Research Article

Alina Alb Lupaş* and Georgia Irina Oros

Sandwich-type results regarding Riemann-Liouville fractional integral of *q*-hypergeometric function

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Abstract: The study presented in this article involves q-calculus connected to fractional calculus applied in the univalent functions theory. Riemann-Liouville fractional integral of q-hypergeometric function is defined here, and investigations are conducted using the theories of differential subordination and superordination. Theorems and corollaries containing new subordination and superordination results are proved for which best dominants and best subordinants are given, respectively. As an application of the results obtained by the means of the two theories, the statement of a sandwich-type theorem concludes the study.

Keywords: fractional integral of q-hypergeometric function, differential subordination, differential superordination, best dominant, best subordinant

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1 Introduction

Applications of the q-calculus in different mathematical arias, physics, or engineering domains are well known. Factional calculus is also known to have multiple applications in many domains of research. The far-reaching paper published by Srivastava [1] gives an overview of the numerous applications of q-calculus and fractional q-calculus in general and in geometric function theory in particular.

The first applications of q-calculus in mathematics were given by Jackson [2,3] who introduced the notions of q-derivative and q-integral. The connection between q-calculus and univalent functions theory was established by Ismail et al. [4] when they studied a class of q-stalike functions. But it was Srivastava [5] who set the basis for the applications of q-calculus in the geometric function theory in the book chapter published in 1989. In that chapter, the q-hypergeometric function was presented as a function with notable applications in the geometric function theory.

Numerous applications of q-calculus on univalent functions appeared by introducing new q-analog operators. q-analog of the Ruscheweyh differential operator was defined by Kanas and Răducanu [6] using convolution. The application of this differential operator was further studied by Mohammed and Darus [7] and Mahmood and Sokół [8]. Following the same pattern, q-analog of Sălăgean differential operator emerged [9] inspiring many applications [10–12]. The q-hypergeometric function was also used in introducing new operators, which were intensely studied and several important results were obtained. Studies presented in [13–15] can be viewed for such applications.

*** Corresponding author: Alina Alb Lupaş,** Department of Mathematics and Computer Science, University of Oradea, 1 Universitatii Street, 410087 Oradea, Romania, e-mail: alblupas@gmail.com
Georgia Irina Oros: Department of Mathematics and Computer Science, University of Oradea, 1 Universitatii Street, 410087 Oradea, Romania, e-mail: georgia_oros_ro@yahoo.co.uk

The research presented in this article uses an operator-defined combining Riemann-Liouville fractional integral and q-hypergeometric function. Riemann-Liouville fractional integral was considered for the study due to its numerous recent applications in defining new operators. Confluent hypergeometric function was combined with it in studies presented [16–18] and Gaussian hypergeometric function in [19].

Before reminding the definitions related to Riemann-Liouville fractional integral and q-hypergeometric function, let us review the basic notations from the geometric function theory.

The class of analytic functions defined on the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ is denoted by $\mathcal{H}(U)$. Taking the complex number a and n a positive integer, the class and $\mathcal{H}[a, n]$ contains functions $f \in \mathcal{H}(U)$ written as $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots$, $z \in U$. Class $\mathcal{A}_n$ is formed of functions $f \in \mathcal{H}(U)$ of the form $f(z) = z + a_{n+1}z^{n+1} + \dots$, $z \in U$, with $\mathcal{A}_1 = \mathcal{A}$.

The definition of Riemann-Liouville fractional integral can be seen in [20,21]:

Definition 1.1. [20,21] The fractional integral of order λ ($\lambda > 0$) is defined for a function f by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} \mathrm{d}t,$$

where f is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z - t)^{\lambda-1}$ is removed by requiring $\log(z - t)$ to be real, when $(z - t) > 0$.

Definition 1.2. [22, p. 5] Let a and b be complex numbers with $b \neq 0, -1, -2, \dots$ and consider

$$\phi(a, b; z) = {}_1F_1(a, b; z) = 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \dots, \quad z \in U. \tag{1}$$

This function is called confluent (Kummer) hypergeometric function, is analytic in $\mathbb{C}$ and satisfies Kummer's differential equation:

$$zw''(z) + (c - z)w'(z) - aw(z) = 0.$$

The q-hypergeometric function $\phi(a, b; q, z)$ is defined by

$$\phi(a, b; q, z) = \sum_{k=0}^{\infty} \frac{(a, q)_k}{(q, q)_k (b, q)_k} z^k, \tag{2}$$

where

$$(a, q)_k = \begin{cases} 1, & k = 0, \\ (1 - a)(1 - aq)(1 - aq^2)\dots(1 - aq^{k-1}), & k \in \mathbb{N}, \end{cases}$$

and $0 < q < 1$.

Definitions regarding the theories of differential subordination and differential superordination are next recalled.

Definition 1.3. [23] Let the functions f and g be analytic in U. We say that the function f is subordinate to g, written $f \prec g$, if there exists a Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$, for all $z \in U$. In particular, if the function g is univalent in U, the aforementioned subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 1.4. [23] Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h be an univalent function in U. If p is analytic in U and satisfies the second-order differential subordination:

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad z \in U, \tag{3}$$

then p is called a solution of the differential subordination. The univalent function g is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec g$ for all p satisfying (3). A dominant $\widetilde{g}$ that satisfies $\widetilde{g} \prec g$ for all dominants g of (3) is said to be the best dominant of (3).

Definition 1.5. [24] Let $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent in U satisfy the (second-order) differential superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \quad z \in U, \tag{4}$$

then p is called a solution of the differential superordination. An analytic function g is called a subordinant of the solutions of the differential superordination or more simply a subordinant, if $g \prec p$ for all p satisfying (4). A subordinant $\widetilde{g}$ that satisfies $g \prec \widetilde{g}$ for all subordinants g of (4) is said to be the best subordinant of (4).

Definition 1.6. [23] Denote by Q the set of all functions f that are analytic and injective on $\overline{U} \backslash E(f)$, where $E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$ and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.

The next two lemmas are tools for proving the results from the Main results section.

Lemma 1.1. [23] *Let the function g be univalent in the unit disc U and θ and η be analytic in a domain D containing $g(U)$ with $\eta(w) \neq 0$ when $w \in g(U)$. Set $G(z) = zg'(z)\eta(g(z))$ and $h(z) = \theta(g(z)) + G(z)$. Suppose that G is starlike univalent in U and $\operatorname{Re}\left(\frac{zh'(z)}{G(z)}\right) > 0$ for $z \in U$. If p is analytic with $p(0) = g(0)$, $p(U) \subseteq D$ and $\theta(p(z)) + zp'(z)\eta(p(z)) \prec \theta(g(z)) + zg'(z)\eta(g(z))$, then $p(z) \prec g(z)$ and g is the best dominant.*

Lemma 1.2. [25] *Let the function g be convex univalent in the open unit disc U and θ and η be analytic in a domain D containing $g(U)$. Suppose that $\operatorname{Re}\left(\frac{\theta'(g(z))}{\eta(g(z))}\right) > 0$ for $z \in U$ and $G(z) = zg'(z)\eta(g(z))$ is starlike univalent in U. If $p(z) \in \mathcal{H}[g(0), 1] \cap Q$, with $p(U) \subseteq D$ and $\theta(p(z)) + zp'(z)\eta(p(z))$ is univalent in U and $\theta(g(z)) + zg'(z)\eta(g(z)) \prec \theta(p(z)) + zp'(z)\eta(p(z))$, then $g(z) \prec p(z)$ and g is the best subordinant.*

2 Results

We begin by using Definitions 1.1 and 1.2 for introducing the new operator, which will be used for obtaining the new results contained in the theorems and corollaries presented in this section.

Definition 2.1. Let a and b be complex numbers with $b \neq 0, -1, -2, \ldots$ and $\lambda > 0$, $0 < q < 1$. We define the Riemann-Liouville fractional integral of q-confluent hypergeometric function:

$$D_z^{-\lambda}\phi(a, b; q, z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{\phi(a, b; q, t)}{(z-t)^{1-\lambda}} \mathrm{d}t = \frac{1}{\Gamma(\lambda)} \sum_{k=0}^{\infty} \frac{(a, q)_k}{(q, q)_k (b, q)_k} \int_0^z \frac{t^k}{(z-t)^{1-\lambda}} \mathrm{d}t. \tag{5}$$

After a simple calculation, the Riemann-Liouville fractional integral of q-confluent hypergeometric function has the following form:

$$D_z^{-\lambda}\phi(a, b; q, z) = \sum_{k=0}^{\infty} \frac{(a, q)_k}{(q, q)_k (b, q)_k (k+1)_\lambda} z^{\lambda+k}. \tag{6}$$

We note that $D_z^{-\lambda}\phi(a, b; q, z) \in \mathcal{H}[0, \lambda]$.

The first subordination result obtained using the operator given by (5) is the following theorem:

Theorem 2.1. *Let* $\left(\frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z}\right)^{\gamma} \in \mathcal{H}(U)$ *and the function g be analytic and univalent in U such that* $g(z) \neq 0$, *for all* $z \in U$, *where a and b be complex numbers with* $b \neq 0, -1, -2, \ldots$ *and* $\lambda, \gamma > 0$, $0 < q < 1$. *Suppose that* $\frac{zg'(z)}{g(z)}$ *is starlike univalent in U. Let*

$$\operatorname{Re}\left(1 + \frac{\rho}{\tau}g(z) + \frac{2\chi}{\tau}(g(z))^2 - \frac{zg'(z)}{g(z)} + \frac{zg''(z)}{g'(z)}\right) > 0, \tag{7}$$

for $\mu, \rho, \chi, \tau \in \mathbb{C}$, $\tau \neq 0$, $z \in U$ *and*

$$\begin{aligned}\psi_{\lambda}^{a,b,q}(\gamma, \mu, \rho, \chi, \tau;\ z) := \mu + \rho\left[\frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z}\right]^{\gamma} + \chi\left[\frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z}\right]^{2\gamma} \\ + \tau\gamma\left[\frac{z(D_z^{-\lambda}\phi(a,b;\ q,z))'}{D_z^{-\lambda}\phi(a,b;\ q,z)} - 1\right].\end{aligned} \tag{8}$$

If g satisfies the following subordination

$$\psi_{\lambda}^{a,b,q}(\gamma, \mu, \rho, \chi, \tau;\ z) \prec \mu + \rho g(z) + \chi(g(z))^2 + \tau\frac{zg'(z)}{g(z)}, \tag{9}$$

for $\mu, \rho, \chi, \tau \in \mathbb{C}$, $\tau \neq 0$, *then*

$$\left(\frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z}\right)^{\gamma} \prec g(z), \quad z \in U, \tag{10}$$

and g is the best dominant.

Proof. Let the function p be defined by $p(z) := \left(\frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z}\right)^{\gamma}$, $z \in U, z \neq 0$. We have $p'(z) = \gamma\left(\frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z}\right)^{\gamma-1}$ $\left[\frac{(D_z^{-\lambda}\phi(a,b;\ q,z))'}{z} - \frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z^2}\right] = \gamma\left(\frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z}\right)^{\gamma-1}\frac{(D_z^{-\lambda}\phi(a,b;\ q,z))'}{z} - \frac{\gamma}{z}p(z)$. Then $\frac{zp'(z)}{p(z)} = \gamma\left[\frac{z(D_z^{-\lambda}\phi(a,b;\ q,z))'}{D_z^{-\lambda}\phi(a,b;\ q,z)} - 1\right]$.

By setting $\theta(w) := \mu + \rho w + \chi w^2$ and $\eta(w) := \frac{\tau}{w}$, it can be easily verified that θ is analytic in $\mathbb{C}$ and η is analytic in $\mathbb{C}\backslash\{0\}$ and that $\eta(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Also, by letting $G(z) = zg'(z)\eta(g(z)) = \tau\frac{zg'(z)}{g(z)}$ and $h(z) = \theta(g(z)) + G(z) = \mu + \rho g(z) + \chi(g(z))^2 + \tau\frac{zg'(z)}{g(z)}$, we find that $R(z)$ is starlike univalent in U.

We have $h'(z) = \tau + g'(z) + 2\chi g(z)g'(z) + \tau\frac{(g'(z)+zg''(z))g(z) - z(g'(z))^2}{(g(z))^2}$ and $\frac{zh'(z)}{G(z)} = \frac{zh'(z)}{\tau\frac{zg'(z)}{g(z)}} = 1 + \frac{\rho}{\tau}g(z) + \frac{2\chi}{\tau}(g(z))^2 - \frac{zg'(z)}{g(z)} + \frac{zg''(z)}{g'(z)}$.

We deduce that $\operatorname{Re}\left(\frac{zh'(z)}{G(z)}\right) = \operatorname{Re}\left(1 + \frac{\rho}{\tau}g(z) + \frac{2\chi}{\tau}(g(z))^2 - \frac{zg'(z)}{g(z)} + \frac{zg''(z)}{g'(z)}\right) > 0$.

We have $\mu + \rho p(z) + \chi(p(z))^2 + \tau\frac{zp'(z)}{p(z)} = \mu + \rho\left[\frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z}\right]^{\delta} + \chi\left[\frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z}\right]^{2\delta} + \tau\gamma\left[\frac{z(D_z^{-\lambda}\phi(a,b;\ q,z))'}{D_z^{-\lambda}\phi(a,b;\ q,z)} - 1\right]$.

By using (9), we obtain $\mu + \rho p(z) + \chi(p(z))^2 + \tau\frac{zp'(z)}{p(z)} \prec \mu + \rho g(z) + \chi(g(z))^2 + \tau\frac{zg'(z)}{g(z)}$.

By applying Lemma 1.1, we obtain $p(z) \prec g(z)$, $z \in U$, i.e., $\left(\frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z}\right)^{\gamma} \prec g(z)$, $z \in U$ and g is the best dominant. □

Corollary 2.2. *Let a and b be complex numbers with* $b \neq 0, -1, -2, \ldots$ *and* $\lambda, \gamma > 0$, $0 < q < 1$. *Assume that* (7) *holds. If*

$$\psi_{\lambda}^{a,b,q}(\gamma, \mu, \rho, \chi, \tau;\ z) \prec \mu + \rho\frac{1+Mz}{1+Nz} + \chi\left(\frac{1+Mz}{1+Nz}\right)^2 + \tau\frac{(M-N)z}{(1+Mz)(1+Nz)},$$

for $\mu, \rho, \chi, \tau \in \mathbb{C}$, $\tau \neq 0$, $-1 \leq N < M \leq 1$, *where* $\psi_{\lambda}^{a,b,q}$ *is defined in* (8), *then*

$$\left(\frac{D_z^{-\lambda}\phi(a,b;\,q,z)}{z}\right)^{\gamma} \prec \frac{1+Mz}{1+Nz}, \quad z \in U,$$

and $\frac{1+Mz}{1+Nz}$ *is the best dominant.*

Proof. For $g(z) = \frac{1+Mz}{1+Nz}$, $-1 \le N < M \le 1$ in Theorem 2.1, we obtain the corollary. □

Corollary 2.3. *Let a and b be complex numbers with* $b \neq 0, -1, -2, \ldots$ *and* $\lambda, \gamma > 0$, $0 < q < 1$. *Assume that* (7) *holds. If*

$$\psi_{\lambda}^{a,b,q}(\gamma,\mu,\rho,\chi,\tau;\,z) \prec \mu + \rho\left(\frac{1+z}{1-z}\right)^{\sigma} + \chi\left(\frac{1+z}{1-z}\right)^{2\sigma} + \tau\frac{2\sigma z}{1-z^2},$$

for $\mu, \rho, \chi, \tau \in \mathbb{C}$, $0 < \sigma \le 1$, $\tau \neq 0$, *where* $\psi_{\lambda}^{a,b,q}$ *is defined in* (8), *then*

$$\left(\frac{D_z^{-\lambda}\phi(a,b;\,q,z)}{z}\right)^{\gamma} \prec \left(\frac{1+z}{1-z}\right)^{\sigma}, \quad z \in U,$$

and $\left(\frac{1+z}{1-z}\right)^{\sigma}$ *is the best dominant.*

Proof. Corollary follows by using Theorem 2.1 for $g(z) = \left(\frac{1+z}{1-z}\right)^{\sigma}$, $0 < \sigma \le 1$. □

Theorem 2.4. *Let g be analytic and univalent in U such that* $g(z) \neq 0$ *and* $\frac{zg'(z)}{g(z)}$ *be starlike univalent in U. Assume that*

$$\operatorname{Re}\left(\frac{2\chi}{\tau}(g(z))^2 + \frac{\rho}{\tau}g(z)\right) > 0, \quad \text{for } \rho,\chi,\tau \in \mathbb{C}, \quad \tau \neq 0. \tag{11}$$

Let a and b be complex numbers with $b \neq 0, -1, -2, \ldots$ *and* $\lambda, \gamma > 0$, $0 < q < 1$. *If* $\left(\frac{D_z^{-\lambda}\phi(a,b;\,q,z)}{z}\right)^{\gamma} \in \mathcal{H}[0,(\lambda-1)\gamma] \cap Q$ *and* $\psi_{\lambda}^{a,b,q}(\gamma,\mu,\rho,\chi,\tau;\,z)$ *is univalent in U, where* $\psi_{\lambda}^{a,b,q}(\gamma,\mu,\rho,\chi,\tau;\,z)$ *is as defined in* (8), *then*

$$\mu + \rho g(z) + \chi(g(z))^2 + \tau\frac{zg'(z)}{g(z)} \prec \psi_{\lambda}^{a,b,q}(\gamma,\mu,\rho,\chi,\tau;\,z) \tag{12}$$

implies

$$g(z) \prec \left(\frac{D_z^{-\lambda}\phi(a,b;\,q,z)}{z}\right)^{\gamma}, \quad z \in U, \tag{13}$$

and g is the best subordinant.

Proof. Let the function p be defined by $p(z) := \left(\frac{D_z^{-\lambda}\phi(a,b;\,q,z)}{z}\right)^{\gamma}$, $z \in U$, $z \neq 0$.

By setting $\theta(w) := \mu + \rho w + \chi w^2$ and $\eta(w) := \frac{\tau}{w}$, it can be easily verified that θ is analytic in $\mathbb{C}$ and η is analytic in $\mathbb{C}\backslash\{0\}$ and that $\eta(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since $\frac{\theta'(g(z))}{\eta(g(z))} = \frac{g'(z)[\rho + 2\chi g(z)]g(z)}{\tau}$, it follows that $\operatorname{Re}\left(\frac{\theta'(g(z))}{\eta(g(z))}\right) = \operatorname{Re}\left(\frac{2\chi}{\tau}(g(z))^2 + \frac{\rho}{\tau}g(z)\right) > 0$, for $\chi, \rho, \tau \in \mathbb{C}$, $\tau \neq 0$.

We obtain

$$\mu + \rho g(z) + \chi(g(z))^2 + \tau\frac{zg'(z)}{g(z)} \prec \mu + \rho p(z) + \chi(p(z))^2 + \tau\frac{zp'(z)}{p(z)}.$$

By using Lemma 1.2, we have

$$g(z) \prec p(z) = \left(\frac{D_z^{-\lambda}\phi(a,b;q,z)}{z}\right)^{\gamma}, z \in U,$$

and g is the best subordinant. □

Corollary 2.5. *Let a and b be complex numbers with $b \neq 0, -1, -2, \dots$ and $\lambda, \gamma > 0$, $0 < q < 1$. Assume that* (11) *holds. If $\left(\frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z}\right)^{\gamma} \in \mathcal{H}[0, (\lambda-1)\gamma] \cap Q$ and*

$$\mu + \rho\frac{1+Mz}{1+Nz} + \chi\left(\frac{1+Mz}{1+Nz}\right)^2 + \tau\frac{(M-N)z}{(1+Mz)(1+NBz)} \prec \psi_\lambda^{a,b,q}(\gamma, \mu, \rho, \chi, \tau; z),$$

for $\mu, \rho, \chi, \tau \in \mathbb{C}$, $\tau \neq 0$, $-1 \leq N < M \leq 1$, where $\psi_\lambda^{a,b,q}$ is defined in (8), *then*

$$\frac{1+Mz}{1+Nz} \prec \left(\frac{D_z^{-\lambda}\phi(a,b;q,z)}{z}\right)^{\gamma}, \quad z \in U,$$

and $\frac{1+Mz}{1+Nz}$ is the best subordinant.

Proof. For $g(z) = \frac{1+Mz}{1+Nz}$, $-1 \leq N < M \leq 1$ in Theorem 2.4, we obtain the corollary. □

Corollary 2.6. *Let a and b be complex numbers with $b \neq 0, -1, -2, \dots$ and $\lambda, \gamma > 0$, $0 < q < 1$. Assume that* (11) *holds. If $\left(\frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z}\right)^{\gamma} \in \mathcal{H}[0, (\lambda-1)\gamma] \cap Q$ and*

$$\mu + \rho\left(\frac{1+z}{1-z}\right)^{\sigma} + \chi\left(\frac{1+z}{1-z}\right)^{2\sigma} + \tau\frac{2\sigma z}{1-z^2} \prec \psi_\lambda^{a,b,q}(\gamma, \mu, \rho, \chi, \tau; z),$$

for $\mu, \rho, \chi, \tau \in \mathbb{C}$, $0 < \sigma \leq 1$, $\tau \neq 0$, where $\psi_\lambda^{a,b,q}$ is defined in (8), *then*

$$\left(\frac{1+z}{1-z}\right)^{\sigma} \prec \left(\frac{D_z^{-\lambda}\phi(a,b;q,z)}{z}\right)^{\gamma}, z \in U,$$

and $\left(\frac{1+z}{1-z}\right)^{\sigma}$ is the best subordinant.

Proof. Corollary follows by using Theorem 2.4 for $g(z) = \left(\frac{1+z}{1-z}\right)^{\sigma}$, $0 < \sigma \leq 1$. □

Combining Theorems 2.1 and 2.4, we state the following sandwich theorem.

Theorem 2.7. *Let g_1 and g_2 be analytic and univalent in U such that $g_1(z) \neq 0$ and $g_2(z) \neq 0$, for all $z \in U$, with $\frac{zg_1'(z)}{g_1(z)}$ and $\frac{zg_2'(z)}{g_2(z)}$ being starlike univalent. Suppose that g_1 satisfies* (7) *and g_2 satisfies* (11). *Let a and b be complex numbers with $b \neq 0, -1, -2, \dots$ and $\lambda, \gamma > 0$, $0 < q < 1$. If $\left(\frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z}\right)^{\gamma} \in \mathcal{H}[0, (\lambda-1)\gamma] \cap Q$ and $\psi_\lambda^{a,b,q}(\gamma, \mu, \rho, \chi, \tau; z)$ is as defined in* (8) *univalent in U, then*

$$\mu + \rho g_1(z) + \chi(g_1(z))^2 + \tau\frac{zg_1'(z)}{g_1(z)} \prec \psi_\lambda^{a,b,q}(\gamma, \mu, \rho, \chi, \tau; z) \prec \mu + \chi g_2(z) + \chi(g_2(z))^2 + \tau\frac{zg_2'(z)}{g_2(z)},$$

for $\mu, \rho, \chi, \tau \in \mathbb{C}$, $\tau \neq 0$, implies

$$g_1(z) \prec \left(\frac{D_z^{-\lambda}\phi(a,b;q,z)}{z}\right)^{\gamma} \prec g_2(z), \quad z \in U,$$

and g_1 and g_2 are, respectively, the best subordinant and the best dominant.

For $g_1(z) = \frac{1+M_1z}{1+N_1z}$, $g_2(z) = \frac{1+M_2z}{1+N_2z}$, where $-1 \leq N_2 < N_1 < M_1 < M_2 \leq 1$, we have the following corollary.

Corollary 2.8. *Let a and b be complex numbers with* $b \neq 0, -1, -2, \ldots$ *and* $\lambda, \gamma > 0$, $0 < q < 1$. *Assume that* (7) *and* (11) *hold. If* $\left(\frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z}\right)^{\gamma} \in \mathcal{H}[0,(\lambda-1)\gamma] \cap Q$ *and*

$$\mu + \rho\frac{1+M_1z}{1+N_1z} + \chi\left(\frac{1+M_1z}{1+N_1z}\right)^2 + \tau\frac{(M_1-N_1)z}{(1+M_1z)(1+N_1z)} \prec \psi_{\lambda}^{a,b,q}(\gamma,\mu,\rho,\chi,\tau;\ z)$$

$$\prec \mu + \rho\frac{1+M_2z}{1+N_2z} + \chi\left(\frac{1+M_2z}{1+N_2z}\right)^2 + \tau\frac{(M_2-N_2)z}{(1+M_2z)(1+N_2z)},$$

for $\mu,\rho,\chi,\tau \in \mathbb{C}$, $\tau \neq 0$, $-1 \leq N_2 \leq N_1 < M_1 \leq M_2 \leq 1$, *where* $\psi_{\lambda}^{a,b,q}$ *is defined in* (8), *then*

$$\frac{1+M_1z}{1+N_1z} \prec \left(\frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z}\right)^{\gamma} \prec \frac{1+M_2z}{1+N_2z},$$

and hence, $\frac{1+M_1z}{1+N_1z}$ *and* $\frac{1+M_2z}{1+N_2z}$ *are the best subordinant and the best dominant, respectively.*

Corollary 2.9. *Let a and b be complex numbers with* $b \neq 0, -1, -2, \ldots$ *and* $\lambda, \gamma > 0$, $0 < q < 1$. *Assume that* (7) *and* (11) *hold. If* $\left(\frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z}\right)^{\gamma} \in \mathcal{H}[0,(\lambda-1)\gamma] \cap Q$ *and*

$$\mu + \rho\left(\frac{1+z}{1-z}\right)^{\sigma_1} + \chi\left(\frac{1+z}{1-z}\right)^{2\sigma_1} + \tau\frac{2\sigma_1z}{1-z^2} \prec \psi_{\lambda}^{a,b,q}(\gamma,\mu,\rho,\chi,\tau;\ z)$$

$$\prec \mu + \rho\left(\frac{1+z}{1-z}\right)^{\sigma_2} + \chi\left(\frac{1+z}{1-z}\right)^{2\sigma_2} + \tau\frac{2\sigma_2z}{1-z^2},$$

for $\mu,\rho,\chi,\tau \in \mathbb{C}$, $0 < \sigma_1, \sigma_2 \leq 1$, $\tau \neq 0$, *where* $\psi_{\lambda}^{a,b,q}$ *is defined in* (8), *then*

$$\left(\frac{1+z}{1-z}\right)^{\sigma_1} \prec \left(\frac{D_z^{-\lambda}\phi(a,b;\ q,z)}{z}\right)^{\gamma} \prec \left(\frac{1+z}{1-z}\right)^{\sigma_2},$$

and hence, $\left(\frac{1+z}{1-z}\right)^{\sigma_1}$ *and* $\left(\frac{1+z}{1-z}\right)^{\sigma_2}$ *are the best subordinant and the best dominant, respectively.*

3 Conclusion

The results presented in this article are obtained as applications in the geometric function theory of q-calculus aspects combined with fractional calculus. Riemann-Liouville fractional integral and q-hypergeometric function are put together to obtain a new operator given in Definition 2.1. The means of differential subordiantion and superordination theories are involved in obtaining new subordination and superordination results concerning the new fractional q-hypergeometric operator introduced in the article. In Theorem 2.1 regarding subordination theory, the best dominant of the differential subordination is provided, and using specific functions well known due to their geometric properties as best dominant, nice corollaries are stated. Similarly, for the differential superordination proved in Theorem 2.4, the best subordinant is found and interesting corollaries are obtained by using particular functions that are known to have nice geometric properties. By using Theorems 2.1 and 2.4, a sandwich-type theorem connects the results regarding the two dual theories of differential subordination and superordination. Corollaries are obtained for the sandwich-type theorem when certain functions are involved as best subordinant and best dominant.

For future studies, q-subclasses of univalent functions could be defined using the new fractional q-hypergeometric operator introduced in Definition 2.1. Univalence conditions for this operator could be investigated using applications of the best dominant of the differential subordination contained in Theorem 2.1 or of the best subordinant of the differential superordination found in Theorem 2.4. Also, having as inspiration the results presented here, Riemann-Liouville fractional integral could be applied to other q-calculus operators or functions for defining new operators.

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