2}x_\mu, x\right)\\ &\le -\frac{1}{2}\log\cos d(x_\lambda, x) - \frac{1}{2}\log\cos d(x_\mu, x) + \log\cos\frac{d(x_\lambda, x_\mu)}{2},\end{aligned}$$

and hence

$$-\log\cos\frac{d(x_\lambda, x_\mu)}{2} \le -\frac{1}{2}\log\cos d(x_\mu, x) + \frac{1}{2}\log\cos d(x_\lambda, x),$$

which is the desired result. □

Theorem 4.4. *Let X be a complete admissible* CAT(1) *space satisfying the convex hull finite property, K a nonempty closed convex subset of X, and $f : K \times K \to \mathbb{R}$ a bifunction satisfying* (E1)–(E4). *Let $x \in X$ and define x_λ dy*

$$x_\lambda = R_{\lambda f}x = \left\{z \in K \,\middle|\, \inf_{y\in K}(\lambda f(z, y) - \log\cos d(y, x) + \log\cos d(z, x)) \ge 0\right\}$$

for a positive number λ. If there exists $\{\mu_n\} \subset]0, \infty[$ such that $\mu_n \to \infty$ and $\sup d(x_{\mu_n}, x) < \pi/2$, then $\mathrm{Equil} f \neq \emptyset$ *and*

$$\lim_{\lambda\to\infty} x_\lambda = P_{\mathrm{Equil} f}x,$$

where P_K is a metric projection onto a nonempty closed convex subset K of X.

Proof. Suppose that there exists a sequence $\{\mu_n\}$ which satisfies the assumption above. In a similar way to the proof of Theorem 3.4, we have

$$\sup_{n\in\mathbb{N}} d\left(Q_{\lambda_n}x, x\right) < \frac{\pi}{2}$$

for all positive increasing sequence $\{\lambda_n\}$. Put $x_n = x_{\lambda_n}$ for $n \in \mathbb{N}$. Let $n, m \in \mathbb{N}$ with $n \le m$. From Lemma 4.3, we have

$$-2\log\cos\frac{d(x_n, x_m)}{2} \le -\log\cos d(x_m, x) + \log\cos d(x_n, x).$$

By Lemma 4.2, $\{-\log\cos d(x_n, x)\}$ is convergent and thus $\{x_n\}$ is Cauchy. From the completeness of K, there exists $p \in K$ such that $x_n \to p$. By the definition of x_n, we obtain

$$\begin{aligned}0 &\le f(x_n, y) + \frac{1}{\lambda_n}\{-\log\cos d(y, x) + \log\cos d(x_n, x)\}\\ &\le -f(y, x_n) + \frac{1}{\lambda_n}\{-\log\cos d(y, x) + \log\cos d(x_n, x)\}\end{aligned}$$

for all $y \in X$. From (E3), we obtain

$$0 \le \limsup_{n\to\infty} f(x_n, y) \le -f(y, p).$$

This implies

$$f(y, p) \le 0$$

for all $y \in K$. Let $w \in K$ and $t \in \,]0, 1[$. Put $w_t = tw \oplus (1-t)p$. Then we obtain $f(w_t, p) \le 0$. From (E1) and (E2), we have

$$0 = f(w_t, w_t) \le tf(w_t, w) + (1-t)f(w_t, p) \le tf(w_t, w).$$

Dividing by t and letting $t \to 0$, we have

$$0 \le f(p, w)$$

for all $w \in K$ from (E4). Therefore, p is a solution of the equilibrium problem for f. For all $z \in \operatorname{Equil} f$, we have

$$\begin{aligned} 0 &\le \lambda_n f(x_n, z) - \log\cos d(z, x) + \log\cos d(x_n, x) \\ &\le -\lambda_n f(z, x_n) - \log\cos d(z, x) + \log\cos d(x_n, x) \\ &\le -\log\cos d(z, x) + \log\cos d(x_n, x), \end{aligned}$$

and hence $d(x_n, x) \le d(z, x)$. From the lower semicontinuity of $d(\cdot, x)$, we obtain

$$d(p, x) \le \liminf_{n\to\infty} d(x_n, x) \le d(z, x)$$

for all $z \in \operatorname{Equil} f$ and this means $p = P_{\operatorname{Equil} f} x$. We have $x_n \to P_{\operatorname{Equil} f} x$ for every increasing sequence $\{\lambda_n\}$ diverging to infinity, and hence we obtain

$$\lim_{\lambda\to\infty} x_\lambda = P_{\operatorname{Equil} f} x.$$

This is the desired result. □

Theorem 4.5. *Let X be a complete admissible* CAT(1) *space satisfying the convex hull finite property, K a nonempty closed convex subset of X, and $f : K \times K \to \mathbb{R}$ a bifunction satisfying* (E1)–(E4). *Let $x \in X$ and define x_λ by*

$$x_\lambda = R_{\lambda f} x = \left\{ z \in K \,\middle|\, \inf_{y\in K} (\lambda f(z, y) - \log\cos d(y, x) + \log\cos d(z, x)) \ge 0 \right\}$$

for a positive number λ. Then

$$\lim_{\lambda\to 0} x_\lambda = P_K x.$$

Proof. Let $\{\lambda_n\}$ be a decreasing positive sequence converging to 0 and put $x_n = x_{\lambda_n}$ for $n \in \mathbb{N}$. By Lemma 4.3,

$$-2\log\cos\frac{d(x_n, x_m)}{2} \le |-\log\cos d(x_m, x) + \log\cos d(x_n, x)|,$$

for $n, m \in \mathbb{N}$. On the other hand, from Lemma 4.2, $\{-\log\cos d(x_n, x)\}$ is decreasing and bounded below. Therefore, $\{x_n\}$ converges to $p \in K$ since K is closed. By the definition of the resolvent, we have

$$0 \le \lambda_n f(x_n, y) - \log\cos d(y, x) + \log\cos d(x_n, x) \le -\lambda_n f(y, x_n) - \log\cos d(y, x) + \log\cos d(x_n, x).$$

From the lower semicontinuity of f for the second argument, letting $n \to \infty$, we obtain $0 \le -\log\cos d(y, x) + \log\cos d(x_n, x)$, and hence

$$d(p, x) \le d(y, x)$$

for all $y \in X$. This implies $p = P_K x$. Therefore, for every decreasing sequence $\{\lambda_n\}$ convergent to 0, $\{x_n\}$ converges to $P_K x$. Thus, we conclude that

$$\lim_{\lambda \to 0} x_\lambda = P_K x,$$

which is the desired result. □

Theorem 4.6. *Let X be a complete admissible* CAT(1) *space satisfying the convex hull finite property, K a nonempty closed convex subset of X, and $f : K \times K \to \mathbb{R}$ a bifunction satisfying* (E1)–(E4). *Let $x \in X$ and define x_λ by*

$$x_\lambda = R_{\lambda f} x = \left\{ z \in K \,\middle|\, \inf_{y \in K} (\lambda f(z, y) - \log\cos d(y, x) + \log\cos d(z, x)) \geq 0 \right\}$$

for a positive number λ. Then

$$\lim_{\lambda \to \lambda_0} x_\lambda = x_{\lambda_0},$$

where λ_0 is a positive number.

Proof. Let $\{\lambda_n\}$ be an increasing or decreasing positive sequence converging to λ_0 and put $x_n = x_{\lambda_n}$ for $n \in \mathbb{N}$. By Lemma 4.3,

$$-2\log\cos d(x_n, x_m) \leq |-\log\cos d(x_m, x) + \log\cos d(x_n, x)|$$

for $n, m \in \mathbb{N}$ and $\{-\log\cos d(x_n, x)\}$ is convergent. Hence, we can find $x_0 \in K$ such that $x_n \to x_0$. From the definition of x_n, we obtain

$$0 \leq \lambda_n f(x_n, y) + d(y, x)^2 - d(x_n, x)^2 \leq -\lambda_n f(y, x_n) + d(y, x)^2 - d(x_n, x)^2.$$

By the lower semicontinuity of $f(y, \cdot)$, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty}\{-\lambda_n f(x_n, y) - \log\cos d(y, x) + \log\cos d(x_n, x)\} \\ &\leq \lambda_0 \limsup_{n\to\infty}(-f(y, x_n)) - \log\cos d(y, x) + \log\cos d(x_0, x) \\ &\leq -\lambda_0 f(y, x_0) - \log\cos d(y, x) + \log\cos d(x_0, x). \end{aligned}$$

for all $y \in K$. Let $w \in K$ and $t \in \,]0, 1[$. Put $w_t = tw \oplus (1-t)x_0$. From (E1) and (E2), we have

$$\begin{aligned} 0 &= \lambda_0 f(w_t, w_t) \\ &\leq \lambda_0 t f(w_t, w) + \lambda_0 (1-t) f(w_t, x_0) \\ &\leq \lambda_0 t f(w_t, w) + (1-t)\{-\log\cos d(w_t, x) + \log\cos d(x_0, x)\} \\ &\leq \lambda_0 t f(w_t, w) + (1-t)\{-t\log\cos d(w, x) - (1-t)\log\cos d(x_0, x) + \log\cos d(x_0, x)\} \\ &= \lambda_0 t f(w_t, w) - t(1-t)\log\cos d(w, x) + t(1-t)\log\cos d(x_0, x). \end{aligned}$$

Dividing by t and letting $t \to 0$, we obtain

$$0 \leq \lambda_0 f(x_0, w) - \log\cos d(w, x) + \log\cos d(x_0, x)$$

for all $w \in K$. This implies $x_0 = x_{\lambda_0}$, and hence we conclude that

$$\lim_{\lambda \to \lambda_0} x_\lambda = x_{\lambda_0}.$$

This completes the proof. □

5 Applications

We discuss applications of our results to a convergence of resolvent for a convex function in minimization problems.

Let X be a complete CAT(0) space satisfying the convex hull finite property and $g : X \to]-\infty, \infty]$ a lower semicontinuous convex function. The effective domain $\mathrm{dom} g$ of g is defined by

$$\mathrm{dom} g = \{x \in X | g(x) \in \mathbb{R}\}.$$

Furthermore, we define a resolvent J_g for g by

$$J_g x = \operatorname*{argmin}_{y \in X} \{g(y) + d(y, x)^2\},$$

for all $x \in X$. In what follows, we suppose $\mathrm{dom} g$ is closed. We assume $f : \mathrm{dom} g \times \mathrm{dom} g \to \mathbb{R}$ satisfies

$$f(z, y) = g(y) - g(z)$$

for all $y, z \in K$. Then $J_g = R_f$. In fact, if $z = J_g x$, it holds that

$$g(z) + d(z, x)^2 \leq g(y) + d(y, x)^2$$

for all $y \in X$, and hence

$$0 \leq g(y) - g(z) + d(y, x^2) - d(z, x)^2.$$

Therefore, $z = J_g x$ implies $z = R_f x$, and *vice versa*.

We remark that it is not easy to calculate the exact value of the resolvent operator, even if a convex function is explicitly given, because it is defined by a solution to another convex minimization problem. However, this problem is much easier to solve than the original minimization problem since the corresponding convex function satisfies more vital conditions, such as coercivity and strict convexity. If the given convex function meets additional requirements, such as differentiability, then the Newton method and other approximation techniques will work well.

From the argument of Section 3, we can derive the following theorems.

Theorem 5.1. (See [16]) *Let X be a complete* CAT(0) *space having the convex hull finite property, $g : X \to]-\infty, \infty]$ a lower semicontinuous convex function, and $x \in X$. Suppose* $\mathrm{dom} f$ *is closed. Define x_λ by*

$$x_\lambda = J_{\lambda g} x = \operatorname*{argmin}_{y \in X} \{\lambda g(y) + d(x, y)^2\}$$

for a positive number λ. If there exists $\{\mu_n\} \subset]0, \infty[$ such that $\mu_n \to \infty$ and $\{d(J_{\mu_n g} x, x)\}$ is bounded, then $\mathrm{Equil} f \neq \emptyset$ *and*

$$\lim_{\lambda \to \infty} x_\lambda = P_{\mathrm{argmin} f} x.$$

Theorem 5.2. (See [16]) *Let X be a complete* CAT(0) *space having the convex hull finite property, $g : X \to]-\infty, \infty]$ a lower semicontinuous convex function, and $x \in X$. Suppose* $\mathrm{dom} f$ *is closed. Define x_λ by*

$$x_\lambda = J_{\lambda g} x = \operatorname*{argmin}_{y \in X} \{\lambda g(y) + d(x, y)^2\}$$

for a positive number λ. Then

$$\lim_{\lambda \to 0} x_\lambda = P_{\mathrm{dom} f} x.$$

Theorem 5.3. (See [16]) *Let X be a complete* CAT(0) *space having the convex hull finite property, $g : X \to]-\infty, \infty]$ a lower semicontinuous convex function, and $x \in X$. Suppose* $\mathrm{dom} f$ *is closed. Define x_λ by*

$$x_\lambda = J_{\lambda g} x = \operatorname*{argmin}_{y \in X} \{\lambda g(y) + d(x, y)^2\}$$

for a positive number λ. Then

$$\lim_{\lambda \to \lambda_0} x_\lambda = x_{\lambda_0},$$

where λ_0 is a positive number.

Strictly speaking, in the minimization problems, we need not assume the convex hull finite property to show the convergence of the resolvent for convex functions (see [16]).

We can consider the convergence of resolvent of a convex function on a complete admissible CAT(1) space. Let X be a complete admissible CAT(1) space satisfying the convex hull finite property, $g : X \to]-\infty, \infty]$ a lower semicontinuous convex function, and $x \in X$. We define $L_{\lambda g}x$ by

$$L_{\lambda g}x = \operatorname*{argmin}_{x \in X}\{\lambda g(y) - \log\cos d(y, x)\}$$

for $x \in X$. We can obtain similar results to the theorems of CAT(0) spaces in this section. The problem of asymptotic behavior of convex functions on a complete CAT(1) space is discussed in [17].

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